AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(11): 30638–30654.
DOI: 10.3934/math.20241479
Received: 10 August 2024
Revised: 21 September 2024
Accepted: 12 October 2024
Published: 28 October 2024

Research article

Asymptotic behavior of the wave equation solution with nonlinear boundary damping and source term of variable exponent-type

Adel M. Al-Mahdi[1,2,*], **Mohammad M. Al-Gharabli**[1,2] **and Mohammad Kafini**[1,2]

[1] Department of Mathematics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

[2] The Interdisciplinary Research Center in Construction and Building Materials, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

* **Correspondence:** Email: almahdi@kfupm.edu.sa.

Abstract: In this study, a nonlinear damped wave equation within a bounded domain was considered. We began by demonstrating the global existence of solutions through the application of the well-depth method. Following this, a general decay rate for the solutions was established using the multiplier method alongside key properties of convex functions. Notably, these results were derived without the imposition of restrictive growth assumptions on the frictional damping, making this work an improvement and extension of previous findings in the field.

Keywords: nonlinear wave equations; stability; boundary damping and source terms; variable exponent; multiplier and perturbed energy methods
Mathematics Subject Classification: 35A02, 35B35, 35B40, 35L20, 93D20

1. Introduction

The wave equation with internal and boundary damping, along with a source term, is described by the system:

$$\begin{cases} \omega_{tt} - \Delta\omega + \psi_1(\omega_t) = \mathcal{F}_1(\omega) & \text{on } \Omega \times \mathcal{R}^+, \\ \omega = 0 & \text{on } \Gamma_0 \times \mathcal{R}^+, \\ \frac{\partial \omega_t}{\partial \eta} + \psi_2(\omega_t) = \mathcal{F}_2(\omega) & \text{on } \Gamma_1 \times \mathcal{R}^+, \\ \omega(x,0) = \omega_0(x),\ \omega_t(x,0) = \omega_1(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

In this problem, the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ are nonlinear source terms on the domain $\Omega \subseteq \mathcal{R}^n$ and the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, respectively, where Γ_0 and Γ_1 are closed and disjoint and meas.$(\Gamma_0) > 0$. The vector η is the unit outer normal to $\partial\Omega$. The functions ω_0 and ω_1 are given data. The functions ψ_1 is a nonlinear damping acting on the domain Ω, while ψ_2 is a nonlinear damping acting on the boundary $\partial\Omega$.

The study of the existence, blow-up, and stability of solutions to wave equations has been extensively explored in previous research. For example, Lasiecka and Tataru [1] studied the following semilinear model of the wave equation with nonlinear boundary conditions and nonlinear boundary velocity feedback:

$$\begin{cases} \omega_{tt} = \Delta\omega - \chi_0(\omega), & \text{in } \Omega \times \mathcal{R}^+, \\ \frac{\partial\omega}{\partial\nu} = -\tilde{\chi}(\omega_t|_{\Gamma_1}) - \chi_1(\omega|_{\Gamma_1}), & \text{on } \Gamma_1 \times \mathcal{R}^+, \\ \omega = 0, & \text{on } \Gamma_0 \times \mathcal{R}^+, \\ \omega(x,0) = \omega_0(x),\ \omega_t(x,0) = \omega_1(x) & \text{in } \Omega. \end{cases} \tag{1.2}$$

Assuming that the velocity boundary feedback is dissipative and the other nonlinear terms are conservative, uniform decay rates for the solutions are derived. Georgiev and Todorova [2] studied system (1.1) with $\psi_1(\omega_t) = |\omega_t|^{\vartheta-2}\omega_t$, $\psi_2(\omega_t) = \mathcal{F}_2(\omega) = 0$ and $\mathcal{F}_1(\omega) = |\omega|^{q-2}\omega$, proving global existence for $q \leq \vartheta$ and a blow-up result when $q > \vartheta$. Levine and Serrin [3] expanded on this by investigating the case of negative energy with $\vartheta > 1$. Rivera and Andrade [4] examined a nonlinear wave equation with viscoelastic boundary conditions, showing the existence and uniform decay under certain initial data restrictions. Santos [5] focused on a one-dimensional wave equation with viscoelastic boundary feedback, demonstrating that under specific assumptions on g' and g'', sufficient dissipation leads to exponential or polynomial decay if the relaxation function follows the same pattern. Vitillaro [6] explored system (1.1) with $\psi_1(\omega_t) = \mathcal{F}_1(\omega) = 0$ and $\psi_2(\omega_t) = |\omega_t|^{\vartheta-2}\omega_t$ and $\mathcal{F}_2(\omega) = |\omega|^{q-2}u$, establishing local and global existence under appropriate conditions on the initial data and exponents. Cavalcanti et al. [7] studied the following problem

$$\begin{cases} u_{tt} - \Delta\omega + \int_0^t g(t-s)\Delta\omega(s)ds = 0, & \text{in } \Omega \times \mathcal{R}^+, \\ \omega = 0, & \text{on } \Gamma_1 \times \mathcal{R}^+, \\ \frac{\partial\omega_t}{\partial\nu} - \int_0^t g(t-s)\frac{\partial\omega_t}{\partial\nu}(s)ds + \psi(\omega_t) = 0, & \text{on } \Gamma_0 \times \mathcal{R}^+, \end{cases} \tag{1.3}$$

where $\psi : \mathcal{R} \to \mathcal{R}$ is a nondecreasing C^1 function such that

$$\psi(s)s > 0, \ \text{ for all } s \neq 0$$

and there exist $C_i > 0$, $i = 1, 2, 3, 4$, such that

$$\begin{cases} C_1|s|^p \leq |\psi(s)| \leq C_2|s|^{\frac{1}{p}}, & \text{if } |s| \leq 1, \\ C_3|s| \leq |\psi(s)| \leq C_4|s|, & \text{if } |s| > 1, \end{cases} \tag{1.4}$$

where $p \geq 1$. They proved global existence of both strong and weak solutions, along with uniform decay rates, under restrictive conditions on the damping function ψ and the kernel g. After that, Cavalcanti et al. [8] relaxed these conditions on ψ and g, demonstrating uniform stability based on their behavior. Al-Gharabli et al. [9] extended this work by considering a large class of relaxation functions and establishing general and optimal decay results. Messaoudi and Mustafa [10] focused on system (1.3), exploring more general relaxation functions, and achieved a general decay result without assuming growth conditions on ψ, with the results depending on both g and ψ. Cavalcanti and Guesmia [11] analyzed the following hyperbolic problem involving memory terms

$$\begin{cases} \omega_{tt} - \Delta\omega + \mathcal{F}(x, t, \nabla\omega) = 0, & \text{in } \Omega \times \mathcal{R}^+, \\ \omega = 0, & \text{on } \Gamma_0, \\ \omega + \int_0^t g(t-s)\frac{\partial\omega}{\partial\mu}(s)ds = 0, & \text{on } \Gamma_1 \times \mathcal{R}^+, \end{cases} \tag{1.5}$$

showing that under certain conditions, the memory term dissipation is sufficient to ensure system stability. Specifically, they demonstrated that if the relaxation function decays exponentially or polynomially, the solution follows the same decay rate.

Liu and Yu [12] investigated the following viscoelastic equation with nonlinear boundary damping and source terms

$$\begin{cases} \omega_{tt} - \Delta\omega + \int_0^t g(t-s)\Delta\omega(s)ds = 0, & \text{in } \Omega \times \mathcal{R}^+, \\ \omega = 0, & \text{on } \Gamma_1 \times \mathcal{R}^+, \\ \frac{\partial\omega}{\partial\nu} - \int_0^t g(t-s)\frac{\partial\omega}{\partial\nu}(s)ds + |\omega_t|^{m-2} = |\omega|^{p-2}\omega, & \text{on } \Gamma_0 \times \mathcal{R}^+, \end{cases} \tag{1.6}$$

proving global existence and general decay of energy under suitable assumptions on the relaxation function and the initial data. Al-Mahdi et al. [13] extended this work by considering system (1.1) with modified terms: $\mathcal{F}_1(u) = 0$, $\mathcal{F}_2(\omega) = |\omega|^{q(x)-2}\omega$, $\psi_1(\omega_t)$ is replaced by $\int_0^t g(t-s)\Delta\omega(s)ds$, $\psi_2(\omega_t)$ is replaced by $\int_0^t g(t-s)\frac{\partial\omega}{\partial n}ds + |\omega_t|^{\vartheta(x)-2}\omega_t$, proving global existence and establishing general and optimal decay estimates under specific conditions on the relaxation function and variable exponents $\vartheta(x)$ and $q(x)$. They also provided numerical tests to validate their theoretical decay results.

Zhang and Huang [14] studied a nonlinear Kirchhoff equation described by the system:

$$\begin{cases} \omega_{tt} - M\left(\|\nabla\omega\|^2\right)\Delta\omega + \alpha\omega_t + \chi(\omega) = 0 & \text{on } \Omega \times \mathcal{R}^+, \\ \omega = 0 & \text{on } \Gamma_1 \times \mathcal{R}^+, \\ \frac{\partial\omega}{\partial\eta} + \psi(\omega_t) = 0 & \text{on } \Gamma_0 \times \mathcal{R}^+, \\ \omega(x,0) = \omega_0(x),\ \omega_t(x,0) = \omega_1(x) & \text{in } \Omega, \end{cases} \tag{1.7}$$

where Ω is a bounded domain of $\mathcal{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$,and α is a positive real constant. The functions $M(s), \chi(\omega), \psi(\omega_t)$ are satisfy some conditions, while η represents the unit outward normal vector. Using the Galerkin approximation, Zhang and Huang established the global existence and uniqueness of the solution. They also addressed challenges posed by the nonlinear terms $M(s)$ and $\psi(\omega_t)$ through a transformation to zero initial data and employed compactness, monotonicity, and perturbed energy method to resolve the problem. Zhang and Ouyang [15] examined a viscoelastic wave equation with a memory term, nonlinear damping, and a source term:

$$\begin{cases} |\omega_t|^\rho\omega_{tt} - \Delta\omega + \alpha|\omega_t|^{p-2}\omega_t + \int_0^t g(t-s)\Delta\omega(s)ds = |\omega|^{q-2}\omega & \text{on } \Omega \times \mathcal{R}^+, \\ \omega = 0 & \text{on } \Gamma \times \mathcal{R}^+, \\ \omega(x,0) = \omega_0(x),\ \omega_t(x,0) = \omega_1(x) & \text{in } \Omega, \end{cases} \tag{1.8}$$

where Ω is a bounded domain of $\mathcal{R}^n$ with a smooth boundary $\partial\Omega$, $\rho, \alpha > 0$, $p \geq 2$, $q > 2$, and $g(t)$ is a positive function that represents the kernel of the memory term. Using the potential well method combined with the Galerkin approximation, they demonstrated the existence of global weak solutions. Additionally, under certain conditions on the damping coefficient and the relaxation function, they established the optimal decay of solutions via the perturbed energy method. They further showed that the solution can blow up for both positive and negative initial energy conditions.

For further results on wave equations, see the works of Aassila [16], Wang and Chen [17], Zuazua [18], Soufyane et al. [19], Zhang et al. [20].

There has been increasing interest among researchers in replacing constant exponents with variable exponents, driven by their practical applications [21] and related references. Variable exponents are commonly used in mathematical models and equations, particularly in damping terms, to better represent a system's diverse behaviors or properties. Damping, which helps dissipate energy and regulate a system's response to external forces, can be more accurately modeled using variable exponents. This allows for a more flexible representation of damping effects tailored to the specific characteristics of the system in question.

Inspired by these studies and the significance of mathematical models involving nonlinear damping and/or source terms with variable exponents, we consider problem (1.1) with $\psi_1(\omega_t) = \psi(\omega_t)$, $\mathcal{F}_1(\omega) = 0$ and $\psi_2(\omega_t) = |\omega_t|^{\vartheta(x)-2}\omega_t$, and $\mathcal{F}_2(\omega) = |\omega|^{\theta(x)-2}\omega$.

More precisely, we consider the following nonlinear wave equation with internal and boundary damping, along with a source term of variable exponent type:

$$\begin{cases} \omega_{tt} - \Delta\omega + \psi(\omega_t) = 0 & \text{on } \Omega \times \mathcal{R}^+, \\ \omega = 0 & \text{on } \Gamma_0 \times \mathcal{R}^+, \\ \frac{\partial \omega}{\partial \eta} + |\omega_t|^{\vartheta(x)-2}\omega_t = |\omega|^{\theta(x)-2}\omega & \text{on } \Gamma_1 \times \mathcal{R}^+, \\ \omega(x,0) = \omega_0(x),\ \omega_t(x,0) = \omega_1(x) & \text{in } \Omega. \end{cases} \tag{1.9}$$

We aim to study the global existence and stability of solutions to problem (1.9). We investigate the interaction between the internal nonlinear frictional damping and the nonlinear boundary damping of variable exponent type. Additionally, we derive general decay rates, including optimal exponential and polynomial decay rates as the special cases.

This paper is organized into five sections. In Section 2, we introduce the notation and necessary background material. In Section 3, we prove the global existence of the solution to the problem. In Sections 4 and 5, we present technical lemmas and decay results, respectively.

2. Preliminaries

In this section, we outline some necessary materials for proving our results. Throughout the paper, we denote a generic positive constant by c. We consider the following assumptions:

$(A1)$ $\vartheta : \Gamma_1 \to [1, \infty)$ is a continuous function such that

$$1 < \vartheta_1 \leq \vartheta(x) \leq \vartheta_2 < q := \begin{cases} \frac{2(n-1)}{n-2}, & n > 2; \\ \infty, & n = 1, 2, \end{cases}$$

where

$$\vartheta_1 := \operatorname*{ess\,inf}_{x\in\Gamma_1} \vartheta(x), \qquad \vartheta_2 := \operatorname*{ess\,inf}_{x\in\Gamma_1} \vartheta(x).$$

$(A2)$ $\theta : \Gamma_1 \to [1, \infty)$ is a continuous function such that

$$1 < \theta_1 \leq \theta(x) \leq \vartheta_2 < q := \begin{cases} \frac{2(n-1)}{n-2}, & n > 2; \\ \infty, & n = 1, 2, \end{cases}$$

where

$$\theta_1 := \operatorname*{ess\,inf}_{x \in \Gamma_1} \theta(x), \quad \theta_2 := \operatorname*{ess\,inf}_{x \in \Gamma_1} \theta(x).$$

Moreover, the variable functions $\vartheta(x)$ and $\theta(x)$ satisfy the log-Hölder continuity condition.

For more details about the Lebesgue and Sobolev spaces with variable exponents (see [22–24]).

$(A3)$ $\psi : \mathcal{R} \to \mathcal{R}$ is a C^0 nondecreasing function satisfying, for $c_1, c_2 > 0$,

$$\begin{aligned} s^2 + \psi^2(s)| &\leq \Psi^{-1}(s\psi(s)) \quad \text{for all } |s| \leq r, \\ c_1|s| \leq |\psi(s)| &\leq c_2|s| \qquad\quad \text{for all } |s| \geq r, \end{aligned}$$

where $\Psi : (0, \infty) \to (0, \infty)$ is C^1 function which is a linear or strictly increasing and strictly convex C^2 function on $(0, r]$ with $\Psi(0) = \Psi'(0) = 0$.

Remark 2.1. Condition $(A3)$ was introduced for the first time in 1993 by Lasiecka and Tataru [1]. Examples of such functions satisfying Condition $(A3)$ are the following:

(1) If $\psi(s) = cs^q$ and $q \geq 1$, then $\Psi(s) = cs^{\frac{q+1}{2}}$ satisfies $(A3)$.
(2) If $\psi(s) = e^{-\frac{1}{s}}$, then $(A3)$ is satisfied for $\Psi(s) = \sqrt{\frac{s}{2}}\, e^{-\sqrt{\frac{2}{s}}}$ near zero.

We define the energy functional $E(t)$ associated to system (1.9) as follows:

$$E(t) := \frac{1}{2}\Big[\|\omega_t\|_2^2 + \|\nabla\omega\|_2^2\Big] - \int_{\Gamma_1} \frac{1}{\theta(x)} |\omega|^{\theta(x)} dx. \tag{2.1}$$

Lemma 2.1. *The energy functional $E(t)$ satisfies*

$$\frac{d}{dt}E(t) = -\int_{\Gamma_1} |\omega_t|^{\vartheta(x)} dx - \int_\Omega \omega_t \psi(\omega_t) dx \leq 0. \tag{2.2}$$

Proof. Multiplying $(1.9)_1$ by ω_t integrating over the interval Ω, we have

$$\int_\Omega \omega\omega_{tt} - \int_\Omega \omega_t \Delta\omega dx + \int_\Omega \omega_t \psi(\omega_t) dx = 0.$$

Using integration by parts, we obtain

$$\int_\Omega \omega\omega_{tt} + \int_\Omega \nabla\omega_t . \nabla\omega dx - \int_{\Gamma_1} \omega_t \frac{\partial\omega_t}{\partial\eta} dx + \int_\Omega \omega_t \psi(\omega_t) dx = 0.$$

Now, using $(1.9)_3$, and doing some modifications, we get

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \omega_t^2 dx + \frac{1}{2}\int_\Omega |\nabla\omega|^2 dx - \int_{\Gamma_1} \frac{1}{\theta(x)} |\omega|^{\theta(x)} dx\right) = -\int_{\Gamma_1} |\omega_t|^{\vartheta(x)} dx - \int_\Omega \omega_t \psi(\omega_t) dx,$$

which gives (2.2). □

For completeness, we present the following existence result, which can be established using the Faedo-Galerkin method and the Banach fixed point theorem, similar to the approaches taken in [2, 25, 26] for analogous problems.

Theorem 2.1. (Local existence) Given $(\omega_0, \omega_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ and assume that $(A1) - (A3)$ hold . Then, there exists $T > 0$, such that problem (1.9) has a weak solution

$$\omega \in L^\infty\left((0,T), H^1_{\Gamma_0}(\Omega)\right) \cap L^{\theta(.)}\left(\Gamma_1 \times (0,T)\right), \quad \omega_t \in L^\infty\left((0,T), L^2(\Omega)\right) \cap L^{\vartheta(.)}\left(\Gamma_1 \times (0,T)\right).$$

3. Global existence

In this section, we state and prove a global existence result under smallness conditions on the initial data (ω_0, ω_1). For this purpose, we define the following functionals:

$$\mathcal{J}(t) = \frac{1}{2}\|\nabla\omega\|_2^2 - \frac{1}{\theta_1}\int_{\Gamma_1} |\omega_t|^{\theta(x)} dx \tag{3.1}$$

and

$$\mathcal{I}(t) = \mathcal{I}(\omega(t)) = \|\nabla\omega\|_2^2 - \int_{\Gamma_1} |\omega_t|^{\theta(x)} dx. \tag{3.2}$$

Clearly, we have

$$E(t) \geq \mathcal{J}(t) + \frac{1}{2}\|\omega_t\|_2^2. \tag{3.3}$$

Lemma 3.1. *Suppose that* $(A1) - (A3)$ *hold and* $(\omega_0, \omega_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$*, such that*

$$c_e^{\theta_2} E^{\frac{\theta_2 - 2}{2}}(0) + c_e^{\theta_2} E^{\frac{\theta_1 - 2}{2}}(0) < 1, \quad I(\omega_0) > 0, \tag{3.4}$$

then

$$\mathcal{I}(\omega(t)) > 0, \quad \forall t > 0.$$

Proof. Since $\mathcal{I}$ is continuous and $\mathcal{I}(\omega_0) > 0$, then there exists $T_m < T$ such that

$$\mathcal{I}(\omega(t)) \geq 0, \ \forall t \in [0, T_m];$$

which gives

$$\begin{aligned} \mathcal{J}(t) &= \frac{1}{\theta_1}\mathcal{I}(t) + \frac{\theta_1 - 2}{2\theta_1}\|\nabla\omega\|_2^2 \\ &\geq \frac{\theta_1 - 2}{2\theta_1}\|\nabla\omega\|_2^2. \end{aligned} \tag{3.5}$$

Now,

$$\|\nabla\omega\|_2^2 \leq \frac{2\theta_1}{\theta_1 - 2}\mathcal{J}(t) \leq \frac{2\theta_1}{\theta_1 - 2}E(t) \leq \frac{2\theta_1}{\theta_1 - 2}E(0). \tag{3.6}$$

Using Young's and Poincaré's inequalities and the trace theorem, we get $\forall t \in [0, T_m]$,

$$\begin{aligned}
\int_{\Gamma_1} |\omega|^{\theta(x)} dx &= \int_{\Gamma_1^+} |\omega|^{\theta(x)} dx + \int_{\Gamma_1^-} |\omega|^{\theta(x)} dx \\
&\leq \int_{\Gamma_1^+} |\omega|^{\theta_2} dx + \int_{\Gamma_1^-} |\omega|^{\theta_1} dx \\
&\leq \int_{\Gamma_1} |\omega|^{\theta_2} dx + \int_{\Gamma_1} |\omega|^{\theta_1} dx \\
&\leq c_e^{\theta_2} \|\nabla\omega\|_2^{\theta_2} + c_e^{\theta_1} \|\nabla\omega\|_2^{\theta_1} \\
&\leq \left(c_e^{\theta_2} \|\nabla\omega\|_2^{\theta_2 - 2} + c_e^{\theta_1} \|\nabla\omega\|_2^{\theta_1 - 2} \right) \|\nabla\omega\|_2^2 \\
&< \|\nabla\omega\|_2^2,
\end{aligned} \tag{3.7}$$

where

$$\Gamma_1^- = \{x \in \Gamma_1 : |\omega(x,t)| < 1\} \text{ and } \Gamma_1^+ = \{x \in \Gamma_1 : |\omega(x,t)| \geq 1\}.$$

Therefore,

$$\mathcal{I}(t) = \|\nabla\omega\|_2^2 - \int_{\Gamma_1} |\omega|^{\theta(x)} > 0.$$

□

Proposition 3.1. *Suppose that* $(A1)$–$(A3)$ *hold. Let* $(\omega_0, \omega_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *be given, satisfying* (3.4). *Then, the solution of* (1.9) *is global and bounded.*

Proof. It suffices to show that $\|\nabla\omega\|_2^2 + \|\omega_t\|_2^2$ is bounded independently of t. To achieve this, we use (2.2), (3.2) and (3.5) to get

$$\begin{aligned}
E(0) \geq E(t) &= \mathcal{J}(t) + \frac{1}{2}\|\omega_t\|_2^2 \\
&\geq \frac{\theta_1 - 2}{2\theta_1}\|\nabla\omega\|_2^2 + \frac{1}{2}\|\omega_t\|_2^2 + \frac{1}{\theta_1}\mathcal{I}(t) \\
&\geq \frac{\theta_1 - 2}{2\theta_1}\|\nabla\omega\|_2^2 + \frac{1}{2}\|\omega_t\|_2^2.
\end{aligned} \tag{3.8}$$

Since $\mathcal{I}(t)$ is positive, Therefore

$$\|\nabla\omega\|_2^2 + \|\omega_t\|_2^2 \leq CE(0),$$

where C is a positive constant, which depends only on θ_1 and the proof is completed. □

Remark 3.1. Using (3.6), we have

$$\|\nabla\omega\|_2^2 \leq \frac{2\theta_1}{\theta_1 - 2} E(0). \tag{3.9}$$

4. Technical lemmas

In this section, we present and prove several essential lemmas for demonstrating the main results.

Lemma 4.1. *The functional defined by*

$$\Delta(t) = \int_\Omega \omega\omega_t dx \tag{4.1}$$

satisfies, along the solutions of (1.9),

$$\begin{aligned}\Delta'(t) \leq & -\frac{1}{2}\int_\Omega |\nabla\omega|^2 dx + \int_\Omega |\omega|^{\theta(x)} dx + c\int_\Omega \omega_t^2 dx + c\int_\Omega \psi^2(\omega_t) dx \\ & + c\int_{\Gamma_1} |\omega_t|^{\vartheta(x)} d\Gamma + c\int_{\Gamma_*} |\omega_t|^{2\vartheta(x)-2} d\Gamma,\end{aligned} \tag{4.2}$$

where $\Gamma_* = \{x \in \Gamma_1 : \vartheta(x) < 2\}$.

Proof.

$$\begin{aligned}\Delta'(t) &= \int_\Omega \omega_t^2 dx + \int_\Omega \omega\Delta\omega dx - \int_\Omega \omega\psi(\omega_t) \\ &= \int_\Omega \omega_t^2 dx - \int_\Omega |\nabla\omega|^2 dx + \int_{\Gamma_1} \omega\frac{\partial\omega}{\partial\eta} d\Gamma - \int_\Omega \omega\psi(\omega_t) \\ &= \int_\Omega \omega_t^2 dx - \int_\Omega |\nabla\omega|^2 dx - \int_{\Gamma_1} \omega|\omega_t|^{\vartheta(x)-2}\omega_t d\Gamma + \int_{\Gamma_1} \omega|\omega|^{\theta(x)-2}\omega d\Gamma \\ &\quad - \int_\Omega \omega\psi(\omega_t).\end{aligned} \tag{4.3}$$

The use of Young's and Poincaré's inequalities and choosing $\varepsilon_1 = \frac{1}{4c_p}$ give

$$\begin{aligned}-\int_\Omega \omega\psi(\omega_t) dx &\leq \varepsilon_1 \int_\Omega \omega^2 dx + \frac{1}{4\varepsilon_1}\int_\Omega \psi^2(\omega_t) dx \\ &\leq c_p\varepsilon_1 \int_\Omega |\nabla\omega|^2 dx + \frac{1}{4\varepsilon_1}\int_\Omega \psi^2(\omega_t) dx \\ &\leq \frac{1}{4}\int_\Omega |\nabla\omega|^2 dx + c_p\int_\Omega \psi^2(\omega_t) dx.\end{aligned} \tag{4.4}$$

Define the following partition of Γ_1:

$$\Gamma_* = \{x \in \Gamma_1 : \vartheta(x) < 2\}, \quad \Gamma_{**} = \{x \in \Gamma_1 : \vartheta(x) \geq 2\}.$$

Now, using Young's and Poincaré's inequalities, we obtain

$$\int_{\Gamma_*} \omega|\omega_t|^{\vartheta(x)-2}\omega_t d\Gamma \leq \lambda c_p \|\nabla\omega\|_2^2 + \frac{1}{4\lambda}\int_{\Gamma_*} |\omega_t|^{2\vartheta(x)-2} d\Gamma, \tag{4.5}$$

choosing $\lambda = \frac{1}{8c_p}$, then we have

$$\int_{\Gamma_*} \omega|\omega_t|^{\vartheta(x)-2}\omega_t d\Gamma \leq \frac{1}{8}\|\nabla\omega\|_2^2 + c\int_{\Gamma_*} |\omega_t|^{2\vartheta(x)-2} d\Gamma. \tag{4.6}$$

Using Young's inequality with $p(x) = \frac{\vartheta(x)}{\vartheta(x)-1}$ and $p'(x) = \vartheta(x)$ so, for all $x \in \Omega$, we have

$$|\omega_t|^{\vartheta(x)-2}\omega_t\omega \leq \varepsilon_2|\omega_t|^{\vartheta(x)} + C_{\varepsilon_2}(x)|\omega_t|^{\vartheta(x)},$$

where

$$C_{\varepsilon_2}(x) = \varepsilon_2^{1-\vartheta(x)}(\vartheta(x))^{-\vartheta(x)}(\vartheta(x)-1)^{\vartheta(x)-1}.$$

Hence, Young's inequality gives

$$\begin{aligned}\int_{\Gamma_{**}} \omega|\omega_t|^{\vartheta(x)-2}\omega_t d\Gamma &\leq \varepsilon_2 \int_{\Gamma_{**}} |\omega|^{\vartheta(x)} d\Gamma + \int_{\Gamma_{**}} C_{\varepsilon_2}(x)|\omega_t|^{\vartheta(x)} d\Gamma \\ &\leq c\varepsilon_2 \left(1 + \left(\frac{2\theta_1}{\theta_1 - 2}E(0)\right)^{\frac{\vartheta_2-2}{2}}\right)\|\nabla\omega\|_2^2 + \int_{\Gamma_{**}} C_{\varepsilon_2}(x)|\omega_t|^{\vartheta(x)} d\Gamma.\end{aligned} \tag{4.7}$$

Choosing $\varepsilon_2 = \frac{1}{8c\left(1+\left(\frac{2\theta_1}{\theta_1-2}E(0)\right)^{\frac{\vartheta_2-2}{2}}\right)}$, then $C_{\varepsilon_2}(x)$ is bounded and noting that $\Gamma_{**} \subset \Gamma_1$, then we have

$$\int_{\Gamma_{**}} \omega|\omega_t|^{\vartheta(x)-2}\omega_t d\Gamma \leq \frac{1}{8}\|\nabla\omega\|_2^2 + c\int_{\Gamma_1} |\omega_t|^{\vartheta(x)} d\Gamma. \tag{4.8}$$

By combining the above estimates, the proof is completed. □

Lemma 4.2. *Let us introduce perturbed energy functional as follows:*

$$\mathcal{M}(t) = NE(t) + \Delta(t)$$

satisfies, for all $t \geq 0$ and for a positive constant N,

$$\mathcal{M}'(t) \leq -cE(t) - cE'(t) + c\int_{\Omega}\left(\omega_t^2 + \psi^2(\omega_t)\right)dx + c\int_{\Gamma_*} |\omega_t|^{2\vartheta(x)-2} d\Gamma. \tag{4.9}$$

Proof. We establish the proof by means of perturbed energy method. Taking the derivative of $\mathcal{M}$ with respect to t, and using the estimates in (4.2), and (2.2), we obtain

$$\begin{aligned}\mathcal{M}'(t) \leq &- N\int_{\Gamma_1} |\omega_t|^{\vartheta(x)} dx - N\int_{\Omega} \omega_t\psi(\omega_t)dx \\ &- \frac{1}{2}\int_{\Omega} |\nabla\omega|^2 dx + \int_{\Omega} |\omega|^{\theta(x)} dx + c\int_{\Omega} \omega_t^2 dx + c\int_{\Omega} \psi^2(\omega_t)dx \\ &+ c\int_{\Gamma_1} |\omega_t|^{\vartheta(x)} d\Gamma + c\int_{\Gamma_*} |\omega_t|^{2\vartheta(x)-2} d\Gamma.\end{aligned} \tag{4.10}$$

Choosing N large enough such that $\mathcal{M} \sim E$, and recalling (2.2), therefore the proof of (4.9) is completed. □

Lemma 4.3. *If $1 < \vartheta_1 < 2$, then the following estimate holds:*

$$\int_{\Gamma_*} |\omega_t|^{2\vartheta(x)-2} d\Gamma \leq cE(t) - \frac{cE'(t)}{(E(t))^{\frac{2-\vartheta_1}{2\vartheta_1-2}}} - cE'(t). \tag{4.11}$$

Proof. First, we define the following partition:

$$\Gamma_{*1} = \{x \in \Gamma_* : |\omega_t(t)| \leq 1\}, \quad \Gamma_{*2} = \{x \in \Gamma_* : |\omega_t(t)| > 1\},$$

and use the fact that $\frac{2\vartheta(x)-2}{\vartheta(x)} \geq \frac{2\vartheta_1-2}{\vartheta_1}$, and Jensen's inequality to obtain

$$\begin{aligned}\int_{\Gamma_*} |\omega_t|^{2\vartheta(x)-2} d\Gamma &= \int_{\Gamma_{*1}} |\omega_t|^{2\vartheta(x)-2} d\Gamma + \int_{\Gamma_{*2}} |\omega_t|^{2\vartheta(x)-2} d\Gamma \\ &= \int_{\Gamma_{*1}} \left[|\omega_t|^{\vartheta(x)}\right]^{\frac{2\vartheta(x)-2}{\vartheta(x)}} d\Gamma + \int_{\Gamma_{*2}} |\omega_t|^{\vartheta(x)+\vartheta(x)-2} d\Gamma \\ &\leq c \int_{\Gamma_{*1}} \left[|\omega_t|^{\vartheta(x)}\right]^{\frac{2\vartheta_1-2}{\vartheta_1}} d\Gamma + c \int_{\Gamma_{*2}} |\omega_t|^{\vartheta(x)} d\Gamma \\ &\leq c\left[-E'(t)\right]^{\frac{2\vartheta_1-2}{\vartheta_1}} - cE'(t).\end{aligned} \tag{4.12}$$

Using Young's inequality, we find that

$$\begin{aligned}\left[-E'(t)\right]^{\frac{2\vartheta_1-2}{\vartheta_1}} &= \frac{(E(t))^{\frac{2-\vartheta_1}{2\vartheta_1-2}}\left[-E'(t)\right]^{\frac{2\vartheta_1-2}{\vartheta_1}}}{(E(t))^{\frac{2-\vartheta_1}{2\vartheta_1-2}}} \\ &\leq \frac{\varepsilon\,(E(t))^{\frac{\vartheta_1}{2\vartheta_1-2}} - C_\varepsilon E'(t)}{(E(t))^{\frac{2-\vartheta_1}{2\vartheta_1-2}}} \\ &= \varepsilon E(t) - \frac{C_\varepsilon E'(t)}{(E(t))^{\frac{2-\vartheta_1}{2\vartheta_1-2}}}.\end{aligned} \tag{4.13}$$

Choosing ε small enough, the proof of (4.11) is completed. □

Remark 4.1. If $\vartheta_1 \geq 2$ and since $meas(\Gamma_*) = 0$ then

$$\int_{\Gamma_*} |\omega_t|^{2\vartheta(x)-2} d\Gamma = 0. \tag{4.14}$$

Lemma 4.4. *Under assumption* (A3)*, the following estimates hold:*

$$\int_\Omega \omega_t \psi(\omega_t) dx \leq -cE'(t), \text{ if } \psi \text{ is linear,} \tag{4.15}$$

$$\int_\Omega \omega_t \psi(\omega_t) dx \leq c\Psi^{-1}(\Lambda(t)) - cE'(t), \text{ if } \psi \text{ is nonlinear,} \tag{4.16}$$

where $\Lambda(t)$ *is defined in the proof.*

Proof. Case 1: ψ is linear, then

$$c\int_\Omega \left(\omega_t^2 + \psi^2(\omega_t)\right) dx \leq -cE'(t).$$

Case 2: ψ is nonlinear, we define the following partition of Ω

$$\Omega_1 = \{x \in \Omega : |\omega_t| \leq r\}, \quad \Omega_2 = \{x \in \Omega : |\omega_t| \geq r\},$$

where r is small enough such that

$$s\psi(s) \leq \min\{r, \psi(r)\}, \quad |s| \leq r.$$

We also define

$$\Lambda(t) = \int_{\Omega_1} \omega_t \psi(\omega_t) dx.$$

Now, using hypothesis ($A3$) and Jensen's inequality, we get

$$\int_{\Omega_1} \left(\omega_t^2 + \psi^2(\omega_t)\right) dx \le \int_{\Omega_1} \Psi^{-1}(\omega_t \psi(\omega_t)) dx \le c\Psi^{-1}(\Lambda). \tag{4.17}$$

□

5. Stability result

In this section, we state and prove the stability result of system (1.9).

Theorem 5.1. *Assume that $\vartheta_1 \ge 2$ and ψ is linear. Then*

$$E(t) \le \kappa_1 e^{-\kappa_2 t}, \tag{5.1}$$

for some positive constants κ_1 and κ_2.

Proof. Combining (4.9), (4.15) with (4.14), we obtain,

$$\mathcal{M}'(t) \le -cE(t) - cE'(t).$$

Therefore, $\mathcal{M} + cE \sim E$ and a simple integration over $(0, t)$ yields, for some $\kappa_1, \kappa_2 > 0$,

$$E(t) \le \kappa_1 e^{-\kappa_2 t}, \quad t \ge 0.$$

□

Theorem 5.2. *Assume that $1 < \vartheta_1 < 2$ and ψ is linear. Then*

$$E(t) \le c\,(1 + t)^{-\frac{1}{\alpha}}, \tag{5.2}$$

where $\alpha = \frac{2-\vartheta_1}{2\vartheta_1 - 2} > 0$.

Proof. From (4.9), (4.11) and (4.15), we have

$$\mathcal{M}_1'(t) \le -cE(t) - \frac{cE'(t)}{(E(t))^{\frac{2-\vartheta_1}{2\vartheta_1-2}}}, \tag{5.3}$$

where $\mathcal{M}_1 = \mathcal{M} + cE \sim E$. Multiply both sides of (5.13) by $(E(t))^{\alpha}$ where $\alpha = \frac{2-\vartheta_1}{2\vartheta_1-2}$, to obtain

$$\mathcal{M}_2'(t) \le -cE^{\alpha+1}(t), \tag{5.4}$$

where $\mathcal{M}_2 = (E(t))^{\alpha} \mathcal{M}_1 + cE \sim E$. Integrating over $(0, t)$ and using the equivalence relation lead to (5.2). □

Theorem 5.3. *Assume that $\vartheta_1 \geq 2$ and ψ is nonlinear. Then, for some positive constants ϱ_1 and ϱ_2, we have*

$$E(t) \leq \Psi_1^{-1}(\varrho_1 t + \varrho_2), \quad \forall t \geq 0, \tag{5.5}$$

where $\Psi_1(t) = \int_t^1 \frac{1}{\Psi_2(s)} ds$ and $\Psi_2(t) = t\Psi'(\varepsilon_0 t)$

Proof. From (4.9), (4.1) and (4.15), we have

$$\mathcal{M}'(t) \leq -cE(t) + c\Psi^{-1}(\Lambda(t)). \tag{5.6}$$

Now, for $\varepsilon_0 < r$, using the fact that $E' \leq 0, \Psi' > 0, \Psi'' > 0$ on $(0, r]$, we find that the functional $\tilde{\mathcal{M}}$, by

$$\tilde{\mathcal{M}}(t) := \Psi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{M}(t) + c_0 E(t),$$

satisfies, for some $\alpha_1, \alpha_1 > 0$,

$$\alpha_1 \tilde{\mathcal{M}}(t) \leq E(t) \leq \alpha_2 \tilde{\mathcal{M}}(t), \tag{5.7}$$

and

$$\begin{aligned} \tilde{\mathcal{M}}'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)}\Psi''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{M}(t) + \Psi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{M}'(t) + c_0 E'(t) \\ &\leq -cE(t)\Psi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\Psi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\Psi^{-1}(\Lambda(t)) + c_0 E'(t). \end{aligned} \tag{5.8}$$

Let Ψ^* be the convex conjugate of Ψ in the sense of Young with $A = \Psi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = \Psi^{-1}(\Lambda(t))$, we arrive at

$$\begin{aligned} \tilde{\mathcal{M}}'(t) &\leq -cE(t)\Psi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\Psi^*\left(\Psi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\Lambda(t) + c_0 E'(t) \\ &\leq -cE(t)\Psi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)}\Psi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t) + c_0 E'(t). \end{aligned}$$

Consequently, with a suitable choice of ε_0 and c_0, we obtain, for all $t \geq 0$,

$$\tilde{\mathcal{M}}'(t) \leq -c\frac{E'(t)}{E(0)}\Psi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = -c\Psi_2\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), \tag{5.9}$$

where $\Psi_2(t) = t\Psi'(\varepsilon_0 t)$. Since $\Psi_2'(t) = \Psi'(\varepsilon_0 t) + \varepsilon_0 t\Psi''(\varepsilon_0 t)$, then, using the strict convexity of Ψ on $(0, r]$, we find that $\Psi_2'(t), \Psi_2(t) > 0$ on $(0, 1]$. Thus, with

$$\Phi(t) = \varepsilon \frac{\alpha_1 \tilde{\mathcal{M}}(t)}{E(0)}, \quad 0 < \varepsilon < 1,$$

taking in account (5.7) and (5.9), we have

$$\Phi(t) \sim E(t), \tag{5.10}$$

and then

$$\Phi'(t) \leq -c\Psi_2(\Phi(t)), \quad \forall t \geq 0.$$

Then, a simple integration gives, for some $\varrho_1, \varrho_2 > 0$,

$$\Phi(t) \leq \Psi_1^{-1}(\varrho_1 t + \varrho_2), \quad \forall t \geq 0, \tag{5.11}$$

where $\Psi_1(t) = \int_t^1 \frac{1}{\Psi_2(s)} ds$. A combination of (5.10) and (5.11) gives (5.5). □

Theorem 5.4. *Assume that* $1 < \vartheta_1 < 2$ *and* ψ *is nonlinear. Then, for some positive constants* ϱ_3 *and* ϱ_4*, we have*

$$E(t) \le \chi_1^{-1}(\varrho_3 t + \varrho_4), \quad \forall t \ge 0, \tag{5.12}$$

where $\chi_1(t) = \int_t^1 \frac{1}{\Psi_2(s)} ds$, $\chi_2(t) = t\chi'(\varepsilon_0 t)$, $\chi = \left(\mathcal{G}^{-1} + \Psi^{-1}\right)^{-1}$ *and* $\mathcal{G}(t) = t^{\frac{\vartheta_1}{2\vartheta_1 - 2}}$.

Proof. From (4.9) and (4.13), we have

$$\mathcal{M}'(t) \le -cE(t) + (-E'(t))^{\frac{2\vartheta_1-2}{\vartheta_1}} + c\Psi^{-1}(\Lambda)(t), \tag{5.13}$$

where $\mathcal{M} = E^\alpha \mathcal{M} + cE \sim E$. Let $\mathcal{G}(t) = t^{\frac{\vartheta_1}{2\vartheta_1 - 2}}$. Then the last inequality can be written as

$$\mathcal{M}'(t) \le -cE(t) + \mathcal{G}^{-1}(-E'(t)) + c\Psi^{-1}(\Lambda)(t). \tag{5.14}$$

Therefore, (5.14) becomes

$$\mathcal{M}'(t) \le -cE(t) + c\chi^{-1}(\xi(t)), \tag{5.15}$$

where $\chi = \left(\mathcal{G}^{-1} + \Psi^{-1}\right)^{-1}$ and $\xi(t) = \max\{-E'(t), \Lambda(t)\}$.

Define the following functional

$$\mathcal{K}(t) := \chi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{M}(t) + c_0 E(t), \tag{5.16}$$

satisfies, for some $\alpha_2, \alpha_3 > 0$,

$$\alpha_2 \mathcal{K}(t) \le E(t) \le \alpha_3 \mathcal{K}(t). \tag{5.17}$$

Combining (5.15) and (5.16), we obtain

$$\mathcal{K}'(t) \le -cE(t)\chi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + \chi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\chi^{-1}(\xi(t)) + c_0 E'(t). \tag{5.18}$$

Let χ^* be the convex conjugate of χ in the sense of Young, then

$$\chi^*(s) = s(\chi')^{-1}(s) - \chi\left[(\chi')^{-1}(s)\right], \quad \text{if } s \in (0, \chi'(r)] \tag{5.19}$$

and χ^* satisfies the following generalized Young inequality

$$AB \le \chi^*(A) + \chi(B), \quad \text{if } A \in (0, \chi'(r)],\ B \in (0, r]. \tag{5.20}$$

Thus, with $A = \chi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = \chi^{-1}(\xi(t))$, we arrive at

$$\mathcal{K}'(t) \le -cE(t)\chi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)}\chi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t) + c_0 E'(t).$$

Choosing c_0, ε_0 small enough, we get

$$\mathcal{K}'(t) \le -c\varepsilon_0 \frac{E(t)}{E(0)}\chi'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = -c\chi_2\left(\varepsilon_0 \frac{E(t)}{E(0)}\right),$$

where $\chi_2(t) = t\chi'(\varepsilon_0 t)$. Letting

$$Y(t) = \varepsilon \frac{\alpha_3 \mathcal{K}(t)}{E(0)}, \quad 0 < \varepsilon < 1,$$

and taking in account (5.7) and (5.9), we have

$$Y(t) \sim E(t), \tag{5.21}$$

and then

$$Y'(t) \leq -c\chi_2(Y(t)), \quad \forall t \geq 0.$$

Then, a simple integration gives, for some $\varrho_3, \varrho_4 > 0$,

$$Y(t) \leq \chi_1^{-1}(\varrho_3 t + \varrho_4), \quad \forall t \geq 0, \tag{5.22}$$

where $\chi_1(t) = \int_t^1 \frac{1}{\chi_2(s)} ds$, which finishes the proof. □

Examples 5.1. The following examples illustrate our results:

(1) If $\psi(t) = ct$ and $\vartheta(x) = 2$, then

$$E(t) \leq c_1 e^{-c_2 t}, \tag{5.23}$$

which is an exponential decay.

(2) If $\psi(t) = ct$ and $\vartheta(x) = 2 - \frac{3}{4+x}$, then $\vartheta_1 = \frac{5}{4}$ and $\vartheta_2 = \frac{7}{5}$, then the energy functional satisfies

$$E(t) \leq c\,(1+t)^{-\frac{2}{3}}. \tag{5.24}$$

(3) If $\psi(t) = ct^2$ and $\vartheta(x) = 2 + \frac{1}{1+x}$, then $\vartheta_1 = \frac{5}{2}$, $\vartheta_2 = 3$ and $\psi(t) = ct^{\frac{3}{2}}$. Then,

$$\Psi_1{}^{-1}(t) = (ct+1)^{-2}.$$

Therefore, we obtain

$$E(t) \leq c\,(1+t)^{-2}. \tag{5.25}$$

(4) If $\psi(t) = ct^5$ and $\vartheta(x) = 2 - \frac{3}{4+x}$, then $\vartheta_1 = \frac{5}{4}$, $\vartheta_2 = \frac{7}{5}$ and $\psi(t) = ct^3$. Then,

$$\chi(s) = (\mathcal{G}^{-1} + \Psi^{-1})^{-1} = \left(\frac{-1 + \sqrt{1+4s}}{2}\right)^3$$

and

$$\begin{aligned}\chi_2(s) &= \frac{3s}{\sqrt{1+4s}} \left(\frac{-1 + \sqrt{1+4s}}{2}\right)^2 \\ &= \frac{3s}{2\sqrt{1+4s}} + \frac{3s^2}{\sqrt{1+4s}} - \frac{3s}{2} \\ &\leq \frac{3s}{2} + \frac{3s^2}{2\sqrt{s}} - \frac{3s}{2} = cs^{\frac{3}{2}}.\end{aligned}$$

Therefore, we obtain

$$E(t) \leq \frac{c}{(1+t)^{\frac{1}{3}}}.$$

6. Conclusions

In this work, we consider a nonlinear wave equation with internal and boundary damping and a source term of variable exponent type. We prove the global existence and stability of solutions to this problem problem. We study the interaction between the internal nonlinear frictional damping and the nonlinear boundary damping of variable exponent type. In addition, we establish general decay rates, including optimal exponential and polynomial decay rates as the special cases.

Author contributions

Adel M. Al-Mahdi: Conceptualization, methodology, formal analysis, writing-original draft; Mohammad M. Al-Gharabli: Formal analysis, validation, writing-reviewing and editing; Mohammad Kafini: Conceptualization, methodology, formal analysis, reviewing. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

The authors would like to acknowledge the support provided by King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia. The support provided by the Interdisciplinary Research Center for Construction & Building Materials (IRC-CBM) at King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia, for funding this work through Project (No. INCB2402), is also greatly acknowledged.

Funding

This work is funded by KFUPM, Grant No. INCB2402.

Conflict of interest

The authors declare no competing interests.

References

1. I. Lasiecka, D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping, *Differ. Integral Equ.*, **6** (1993), 507–533. https://doi.org/10.57262/die/1370378427
2. V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Differ. Equ.*, **109** (1994), 295–308. https://doi.org/10.1006/jdeq.1994.1051
3. H. A. Levine, J. Serrin, Global nonexistence theorem for quasilinear evolution equations with dissipation, *Arch. Rational Mech. Anal.*, **137** (1997), 341–361. https://doi.org/10.1007/s002050050032
4. J. E. M. Rivera, D. Andrade, Exponential decay of non-linear wave equation with a viscoelastic boundary condition, *Math. Method Appl. Sci.*, **23** (2000), 41–61.

5. M. d. L. Santos, Asymptotic behavior of solutions to wave equations with a memory condition at the boundary, *Electron. J. Differ. Equ.*, **2001** (2001), 1–11.

6. E. Vitillaro, Global existence for the wave equation with nonlinear boundary damping and source terms, *J. Differ. Equ.*, **186** (2002), 259–298. https://doi.org/10.1016/S0022-0396(02)00023-2

7. M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. S. Prates Filho, J. A. Soriano, Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping, *Differ. Integral Equ.*, **14** (2001), 85–116. https://doi.org/10.57262/die/1356123377

8. M. M. Cavalcanti, V. N. D. Cavalcanti, P. Martinez, General decay rate estimates for viscoelastic dissipative systems, *Nonlinear Anal. Theor.*, **68** (2008), 177–193. https://doi.org/10.1016/j.na.2006.10.040

9. M. M. Al-Gharabli, A. M. Al-Mahdi, S. A. Messaoudi, General and optimal decay result for a viscoelastic problem with nonlinear boundary feedback, *J. Dyn. Control Syst.*, **25** (2019), 551–572. https://doi.org/10.1007/s10883-018-9422-y

10. S. A. Messaoudi, M. I. Mustafa, On convexity for energy decay rates of a viscoelastic equation with boundary feedback, *Nonlinear Anal. Theor.*, **72** (2010), 3602–3611. https://doi.org/10.1016/j.na.2009.12.040

11. M. M. Cavalcanti, A. Guesmia, General decay rates of solutions to a nonlinear wave equation with boundary condition of memory type, *Differ. Integral Equ.*, **18** (2005), 583–600. https://doi.org/10.57262/die/1356060186

12. W. Liu, J. Yu, On decay and blow-up of the solution for a viscoelastic wave equation with boundary damping and source terms, *Nonlinear Anal. Theor.*, **74** (2011), 2175–2190. https://doi.org/10.1016/j.na.2010.11.022

13. A. M. Al-Mahdi, M. M. Al-Gharabli, M. Nour, M. Zahri, Stabilization of a viscoelastic wave equation with boundary damping and variable exponents: Theoretical and numerical study, *AIMS Mathematics*, **7** (2022), 15370–15401. https://doi.org/10.3934/math.2022842

14. Z. Y. Zhang, J. H. Huang, On solvability of the dissipative kirchhoff equation with nonlinear boundary damping, *B. Korean Math. Soc.*, **51** (2014), 189–206. https://doi.org/10.4134/BKMS.2014.51.1.189

15. Z. Zhang, Q. Ouyang, Global existence, blow-up and optimal decay for a nonlinear viscoelastic equation with nonlinear damping and source term, *Discrete Cont. Dyn. B*, **28** (2023), 4735–4760. https://doi.org/10.3934/dcdsb.2023038

16. M. Aassila, A note on the boundary stabilization of a compactly coupled system of wave equations, *Appl. Math. Lett.*, **12** (1999), 19–24.

17. H. K. Wang, G. Chen, Asymptotic behaviour of solutions of the one-dimensional wave equation with a nonlinear boundary stabilizer, *SIAM J. Control Optim.*, **27** (1989), 758–775. https://doi.org/10.1137/0327040

18. E. Zuazua, Uniform stabilization of the wave equation by nonlinear boundary feedback, *SIAM J. Control Optim.*, **28** (1990), 466–477. https://doi.org/10.1137/0328025

19. A. M. Al-Mahdi, M. M. Al-Gharabli, I. Kissami, A. Soufyane, M. Zahri, Exponential and polynomial decay results for a swelling porous elastic system with a single nonlinear variable exponent damping: Theory and numerics, *Z. Angew. Math. Phys.*, **74** (2023), 72. https://doi.org/10.1007/s00033-023-01962-6

20. Z. Zhang, J. Huang, Z. Liu, M. Sun, Boundary stabilization of a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback, *Abstr. Appl. Anal.*, **2014** (2014), 102594. https://doi.org/10.1155/2014/102594

21. M. Ruzicka, *Electrorheological fluids: Modeling and mathematical theory*, Berlin, Heidelberg: Springer, 2000. https://doi.org/10.1007/BFb0104029

22. S. Antontsev, S. Shmarev, *Evolution PDEs with nonstandard growth conditions*, Paris: Atlantis Press, 2015. https://doi.org/10.2991/978-94-6239-112-3

23. L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Berlin, Heidelberg: Springer, 2011. https://doi.org/10.1007/978-3-642-18363-8

24. V. D. Radulescu, D. D. Repovs, *Partial differential equations with variable exponents: Variational methods and qualitative analysis*, New York: CRC Press, 2015. https://doi.org/10.1201/b18601

25. S. Antontsev, Wave equation with $p(x,t)$-laplacian and damping term: Existence and blow-up, *Differ. Equ. Appl.*, **3** (2011), 503–525. https://doi.org/10.7153/dea-03-32

26. S. A. Messaoudi, A. A. Talahmeh, J. H. Al-Smail, Nonlinear damped wave equation: Existence and blow-up, *Comput. Math. Appl.*, **74** (2017), 3024–3041. https://doi.org/10.1016/j.camwa.2017.07.048