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Research article

Two fixed point theorems in complete metric spaces

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Abstract: Two new classes of self-mappings defined on a complete metric space (M, d) are introduced. The first one, called the class of p-contractions with respect to a family of mappings, includes mappings $F : M \to M$ satisfying a contraction involving a finite number of mappings $S_i : M \times M \to M$. The second one, called the class of (ψ, Γ, α)-contractions, includes mappings $F : M \to M$ satisfying a contraction involving the famous ratio $\psi\left(\frac{\Gamma(t+1)}{\Gamma(t+\alpha)}\right)$, where $\psi : [0, \infty) \to [0, \infty)$ is a function, Γ is the Euler Gamma function, and $\alpha \in (0, 1)$ is a given constant. For both classes, under suitable conditions, we establish the existence and uniqueness of fixed points of F. Our results are supported by some examples in which the Banach fixed point theorem is inapplicable. Moreover, the paper includes some interesting questions related to our work for further studies in the future. These questions will push forward the development of fixed point theory and its applications.

Keywords: complete metric space; weakly Picard continuous; p-contraction; (ψ, Γ, α)-contraction; fixed point; Euler Gamma function
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1. Introduction

A contraction is a self-mapping F defined on a metric space (M, d) satisfying the inequality $d(Fu, Fv) \leq \xi d(u, v)$ for every $u, v \in M$, where $\xi \in [0, 1)$ is a constant. The famous Banach fixed point theorem [1], also called the Banach contraction principle, states that if (M, d) is a complete metric space and F is a contraction defined on M, then F possesses a unique fixed point. Moreover, starting from any element $u_0 \in M$, the Picard sequence $\{F^n u_0\}$ converges to the fixed point. This theorem provides an important tool for studying the existence of solutions for various kinds of nonlinear problems such as integral equations, differential equations, partial differential equations, and evolution equations.

A natural question to ask is whether it is possible to obtain a similar result to that of Banach for non-contraction mapping $F : M \to M$, where (M, d) is a metric space. For instance, if F is not continuous on (M, d), then F is not a contraction. This question attracted the attentions of several mathematicians. For instance, Kannan [2] considered the class of mappings $F : M \to M$ satisfying the inequality

$$d(Fu, Fv) \leq \xi\,[d(u, Fu) + d(v, Fv)]$$

for every $u, v \in M$, where $\xi \in \left[0, \frac{1}{2}\right)$ is a constant. Namely, Kannan proved that Banach's fixed point result holds true for this class of mappings. Motivated by the work of Kannan, Chatterjea [3] established the same result for the class of mappings $F : M \to M$ satisfying the inequality

$$d(Fu, Fv) \leq \xi\,[d(u, Fv) + d(v, Fu)]$$

for every $u, v \in M$, where $\xi \in \left[0, \frac{1}{2}\right)$ is a constant. Reich [4, 5] investigated the class of mappings $F : M \to M$ satisfying the inequality

$$d(Fu, Fv) \leq \xi_1 d(u, v) + \xi_2 d(Fu, u) + \xi_3 d(Fv, v)$$

for every $u, v \in M$, where $\xi_1, \xi_2, \xi_3 \geq 0$ and $\xi_1 + \xi_2 + \xi_3 < 1$. In [6], Ćirić introduced and studied the class of quasi-contraction mappings. Namely, the class of mappings $F : M \to M$ satisfying the inequality

$$d(Fu, Fv) \leq \xi \max\{d(u, v), d(u, Fu), d(v, Fv), d(u, Fv), d(v, Fu)\}$$

for every $u, v \in M$, where $\xi \in [0, 1)$ is a constant. Berinde [7] introduced and studied the class of almost contractions. Namely, the class of mappings $F : M \to M$ satisfying the inequality

$$d(Fu, Fv) \leq \xi d(u, v) + L d(Fu, v)$$

for every $u, v \in M$, where $\xi \in [0, 1)$ and $L \geq 0$ are constants. In [8], Khojasteh et al. unified several fixed point theorems by introducing the class of contractions involving simulation functions. In [9], Górnicki established various extensions of Kannan's fixed point theorem. Petrov [10] introduced and studied the class of mappings contracting perimeters of triangles. Recently, Păcurar and Popescu [11] investigated a new class of generalized Chatterjea-type mappings. We also refer to Branga and Olaru [12], where new fixed point results for generalized contractions in spaces with altering metrics were established.

Another kind of contribution was concerned with the study of fixed points when the set M is equipped with a generalized metric, such as a b-metric [13], cone E_b-metric [14], G-metric [15], F-metric [16], hemi metric [17], non-triangular metric [18, 19], or suprametric [20].

Assume now that (M, d) is a metric space and $F : M \to M$ is a contraction. Then, for all $p \geq 1$ and $u, v \in M$, we have

$$\begin{aligned} d^p(Fu, F^2u) + d^p(F^2u, Fv) &\leq [\xi d(u, Fu)]^p + [\xi d(Fu, v)]^p \\ &= \xi^p\,[d^p(u, Fu) + d^p(Fu, v)], \end{aligned}$$

that is,

$$d^p(Fu, F^2u) + d^p(F^2u, Fv) \leq \xi_p\,[d^p(u, Fu) + d^p(Fu, v)], \tag{1.1}$$

where $\xi_p = \xi^p \in [0, 1)$ is a constant. We remark that (1.1) provides a new type of contractions. Notice that a mapping F satisfying (1.1) is not necessarily a contraction (see Example 2.12). Based on this observation, for a metric space (M, d), we introduce in Section 2 the class of mappings $F : M \to M$ satisfying the inequality

$$\begin{aligned} & d^p(Fu, S_1(Fu, Fv)) + \sum_{i=1}^{k-1} d^p(S_i(Fu, Fv), S_{i+1}(Fu, Fv)) + d^p(S_k(Fu, Fv), Fv) \\ & \le \xi \left[d^p(u, S_1(u, v)) + \sum_{i=1}^{k-1} d^p(S_i(u, v), S_{i+1}(u, v)) + d^p(S_k(u, v), v) \right] \end{aligned}$$

for every $u, v \in M$, where $\xi \in [0, 1)$ is a constant and $\{S_i\}_{i=1}^k$ is a family of mappings $S_i : M \times M \to M$. A mapping F satisfying the above condition is called a p-contraction with respect to $\{S_i\}_{i=1}^k$. Notice that in the special case $k = 1$ and $S_1(u, v) = Fu$ for every $u, v \in M$, a p-contraction with respect to $\{S_1\}$ is a mapping F satisfying (1.1). A fixed point theorem for p-contractions with respect to a family of mappings is established in Section 2. Our obtained result is a generalization of the Banach fixed point theorem. The result is supported by some examples where the Banach fixed point theorem is not applicable. We also provide an application to the study of fixed points for single-valued mappings defined on the set of positive integers.

On the other hand, it is well known that special functions arise in numerous applications. Indeed, various classical problems of physics can be solved by making use of such functions. In particular, the famous Euler Gamma function is one of the most important special functions, which plays a crucial role in various branches of mathematics; see, e.g., [21–23]. Motivated by this fact, in Section 3, we are concerned with the study of fixed points for mappings $F : M \to M$ satisfying contractions involving the ratio

$$\psi\left(\frac{\Gamma(t+1)}{\Gamma(t+\alpha)}\right), \quad t > 0,$$

where (M, d) is a metric space, Γ is the Euler Gamma function, $\psi : [0, \infty) \to [0, \infty)$ is a function satisfying a certain condition, and $\alpha \in (0, 1)$ is a constant. We call this class of mappings as the class of (ψ, Γ, α)-contractions. Our obtained result is supported by an example where the Banach fixed point theorem is not applicable.

We end this section by fixing some notations that will be used throughout this paper. By $\mathbb{N}$ (resp. $\mathbb{N}^*$), we mean the set of nonnegative (resp. positive) integers. We denote by M an arbitrary nonempty set. For a given mapping $F : M \to M$, we denote by $\{F^n\}$ the sequence of mappings $F^n : M \to M$ defined by

$$F^0 u = u, \quad F^{n+1} u = F(F^n u), \quad n \in \mathbb{N}$$

for all $u \in M$. By Fix(F), we mean the set of fixed points of F, that is,

$$\text{Fix}(F) = \{u \in M : Fu = u\}.$$

Similarly, for a mapping $S : M \times M \to M$, we denote by Fix(S) the set of fixed points of S, that is,

$$\text{Fix}(S) = \{u \in M : S(u, u) = u\}.$$

2. The class of p-contractions with respect to a family of mappings

In this section, we introduce the class of p-contractions with respect to a family of mappings, and study the existence and uniqueness of fixed points for such mappings.

Let (M, d) be a metric space. Let $k \in \mathbb{N}^*$, $p \geq 1$ be constants, and $\{S_i\}_{i=1}^k$ be a family of mappings $S_i : M \times M \to M$.

Definition 2.1. A mapping $F : M \to M$ is called a p-contraction with respect to $\{S_i\}_{i=1}^k$ if there exists $\xi \in [0, 1)$ such that

$$\begin{aligned} & d^p(Fu, S_1(Fu, Fv)) + \sum_{i=1}^{k-1} d^p(S_i(Fu, Fv), S_{i+1}(Fu, Fv)) \\ & + d^p(S_k(Fu, Fv), Fv) \\ \leq\ & \xi \left[d^p(u, S_1(u, v)) + \sum_{i=1}^{k-1} d^p(S_i(u, v), S_{i+1}(u, v)) + d^p(S_k(u, v), v) \right] \end{aligned} \tag{2.1}$$

for every $u, v \in M$.

Definition 2.2. We say that a mapping $F : M \to M$ is weakly Picard continuous on (M, d) if the following condition holds: If for $u, v \in M$ and

$$\lim_{n \to \infty} d(F^n u, v) = 0,$$

then there exists a subsequence $\{F^{n_j} u\}$ of $\{F^n u\}$ such that

$$\lim_{j \to \infty} d(F(F^{n_j} u), Fv) = 0.$$

Remark 2.3. It can be easily seen that if $F : M \to M$ is continuous on (M, d), then F is weakly Picard continuous on (M, d). However, the converse is not necessarily true, as demonstrated in the following example.

Example 2.4. Let d be the Euclidean metric on $[0, 1]$, that is,

$$d(u, v) = |u - v|, \quad u, v \in [0, 1].$$

Suppose that $F : [0, 1] \to [0, 1]$ is the function defined by

$$Fu = \begin{cases} \dfrac{u}{2}, & \text{if } 0 \leq u < 1, \\ \dfrac{1}{4}, & \text{if } u = 1. \end{cases}$$

It is clear that F is not continuous at $u = 1$. Notice that for every $n \in \mathbb{N}^*$, it follows that

$$F^n u = \begin{cases} \dfrac{u}{2^n}, & \text{if } 0 \leq u < 1, \\ \dfrac{1}{2^{n+1}}, & \text{if } u = 1, \end{cases}$$

which shows that

$$\lim_{n \to \infty} d(F^n u, 0) = 0, \quad u \in [0, 1].$$

Thus, F is weakly Picard continuous on $([0, 1], d)$.

We recall below the famous Jensen inequality (see [24]) that will be used later.

Lemma 2.5. *Let $J : [0,\infty) \to \mathbb{R}$ be a convex function. Then, for every $q \in \mathbb{N}^*$ and $\{x_1, x_2, \cdots, x_q\}$, $\{a_1, a_2, \cdots, a_q\} \subset [0,\infty)$ with $\sum_{i=1}^{q} a_i > 0$, we have*

$$J\left(\frac{\sum_{i=1}^{q} a_i x_i}{\sum_{i=1}^{q} a_i}\right) \leq \frac{\sum_{i=1}^{q} a_i J(x_i)}{\sum_{i=1}^{q} a_i}.$$

2.1. Fixed point results

In this subsection, we give a fixed point theorem and several corollaries. Moreover, we provide some nontrivial examples to illustrate that our results can be used, but that Banach fixed point theorem is not applicable.

Our first main result is the following fixed point theorem.

Theorem 2.6. *Let (M, d) be a complete metric space and $F : M \to M$ a mapping. Suppose that the following conditions hold:*

(i) *F is a p-contraction with respect to $\{S_i\}_{i=1}^k$;*
(ii) *F is weakly Picard continuous on (M, d).*

Then:

(I) *For every $u_0 \in M$, the sequence $\{F^n u_0\}$ converges to a fixed point of F;*
(II) *F possesses a unique fixed point u^* in M;*
(III) $u^* \in \bigcap_{i=1}^{k} \mathrm{Fix}(S_i)$.

Proof. (I) For an arbitrary $u_0 \in M$, let us consider the sequence $\{u_n\} \subset M$ defined by

$$u_n = F^n u_0, \quad n \in \mathbb{N}.$$

By (i), taking $(u, v) = (u_0, u_1)$ in (2.1), we obtain

$$\begin{aligned} & d^p(Fu_0, S_1(Fu_0, Fu_1)) + \sum_{i=1}^{k-1} d^p(S_i(Fu_0, Fu_1), S_{i+1}(Fu_0, Fu_1)) \\ & + d^p(S_k(Fu_0, Fu_1), Fu_1) \\ \leq\ & \xi \left[d^p(u_0, S_1(u_0, u_1)) + \sum_{i=1}^{k-1} d^p(S_i(u_0, u_1), S_{i+1}(u_0, u_1)) + d^p(S_k(u_0, u_1), u_1) \right], \end{aligned}$$

that is,

$$d^p(u_1, S_1(u_1, u_2)) + \sum_{i=1}^{k-1} d^p(S_i(u_1, u_2), S_{i+1}(u_1, u_2)) + d^p(S_k(u_1, u_2), u_2)$$

$$\leq \xi \left[d^p(u_0, S_1(u_0,u_1)) + \sum_{i=1}^{k-1} d^p(S_i(u_0,u_1), S_{i+1}(u_0,u_1)) + d^p(S_k(u_0,u_1),u_1) \right]. \tag{2.2}$$

Repeating the same process with $(u, v) = (u_1, u_2)$, we obtain

$$\begin{aligned} & d^p(Fu_1, S_1(Fu_1,Fu_2)) + \sum_{i=1}^{k-1} d^p(S_i(Fu_1,Fu_2), S_{i+1}(Fu_1,Fu_2)) \\ & + d^p(S_k(Fu_1,Fu_2),Fu_2) \\ & \leq \xi \left[d^p(u_1, S_1(u_1,u_2)) + \sum_{i=1}^{k-1} d^p(S_i(u_1,u_2), S_{i+1}(u_1,u_2)) + d^p(S_k(u_1,u_2),u_2) \right], \end{aligned}$$

that is,

$$\begin{aligned} & d^p(u_2, S_1(u_2,u_3)) + \sum_{i=1}^{k-1} d^p(S_i(u_2,u_3), S_{i+1}(u_2,u_3)) + d^p(S_k(u_2,u_3),u_3) \\ & \leq \xi \left[d^p(u_1, S_1(u_1,u_2)) + \sum_{i=1}^{k-1} d^p(S_i(u_1,u_2), S_{i+1}(u_1,u_2)) + d^p(S_k(u_1,u_2),u_2) \right]. \end{aligned} \tag{2.3}$$

Thus, in view of (2.2) and (2.3), we have

$$\begin{aligned} & d^p(u_2, S_1(u_2,u_3)) + \sum_{i=1}^{k-1} d^p(S_i(u_2,u_3), S_{i+1}(u_2,u_3)) + d^p(S_k(u_2,u_3),u_3) \\ & \leq \xi^2 \left[d^p(u_0, S_1(u_0,u_1)) + \sum_{i=1}^{k-1} d^p(S_i(u_0,u_1), S_{i+1}(u_0,u_1)) + d^p(S_k(u_0,u_1),u_1) \right]. \end{aligned}$$

Continuing in the same way, we obtain

$$\begin{aligned} & d^p(u_n, S_1(u_n,u_{n+1})) + \sum_{i=1}^{k-1} d^p(S_i(u_n,u_{n+1}), S_{i+1}(u_n,u_{n+1})) \\ & + d^p(S_k(u_n,u_{n+1}),u_{n+1}) \\ & \leq \lambda_0 \xi^n \end{aligned} \tag{2.4}$$

for every $n \in \mathbb{N}$, where

$$\lambda_0 = d^p(u_0, S_1(u_0,u_1)) + \sum_{i=1}^{k-1} d^p(S_i(u_0,u_1), S_{i+1}(u_0,u_1)) + d^p(S_k(u_0,u_1),u_1).$$

On the other hand, by the triangle inequality, we obtain that, for all $n \in \mathbb{N}$,

$$\begin{aligned} d(u_n,u_{n+1}) \leq\ & d(u_n, S_1(u_n,u_{n+1})) + d(S_1(u_n,u_{n+1}), S_2(u_n,u_{n+1})) + \cdots \\ & + d(S_{k-1}(u_n,u_{n+1}), S_k(u_n,u_{n+1})) + d(S_k(u_n,u_{n+1}),u_{n+1}), \end{aligned}$$

which yields

$$d^p(u_n,u_{n+1}) \le \Big[d(u_n,S_1(u_n,u_{n+1})) + \sum_{i=1}^{k-1} d(S_i(u_n,u_{n+1}),S_{i+1}(u_n,u_{n+1})) \\ +d(S_k(u_n,u_{n+1}),u_{n+1})\Big]^p. \tag{2.5}$$

Furthermore, since the function $x \mapsto x^p$ is convex on $[0,\infty)$, using Lemma 2.5, we obtain

$$\begin{aligned}
&\left[d(u_n,S_1(u_n,u_{n+1})) + \sum_{i=1}^{k-1} d(S_i(u_n,u_{n+1}),S_{i+1}(u_n,u_{n+1})) + d(S_k(u_n,u_{n+1}),u_{n+1})\right]^p \\
&= (k+1)^p\Big[\frac{d(u_n,S_1(u_n,u_{n+1}))}{k+1} + \sum_{i=1}^{k-1}\frac{d(S_i(u_n,u_{n+1}),S_{i+1}(u_n,u_{n+1}))}{k+1} \\
&\qquad + \frac{d(S_k(u_n,u_{n+1}),u_{n+1})}{k+1}\Big]^p \\
&\le \frac{(k+1)^p}{k+1}\Big[d^p(u_n,S_1(u_n,u_{n+1})) + \sum_{i=1}^{k-1} d^p(S_i(u_n,u_{n+1}),S_{i+1}(u_n,u_{n+1})) \\
&\qquad + d^p(S_k(u_n,u_{n+1}),u_{n+1})\Big],
\end{aligned}$$

which implies by (2.5) that

$$\begin{aligned}
&d^p(u_n,u_{n+1}) \\
&\le (k+1)^{p-1}\Big[d^p(u_n,S_1(u_n,u_{n+1})) + \sum_{i=1}^{k-1} d^p(S_i(u_n,u_{n+1}),S_{i+1}(u_n,u_{n+1})) \\
&\qquad + d^p(S_k(u_n,u_{n+1}),u_{n+1})\Big].
\end{aligned}$$

The above inequality yields

$$\begin{aligned}
&\frac{1}{(k+1)^{p-1}}d^p(u_n,u_{n+1}) \\
&\le d^p(u_n,S_1(u_n,u_{n+1})) + \sum_{i=1}^{k-1} d^p(S_i(u_n,u_{n+1}),S_{i+1}(u_n,u_{n+1})) \\
&\quad + d^p(S_k(u_n,u_{n+1}),u_{n+1}).
\end{aligned} \tag{2.6}$$

Thus, it follows from (2.4) and (2.6) that

$$d(u_n,u_{n+1}) \le \Big[\lambda_0(k+1)^{p-1}\Big]^{\frac{1}{p}}\,\xi_p^n, \quad n \in \mathbb{N}, \tag{2.7}$$

where

$$0 \le \xi_p := \xi^{\frac{1}{p}} < 1. \tag{2.8}$$

Next, thanks to (2.7) and (2.8), we obtain by the triangle inequality that, for all $n \in \mathbb{N}$ and any $m \in \mathbb{N}^*$,

$$\begin{aligned} d(u_n, u_{n+m}) &\leq d(u_n, u_{n+1}) + \cdots + d(u_{n+m-1}, u_{n+m}) \\ &\leq \left[\lambda_0 (k+1)^{p-1}\right]^{\frac{1}{p}} \left(\xi_p^n + \cdots + \xi_p^{n+m-1}\right) \\ &= \left[\lambda_0 (k+1)^{p-1}\right]^{\frac{1}{p}} \xi_p^n \frac{1-\xi_p^m}{1-\xi_p} \\ &\leq \frac{\left[\lambda_0 (k+1)^{p-1}\right]^{\frac{1}{p}}}{1-\xi_p} \xi_p^n \to 0 \ \ (n \to \infty), \end{aligned}$$

which implies that $\{u_n\}$ is a Cauchy sequence. Then, due to the completeness of (M, d), we infer that there exists $u^* \in M$ such that

$$\lim_{n\to\infty} d(F^n u_0, u^*) = \lim_{n\to\infty} d(u_n, u^*) = 0, \tag{2.9}$$

and by (ii) it follows that there exists a subsequence $\{F^{n_j} u_0\}$ of $\{F^n u_0\}$ such that

$$\lim_{j\to\infty} d(u_{n_j+1}, Fu^*) = \lim_{j\to\infty} d(F(F^{n_j} u_0), Fu^*) = 0. \tag{2.10}$$

Thus, in view of (2.9) and (2.10), we have $u^* = Fu^*$, that is, $u^* \in \mathrm{Fix}(F)$. This proves part (I) of Theorem 2.6.

(II) By (I), it follows that $u^* \in \mathrm{Fix}(F)$. If there still exists $v^* \in \mathrm{Fix}(F)$, then by (2.1), we obtain

$$\begin{aligned} & d^p(Fu^*, S_1(Fu^*, Fv^*)) + \sum_{i=1}^{k-1} d^p(S_i(Fu^*, Fv^*), S_{i+1}(Fu^*, Fv^*)) \\ & \quad + d^p(S_k(Fu^*, Fv^*), Fv^*) \\ & \leq \xi \left[d^p(u^*, S_1(u^*, v^*)) + \sum_{i=1}^{k-1} d^p(S_i(u^*, v^*), S_{i+1}(u^*, v^*)) + d^p(S_k(u^*, v^*), v^*) \right], \end{aligned}$$

that is,

$$\begin{aligned} & d^p(u^*, S_1(u^*, v^*)) + \sum_{i=1}^{k-1} d^p(S_i(u^*, v^*), S_{i+1}(u^*, v^*)) + d^p(S_k(u^*, v^*), v^*) \\ & \leq \xi \left[d^p(u^*, S_1(u^*, v^*)) + \sum_{i=1}^{k-1} d^p(S_i(u^*, v^*), S_{i+1}(u^*, v^*)) + d^p(S_k(u^*, v^*), v^*) \right]. \end{aligned}$$

Since $\xi \in [0, 1)$, the above inequality implies that

$$\begin{aligned} & d(u^*, S_1(u^*, v^*)) = d(S_1(u^*, v^*), S_2(u^*, v^*)) = \cdots = d(S_{k-1}(u^*, v^*), S_k(u^*, v^*)) \\ & = d(S_k(u^*, v^*), v^*) = 0, \end{aligned}$$

which yields $u^* = v^*$. Consequently, F admits a unique fixed point u^* in M, which proves part (II) of Theorem 2.6.

(III) From (II), we know that F admits a unique fixed point $u^* \in M$. Taking $(u, v) = (u^*, u^*)$ in (2.1), we obtain

$$\begin{aligned}
&d^p(Fu^*, S_1(Fu^*, Fu^*)) + \sum_{i=1}^{k-1} d^p(S_i(Fu^*, Fu^*), S_{i+1}(Fu^*, Fu^*)) \\
&+ d^p(S_k(Fu^*, Fu^*), Fu^*) \\
\leq\ & \xi \left[d^p(u^*, S_1(u^*, u^*)) + \sum_{i=1}^{k-1} d^p(S_i(u^*, u^*), S_{i+1}(u^*, u^*)) + d^p(S_k(u^*, u^*), u^*) \right],
\end{aligned}$$

that is,

$$\begin{aligned}
&d^p(u^*, S_1(u^*, u^*)) + \sum_{i=1}^{k-1} d^p(S_i(u^*, u^*), S_{i+1}(u^*, u^*)) + d^p(S_k(u^*, u^*), u^*) \\
\leq\ & \xi \left[d^p(u^*, S_1(u^*, u^*)) + \sum_{i=1}^{k-1} d^p(S_i(u^*, u^*), S_{i+1}(u^*, u^*)) + d^p(S_k(u^*, u^*), u^*) \right].
\end{aligned}$$

Since $\xi \in [0, 1)$, the above inequality implies that

$$\begin{aligned}
d(u^*, S_1(u^*, u^*)) &= d(S_1(u^*, u^*), S_2(u^*, u^*)) = \cdots \\
&= d(S_{k-1}(u^*, u^*), S_k(u^*, u^*)) = d(S_k(u^*, u^*), u^*) = 0.
\end{aligned}$$

Consequently, we obtain

$$u^* = S_1(u^*, u^*) = S_2(u^*, u^*) = \cdots = S_k(u^*, u^*),$$

that is, $u^* \in \bigcap_{i=1}^{k} \mathrm{Fix}(S_i)$. This proves part (III) of Theorem 2.6. □

We now discuss some special cases of Theorem 2.6. Taking $k = 1$ in Theorem 2.6, we obtain the following result.

Corollary 2.7. *Let (M, d) be a complete metric space, $p \geq 1$ a constant, and $S_1 : M \times M \to M$ a mapping. Suppose that $F : M \to M$ is a mapping satisfying the following conditions:*

(i) *There exists a constant $\xi \in [0, 1)$ such that*

$$\begin{aligned}
&d^p(Fu, S_1(Fu, Fv)) + d^p(S_1(Fu, Fv), Fv) \\
\leq\ & \xi\, [d^p(u, S_1(u, v)) + d^p(S_1(u, v), v)]
\end{aligned} \tag{2.11}$$

for every $u, v \in M$;

(ii) *F is weakly Picard continuous on (M, d).*

Then:

(I) *For every $u_0 \in M$, the sequence $\{F^n u_0\}$ converges to a fixed point of F;*

(II) *F possesses a unique fixed point u^* in M;*

(III) *$u^* \in \mathrm{Fix}(S_1)$.*

Taking $S_1(u,v) = Fu$ for every $u, v \in M$, we deduce from Corollary 2.7 the following result.

Corollary 2.8. *Let (M,d) be a complete metric space, $p \geq 1$ a constant, and $F : M \to M$ a mapping. Suppose that the following conditions hold:*

(i) *There exists a constant $\xi \in [0,1)$ such that*

$$d^p(Fu, F^2u) + d^p(F^2u, Fv) \leq \xi\,[d^p(u,Fu) + d^p(Fu,v)] \tag{2.12}$$

for every $u, v \in M$;

(ii) *F is weakly Picard continuous on (M,d).*

Then:

(I) *For every $u_0 \in M$, the sequence $\{F^n u_0\}$ converges to a fixed point of F;*

(II) *F possesses a unique fixed point u^* in M.*

Taking $k = 2$ in Theorem 2.6, we obtain the following result.

Corollary 2.9. *Let (M,d) be a complete metric space, $S_1, S_2 : M \times M \to M$ two mappings and $p \geq 1$ a constant. Suppose that $F : M \to M$ is a mapping satisfying the following conditions:*

(i) *There exists a constant $\xi \in [0,1)$ such that*

$$\begin{aligned} &d^p(Fu, S_1(Fu,Fv)) + d^p(S_1(Fu,Fv), S_2(Fu,Fv)) + d^p(S_2(Fu,Fv), Fv) \\ &\leq \xi\,[d^p(u, S_1(u,v)) + d^p(S_1(u,v), S_2(u,v)) + d^p(S_2(u,v), v)] \end{aligned}$$

for every $u, v \in M$;

(ii) *F is weakly Picard continuous on (M,d).*

Then:

(I) *For every $u_0 \in M$, the sequence $\{F^n u_0\}$ converges to a fixed point of F;*

(II) *F possesses a unique fixed point u^* in M;*

(III) $u^* \in \text{Fix}(S_1) \cap \text{Fix}(S_2)$.

If F is continuous on (M,d) (see Remark 2.3), then we deduce from Theorem 2.6 the following result.

Corollary 2.10. *Let (M,d) be a complete metric space and $F : M \to M$ a mapping. Suppose that the following conditions hold:*

(i) *F is a p-contraction with respect to $\{S_i\}_{i=1}^k$;*

(ii) *F is continuous.*

Then:

(I) *For every $u_0 \in M$, the sequence $\{F^n u_0\}$ converges to a fixed point of F;*

(II) *F possesses a unique fixed point u^* in M;*

(III) $u^* \in \bigcap_{i=1}^{k} \text{Fix}(S_i)$.

We now show that Theorem 2.6 includes the Banach fixed point theorem.

Corollary 2.11. *(Banach fixed point theorem) Let (M, d) be a complete metric space and $F : M \to M$ a mapping. Suppose that there exists a constant $\xi \in [0, 1)$ such that*

$$d(Fu, Fv) \leq \xi d(u, v) \tag{2.13}$$

for every $u, v \in M$. Then, F possesses a unique fixed point u^* in M. Moreover, for every $u_0 \in M$, the sequence $\{F^n u_0\}$ converges to u^*.

Proof. Let $k = p = 1$ and $S_1 : M \times M \to M$ be the projection mapping defined by

$$S_1(u, v) = u, \quad (u, v) \in M \times M.$$

In this case, (2.1) reduces to

$$d(Fu, Fu) + d(Fu, Fv) \leq \xi \left[d(u, u) + d(u, v)\right],$$

that is, (2.13). This shows that, if F satisfies (2.13), then F satisfies (2.1) with $k = p = 1$ and S_1 is the mapping defined above. On the other hand, due to (2.13), the mapping F is continuous. Thus, by Corollary 2.10, the desired result is completed. □

We provide below three examples illustrating our obtained results. In all the presented examples, the Banach fixed point theorem is not applicable.

Example 2.12. Let $F : [0, 1] \to [0, 1]$ be the mapping defined by

$$Fu = \begin{cases} 0, & \text{if } 0 \leq u < 1, \\ \dfrac{1}{4}, & \text{if } u = 1. \end{cases}$$

We consider the Euclidean metric on $[0, 1]$, that is,

$$d(u, v) = |u - v|, \quad u, v \in [0, 1].$$

Notice that F is not continuous at $u = 1$, which implies that F is not a contraction. Thus, the Banach fixed point theorem is not applicable in this example. Let us now estimate the ratio

$$R(u, v) = \frac{d(Fu, F^2u) + d(F^2u, Fv)}{d(u, Fu) + d(Fu, v)}, \quad u, v \in [0, 1].$$

We distinguish four cases as follows.

Case 1. $0 \leq u, v < 1$. In this case, we obtain

$$d(Fu, F^2u) + d(F^2u, Fv) = d(0, F0) + d(F0, 0) = 2d(0, 0) = 0.$$

Case 2. $u = v = 1$. In this case, we have

$$R(u, v) = \frac{d(F1, F^2 1) + d(F^2 1, F1)}{d(1, F1) + d(F1, 1)} = \frac{d(\frac{1}{4}, 0) + d(0, \frac{1}{4})}{d(1, \frac{1}{4}) + d(\frac{1}{4}, 1)} = \frac{1}{3}.$$

Case 3. $0 \le u < 1$ and $v = 1$. In this case, we have

$$R(u,v) = \frac{d(Fu,F^2u)+d(F^2u,F1)}{d(u,Fu)+d(Fu,1)} = \frac{d(0,F0)+d(F0,F1)}{d(u,0)+d(0,1)} = \frac{d(0,0)+d(0,\frac{1}{4})}{u+1} \le \frac{1}{4}.$$

Case 4. $u = 1$ and $0 \le v < 1$. In this case, we have

$$R(u,v) = \frac{d(F1,F^21)+d(F^21,Fv)}{d(1,F1)+d(F1,v)} = \frac{d(\frac{1}{4},0)+d(0,0)}{d(1,\frac{1}{4})+d(\frac{1}{4},v)} = \frac{\frac{1}{4}}{\frac{3}{4}+d(\frac{1}{4},v)} \le \frac{1}{3}.$$

Generally speaking, from the above estimates, we deduce that

$$d(Fu,F^2u)+d(F^2u,Fv) \le \frac{1}{3}\left[d(u,Fu)+d(Fu,v)\right]$$

for every $u,v \in M$, which shows that F satisfies (2.12) with $p = 1$.

On the other hand, for all $u \in [0,1]$, we have

$$F^nu = 0, \quad n \ge 2,$$

which shows that F is weakly Picard continuous. Consequently, Corollary 2.8 applies. Notice that

$$\mathrm{Fix}(F) = \{0\},$$

which confirms Corollary 2.8.

Example 2.13. Let $M = \{m_1, m_2, m_3\}$ and $F : M \to M$ be the mapping defined by

$$Fm_1 = m_1,\ Fm_2 = m_1,\ Fm_3 = m_2.$$

Let d be the discrete metric on M, that is,

$$d(m_i,m_j) = \begin{cases} 1, & \text{if } i \ne j, \\ 0, & \text{if } i = j. \end{cases} \tag{2.14}$$

Notice that

$$\frac{d(Fm_1,Fm_3)}{d(m_1,m_3)} = \frac{d(m_1,m_2)}{d(m_1,m_3)} = 1,$$

which shows that there is no $\xi \in [0,1)$ such that (2.13) holds for every $u,v \in M$. Consequently, the Banach fixed point theorem is not applicable in this example.

Let us introduce the mapping $S_1 : M \times M \to M$ defined by

$$S_1(m_i,m_i) = m_i,\ S_1(m_i,m_j) = S_1(m_j,m_i), \quad i,j \in \{1,2,3\}$$

and

$$S_1(m_1,m_2) = S_1(m_2,m_3) = m_1,\ S_1(m_1,m_3) = m_2.$$

We claim that F satisfies (2.11) with $p = 1$ and $\xi = \frac{1}{2}$, that is,

$$d(Fm_i, S_1(Fm_i,Fm_j)) + d(S_1(Fm_i,Fm_j),Fm_j)$$

$$\leq \frac{1}{2}\left[d(m_i, S_1(m_i, m_j)) + d(S_1(m_i, m_j), m_j)\right] \tag{2.15}$$

for every $i, j \in \{1, 2, 3\}$.

Indeed, notice that for all $i \in \{1, 2, 3\}$, we have

$$\begin{aligned} & d(Fm_i, S_1(Fm_i, Fm_i)) + d(Fm_i, S_1(Fm_i, Fm_i)) \\ = & d(Fm_i, Fm_i) + d(Fm_i, Fm_i) = 0, \end{aligned}$$

which shows that (2.15) holds for all $i = j \in \{1, 2, 3\}$.

On the other hand, due to the symmetry of S_1, we just have to show that (2.15) holds for $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$. For this purpose, we divide it into three cases below.

Case 1. $(i, j) = (1, 2)$. In this case, we have

$$\begin{aligned} & d(Fm_i, S_1(Fm_i, Fm_j)) + d(Fm_j, S_1(Fm_i, Fm_j)) \\ = & d(Fm_1, S_1(Fm_1, Fm_2)) + d(Fm_2, S_1(Fm_1, Fm_2)) \\ = & d(m_1, S_1(m_1, m_1)) + d(m_1, S_1(m_1, m_1)) \\ = & d(m_1, m_1) + d(m_1, m_1) \\ = & 0, \end{aligned}$$

which shows that (2.15) is satisfied for $(i, j) = (1, 2)$.

Case 2. $(i, j) = (1, 3)$. In this case, we have

$$\begin{aligned} & \frac{d(Fm_i, S_1(Fm_i, Fm_j)) + d(Fm_j, S_1(Fm_i, Fm_j))}{d(m_i, S_1(m_i, m_j)) + d(m_j, S_1(m_i, m_j))} \\ = & \frac{d(Fm_1, S_1(Fm_1, Fm_3)) + d(Fm_3, S_1(Fm_1, Fm_3))}{d(m_1, S_1(m_1, m_3)) + d(m_3, S_1(m_1, m_3))} \\ = & \frac{d(m_1, S_1(m_1, m_2)) + d(m_2, S_1(m_1, m_2))}{d(m_1, S_1(m_1, m_3)) + d(m_3, S_1(m_1, m_3))} \\ = & \frac{d(m_1, m_1) + d(m_2, m_1)}{d(m_1, m_2) + d(m_3, m_2)} \\ = & \frac{1}{2}, \end{aligned}$$

which confirms (2.15).

Case 3. $(i, j) = (2, 3)$. In this case, we have

$$\begin{aligned} & \frac{d(Fm_i, S_1(Fm_i, Fm_j)) + d(Fm_j, S_1(Fm_i, Fm_j))}{d(m_i, S_1(m_i, m_j)) + d(m_j, S_1(m_i, m_j))} \\ = & \frac{d(Fm_2, S_1(Fm_2, Fm_3)) + d(Fm_3, S_1(Fm_2, Fm_3))}{d(m_2, S_1(m_2, m_3)) + d(m_3, S_1(m_2, m_3))} \\ = & \frac{d(m_1, S_1(m_1, m_2)) + d(m_2, S_1(m_1, m_2))}{d(m_2, S_1(m_2, m_3)) + d(m_3, S_1(m_2, m_3))} \\ = & \frac{d(m_1, m_1) + d(m_2, m_1)}{d(m_2, m_1) + d(m_3, m_1)} \\ = & \frac{1}{2}, \end{aligned}$$

which shows that (2.15) is also satisfied for $(i, j) = (2, 3)$.

From the above discussions, we deduce that (2.15) holds for every $i, j \in \{1, 2, 3\}$.

Since F is continuous (because M is a finite set), then F is weakly Picard continuous on (M, d). Consequently, Corollary 2.7 applies. Notice that

$$\mathrm{Fix}(F) = \{m_1\}$$

and

$$S_1(m_1, m_1) = m_1,$$

which confirms Corollary 2.7.

Example 2.14. Let us consider the mapping $F : [0, 1] \to [0, 1]$ defined by

$$Fu = \begin{cases} \frac{1}{2}e^{-u}, & \text{if} \quad 0 \le u < 1, \\ \frac{1}{2}, & \text{if} \quad u = 1. \end{cases}$$

Let d be the metric on $[0, 1]$ defined by

$$d(u, v) = |u - v|, \quad u, v \in [0, 1].$$

Notice that F is not continuous at $u = 1$, which implies that F is not a contraction. Then, the Banach fixed point theorem is not applicable in this case.

Let us introduce the mapping $S : [0, 1] \times [0, 1] \to [0, 1]$ defined by

$$S(u, v) = Fu, \quad u, v \in [0, 1].$$

We consider the following four cases.

Case 1. $0 \le u, v < 1$. Taking into consideration that

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}e^{-x}\right)\right| = \frac{1}{2}e^{-x} \le \frac{1}{2}, \quad 0 \le x \le 1 \tag{2.16}$$

and $F([0, 1)) \subset [0, 1)$, we obtain by the mean value theorem that

$$\begin{aligned} & d(Fu, S(Fu, Fv)) + d(S(Fu, Fv), Fv) \\ &= |Fu - F(Fu)| + |F(Fu) - Fv| \\ &\le \frac{1}{2}|u - Fu| + \frac{1}{2}|Fu - v| \\ &= \frac{1}{2}\left(|u - S(u, v)| + |S(u, v) - v|\right) \\ &= \frac{1}{2}\left(d(u, S(u, v)) + d(S(u, v), v)\right). \end{aligned}$$

Case 2. $0 \le u < 1$, $v = 1$. In this case, on the one hand, using that $F1 = F0$ and (2.16), we obtain by the mean value theorem that

$$d(Fu, S(Fu, F1)) + d(S(Fu, F1), F1)$$

$$
\begin{aligned}
&= |Fu - F(Fu)| + |F(Fu) - F0| \\
&\leq \frac{1}{2}|u - Fu| + \frac{1}{2}|Fu - 0| \\
&= \frac{1}{2}\left(d(u, S(u,1)) + Fu\right).
\end{aligned}
$$

On the other hand,

$$2Fu = \mathrm{e}^{-u} \leq 1,$$

which implies that

$$Fu \leq 1 - Fu = |Fu - 1| = d(S(u,1), 1).$$

Consequently, we have

$$d(Fu, S(Fu, F1)) + d(S(Fu, F1), F1) \leq \frac{1}{2}\left(d(u, S(u,1)) + d(S(u,1), 1)\right).$$

Case 3. $u = 1$, $0 \leq v < 1$. In this case, we obtain

$$
\begin{aligned}
&d(F1, S(F1, Fv)) + d(S(F1, Fv), Fv) \\
&= |F1 - F^2 1| + |F^2 1 - Fv| \\
&= |F0 - F(F0)| + |F(F0) - Fv| \\
&\leq \frac{1}{2}\left(|0 - F0| + |F0 - v|\right) \\
&= \frac{1}{2}\left(F1 + |F1 - v|\right) \\
&= \frac{1}{2}\left(F1 + d(S(1,v), v)\right).
\end{aligned}
$$

Moreover, we have

$$F1 = \frac{1}{2} = 1 - F1 = d(1, S(1,v)).$$

Consequently, we arrive at

$$d(F1, S(F1, Fv)) + d(S(F1, Fv), Fv) \leq \frac{1}{2}\left(d(1, S(1,v)) + d(S(1,v), v)\right).$$

Case 4. $u = v = 1$. In this case, we have

$$
\begin{aligned}
&d(F1, S(F1, F1)) + d(S(F1, F1), F1) \\
&= d(F1, F^2 1) + d(F^2 1, F1) \\
&= 2d(F0, F(F0)) \\
&\leq 2 \cdot \frac{1}{2}|0 - F0| \\
&= F0 \\
&= \frac{1}{2}
\end{aligned}
$$

$$= \frac{1}{2}(d(1, S(1,1)) + d(S(1,1),1)).$$

Thus, from the above discussion, we deduce that F satisfies (2.11) with $p = 1$, $S_1 = S$, and $\xi = \frac{1}{2}$. Furthermore, for all $u \in [0,1]$, we have $\{F^n u\} \subset [0, \frac{1}{2}]$. Then, by the continuity of F on $[0, \frac{1}{2}]$, if for some $u, v \in [0,1]$ we have

$$\lim_{n\to\infty} d(F^n u, v) = 0,$$

so $v \in [0, \frac{1}{2}]$, and

$$\lim_{n\to\infty} d(F(F^n u), Fv) = 0.$$

This shows that F is weakly Picard continuous on $([0,1], d)$. Finally, by Corollary 2.7, we deduce that F has one and only one fixed point $u^* \approx 0.3517$. Figure 1 confirms our conclusion.

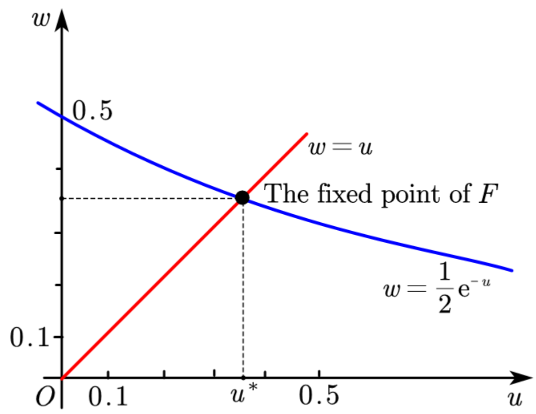

Figure 1. The unique fixed point of F.

2.2. An application: A discrete fixed point result

In this subsection, as an application of Theorem 2.6, we obtain sufficient conditions under which a mapping $F : \mathbb{N}^* \to \mathbb{N}^*$ admits a unique fixed point.

Theorem 2.15. *Let $F : \mathbb{N}^* \to \mathbb{N}^*$ be a mapping satisfying the inequality*

$$2^{F(n)+F(m)+1} - F(n) - F(m) \leq \xi \left(2^{n+m+1} - n - m\right) \tag{2.17}$$

for every $n, m \in \mathbb{N}^*$ with $F(n) \neq F(m)$, where $\xi \in [0,1)$ is a constant. Then, F possesses a unique fixed point.

Proof. Let us consider the mapping $S_1 : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ defined by

$$S_1(n,m) = \begin{cases} 2^{n+m}, & \text{if } n \neq m, \\ n, & \text{if } n = m. \end{cases}$$

We claim that

$$\begin{aligned} &|F(n) - S_1(F(n), F(m))| + |S_1(F(n), F(m)) - F(m)| \\ &\leq \xi \left(|n - S_1(n,m)| + |S_1(n,m) - m|\right) \end{aligned} \tag{2.18}$$

for every $n, m \in \mathbb{N}^*$. To this end, we discuss two cases.

Case 1. $F(n) = F(m)$. In this case, by the definition of S_1, we obtain

$$\begin{aligned}
&|F(n) - S_1(F(n), F(m))| + |S_1(F(n), F(m)) - F(m)| \\
&= |F(n) - S_1(F(n), F(n))| + |S_1(F(n), F(n)) - F(n)| \\
&= 2|F(n) - S_1(F(n), F(n))| \\
&= 2|F(n) - F(n)| \\
&= 0,
\end{aligned}$$

which shows that (2.18) holds in this case.

Case 2. $F(n) \neq F(m)$. In this case, $n \neq m$. By the definition of S_1, we obtain

$$\begin{aligned}
&|F(n) - S_1(F(n), F(m))| + |S_1(F(n), F(m)) - F(m)| \\
&= |F(n) - 2^{F(n)+F(m)}| + |2^{F(n)+F(m)} - F(m)| \\
&= 2^{F(n)+F(m)} - F(n) + 2^{F(n)+F(m)} - F(m) \\
&= 2^{F(n)+F(m)+1} - F(n) - F(m),
\end{aligned}$$

which implies by (2.17) that

$$|F(n) - S_1(F(n), F(m))| + |S_1(F(n), F(m)) - F(m)| \leq \xi \left(2^{n+m+1} - n - m\right). \tag{2.19}$$

Note that

$$\begin{aligned}
2^{n+m+1} - n - m &= 2^{n+m} - n + 2^{n+m} - m \\
&= |2^{n+m} - n| + |2^{n+m} - m|,
\end{aligned}$$

that is,

$$2^{n+m+1} - n - m = |n - S_1(n, m)| + |S_1(n, m) - m|. \tag{2.20}$$

Thus, by (2.19) and (2.20), we get (2.18). Consequently, F satisfies (2.11) with $k = p = 1$, $M = \mathbb{N}^*$, and

$$d(n, m) = |n - m|, \quad n, m \in M.$$

On the other hand, since F is defined on $\mathbb{N}^*$, then F is continuous on (M, d), which shows that condition (ii) of Corollary 2.7 is satisfied. Therefore, by Corollary 2.7, we obtain that F possesses a unique fixed point. This completes the proof of Theorem 2.15. □

We now illustrate Theorem 2.15 by an example in which the Banach fixed point theorem is not applicable.

Example 2.16. Let us consider the mapping $F : \mathbb{N}^* \to \mathbb{N}^*$ defined by

$$F(n) = \begin{cases} 1, & \text{if } n = 1, \\ n - 1, & \text{if } n \geq 2. \end{cases}$$

First, notice that

$$\lim_{n \to \infty} \frac{|F(n) - F(1)|}{n - 1} = \lim_{n \to \infty} \frac{n - 2}{n - 1} = 1,$$

which shows that there is no $\xi \in [0,1)$ such that

$$|F(n) - F(m)| \le \xi |n-m|$$

for every $n, m \in \mathbb{N}^*$. Consequently, F is not a contraction on $(\mathbb{N}^*, d)$, where $d(n,m) = |n-m|$ for every $n, m \in \mathbb{N}^*$. Thus, the Banach fixed point theorem is not applicable in this example.

Now, we show that F satisfies (2.17) for every $n, m \in \mathbb{N}^*$ with $F(n) \neq F(m)$. Notice that for every $n, m \in \mathbb{N}^*$, we have

$$F(n) \neq F(m) \Longleftrightarrow (n,m) \in V \cup V',$$

where

$$V = \{(n,m) \in \mathbb{N}^* \times \mathbb{N}^* : n = 1,\ m \ge 3;\ \text{or } n > m \ge 2\}$$

and

$$V' = \{(n,m) \in \mathbb{N}^* \times \mathbb{N}^* : (m,n) \in V\}.$$

Due to the symmetry of (2.17), without restriction of the generality, we may assume that $(n,m) \in V$. So, we have two possible cases.

Case 1. $n = 1$ and $m \ge 3$. In this case, we have

$$\begin{aligned}\frac{2^{F(n)+F(m)+1} - F(n) - F(m)}{2^{n+m+1} - n - m} &= \frac{2^{F(1)+F(m)+1} - F(1) - F(m)}{2^{1+m+1} - 1 - m}\\ &= \frac{2^{m+1} - m}{2^{m+2} - 1 - m}\\ &= \frac{2^{m+2} - 2m}{2\left(2^{m+2} - 1 - m\right)}\\ &\le \frac{1}{2}.\end{aligned}$$

Case 2. $n > m \ge 2$. In this case, we have

$$\begin{aligned}\frac{2^{F(n)+F(m)+1} - F(n) - F(m)}{2^{n+m+1} - n - m} &= \frac{2^{n+m-1} - n - m + 2}{2^{n+m+1} - n - m}\\ &= \frac{2^{n+m} - 2n - 2m + 4}{2(2^{n+m+1} - n - m)}\\ &\le \frac{1}{2}.\end{aligned}$$

From the above estimates, we deduce that

$$2^{F(n)+F(m)+1} - F(n) - F(m) \le \frac{1}{2}\left(2^{n+m+1} - n - m\right)$$

for every $n, m \in \mathbb{N}^*$ with $F(n) \neq F(m)$. Thus, F satisfies (2.17) with $\xi = \frac{1}{2}$, and Theorem 2.15 applies. On the other hand, we have $\mathrm{Fix}(F) = \{1\}$, which confirms Theorem 2.15.

3. The class of (ψ, Γ, α)-contractions

Before introducing the class of (ψ, Γ, α)-contractions, let us briefly recall some properties related to the Euler gamma function. For more details, we refer to Abramowitz and Stegun [21].

The Euler gamma function is the function Γ defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} \mathrm{e}^{-t}\, \mathrm{d}t, \quad s > 0.$$

An integration by parts shows that

$$\Gamma(s+1) = s\Gamma(s), \quad s > 0. \tag{3.1}$$

In particular, when $s = k \in \mathbb{N}^*$, we obtain

$$\Gamma(k+1) = k!.$$

For all $n \in \mathbb{N}$, we have

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!} \sqrt{\pi}. \tag{3.2}$$

The function Γ is ln-convex, i.e.,

$$\Gamma(\alpha s + (1-\alpha)t) \leq \Gamma^\alpha(s)\Gamma^{1-\alpha}(t), \quad s, t > 0,\ \alpha \in (0,1). \tag{3.3}$$

Let us denote by Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying

$$\psi(t) \geq ct^\tau, \quad t \geq 0, \tag{3.4}$$

where $c, \tau > 0$ are constants. Remark that by (3.4), if $\psi \in \Psi$, then

$$\psi(t) > 0, \quad t > 0. \tag{3.5}$$

Notice that no continuity assumption is imposed on $\psi \in \Psi$.

Definition 3.1. Let (M, d) be a metric space and $F : M \to M$ a mapping. We say that F is a (ψ, Γ, α)-contraction, if there exist $\alpha, \beta \in (0,1)$ and $\psi \in \Psi$ such that

$$\psi\left(\frac{\Gamma(d(Fu, Fv) + 1)}{\Gamma(d(Fu, Fv) + \alpha)}\right) \leq \beta\psi\left(\frac{\Gamma(d(u, v) + 1)}{\Gamma(d(u, v) + \alpha)}\right) \tag{3.6}$$

for every $u, v \in M$ with $Fu \neq Fv$.

Our second main result is the following fixed point theorem.

Theorem 3.2. *Let (M, d) be a complete metric space and $F : M \to M$ a mapping. Suppose that the following conditions hold:*

(i) *F is a (ψ, Γ, α)-contraction;*
(ii) *F is weakly Picard continuous on (M, d).*

Then:

(I) *For every* $u_0 \in M$*, the sequence* $\{F^n u_0\}$ *converges to a fixed point of* F*;*
(II) F *possesses a unique fixed point.*

Proof. (I) For an arbitrary $u_0 \in M$, let

$$u_n = F^n u_0, \quad n \in \mathbb{N}.$$

We distinguish two cases as follows:

Case 1. There exists $m \in \mathbb{N}$ such that

$$u_m = u_{m+1}.$$

In this case, we obtain $u_m = Fu_m$ and

$$u_n = u_m, \quad n \geq m,$$

which shows that $u_m \in \mathrm{Fix}(F)$ and $\{u_n\}$ converges to u_m.

Case 2. For all $n \in \mathbb{N}$, we have

$$u_n \neq u_{n+1},$$

that is,

$$d(Fu_{n-1}, Fu_n) > 0, \quad n \in \mathbb{N}^*.$$

In this case, using (3.6) with $(u, v) = (u_0, u_1)$, we obtain

$$\psi\left(\frac{\Gamma(d(Fu_0, Fu_1) + 1)}{\Gamma(d(Fu_0, Fu_1) + \alpha)}\right) \leq \beta\psi\left(\frac{\Gamma(d(u_0, u_1) + 1)}{\Gamma(d(u_0, u_1) + \alpha)}\right),$$

that is,

$$\psi\left(\frac{\Gamma(d(u_1, u_2) + 1)}{\Gamma(d(u_1, u_2) + \alpha)}\right) \leq \beta\psi\left(\frac{\Gamma(d(u_0, u_1) + 1)}{\Gamma(d(u_0, u_1) + \alpha)}\right). \tag{3.7}$$

Again using (3.6) with $(u, v) = (u_1, u_2)$, we obtain

$$\psi\left(\frac{\Gamma(d(Fu_1, Fu_2) + 1)}{\Gamma(d(Fu_1, Fu_2) + \alpha)}\right) \leq \beta\psi\left(\frac{\Gamma(d(u_1, u_2) + 1)}{\Gamma(d(u_1, u_2) + \alpha)}\right),$$

that is,

$$\psi\left(\frac{\Gamma(d(u_2, u_3) + 1)}{\Gamma(d(u_2, u_3) + \alpha)}\right) \leq \beta\psi\left(\frac{\Gamma(d(u_1, u_2) + 1)}{\Gamma(d(u_1, u_2) + \alpha)}\right). \tag{3.8}$$

Then, it follows from (3.7) and (3.8) that

$$\psi\left(\frac{\Gamma(d(u_2, u_3) + 1)}{\Gamma(d(u_2, u_3) + \alpha)}\right) \leq \beta^2\psi\left(\frac{\Gamma(d(u_0, u_1) + 1)}{\Gamma(d(u_0, u_1) + \alpha)}\right).$$

Repeating the same argument, we obtain by induction that

$$\psi\left(\frac{\Gamma(d(u_n, u_{n+1}) + 1)}{\Gamma(d(u_n, u_{n+1}) + \alpha)}\right) \leq \beta^n\psi\left(\frac{\Gamma(d(u_0, u_1) + 1)}{\Gamma(d(u_0, u_1) + \alpha)}\right), \quad n \in \mathbb{N}. \tag{3.9}$$

On the other hand, by (3.4), we have

$$\psi\left(\frac{\Gamma(d(u_n,u_{n+1})+1)}{\Gamma(d(u_n,u_{n+1})+\alpha)}\right) \geq c\left(\frac{\Gamma(d(u_n,u_{n+1})+1)}{\Gamma(d(u_n,u_{n+1})+\alpha)}\right)^{\tau},$$

which implies together with (3.9) that

$$\frac{\Gamma(d(u_n,u_{n+1})+1)}{\Gamma(d(u_n,u_{n+1})+\alpha)} \leq \beta^{\frac{n}{\tau}}\left[\frac{1}{c}\psi\left(\frac{\Gamma(d(u_0,u_1)+1)}{\Gamma(d(u_0,u_1)+\alpha)}\right)\right]^{\frac{1}{\tau}}, \quad n\in\mathbb{N}. \tag{3.10}$$

Furthermore, by the ln-convexity of Γ (see (3.3)), for any $n\in\mathbb{N}$, we have

$$\begin{aligned}\Gamma(d(u_n,u_{n+1})+\alpha) &= \Gamma\left((1-\alpha)d(u_n,u_{n+1})+\alpha(d(u_n,u_{n+1})+1)\right)\\ &\leq \Gamma^{1-\alpha}(d(u_n,u_{n+1}))\Gamma^{\alpha}(d(u_n,u_{n+1})+1).\end{aligned} \tag{3.11}$$

Subsequently, by (3.1), we have

$$\Gamma(d(u_n,u_{n+1})+1) = d(u_n,u_{n+1})\Gamma(d(u_n,u_{n+1})),$$

which yields

$$\Gamma^{1-\alpha}(d(u_n,u_{n+1})) = [d(u_n,u_{n+1})]^{\alpha-1}\Gamma^{1-\alpha}(d(u_n,u_{n+1})+1).$$

Hence, from (3.11), we deduce that

$$\begin{aligned}\Gamma(d(u_n,u_{n+1})+\alpha) &\leq [d(u_n,u_{n+1})]^{\alpha-1}\Gamma^{1-\alpha}(d(u_n,u_{n+1})+1)\Gamma^{\alpha}(d(u_n,u_{n+1})+1)\\ &= [d(u_n,u_{n+1})]^{\alpha-1}\Gamma(d(u_n,u_{n+1})+1),\end{aligned}$$

which implies that

$$\frac{\Gamma(d(u_n,u_{n+1})+1)}{\Gamma(d(u_n,u_{n+1})+\alpha)} \geq d^{1-\alpha}(u_n,u_{n+1}), \quad n\in\mathbb{N}. \tag{3.12}$$

Now, using both inequalities (3.10) and (3.12), we obtain

$$d^{1-\alpha}(u_n,u_{n+1}) \leq \beta^{\frac{n}{\tau}}\left[\frac{1}{c}\psi\left(\frac{\Gamma(d(u_0,u_1)+1)}{\Gamma(d(u_0,u_1)+\alpha)}\right)\right]^{\frac{1}{\tau}}, \quad n\in\mathbb{N},$$

which is equivalent to

$$d(u_n,u_{n+1}) \leq \sigma^n\psi_0, \quad n\in\mathbb{N}, \tag{3.13}$$

where

$$\sigma = \beta^{\frac{1}{\tau(1-\alpha)}} \in (0,1)$$

and

$$\psi_0 = \left[\frac{1}{c}\psi\left(\frac{\Gamma(d(u_0,u_1)+1)}{\Gamma(d(u_0,u_1)+\alpha)}\right)\right]^{\frac{1}{\tau(1-\alpha)}}.$$

Next, using (3.13) and the triangle inequality, we obtain that for all $n\in\mathbb{N}$ and $q\in\mathbb{N}^*$,

$$\begin{aligned}d(u_n,u_{n+q}) &\leq d(u_n,u_{n+1})+\cdots+d(u_{n+q-1},u_{n+q})\\ &\leq \left(\sigma^n+\cdots+\sigma^{n+q-1}\right)\psi_0\\ &= \frac{\sigma^n(1-\sigma^q)}{1-\sigma}\psi_0\\ &\leq \frac{\sigma^n}{1-\sigma}\psi_0,\end{aligned}$$

which shows (since $0<\sigma<1$) that $\{u_n\}$ is a Cauchy sequence. Then, by the completeness of (M,d), we infer that there exists $u^*\in M$ such that

$$\lim_{n\to\infty} d(F^n u_0, u^*) = 0.$$

Thus, using the weakly Picard continuity of F and proceeding as in the proof of Theorem 2.6, we obtain that $u^* \in \mathrm{Fix}(F)$. This proves part (I) of Theorem 3.2.

(II) From part (I), we know that $u^* \in \mathrm{Fix}(F)$. Assume that there exists another different $v^* \in \mathrm{Fix}(F)$ (so, $d(Fu^*, Fv^*) > 0$). Then, using (3.6), we obtain

$$\psi\left(\frac{\Gamma(d(Fu^*, Fv^*) + 1)}{\Gamma(d(Fu^*, Fv^*) + \alpha)}\right) \le \beta\psi\left(\frac{\Gamma(d(u^*, v^*) + 1)}{\Gamma(d(u^*, v^*) + \alpha)}\right),$$

that is,

$$\psi\left(\frac{\Gamma(d(u^*, v^*) + 1)}{\Gamma(d(u^*, v^*) + \alpha)}\right) \le \beta\psi\left(\frac{\Gamma(d(u^*, v^*) + 1)}{\Gamma(d(u^*, v^*) + \alpha)}\right). \tag{3.14}$$

Moreover, since

$$\frac{\Gamma(d(u^*, v^*) + 1)}{\Gamma(d(u^*, v^*) + \alpha)} > 0,$$

it follows immediately from (3.5) that

$$\psi\left(\frac{\Gamma(d(u^*, v^*) + 1)}{\Gamma(d(u^*, v^*) + \alpha)}\right) > 0.$$

Then, from (3.14), we reach a contradiction with $\beta \in (0, 1)$. Therefore, u^* is the unique fixed point of F. This proves part (II) of Theorem 3.2. □

In the special case when F is continuous on (M, d), we deduce from Theorem 3.2 the following result.

Corollary 3.3. *Let (M, d) be a complete metric space and $F : M \to M$ a mapping. Suppose that the following conditions hold:*

(i) *F is a (ψ, Γ, α)-contraction;*
(ii) *F is continuous.*

Then:

(I) *For every $u_0 \in M$, the sequence $\{F^n u_0\}$ converges to a fixed point of F;*
(II) *F possesses a unique fixed point.*

We now give an example to illustrate Theorem 3.2.

Example 3.4. Let $M = \{q_1, q_2, q_3\}$ and $F : M \to M$ be the mapping defined by

$$Fq_1 = q_1, \;\; Fq_2 = q_3, \;\; Fq_3 = q_1.$$

Let $d : M \times M \to [0, \infty)$ be the mapping defined by

$$d(q_i, q_i) = 0, \;\; d(q_i, q_j) = d(q_j, q_i), \quad i, j \in \{1, 2, 3\}$$

and

$$d(q_1, q_2) = 1, \ d(q_1, q_3) = 2, \ d(q_2, q_3) = 3. \tag{3.15}$$

Observe that

$$\begin{aligned} d(q_1, q_2) &= 1 < 5 = d(q_1, q_3) + d(q_3, q_2), \\ d(q_1, q_3) &= 2 < 4 = d(q_1, q_2) + d(q_2, q_3), \\ d(q_2, q_3) &= 3 = d(q_2, q_1) + d(q_1, q_3), \end{aligned}$$

which shows that d is a metric on M.

Notice that

$$\frac{d(Fq_1, Fq_2)}{d(q_1, q_2)} = \frac{d(q_1, q_3)}{d(q_1, q_2)} = 2 > 1,$$

which shows that there is no $\xi \in [0, 1)$ such that

$$d(Fq_i, Fq_j) \le \xi d(q_i, q_j)$$

for all $i, j \in \{1, 2, 3\}$. Consequently, the Banach fixed point theorem is not applicable in this case.

Consider now the function $\psi : [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = \begin{cases} \frac{\sqrt{\pi}}{2} t + 1, & \text{if } \ 0 \le t \le \frac{2}{\sqrt{\pi}}, \\ \frac{3\sqrt{\pi}}{8} t, & \text{if } \ \frac{2}{\sqrt{\pi}} < t \le \frac{8}{3\sqrt{\pi}}, \\ \frac{5\sqrt{\pi}}{32} t + \frac{5}{2}, & \text{if } \ t > \frac{8}{3\sqrt{\pi}}. \end{cases}$$

It can easily be seen that

$$\psi(t) \ge \frac{5\sqrt{\pi}}{32} t, \quad t \ge 0,$$

which shows that $\psi \in \Psi$ (indeed, ψ satisfies (3.4) with $c = \frac{5\sqrt{\pi}}{32}$ and $\tau = 1$).

From the definition of F, for all $i, j \in \{1, 2, 3\}$, we have

$$d(Fq_i, Fq_j) > 0 \Longleftrightarrow (i, j) \in \{(1, 2), (2, 1), (2, 3), (3, 2)\}. \tag{3.16}$$

Moreover, by (3.15), the definition of ψ, and (3.2), we obtain

$$\frac{\psi\left(\frac{\Gamma(d(Fq_1, Fq_2) + 1)}{\Gamma(d(Fq_1, Fq_2) + \frac{1}{2})}\right)}{\psi\left(\frac{\Gamma(d(q_1, q_2) + 1)}{\Gamma(d(q_1, q_2) + \frac{1}{2})}\right)} = \frac{\psi\left(\frac{\Gamma(3)}{\Gamma(2 + \frac{1}{2})}\right)}{\psi\left(\frac{\Gamma(2)}{\Gamma(1 + \frac{1}{2})}\right)} = \frac{\psi\left(\frac{8}{3\sqrt{\pi}}\right)}{\psi\left(\frac{2}{\sqrt{\pi}}\right)} = \frac{1}{2}$$

and

$$\frac{\psi\left(\frac{\Gamma(d(Fq_2, Fq_3) + 1)}{\Gamma(d(Fq_2, Fq_3) + \frac{1}{2})}\right)}{\psi\left(\frac{\Gamma(d(q_2, q_3) + 1)}{\Gamma(d(q_2, q_3) + \frac{1}{2})}\right)} = \frac{\psi\left(\frac{\Gamma(3)}{\Gamma(2 + \frac{1}{2})}\right)}{\psi\left(\frac{\Gamma(4)}{\Gamma(3 + \frac{1}{2})}\right)} = \frac{\psi\left(\frac{8}{3\sqrt{\pi}}\right)}{\psi\left(\frac{16}{5\sqrt{\pi}}\right)} = \frac{1}{3}.$$

Then, by (3.16), the above calculations, and the symmetry of d, we deduce that for all $i, j \in \{1,2,3\}$ and $\frac{1}{2} \leq \beta < 1$,

$$d(Fq_i, Fq_j) > 0 \implies \psi\left(\frac{\Gamma(d(Fq_i, Fq_j)+1)}{\Gamma(d(Fq_i, Fq_j)+\frac{1}{2})}\right) \leq \beta\psi\left(\frac{\Gamma(d(q_i, q_j)+1)}{\Gamma(d(q_i, q_j)+\frac{1}{2})}\right),$$

which means that F satisfies (3.6) with $\alpha = \frac{1}{2}$ and $\frac{1}{2} \leq \beta < 1$. Furthermore, since M is a finite set of elements, then F is continuous on M with respect to the metric d. This shows that Theorem 3.2 (or, more precisely, Corollary 3.3) applies. On the other hand, we have

$$\mathrm{Fix}(F) = \{q_1\},$$

which confirms Theorem 3.2.

4. Conclusions

The Banach fixed point theorem is a very important result in fixed point theory that has numerous applications in nonlinear analysis. However, when the mapping does not belong to the class of contractions, the theorem is inapplicable. Thus, the development of fixed point theory for non-contraction mappings is of great importance. Motivated by this fact, two new classes of self-mappings defined on a complete metric space (M, d) are introduced in this work. The first one is the class of p-contractions with respect to a family of mappings $\{S_i\}_{i=1}^k$, where $S_i : M \times M \to M$. For such a class of mappings, a fixed point theorem was established (see Theorem 2.6). Namely, we proved that, if $F : M \to M$ is weakly Picard continuous on (M, d) and F is a p-contraction with respect to $\{S_i\}_{i=1}^k$, then F possesses a unique fixed point, which is also a common fixed point of the mappings S_i $(i = 1, 2, \cdots, k)$. Moreover, for any $u_0 \in M$, the Picard sequence $\{F^n u_0\}$ converges to this fixed point. We also proved that our obtained result recovers the Banach fixed point theorem. The second class introduced in this work is the class of (ψ, Γ, α)-contractions, where Γ is the Euler gamma function, $\psi : [0, \infty) \to [0, \infty)$ satisfies condition (3.4), and $\alpha \in (0, 1)$. A fixed point theorem was proved for this class of mappings (see Theorem 3.2). Namely, as for the previous class of mappings, we proved that, if $F : M \to M$ is weakly Picard continuous on (M, d) and F is a (ψ, Γ, α)-contraction, then F admits a unique fixed point, and for any $u_0 \in M$, the sequence $\{F^n u_0\}$ converges to this fixed point.

Some questions related to this work need to be investigated. For this purpose, we provide below some interesting questions for further studies:

I. Consider the class of mappings $F : (M, d) \to (M, d)$ satisfying the inequality

$$\psi\left(\frac{\Gamma(d(Fu, Fv)+1)}{\Gamma(d(Fu, Fv)+\alpha)}\right) \leq \beta\left[\psi\left(\frac{\Gamma(d(u, Fu)+1)}{\Gamma(d(u, Fu)+\alpha)}\right) + \psi\left(\frac{\Gamma(d(v, Fv)+1)}{\Gamma(d(v, Fv)+\alpha)}\right)\right]$$

for all $u, v \in M$ with $d(Fu, Fv) > 0$, where $\alpha \in (0, 1)$ and $\beta > 0$ are constants. Is it possible to obtain sufficient conditions on β and ψ so that F admits one and only one fixed point?

II. We reiterate the previous question for the class of mappings $F : (M, d) \to (M, d)$ satisfying

$$\psi\left(\frac{\Gamma(d(Fu, Fv)+1)}{\Gamma(d(Fu, Fv)+\alpha)}\right) \leq \beta\left[\psi\left(\frac{\Gamma(d(u, Fv)+1)}{\Gamma(d(u, Fv)+\alpha)}\right) + \psi\left(\frac{\Gamma(d(v, Fu)+1)}{\Gamma(d(v, Fu)+\alpha)}\right)\right].$$

III. It would be also interesting to study the multi-valued version of Theorem 3.2 by considering the class of multi-valued mappings $F : M \to CB(M)$, where $CB(M)$ denotes the family of all nonempty bounded and closed subsets of M, satisfying the inequality

$$\psi\left(\frac{\Gamma(H(Fu,Fv)+1)}{\Gamma(H(Fu,Fv)+\alpha)}\right) \leq \beta\psi\left(\frac{\Gamma(d(u,v)+1)}{\Gamma(d(u,v)+\alpha)}\right),$$

where H is the Hausdorff-Pompeiu metric on $CB(M)$.

Author contributions

All authors contributed equally in this work. All authors have read and agreed to the submitted version of the manuscript.

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Conflict of interest

The authors declare no conflict of interest.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181
2. R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **10** (1968), 71–76. https://doi.org/10.2307/2316437
3. S. K. Chatterjea, Fixed-point theorems, *C. R. Acad. Bulg. Sci.*, **25** (1972), 727–730.
4. S. Reich, Kannan's fixed point theorem, *Boll. Unione. Mat. Ital.*, **4** (1971), 1–11.
5. S. Reich, Some remarks concerning contraction mappings, *Can. Math. Bull.*, **14** (1971), 121–124. https://doi.org/10.4153/CMB-1971-024-9
6. L. Ćirić, A generalization of Banach's contraction principle, *P. Am. Math. Soc.*, **45** (1974), 267–273. https://doi.org/10.1090/S0002-9939-1974-0356011-2
7. V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, *Nonlinear Anal. Forum*, **9** (2004), 43–53.
8. F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theorems via simulation functions, *Filomat*, **29** (2015), 1189–1194. https://doi.org/10.2298/FIL1506189K

9. J. Górnicki, Various extensions of Kannan's fixed point theorem, *J. Fix. Point Theory Appl.*, **20** (2018), 1–12. https://doi.org/10.1007/s11784-018-0500-2
10. E. Petrov, Fixed point theorem for mappings contracting perimeters of triangles, *J. Fix. Point Theory Appl.*, **25** (2023). https://doi.org/10.1007/s11784-023-01078-4
11. C. M. Păcurar, O. Popescu, Fixed point theorem for generalized Chatterjea type mappings, *Acta Math. Hung.*, **2024** (2024). https://doi.org/10.1007/s10474-024-01455-6
12. A. N. Branga, I. M. Olaru, Generalized contractions and fixed point results in spaces with altering metrics, *Mathematics*, **10** (2022). https://doi.org/10.3390/math10214083
13. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis*, **1** (1993), 5–11. Available from: `http://dml.cz/dmlcz/120469`.
14. Z. Djedid, S. Al-Sharif, M. Al-Khaleel, J. Jawdat, On solutions of differential and integral equations using new fixed point results in cone E_b-metric spaces, *Part. Differ. Equ. Appl. Math.*, **8** (2023). https://doi.org/10.1016/j.padiff.2023.100559
15. Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.*, **7** (2006), 289–297.
16. I. Altun, A. Erduran, Two fixed point results on F-metric spaces, *Topol. Algebra Appl.*, **10** (2022), 61–67. https://doi.org/10.1515/taa-2022-0114
17. V. Ozturk, S. Radenović, Hemi metric spaces and Banach fixed point theorem, *Appl. Gen. Topol.*, **25** (2024), 175–182. https://doi.org/10.4995/agt.2024.19780
18. F. Khojasteh, H. Khandani, Scrutiny of some fixed point results by S-operators without triangular inequality, *Math. Slovaca*, **70** (2020), 467–476. https://doi.org/10.1515/ms-2017-0364
19. V. Rakočević, K. Roy, M. Saha, Wardowski and Ćirić type fixed point theorems over non-triangular metric spaces, *Quaest. Math.*, **45** (2022), 1759–1769. https://doi.org/10.2989/16073606.2021.1970044
20. M. Berzig, First results in suprametric spaces with applications, *Mediterr. J. Math.*, **19** (2022), 1–18. https://doi.org/10.1007/s00009-022-02148-6
21. M. Abramowitz, I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications, New York, 1992.
22. I. V. Blagouchine, Two series expansions for the logarithm of the gamma function involving Stirling numbers and containing only rational coefficients for certain arguments related to π^{-1}, *J. Math. Anal. Appl.*, **442** (2016), 404–434. https://doi.org/10.1016/j.jmaa.2016.04.032
23. L. Gordon, A stochastic approach to the gamma function, *Am. Math. Mon.*, **101** (1994), 858–865. https://doi.org/10.1080/00029890.1994.11997039
24. S. S. Dragomir, *Operator inequalities of the Jensen, Cěbysěv and Grüss type*, Springer Briefs in Mathematics, Springer, New York, 2012. https://doi.org/10.1007/978-1-4614-1521-3