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Research article

Analysis of stress-strength reliability with m-step strength levels under type I censoring and Gompertz distribution

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Abstract: Because of modern technology, product reliability has increased, making it more challenging to evaluate products in real-world settings and raising the cost of gathering sufficient data about a product's lifetime. Instead of using stress to accelerate failures, the most practical way to solve this problem is to use accelerated life tests, in which test units are subjected to varying degrees of stress. This paper deals with the analysis of stress-strength reliability when the strength variable has changed m levels at predetermined times. It is common for the observed failure time data of items to be partially unavailable in numerous reliability and life-testing studies. In statistical analyses where data is censored, lowering the time and expense involved is vital. Maximum likelihood estimation when the stress and strength variables follow the Gompertz distribution was introduced under type I censoring data. The bootstrap confidence intervals were deduced for stress-strength reliability under m levels of strength variable and applying the Gompertz distribution to model time. A simulation study was introduced to find the maximum likelihood estimates, bootstrapping, and credible intervals for stress-strength reliability. Real data was presented to show the application of the model in real life.

Keywords: stress-strength reliability; Gompertz distribution; maximum likelihood estimation; type I censoring; bootstrap interval; Bayesian estimation; credible interval; MCMC method; Akaike information criterion; Bayesian information criterion; Hannan-Quinn information criterion; consistent AIC; Kolmogorov-Smirnov test
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Symbols and abbreviations: X: strength variable; Y: stress variable; m: levels of strength variable;

τ_k: predetermined time points for the step strength levels; ϑ_j: scale parameter of the Gompertz distribution at strength level s_j; λ: shape parameter of the Gompertz distribution; ϑ_y: scale parameter of the Gompertz distribution in case of the stress variable; R: stress-strength reliability; n: total number of the units in case of the strength variable; r: number of the censored units in case of the strength variable; l: total number of the units in case of the stress variable; h: number of the censored units in case of the stress variable; L: likelihood function; $\log L$: logarithm of the likelihood function; MLE: maximum likelihood estimate; MSE: mean squared error; MCMC: Markov chain Monte Carlo; $\hat{R}_{MLE}$: maximum likelihood estimate of the stress-strength reliability; $\tilde{R}_{Bayes}$: Bayesian estimate of the stress-strength reliability; P.B.I: parametric bootstrap interval; N.B.I: nonparametric bootstrap interval; AIC: Akaike information criterion; BIC: Bayesian information criterion; HQIC: Hannan-Quinn information criterion; CAIC: consistent Akaike information criterion; K-S: Kolmogorov-Smirnov test

1. Introduction

Stress-strength models are used in critical tasks in many fields, including engineering, mechanics, computer science, and quality control. The reliability of a component or system with strength X subjected to random stress Y can be defined as the probability of the strength exceeding the stress, i.e., X > Y. The estimation of a component's reliability characteristics is critical in this setup, aiding in the evaluation of the efficiency of a product's operation process and allowing to take precautions to avoid interruptions in the production process. Extensive work has been carried out in recent years related to the problem of estimating reliability with different sampling schemes and distributions for (X, Y).

In survival analysis, there are tools to handle missing data in experiments. There are many approaches that deal with partial information or censored data, such as type I and type II censoring. In type I censoring, the experiment will end after a fixed time. On the other hand, in type II censoring, the experiment will stop after a predetermined number of failures.

In systems with highly reliable components that have very long lifetimes under typical test conditions, it is difficult to observe the lifetime of these components since very few failures occur during the short period of time of testing. Testing the systems under normal conditions takes a long time, so the development of partially accelerated life tests and accelerated life testing is necessary. In these tests, units are subjected to a more severe environment (increased or decreased levels) than the normal operating environment in order to quickly induce failures. In this instance, higher stress levels can be controlled by experimenters using accelerated life tests or partially accelerated life tests.

Çetinkaya [1] discussed the stress-strength reliability model with component strength under a partially accelerated life test. Yousef et al. [2] introduced a parametric inference on partially accelerated life testing for the inverted Kumaraswamy distribution based on Type II progressive censoring data. Yousef et al. [3] examined simulation techniques for the partially accelerated strength component to analyze the stress-strength model. Akgul et al. [4] examined classical and Bayesian estimations of step-stress partially accelerated life testing in the case of inverse Weibull lifetime distribution based on type-I censoring. Pandey et al. [5] discussed the estimation procedure for step-stress partially accelerated life testing based on the generalized progressive hybrid censoring scheme. Pathak et al. [6] considered the estimation problem in step-stress partially accelerated life testing of Maxwell-Boltzmann distribution in the presence of progressive type-II censoring with binomial removals. Abd-Elfattah et al. [7] presented an estimation in step-stress partially accelerated life tests for the Burr type XII distribution using type I censoring. Rahman et al. [8] introduced a statistical analysis for type-I progressive hybrid censored data from Burr type XII distribution under a step-stress partially accelerated life test model. Sarhan and Tolba [9] considered the inference of stress-strength reliability when X and Y are independent random variables from two-parameter Weibull distributions, and the strength variable X is subjected to step-stress partially accelerated life testing. El-Sagheer et al. [10] estimated the stress-strength reliability model when the strength variable is subjected to the step-stress partially accelerated life test based on the Weibull distribution. Kamal et al. [11] estimated the parameters in step-stress partially accelerated life testing under different types of censored data. Nassar et al. [12] presented an analysis of Burr type XII distribution under step-stress partially accelerated life tests with type I and adaptive type II progressively hybrid censoring schemes. Abd-Elfattah et al. [13] introduced the Bayesian estimation in step-stress partially accelerated life tests for the Burr type XII parameters using type I censoring. Alrashidi et al. [14] discussed the constant-stress partially accelerated life tests with unified hybrid censored data under exponentiated gamma distribution.

Bhattacharyya and Soejoeti [15] proposed the tampered failure rate model, which generalized the simple two-step stress model to the multiple k-step (k > 2) stress model. In multiple-step-stress life testing experiments, n units are simultaneously put on test at a time $t_0 = 0$ to a stress level s_1. At time $t_1 > 0$, the surviving units are subjected to a higher level of stress s_2 in the time interval $[t_1, t_2)$. At time t_2, the stress is increased on the surviving units to s_3 over the time interval $[t_2, t_3)$ and so on until the k-th and last time interval $[t_{k-1}, \infty)$, where the remaining units are subjected to s_k until they all fail (Madi [16]). Bobotas and Kateri [17] discussed the step-stress tampered failure rate model under interval monitoring. Koley et al. [18] presented a parametric analysis of a tampered random variable model for multiple-step-stress life test.

The Gompertz distribution is a significant and commonly used lifetime distribution, which plays an important role in reliability engineering. Lv et al. [19] introduced a statistical inference for Gompertz distribution under adaptive type II progressive hybrid censoring. The probability density function and the cumulative distribution function of a random variable X following Gompertz distribution are, respectively, given by

$$f(x) = \vartheta e^{\lambda x-(\vartheta/\lambda)(e^{\lambda x}-1)},\ x > 0, \vartheta, \lambda > 0.$$

$$F(x) = 1 - e^{-(\vartheta/\lambda)(e^{\lambda x}-1)},\ x > 0, \vartheta, \lambda > 0.$$

In this paper, an analysis of stress-strength reliability is presented, assuming that the strength variable is subjected to m-step levels. The stress and strength variables are assumed to be independent and follow a Gompertz distribution. The maximum likelihood estimation of the stress-strength reliability under m levels of strength is deduced under type I censoring data. Algorithms for the bootstrap and credible intervals for stress-strength reliability are introduced. A simulation study is introduced to find the maximum likelihood estimates, bootstrapping, and credible intervals for stress-strength reliability. Real data is presented to show the application of the model in real life.

2. Model assumption

In a testing model with m levels of the strength variable, the units are put on an initial strength level, s_1, and at a pre-specified time, τ_0. Then, strength levels are changed to s_2 at the pre-specified point time τ_1 and so on, where $\tau_0 < \tau_1 < \cdots$ and $s_1 < s_2 < \cdots$. According to the considerations that the model will have m-step strength levels and assuming that $\tau_0, \tau_1, \tau_2, \dots, \tau_{m-1}, \tau_m$, where $\tau_0 = 0 <$

$\tau_1 < \tau_2 < \cdots < \tau_{m-1} < \tau_m = \infty$ are the pre-specified times for the m-step strength levels, then the cumulative distribution function for the model will be given as follows:

$$F(x) = \begin{cases} F_1(x), x \le \tau_1, \\ 1 - \prod_{j=1}^{k-1} \frac{1-F_j(x)}{1-F_{j+1}(x)} \left(1 - F_k(x)\right), \tau_{k-1} < x < \tau_k, 2 \le k \le m. \end{cases}$$

Now, using the Gompertz distribution to model the time, it yields:

$$F(x) = \begin{cases} 1 - e^{-\frac{\vartheta_1}{\lambda}(e^{\lambda x}-1)}, x \le \tau_1, \\ 1 - \prod_{j=1}^{k-1} \frac{e^{-\frac{\vartheta_j}{\lambda}\left(e^{\lambda \tau_j}-1\right)}}{e^{-\frac{\vartheta_{j+1}}{\lambda}\left(e^{\lambda \tau_j}-1\right)}} e^{-\frac{\vartheta_k}{\lambda}(e^{\lambda x}-1)}, \tau_{k-1} < x < \tau_k, 2 \le k \le m. \end{cases}$$

After simplifying, it yields:

$$F(x) = \begin{cases} 1 - e^{-\frac{\vartheta_1}{\lambda}(e^{\lambda x}-1)}, x \le \tau_1, \\ 1 - e^{-\left[\sum_{j=1}^{k-1}\left(\frac{\vartheta_j - \vartheta_{j+1}}{\lambda}\right)\left(e^{\lambda \tau_j}-1\right) + \frac{\vartheta_k}{\lambda}(e^{\lambda x}-1)\right]}, \tau_{k-1} < x < \tau_k, 2 \le k \le m, \end{cases}$$

and hence, the probability density function can be obtained as:

$$f(x) = \begin{cases} 1 - e^{-\frac{\vartheta_1}{\lambda}(e^{\lambda x}-1)}, x \le \tau_1, \\ 1 - e^{-\left[\sum_{j=1}^{k-1}\left(\frac{\vartheta_j - \vartheta_{j+1}}{\lambda}\right)\left(e^{\lambda \tau_j}-1\right) + \frac{\vartheta_k}{\lambda}(e^{\lambda x}-1)\right]}, \tau_{k-1} < x < \tau_k, 2 \le k \le m. \end{cases}$$

3. Stress-strength reliability with m-step strength levels

The stress-strength reliability of the model under the m-step level of the strength variable will be deduced as follows:

$$\begin{aligned} R &= \int_0^\infty f_x(x) F_y(x) dx \\ &= \int_0^{\tau_1} f_1(x) F_y(x) dx + \int_{\tau_1}^{\tau_2} f_2(x) F_y(x) dx + \cdots + \int_{\tau_{k-1}}^{\tau_k} f_k(x) F_y(x) dx + \cdots + \int_{\tau_{m-1}}^{\infty} f_2(x) F_y(x) dx. \end{aligned}$$

After calculations and since X and Y follow the Gompertz distribution, stress-strength reliability will take the following formula:

$$\begin{aligned} R = \sum_{k=1}^{m} e^{-\sum_{j=0}^{k-1} \delta_j \left(\frac{\vartheta_j - \vartheta_{j+1}}{\lambda}\right)\left(e^{\lambda \tau_j}-1\right)} \Bigg\{ & e^{-\frac{\vartheta_k}{\lambda}\left(e^{\lambda \tau_{k-1}}-1\right)} - e^{-\frac{\vartheta_k}{\lambda}\left(e^{\lambda \tau_k}-1\right)} \\ & - \left(\frac{\vartheta_k}{\vartheta_k + \vartheta_y}\right)\left[e^{-\left(\frac{\vartheta_k+\vartheta_y}{\lambda}\right)\left(e^{\lambda \tau_{k-1}}-1\right)} - e^{-\left(\frac{\vartheta_k+\vartheta_y}{\lambda}\right)\left(e^{\lambda \tau_k}-1\right)}\right]\Bigg\}, \end{aligned} \tag{1}$$

where

$$\delta_j = \begin{cases} 1, j = 0, \\ 0, j \neq 0, \end{cases} for\ \tau_0 = 0, \vartheta_0 = 0, \tau_m = \infty.$$

4. Maximum likelihood estimation for the stress-strength reliability

The likelihood function for the stress-strength variables X and Y under type I censoring and in case of the step life testing on the strength variable can be formulated as follows:

$$L = \frac{n!}{(n-r)!} \prod_{j=1}^{m} \vartheta_j^{n_j} \prod_{j=1}^{m} \left(\prod_{i=\bar{n}_{j-1}+1}^{\bar{n}_j} f_j(x_i) \right) [1 - F_m(\tau_m)]^{n-r} \frac{l!}{(l-h)!} \prod_{j=1}^{h} f_y(y_j) \left[1 - F_y(\varepsilon)\right]^{l-h},$$

where

$$\bar{n}_j = \sum_{i=1}^{j} n_i \,, 0 < x_1 < x_2 < \cdots < x_r < \tau_m, \quad 0 < y_1 < y_2 < \cdots < y_h < \varepsilon.$$

After applying the Gompertz distribution, the following formula will be obtained:

$$L = \frac{n!}{(n-r)!} \prod_{j=1}^{m} \vartheta_j^{n_j} e^{-\sum_{j=1}^{m} \sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j} \left[\frac{\vartheta_j}{\lambda} (e^{\lambda x_k} - 1) - \lambda x_k \right]} e^{-\sum_{j=1}^{m} \frac{\vartheta_j}{\lambda} \left[(n-\bar{n}_j)\left(e^{\lambda \tau_j} - 1\right) - (n-\bar{n}_{j-1})\left(e^{\lambda \tau_{j-1}} - 1\right) \right]},$$

$$\frac{l!}{(l-h)!} \vartheta_y^h e^{-\sum_{j=1}^{h} \left[\frac{\vartheta_y}{\lambda}\left(e^{\lambda y_j} - 1\right) - \lambda y_j \right]} e^{-(l-h)\frac{\vartheta_y}{\lambda}(e^{\lambda \varepsilon} - 1)}.$$

The logarithm of the likelihood function is applied, and the result will be given by

$$\begin{aligned} \log L = \log \frac{n!}{(n-r)!} + n_j \sum_{j=1}^{m} \log \vartheta_j - \sum_{j=1}^{m} \sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j} \left[\frac{\vartheta_j}{\lambda} \left(e^{\lambda x_k} - 1\right) - \lambda x_k \right] \\ - \sum_{j=1}^{m} \frac{\vartheta_j}{\lambda} \left[(n - \bar{n}_j)\left(e^{\lambda \tau_j} - 1\right) - (n - \bar{n}_{j-1})\left(e^{\lambda \tau_{j-1}} - 1\right) \right] \\ + \log \frac{l!}{(l-h)!} + h \log \vartheta_y - \sum_{j=1}^{h} \left[\frac{\vartheta_y}{\lambda}\left(e^{\lambda y_j} - 1\right) - \lambda y_j \right] - (l-h)\frac{\vartheta_y}{\lambda}\left(e^{\lambda \varepsilon} - 1\right). \end{aligned}$$

Differentiating $\log L$ with respect to the parameters $\vartheta_j, j = 1, \ldots, m, \lambda, \vartheta_y$ gives

$$\frac{\partial \log L}{\partial \vartheta_j} = \frac{n_j}{\vartheta_j} - \sum_{j=1}^{m} \sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j} \frac{1}{\lambda}\left(e^{\lambda x_k} - 1\right) - \sum_{j=1}^{m} \frac{1}{\lambda} \left[(n - \bar{n}_j)\left(e^{\lambda \tau_j} - 1\right) - (n - \bar{n}_{j-1})\left(e^{\lambda \tau_{j-1}} - 1\right) \right],$$

$$j = 1, \ldots m,$$

$$\frac{\partial \log l}{\partial \vartheta_y} = \frac{h}{\vartheta_y} - \sum_{j=1}^{h} \frac{1}{\lambda}\left(e^{\lambda y_j} - 1\right) - (l-h)\frac{1}{\lambda}\left(e^{\lambda \varepsilon} - 1\right).$$

After equating to zero, the following formulas are obtained:

$$\hat{\vartheta}_j = \frac{\lambda n_j}{\sum_{j=1}^{m}\sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j}\left(e^{\lambda x_k}-1\right)+\sum_{j=1}^{m}\left[(n-\bar{n}_j)\left(e^{\lambda\tau_j}-1\right)-(n-\bar{n}_{j-1})\left(e^{\lambda\tau_{j-1}}-1\right)\right]}, j = 1, \dots m, \tag{2}$$

$$\hat{\vartheta}_y = \frac{\lambda h}{\sum_{j=1}^{h}\left(e^{\lambda y_j}-1\right)+(l-h)\left(e^{\lambda\varepsilon}-1\right)}, \tag{3}$$

$$\begin{aligned}\frac{\partial \log l}{\partial \lambda} = &\sum_{j=1}^{m}\sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j}\left[\frac{\vartheta_j}{\lambda^2}\left(e^{\lambda x_k}-1\right)-x_k\left(\frac{\vartheta_j}{\lambda}e^{\lambda x_k}-1\right)\right] \\ &+\sum_{j=1}^{m}\left\{\frac{\vartheta_j}{\lambda^2}\left[(n-\bar{n}_j)\left(e^{\lambda\tau_j}-1\right)-(n-\bar{n}_{j-1})\left(e^{\lambda\tau_{j-1}}-1\right)\right]\right. \\ &\left.-\frac{\vartheta_j}{\lambda}\left[(n-\bar{n}_j)\tau_j\left(e^{\lambda\tau_j}-1\right)-(n-\bar{n}_{j-1})\tau_{j-1}\left(e^{\lambda\tau_{j-1}}-1\right)\right]\right\} \\ &+\sum_{j=1}^{h}\left[\frac{\vartheta_y}{\lambda^2}\left(e^{\lambda y_j}-1\right)-y_j\left(\frac{\vartheta_y}{\lambda}e^{\lambda y_j}-1\right)\right]+(l-h)\frac{\vartheta_y}{\lambda^2}\left(e^{\lambda\varepsilon}-1\right).\end{aligned} \tag{4}$$

It can be noticed that no analytical solutions can be found for the parameter λ; therefore, the Newton-Raphson method will be used with the aid of R software [20]. Then, the maximum likelihood estimate for the step-stress strength reliability of the model is obtained by substituting in Eq (1).

5. Bootstrap intervals for the stress-strength reliability

Efron and Tibshirani [21] presented an introduction to the bootstrap. The steps for calculating parametric and nonparametric bootstrap intervals for the stress-strength reliability with m-step level of the strength variable are explained in the following subsections.

5.1. Parametric bootstrap interval

To obtain the parametric bootstrap interval for the stress-strength reliability, the following steps can be followed:

Step 1. From given samples $(x_1, \dots, x_r)$ and $(y_1, \dots, y_h)$, compute the estimates $\hat{\vartheta}_j, j = 1, \dots, m, \hat{\vartheta}_y, \hat{\lambda}$ of $\vartheta_j, j = 1, \dots, m, \lambda, \vartheta_y$.

Step 2. Generate a bootstrap sample of size r $(x_1^*, \dots, x_r^*)$ from the Gompertz distribution with parameters $\hat{\vartheta}_j, j = 1, \dots, m, \hat{\lambda}$ and generate a bootstrap sample of size h $(y_1^*, \dots, y_h^*)$ from Gompertz distribution with parameters $\hat{\vartheta}_y, \hat{\lambda}$.

Step 3. Compute the estimates $\hat{\vartheta}_j^* (for\ j = 1, \dots, m), \hat{\vartheta}_y^*, \hat{\lambda}^*$ of $\vartheta_j, j = 1, \dots, m, \lambda, \vartheta_y$ and then compute the bootstrap estimates $\hat{R}^*$ of R using Eq (1).

Step 4. Repeat Steps 2 and 3 10000 times to obtain a set of bootstrap samples of $\hat{R}^*$

$$\left\{\hat{R}^{*(j)}, j = 1, \dots, 10000\right\}.$$

Step 5. Rearrange $\hat{R}^{*(j)}, j = 1, \dots, 10000$ in ascending order such that $\hat{R}^{*(1)} < \dots < \hat{R}^{*(10000)}$. A

$100(1-\alpha)\%$ bootstrap confidence interval will be given by:

$$\left[\hat{R}^{*10000(\alpha/2)}, \hat{R}^{*10000(1-\alpha/2)}\right].$$

5.2. Nonparametric bootstrap interval

To obtain the nonparametric bootstrap interval for the stress-strength reliability, the following steps can be followed:

Step 1. From initial samples $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_h)$, generate new samples $(x_1^*, \ldots, x_r^*)$ and $(y_1^*, \ldots, y_h^*)$ by sampling with replacement.

Step 2. Compute the estimates $\hat{\vartheta}_j^*(for\ j = 1, \ldots, m), \hat{\vartheta}_y^*, \hat{\lambda}^*$ and then compute the bootstrap estimates $\hat{R}^*$ of R using Eq (1).

Step 3. Repeat Steps 1 and 2, 10,000 times to obtain a set of bootstrap samples of R

$$\left\{\hat{R}^{*(j)}, j = 1, \ldots, 10000\right\}.$$

Step 4. Rearrange $\hat{R}^{*(j)}, j = 1, \ldots, 10000$ in ascending order such that $\hat{R}^{*(1)} < \cdots < \hat{R}^{*(10000)}$. A $100(1-\alpha)\%$ bootstrap confidence interval will be given by:

$$\left[\hat{R}^{*10000(\alpha/2)}, \hat{R}^{*10000(1-\alpha/2)}\right].$$

5.3. Bayesian credible interval

Chen and Shao [22] introduced a Monte Carlo estimation of Bayesian credible and HPD intervals. Al-Babtain et al. [23] presented the Bayesian and non-Bayesian reliability estimation of stress-strength model for power-modified Lindley distribution. In order to obtain the Bayesian credible interval for the stress-strength reliability, the parameters $\vartheta_j, j = 1, \ldots, m, \lambda, \vartheta_y$ are assumed to be independent random variables with prior distributions that follow a gamma distribution, as follows:

$$\vartheta_j \sim Gamma(\rho_j, \eta_j), j = 1, \ldots, m,$$

$$\vartheta_y \sim Gamma(\sigma, \upsilon),$$

$$\lambda \sim Gamma(\omega, \zeta).$$

The joint prior density function of the parameters $\vartheta_j, j = 1, \ldots, m, \lambda, \vartheta_y$ can be written as:

$$\pi(\vartheta_j, \lambda, \vartheta_y) \propto \prod_{j=1}^{m}\left(\vartheta_j^{\rho_j - 1} e^{-\eta_j \vartheta_j}\right) \vartheta_y^{\sigma-1} e^{-\upsilon\vartheta_y} \lambda^{\omega-1} e^{-\zeta\lambda}.$$

The joint posterior density function of $\vartheta_j, j = 1, \ldots, m, \vartheta_y\ and\ \lambda$ given the data (x, y) is given by:

$$\pi(\vartheta_j, \lambda, \vartheta_y | x, y) \propto \pi(\vartheta_j, \lambda, \vartheta_y) L(\vartheta_j, \lambda, \vartheta_y | x, y),$$

$$\pi(\vartheta_j, \lambda, \vartheta_y | x, y) \propto$$

$$\prod_{j=1}^{m}\left(\vartheta_j^{\rho_j-1} e^{-\eta_j\vartheta_j}\right) \vartheta_y^{\sigma-1} e^{-\upsilon\vartheta_y} \lambda^{\omega-1} e^{-\zeta\lambda} \prod_{j=1}^{m} \vartheta_j^{n_j}\, e^{-\sum_{j=1}^{m}\sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j}\left[\frac{\vartheta_j}{\lambda}\left(e^{\lambda x_k}-1\right)-\lambda x_k\right]},$$

$$e^{-\sum_{j=1}^{m}\frac{\vartheta_j}{\lambda}\left[(n-\bar{n}_j)\left(e^{\lambda\tau_j}-1\right)-(n-\bar{n}_{j-1})\left(e^{\lambda\tau_{j-1}}-1\right)\right]} \vartheta_y^{h} e^{-\sum_{j=1}^{h}\left[\frac{\vartheta_y}{\lambda}\left(e^{\lambda y_j}-1\right)-\lambda y_j\right]} e^{-(l-h)\frac{\vartheta_y}{\lambda}\left(e^{\lambda\varepsilon}-1\right)}$$

$$\propto \prod_{j=1}^{m} \vartheta_j^{n_j+\rho_j-1} e^{-\sum_{j=1}^{m}\vartheta_j\left\{\eta_j+\frac{(n-\bar{n}_j)\left(e^{\lambda\tau_j}-1\right)-(n-\bar{n}_{j-1})\left(e^{\lambda\tau_{j-1}}-1\right)+\sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j}\left(e^{\lambda x_k}-1\right)}{\lambda}\right\}}$$

$$\vartheta_y^{h+\sigma-1} e^{-\vartheta_y\left\{\upsilon+\frac{\sum_{j=1}^{h}\left(e^{\lambda y_j}-1\right)+(l-h)\left(e^{\lambda\varepsilon}-1\right)}{\lambda}\right\}} \lambda^{\omega-1} e^{-\lambda\left\{\zeta-\sum_{j=1}^{m}\sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j} x_k-\sum_{j=1}^{h} y_j\right\}}.$$

It can be observed that the marginal posterior distributions of $\vartheta_j, j = 1, \dots, m, \vartheta_y$ *and* λ are given as:

$$\pi(\vartheta_j|\lambda, x) \propto \text{gamma}\left(n_j + \rho_j, \eta_j + \frac{(n-\bar{n}_j)\left(e^{\lambda\tau_j}-1\right)-(n-\bar{n}_{j-1})\left(e^{\lambda\tau_{j-1}}-1\right)+\sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j}\left(e^{\lambda x_k}-1\right)}{\lambda}\right),$$

$$\pi(\vartheta_y|\lambda, y) \propto \text{gamma}\left(h + \sigma, \upsilon + \frac{\sum_{j=1}^{h}\left(e^{\lambda y_j}-1\right)+(l-h)\left(e^{\lambda\varepsilon}-1\right)}{\lambda}\right),$$

$$\pi(\lambda|x, y) \propto \text{gamma}\left(\omega, \zeta - \sum_{j=1}^{m}\sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j} x_k - \sum_{j=1}^{h} y_j\right).$$

The Markov chain Monte Carlo simulation method will be given to obtain the Bayesian credible interval of stress-strength reliability as follows:

Step 1. Set initial values $\vartheta_j^0, j = 1, \dots, m, \vartheta_y^0$ *and* λ^0 for the parameters $\vartheta_j, j = 1, \dots, m, \vartheta_y$ *and* λ.

Step 2. Generate ϑ_j from

$$\text{gamma}\left(n_j + \rho_j, \eta_j + \frac{(n-\bar{n}_j)\left(e^{\lambda\tau_j}-1\right)-(n-\bar{n}_{j-1})\left(e^{\lambda\tau_{j-1}}-1\right)+\sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j}\left(e^{\lambda x_k}-1\right)}{\lambda}\right), \text{ for } j = 1, \dots, m.$$

Step 3. Generate ϑ_y from $\text{gamma}\left(h + \sigma, \upsilon + \frac{\sum_{j=1}^{h}\left(e^{\lambda y_j}-1\right)+(l-h)\left(e^{\lambda\varepsilon}-1\right)}{\lambda}\right)$.

Step 4. Generate λ from $\text{gamma}\left(\omega, \zeta - \sum_{j=1}^{m}\sum_{k=\bar{n}_{j-1}+1}^{\bar{n}_j} x_k - \sum_{j=1}^{h} y_j\right)$.

Step 5. Compute stress-strength reliability $\tilde{R}_{Bayes}$ using Eq (1).

Step 6. Repeat Steps 2–5 10,000 times.

Step 7. Sort the obtained values of $\tilde{R}_{Bayes}$ in an ascending order such that $\tilde{R}_{Bayes}^{*(1)} < \cdots < \tilde{R}_{Bayes}^{*(10000)}$.

Step 8. The $100(1-\alpha)\%$ credible interval for R is given as $\left[\tilde{R}_{Bayes}^{*(10000\alpha/2)}, \tilde{R}_{Bayes}^{*(10000(1-\alpha/2))}\right]$.

6. Simulation study

The steps that are used to make a simulation to calculate the maximum likelihood estimate for the stress-strength reliability of the model in the R software program are explained below:

Step 1. Set the initial values for n, r, l, h, ε.

Step 2. Set the initial values of parameters $\vartheta_j, j = 1, \dots, m, \lambda, \vartheta_y$.

Step 3. Set the predetermined points $\tau_1, \tau_2, \ldots, \tau_{m-1}$.

Step 4. Choose $n_j, j = 1, \ldots, m$, where $\sum_{j=1}^{m} n_j = r$.

Step 5. Generate the random values of the random variables X_k by applying the inversion formula as follows:

$$X_k = \tau_{k-1} + \frac{\ln\left[1 - \frac{\lambda}{\vartheta_k}\ln(1-u_k)\right]}{\lambda}, 0 < u_k < 1, k = 1,2, \ldots, m.$$

Step 6. Generate the random values of the random variables Y by applying the inversion formula as follows:

$$Y = \frac{\ln\left[1 - \frac{\lambda}{\vartheta_y}\ln(1-u)\right]}{\lambda}, \quad 0 < u < 1.$$

Step 7. Use the Newton-Raphson method in R to solve Eqs (2)–(4) and find the estimates for the parameters $\vartheta_j, j = 1, \ldots, m, \vartheta_y$ and λ.

Step 8. Substitute in Eq (1) to find the maximum likelihood estimate for the stress-strength reliability of the model.

Step 9. Repeat Steps 5–8 10,000 times.

Step 10. Calculate the bias and mean squared error (MSE) from the following relations:

$$bias = \frac{\sum_{i=1}^{10000}\left(\hat{R}_i - R_{true}\right)}{10000},$$

$$MSE = \frac{\sum_{i=1}^{10000}(\hat{R}_i - R_{true})^2}{10000}.$$

The simulation is performed where the experiment is repeated 10,000 times assuming five levels $(m = 5)$ of strength. Different values for (n, r) and (l, h) are assumed and the values for $n_j, j = 1, 2, 3, 4, 5$ will be given as follows:
For $n = 10, r = 5$: $n_1 = 1, n_2 = 1, n_3 = 1, n_4 = 1, n_5 = 1$.
For $n = 30, r = 15$: $n_1 = 4, n_2 = 3, n_3 = 3, n_4 = 2, n_5 = 3$.
For $n = 50, r = 30$: $n_1 = 8, n_2 = 7, n_3 = 4, n_4 = 6, n_5 = 5$.

Tables 1 and 2 show the results for the maximum likelihood estimates $\hat{R}_{MLE}$, with bias and the MSE, and Bayesian estimates $\tilde{R}_{Bayes}$ for stress-strength reliability. Also, the results for the parametric (P.B.I), nonparametric bootstrap (N.B.I), and credible intervals for stress-strength reliability with m-step levels of strength are obtained with the interval lengths. In Table 1, the following initial values for the parameters are assumed: $\vartheta_1 = 1.5$, $\vartheta_2 = 2.5$, $\vartheta_3 = 4.5$, $\vartheta_4 = 1.5$, $\vartheta_5 = 2.5$, $\lambda = 1.5$, $\vartheta_y = 3.5$. Also, the following predetermined points are assumed: $\tau_1 = 1.5$, $\tau_2 = 3.5$, $\tau_3 = 5$, $\tau_4 = 10$, $\tau_5 = 15$ and $\varepsilon = 10$. The true value of stress-strength reliability is obtained as R_{true} =0.70000. For Bayesian estimation, the following values are assumed: $\rho_j = \eta_j = 0.5\ (for\ j = 1,2,3,4,5)$, $\omega = 1000$, $\zeta = \sigma = \upsilon = 0.5$.

In Table 2, the following initial values for the parameters are assumed: $\vartheta_1 = 0.01, \vartheta_2 = 0.01, \vartheta_3 = 0.04, \vartheta_4 = 0.02, \vartheta_5 = 0.05, \lambda = 0.05, \vartheta_y = 0.03$. Also, the following predetermined points are assumed: $\tau_1 = 50, \tau_2 = 100, \tau_3 = 150, \tau_4 = 200, \tau_5 = 250$ and $\varepsilon = 100$. The true value of stress-strength reliability is obtained as R_{true} =0.75000. For Bayesian estimation, the following values are assumed: $\rho_j = \eta_j = 1\ (for\ j = 1,2,3,4,5), \omega = 5000, \zeta = \sigma = \upsilon = 1$.

Table 1. Results for $\widehat{R}_{MLE}$ (with bias and MSE), parametric and nonparametric bootstrap intervals, $\widetilde{R}_{Bayes}$, and credible intervals.

$(\boldsymbol{n}, \boldsymbol{r})$	$(\boldsymbol{l}, \boldsymbol{h})$	$\widehat{\boldsymbol{R}}_{MLE}$	**Bias**	**MSE**	**P.B.I**	**N.B.I**	$\widetilde{\boldsymbol{R}}_{Bayes}$	**Credible interval**
(10, 5)	(10, 5)	0.55837	-0.13570	0.01951	[0.51140, 0.61522] (0.10382)	[0.51840, 0.62405] (0.10565)	0.56277	[0.33955, 0.76700] (0.42744)
	(30, 15)	0.54920	-0.11974	0.01514	[0.52902, 0.62341] (0.09438)	[0.52661, 0.61136] (0.08474)	0.61379	[0.37817, 0.74349] (0.36532)
	(50, 30)	0.68997	-0.00521	0.00053	[0.65874, 0.73055] (0.07180)	[0.42389, 0.47042] (0.04653)	0.701835	[0.46673, 0.79611] (0.32938)
(30, 15)	(10, 5)	0.53344	-0.16538	0.02892	[0.46417, 0.59537] (0.13119)	[0.47524, 0.58896] (0.11371)	0.58951	[0.40268, 0.74675] (0.34407)
	(30, 15)	0.56225	-0.14830	0.02349	[0.48454, 0.61203] (0.12749)	[0.49921, 0.60207] (0.10286)	0.59599	[0.45900, 0.72189] (0.26289)
	(50, 30)	0.67188	-0.03612	0.00169	[0.63037, 0.69389] (0.06352)	[0.69675, 0.72519] (0.02844)	0.74106	[0.55146, 0.77515] (0.22369)
(50, 30)	(10, 5)	0.49790	-0.16735	0.04063	[0.39587, 0.75281] (0.35693)	[0.43598, 0.68031] (0.24433)	0.53423	[0.35848, 0.68989] (0.33140)
	(30, 15)	0.44917	-0.16261	0.02828	[0.46520, 0.60883] (0.14363)	[0.46587, 0.61032] (0.14444)	0.54051	[0.41547, 0.65437] (0.23889)
	(50, 30)	0.64868	-0.05131	0.00348	[0.59939, 0.69709] (0.09770)	[0.59800, 0.69438] (0.09638)	0.65250	[0.51476, 0.71140] (0.19664)

Table 2. Results for $\hat{R}_{MLE}$ (with bias and MSE), parametric and nonparametric bootstrap intervals, $\tilde{R}_{Bayes}$, and credible intervals.

(n, r)	(l, h)	$\hat{R}_{MLE}$	Bias	MSE	P.B.I	N.B.I	$\tilde{R}_{Bayes}$	Credible interval
(10, 5)	(10, 5)	0.52316	-0.18581	0.03675	[0.50221, 0.64902] (0.14681)	[0.50659, 0.65212] (0.14552)	0.67542	[0.44761, 0.85734] (0.40973)
	(30, 15)	0.55960	-0.17929	0.03285	[0.53191, 0.615036] (0.08312)	[0.51090, 0.61376] (0.10286)	0.62406	[0.48694, 0.81526] (0.32831)
	(50, 30)	0.67632	-0.08581	0.00799	[0.62353, 0.70331] (0.07978)	[0.60602, 0.70801] (0.10199)	0.78835	[0.58226, 0.86349] (0.28122)
(30, 15)	(10, 5)	0.48638	-0.21426	0.05067	[0.45338, 0.65881] (0.20542)	[0.44763, 0.68096] (0.23333)	0.66799	[0.54292, 0.86389] (0.32097)
	(30, 15)	0.61415	-0.18407	0.03556	[0.51074, 0.63991] (0.12916)	[0.50317, 0.64186] (0.13869)	0.716726	[0.58600, 0.82707] (0.24106)
	(50, 30)	0.62683	-0.10142	0.01136	[0.59488, 0.69984] (0.10496)	[0.59547, 0.70743] (0.11196)	0.73061	[0.67918, 0.86517] (0.18598)
(50, 30)	(10, 5)	0.55550	-0.16434	0.04781	[0.42957, 0.90070] (0.47112)	[0.55121, 0.71645] (0.16523)	0.76049	[0.51044, 0.84084] (0.33040)
	(30, 15)	0.54872	-0.13861	0.02188	[0.51955, 0.68331] (0.16375)	[0.46701, 0.63014] (0.16313)	0.55126	[0.57270, 0.79976] (0.22706)
	(50, 30)	0.60591	-0.05507	0.00548	[0.61033, 0.76616] (0.15582)	[0.59318, 0.71200] (0.11882)	0.71932	[0.67168, 0.83758] (0.16589)

From the results obtained in Tables 1 and 2, it can be observed that

1) The mean squared errors of the stress-strength reliability decrease as the values of l and h increase.
2) The lengths of parametric bootstrap intervals for the stress-strength reliability decrease as the values of l and h increase.
3) The lengths of nonparametric bootstrap intervals for the stress-strength reliability decrease as the values of l and h increase.
4) The lengths of credible intervals for the stress-strength reliability decrease as the values of l and h increase.

7. Real data application

We consider two data sets that were introduced by Badar and Priest [24]. The first data set (X) represents the strength measured in GPA (Giga-Pascals) for single carbon fibers tested under tension at gauge lengths of 20 mm. The second data set (Y) represents the strength measured in GPA for single carbon fibers tested under tension at gauge lengths of 10 mm. These data were investigated in many papers in the literature. These datasets are listed below:

Data set (X): 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382,2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Data set (Y): 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Results for Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and consistent AIC (CAIC) for the two datasets (X) and (Y) are shown in Table 3 and given by the following formulas:

$$\mathrm{AIC} = 2\xi - 2\log L,$$

$$\mathrm{BIC} = \xi \log n - 2\log L,$$

$$\mathrm{HQIC} = 2\xi \log(\log n) - 2\log L,$$

$$\mathrm{CAIC} = \sigma \mathrm{AIC} + (1-\sigma)\mathrm{BIC},$$

where ξ is the number of model parameters and σ is a weight factor between 0 and 1.

Table 3. Results for AIC, BIC, HQIC, CAIC, and K-S test (with p-value) for different distributions.

Distribution	Dataset	AIC	BIC	HQIC	CAIC	K-S	p-value
Gompertz	**X**	111.2497	115.7179	113.0224	113.4838	0.08478	0.7041
	Y	142.2960	146.5823	143.9818	144.4391	0.13896	0.1754
Chen (Kayal	**X**	114.1069	118.5751	115.8796	116.3410	0.09510	0.5605
et al. [25])	**Y**	144.0265	148.3128	145.7124	146.1696	0.15338	0.1032
Lomax exponential	**X**	125.7011	130.1693	127.4738	255.8704	0.20465	0.0061
(Ijaz et al. [26])	**Y**	148.1101	152.3964	149.796	150.2532	0.16170	0.07416

The results for the Kolmogorov-Smirnov (KS) test with the p-value and a comparison between the results obtained for the Gompertz distribution and other distributions discussed in the literature for the two datasets (X) and (Y) are shown in Table 3. From the results obtained in Table 3, it can be seen that the two datasets fit the Gompertz distribution well. The graphical representation of the probability density function by a histogram, the empirical and theoretical cumulative distribution function, the Q-Q plot, and the P-P plot for the Gompertz distribution subject to dataset (X) and dataset (Y) are presented in Figures 1 and 2, respectively. The histogram of the stress-strength reliability, the MCMC output of the Bayesian stress-strength reliability with 10,000 iterations, and the plot of the stress-strength reliability function are shown in Figures 3–5, respectively.

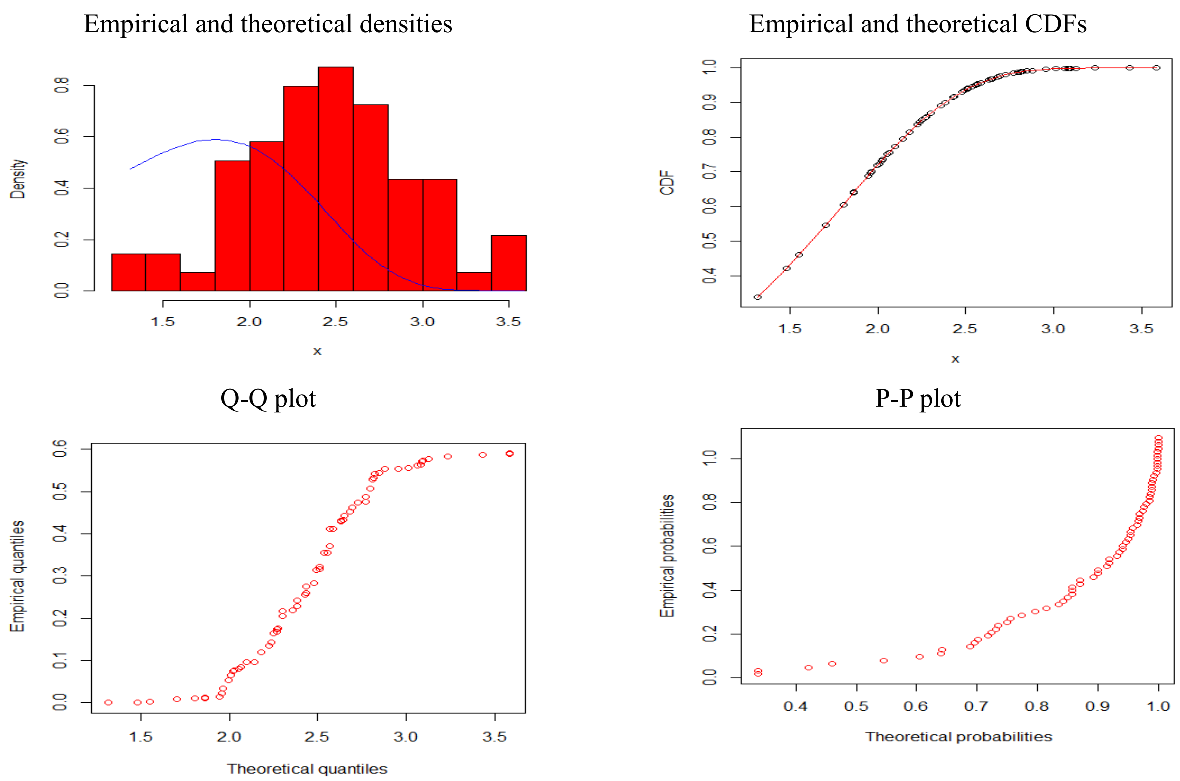

Figure 1. Density, CDF, Q-Q, and P-P plots of Gompertz distribution for dataset (X).

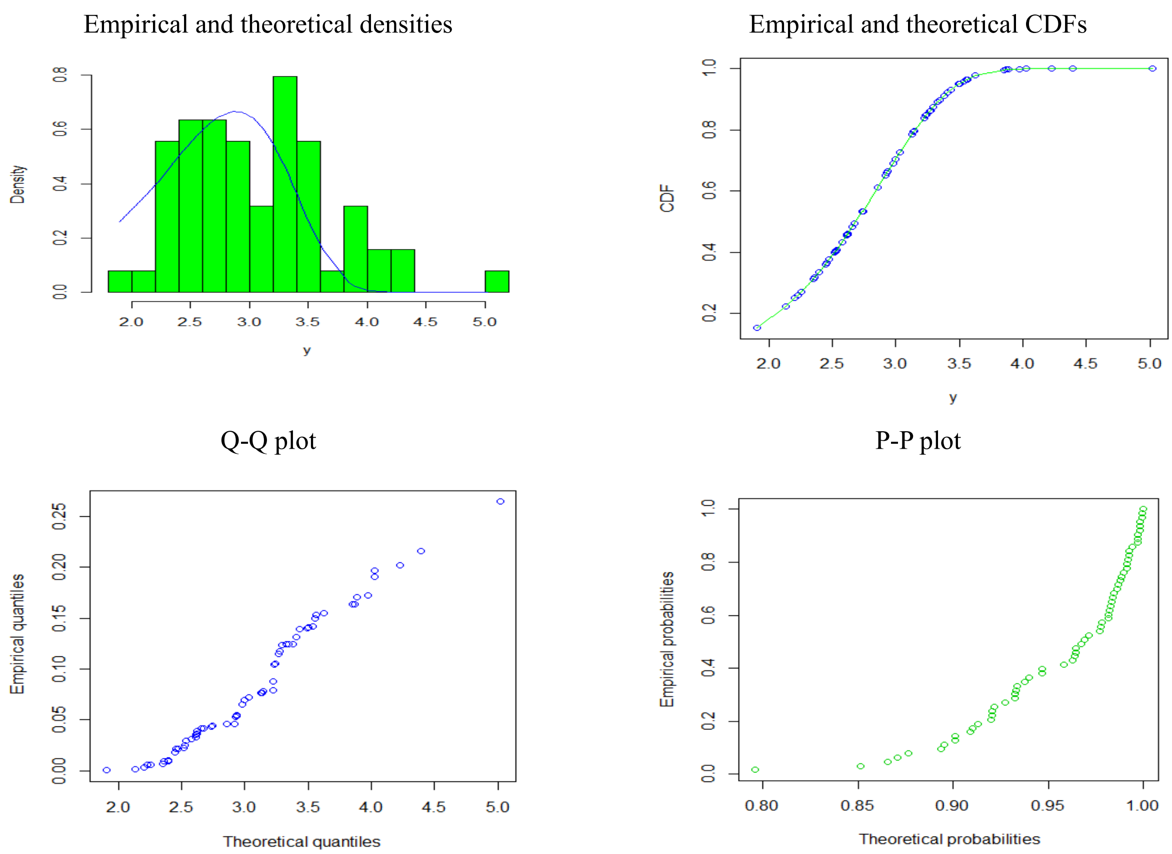

Figure 2. Density, CDF, Q-Q, and P-P plots of Gompertz distribution for dataset (Y).

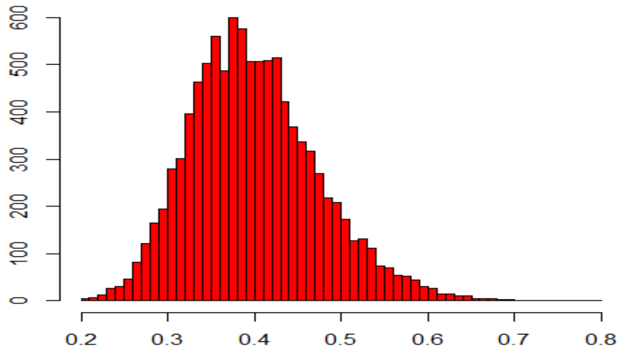

Figure 3. Histogram of R.

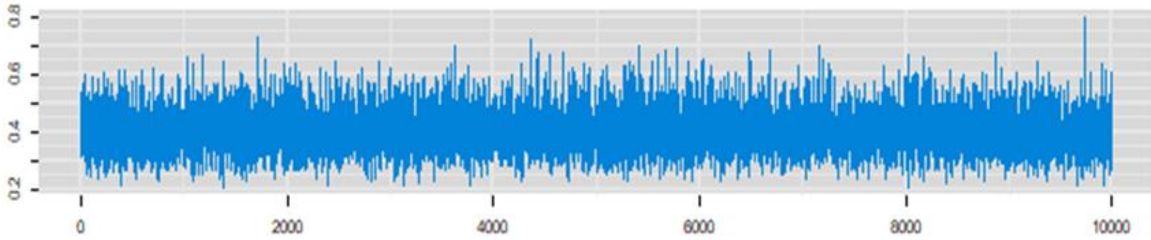

Figure 4. MCMC output of R.

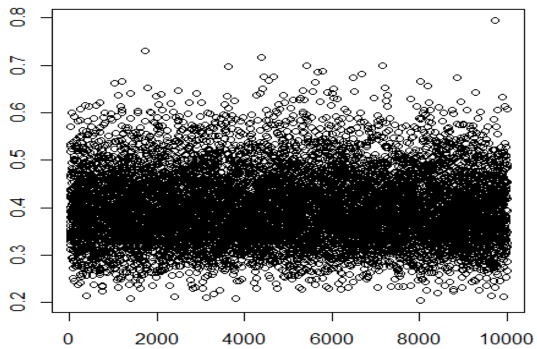

Figure 5. Plot of R.

Now, we consider the following data: $n = 69, r = 30, l = 63, h = 40, m = 5, n_1 = n_2 = n_3 = n_4 = n_5 = 6$, $\varepsilon = 3.2$, $\tau_1 = 1.85$, $\tau_2 = 2$, $\tau_3 = 2.1$, $\tau_4 = 2.271, \tau_5 = 2.5$, $\rho_j = \eta_j = 0.5$ $(for\ j = 1, 2, 3, 4, 5)$, $\sigma = \upsilon = 0.5$, $\omega = 1000$, $\zeta = 1$.

Then, the stress-strength reliability of the model is calculated as 0.4511373; bias is -0.2488611, and MSE is 0.06193185. The nonparametric bootstrap interval for the stress-strength reliability of the model is calculated as (0.2280663, 0.440059) with length 0.2119927. The credible interval of the stress-strength reliability of the model is calculated as (0.2906386, 0.5319297) with a length of 0.2412911.

8. Conclusions

The issue of analyzing stress-strength reliability with m levels of the strength variable was investigated. The units begin with an initial level of strength, which changes after a period of time when a number of units fail; this proceeds until all r units fail. The maximum likelihood estimation was introduced under type I censoring and applying the Gompertz distribution to model the lifetime of the system. Algorithms for obtaining bootstrap confidence intervals were introduced. The MCMC method was given to find the credible interval for stress-strength reliability. A simulation study was presented to apply the model to given data, and the numerical results for the estimates for stress-strength reliability, bootstrap confidence intervals, and credible intervals were obtained. Real datasets were presented to obtain stress-strength reliability when applying m levels of the strength variable. Future research can deal with different distributions and more cases of stress and strength variables.

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Conflict of interest

The author declares no conflict of interest.

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