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Research article

Interval edge colorings of the generalized lexicographic product of some graphs

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Abstract: An edge-coloring of a graph G with colors $1, \dots, t$ is an interval t-coloring if all colors are used and the colors of edges incident to each vertex of G are distinct and form an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. For an interval colorable graph G, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$. Let G be a graph with vertex set $V(G) = \{u_1, \dots, u_m\}$, $m \geq 2$, and let $h_m = (H_i)_{i\in\{1,\dots,m\}}$ be a sequence of vertex-disjoint with $V(H_i) = \{x_1^{(i)}, \dots, x_{n_i}^{(i)}\}$, $n_i \geq 1$. The generalized lexicographic products $G[h_m]$ of G and h_m is a simple graph with vertex set $\cup_{i=1}^{m} V(H_i)$, in which $x_p^{(i)}$ is adjacent to $x_q^{(j)}$ if and only if either $u_i = u_j$ and $x_p^{(i)} x_q^{(i)} \in E(H_i)$ or $u_i u_j \in E(G)$. In this paper, we obtain several sufficient conditions for the generalized lexicographic product $G[h_m]$ to have interval colorings. We also present some sharp bounds on $w(G[h_m])$ and $W(G[h_m])$ of $G[h_m]$.

Keywords: generalized lexicographic product; interval edge coloring; path; empty graph; regular graph
Mathematics Subject Classification: 05C15

1. Introduction

All graphs considered in this paper are finite, undirected, and simple graphs. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of graph G, respectively. The degree of a vertex v in G is denoted by $d_G(v)$, the maximum degree of G by $\Delta(G)$, and the complement of the graph G by $\overline{G}$. We denote by $[a, b]$ the interval of integers $\{a, \dots, b\}$, and by $(x)_k$ the $x \pmod k$, where x is an integer and k is a positive integer. The terms and concepts that we do not define can be found in [3].

The notion of interval colorings was introduced in 1987 by Asratian and Kamalian [1]. This concept was introduced for studying the problems that are related to constructing timetables without "gaps" for teachers and classes. Hansen [6] presented a similar scenario: a school can schedule parent-teacher conferences in consecutive time slots if the bipartite graph, with parents and teachers as vertices and meetings as edges, has an interval coloring.

A proper edge coloring of a graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. The chromatic index $\chi'(G)$ of a graph G is the minimum number of colors used in a proper edge coloring of G. If α is a proper edge coloring of G and $v \in V(G)$, we denote by $S(v, \alpha)$ the set of colors of edges incident to v.

A proper edge coloring of a graph G with colors $1, \ldots, t$ is an interval t-coloring if all colors are used, and for any vertex v of G, the set $S(v, \alpha)$ is an interval of consecutive integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively.

The Vizing Theorem [19] states that $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. Graphs with $\chi'(G) = \Delta(G)$ are said to be Class 1; graphs with $\chi'(G) = \Delta(G) + 1$ are said to be Class 2. Asratian and Kamalian [1] proved that if $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$, that is, G is said to be Class 1. Not all Class 1 graphs are interval colorable; even a simple graph such as wheel $W_n(n \geq 3)$ is not necessarily interval edge colorable. Giaro et al. [4] proved that $W_n \in \mathfrak{N}$ if and only if $n = 4, 7, 10$; otherwise, $W_n \notin \mathfrak{N}$. Asratian et al. [2], Holyer [7], and Sevastjanov [16] proved that it is a NP-complete problem to decide whether a regular graph or a bipartite graph has an interval coloring or not. For a graph $G \in \mathfrak{N}$, the exact values of the parameters $W(G)$ and $w(G)$ are known only for paths, even cycles, trees, complete bipartite graphs [1, 9], and Möbius ladders [12]. However, for some common interval-colorable graphs, such as complete graphs and n-dimensional cubes [13], the exact upper and lower bounds on the number of colors remain unknown.

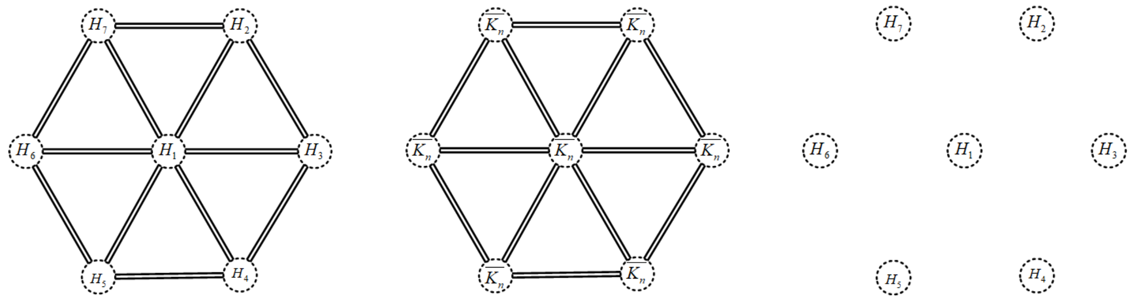
We know that the generalized lexicographic product of graphs is one of the important tools for constructing classes of graphs in graph theory. Common graph operations, such as the join of graphs and the lexicographic product of graphs, are essentially special cases of the generalized lexicographic product. However, to the best knowledge of the author, there are no studies related to the interval edge coloring of generalized lexicographic products. This fact motivates us to begin an exploration of the interval edge coloring of the generalized lexicographic product of graphs.

The generalized lexicographic product of graphs is defined as follows [17]: let G be a graph with vertex set $V(G) = \{u_1, \ldots, u_m\}$, $m \geq 2$, and let $h_m = (H_i)_{i \in \{1,\ldots,m\}}$ be a sequence of vertex-disjoint with $V(H_i) = \{x_1^{(i)}, \ldots, x_{n_i}^{(i)}\}$, $n_i \geq 1$. The generalized lexicographic products $G[h_m]$ of G and h_m is a simple graph with vertex set $\cup_{i=1}^{m} V(H_i)$, in which $x_p^{(i)}$ is adjacent to $x_q^{(j)}$ if and only if either $u_i = u_j$ and $x_p^{(i)} x_q^{(i)} \in E(H_i)$ or $u_i u_j \in E(G)$. If $H_i \cong H$ for every $i \in \{1, \ldots, m\}$, then $G[h_m]$ becomes the lexicographic product of G and H, denoted by $G[H]$. If $G \cong K_2$, then $G[h_2]$ denotes a join $H_1 + H_2$ of vertex disjoint graphs H_1 and H_2. If $G \cong K_m$, and every graph H_i is the complement of a complete graph with n_i vertices (that is, $H_i \cong \overline{k_{n_i}}$), then $G[h_m]$ denotes a complete m-partite graph $K_{n_1,\ldots,n_m}$. For convenience, we write $\widetilde{G}$ to denote $G[h_m]$.

Let $G^{(1)} = G[\overline{K_n}]$ and $G^{(2)} = \bigcup_{i=1}^{m} H_i$, then $\widetilde{G}$ can be decomposed into the union of two edge-disjoint subgraphs $G^{(1)}$ and $G^{(2)}$, that is,

$$\widetilde{G} = G^{(1)} \cup G^{(2)}.$$

Figure 1 shows the decomposition of $\widetilde{G} = W_7[h_7]$, where $h_7 = (H_i)_{i\in\{1,\dots,7\}}$ is a sequence of vertex-disjoint graphs, each with n vertices.



(1) $\widetilde{G} = W_7[h_7]$ (2) The subgraph $G^{(1)}$ of $\widetilde{G}$. (3) The subgraph $G^{(2)}$ of $\widetilde{G}$.

Figure 1. The decomposition of $\widetilde{G} = W_7[h_7]$ and its decomposition (each double line denotes the edges of $\widetilde{G}$ that join the vertices of H_i to the vertices of H_j).

Kamalian [10] proved that the complete bipartite graph $K_{m,n}$ has an interval t-coloring if and only if $m + n - \gcd(m, n) \le t \le m + n - 1$. Grzesik and Khachatrian [5] proved that complete tripartite graphs $K_{1,m,n}$ are interval colorable if and only if $\gcd(m+1, n+1) = 1$. Puning Jing et al. [8] extended the result and obtained several sufficient conditions for a complete tripartite graph $K_{l,m,n}$ to admit an interval coloring, where $\gcd(m, n)$ is the greatest common divisor of m and n. Petrosyan [14] investigated interval edge colorings of lexicographic products and obtained the following two results.

Theorem 1.1. *If $G \in \mathfrak{N}$, then $G[\overline{k_n}] \in \mathfrak{N}$, and $w(G[\overline{k_n}]) \le w(G)n$, $W(G[\overline{k_n}]) \ge (W(G) + 1)n - 1$.*

Theorem 1.2. *If $G, H \in \mathfrak{N}$ and H is a r-regular graph, then $G[H] \in \mathfrak{N}$, and $w(G[H]) \le w(G)|V(H)| + r$, $W(G[H]) \ge W(G)|V(H)| + r$.*

Yepremyan et al. [15] proved that if G is a tree and H is a path or a star, then $G[H] \in \mathfrak{N}$. Tepanyan et al. [18] proved that if $G \in \mathfrak{N}$, and H is a regular graph, complete bipartite graph, or tree, then $G[H] \in \mathfrak{N}$. They obtained the following results:

Theorem 1.3. *For all positive integers $n \ge 2$, if $G \in \mathfrak{N}$, then $G[C_{2n}] \in \mathfrak{N}$, and $w(G[C_{2n}]) \le 2(w(G)n+1)$, $W(G[C_{2n}]) \ge (2W(G) + 1)n + 1$.*

In this paper, our goal is to find sufficient conditions for a generalized lexicographic product $\widetilde{G}$ to admit an interval coloring. Moreover, we also hope to present some sharp bounds on $w(\widetilde{G})$ and $W(\widetilde{G})$, where G is an interval colorable graph with m vertices and all graphs in h_m have n vertices, $n \ge 4$.

The following two lemmas will be used later:

Lemma 1.1. *[11] If α is an edge-coloring of a connected graph G with colors $1, \dots, t$ such that the edges incident to each vertex $v \in V(G)$ are colored by distinct and consecutive colors and $\min\{\alpha(u_iu_j)|u_iu_j \in E(G)\} = 1$, $\max\{\alpha(u_iu_j)|u_iu_j \in E(G)\} = t$, then α is an interval t-coloring of G.*

Lemma 1.2. *Let α be an interval t-coloring of G, and let $\beta(u_iu_j) = t - \alpha(u_iu_j) + 1$ for every edge $u_iu_j \in E(G)$, then β is also an interval t-coloring of G.*

Proof. For any vertex $u_i \in V(G)$, we assume that $S(u_i, \alpha) = [a, a + d(u_i) - 1]$. By the definition of β, we have $S(u_i, \beta) = [t - a - d(u_i) + 2, t - a + 1]$, that is, β is an interval t-coloring of G. □

In Section 2, we obtained several sufficient conditions for $G[h_m]$ to admit an interval edge coloring, and we shall present the bounds of $w(\widetilde{G})$ and $W(\widetilde{G})$.

2. Main results

Let $G \in \mathfrak{N}$. If there exists a graph H_i in h_m that is not isomorphic to an empty graph, then let $w(H_{k_0}) = \max\{w(H_r)|H_r \in \mathfrak{N}, r = 1, \dots, m\}$. If H_i is isomorphic to a path, then let $H_i = x_1^{(i)} \dots x_n^{(i)}$; if H_i is isomorphic to a cycle, then let $H_i = x_1^{(i)} \dots x_n^{(i)} x_1^{(i)}$.

We define the following three classes of graphs:

(i) $\mathcal{G}_1$: each graph H_i in h_m is isomorphic to a path or an empty graph;

(ii) $\mathcal{G}_2$: each graph H_i in h_m is isomorphic to an empty graph or an interval-colorable regular graph, but not all graphs in h_m are empty graphs;

(iii) $\mathcal{G}_3$: each graph H_i in h_m is isomorphic to a path or a cycle or an empty graph of even order, and H_{k_0} is a cycle.

Let $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. And let $\widetilde{G} \in \mathcal{G}_z$, where $z = 1, 2, 3$. If there exists an interval t-coloring α satisfying the condition I, then we say that $\widetilde{G}$ belongs to the subclass $\mathcal{G}_z^1$ of $\mathcal{G}_z$, otherwise, we say that $\widetilde{G}$ belongs to the subclass $\mathcal{G}_z^2$ of $\mathcal{G}_z$. Clearly, we have

$$\mathcal{G}_z = \mathcal{G}_z^1 \cup \mathcal{G}_z^2, z = 1, 2, 3.$$

Condition I. There exists an edge $u_{i_0}u_{j_0} \in E(G)$ with $\alpha(u_{i_0}u_{j_0}) = 1$ or $\alpha(u_{i_0}u_{j_0}) = t$, such that $H_{i_0} \cong H_{k_0}$ or $H_{j_0} \cong H_{k_0}$.

Since $G \in \mathfrak{N}$, there exists an interval $w(G)$-coloring α_1 and an interval $W(G)$-coloring α_2 of G. For any vertex $u_i \in V(G)$, denote by $\tau_i(\alpha_l)$ the $\min S(u_i, \alpha_l)$, and by $\tau_i'(\alpha_l)$ the $\max S(u_i, \alpha_l)$, where $l = 1, 2$, $i = 1, \dots, m$.

We extend Theorem 1.1 from the lexicographic product of graphs to the generalized lexicographic product of graphs and obtain the following result:

Theorem 2.1. *If $\widetilde{G} \in \mathcal{G}_1$, then $\widetilde{G} \in \mathfrak{N}$. Moreover,*

$$w(\widetilde{G}) \le w(G)n + 2(\lambda)_2 + (\lambda + 1)_2(n)_2, W(\widetilde{G}) \ge (W(G) + 1)n - \lambda,$$

where, $\lambda = 1$ if $\widetilde{G} \in \mathcal{G}_1^1$, $\lambda = 2$ if $\widetilde{G} \in \mathcal{G}_1^2$.

Proof. If H_i is isomorphic to a path, then let $H_i = x_1^{(i)} \dots x_n^{(i)}$. For the proof, we consider the following two cases:

Case 1. $\widetilde{G} \in \mathcal{G}_1^1$**.**

Then Condition I holds and $\lambda = 1$. By Lemma 1.2, we can assume that $\alpha(u_{i_0}u_{j_0}) = 1$. Obviously, $H_{k_0} \cong P_n$, $w(G)n + 2(\lambda)_2 + (\lambda + 1)_2(n)_2 = w(G)n + 2, (W(G) + 1)n - \lambda = (W(G) + 1)n - 1$.

Now, we prove that $w(\widetilde{G}) \le w(G)n + 2$. For this, we define an edge-coloring β_1 of the graph $\widetilde{G}$ in the following two steps:

Step 1. For every edge $x_p^{(i)}x_q^{(j)} \in E(G^{(1)})$, $1 \le p, q \le n$, if $(n)_2 = 1$, then let

$$\beta_1(x_p^{(i)}x_q^{(j)}) = \begin{cases} (\alpha_1(u_iu_j) - 1)n + q + 1, & p = 1; \\ (\alpha_1(u_iu_j) - 1)n + (p + q - 3)_n + 3, & 2 \le p \le n. \end{cases}$$

Otherwise, let

$$\beta_1(x_p^{(i)}x_q^{(j)}) = \begin{cases}(\alpha_1(u_iu_j)-1)n+(p+q-1)_n+2, & p+q\neq n+1;\\ \alpha_1(u_iu_j)n+2, & p+q=n+1.\end{cases}$$

Step 2. For every edge $x_p^{(i)}x_{p+1}^{(i)} \in E(G^{(2)})$, $1 \le p \le n-1$, let

$$\beta_1(x_p^{(i)}x_{p+1}^{(i)}) = \begin{cases}(\tau_i(\alpha_1)-1)n+(p+1)_2+1, (n)_2=1;\\ (\tau_i(\alpha_1)-1)n+(p)_2+1, (n)_2=0.\end{cases}$$

Let us prove that β_1 is an interval edge coloring of the graph $\widetilde{G}$.

First, we prove that the set $S(x_p^{(i)},\beta_1)$ is an interval for each vertex $x_p^{(i)} \in V(\widetilde{G})$, where $1 \le i \le m$, $1 \le p \le n$.

If $(n)_2 = 1$ and H_i is isomorphic to a path, by the definition of β_1, we have

$$\begin{aligned}S(x_1^{(i)},\beta_1) &= \{(\tau_i(\alpha_1)-1)n+1+1,\dots,(\tau_i'(\alpha_1)-1)n+n+1\}\cup\{(\tau_i(\alpha_1)-1)n+1\}\\ &= [(\tau_i(\alpha_1)-1)n+1,\tau_i'(\alpha_1)n+1];\end{aligned}$$

$$\begin{aligned}S(x_p^{(i)},\beta_1) &= \{(\tau_i(\alpha_1)-1)n+3,\dots,(\tau_i'(\alpha_1)-1)n+(n-1)+3\}\\ &\quad\cup\{(\tau_i(\alpha_1)-1)n+(p+1)_2+1\}\cup\{(\tau_i(\alpha_1)-1)n+(p)_2+1\\ &= [(\tau_i(\alpha_1)-1)n+1,\tau_i'(\alpha_1)n+2], 2\le p\le n-1;\end{aligned}$$

$$\begin{aligned}S(x_n^{(i)},\beta_1) &= \{(\tau_i(\alpha_1)-1)n+3,\dots,(\tau_i'(\alpha_1)-1)n+(n-1)+3\}\cup\{(\tau_i(\alpha_1)-1)n+(n+1)_2+1\}\\ &= [(\tau_i(\alpha_1)-1)n+2,\tau_i'(\alpha_1)n+2];\end{aligned}$$

If $(n)_2 = 1$ and $H_i \cong \overline{K_n}$, by the definition of β_1, we have

$$S(x_1^{(i)},\beta_1) = \{(\tau_i(\alpha_1)-1)n+1+1,\dots,(\tau_i'(\alpha_1)-1)n+n+1\} = [(\tau_i(\alpha_1)-1)n+2,\tau_i'(\alpha_1)n+1];$$

$$S(x_p^{(i)},\beta_1) = \{(\tau_i(\alpha_1)-1)n+3,\dots,(\tau_i'(\alpha_1)-1)n+(n-1)+3\} = [(\tau_i(\alpha_1)-1)n+3,\tau_i'(\alpha_1)n+2], 2\le p\le n.$$

If $(n)_2 = 0$ and H_i is isomorphic to a path, by the definition of β_1, we have

$$\begin{aligned}S(x_1^{(i)},\beta_1) = S(x_n^{(i)},\beta_1) &= \{(\tau_i(\alpha_1)-1)n+1+1,\dots,\tau_i'(\alpha_1)n+2\}\cup\{(\tau_i(\alpha_1)-1)n+2\}\\ &= [(\tau_i(\alpha_1)-1)n+2,\tau_i'(\alpha_1)n+2];\end{aligned}$$

$$\begin{aligned}S(x_p^{(i)},\beta_1) &= \{(\tau_i(\alpha_1)-1)n+2,\dots,\tau_i'(\alpha_1)n+2\}\\ &\quad\cup\{(\tau_i(\alpha_1)-1)n+(p-1)_2+1\}\cup\{(\tau_i(\alpha_1)-1)n+(p)_2+1\\ &= [(\tau_i(\alpha_1)-1)n+1,\tau_i'(\alpha_1)n+2],\ 2\le p\le n-1.\end{aligned}$$

If $(n)_2 = 0$ and $H_i \cong \overline{K_n}$, by the definition of β_1, we have

$$S(x_1^{(i)},\beta_1) = S(x_n^{(i)},\beta_1) = \{(\tau_i(\alpha_1)-1)n+1+2,\dots,\tau_i'(\alpha_1)n+2\} = [(\tau_i(\alpha_1)-1)n+3,\tau_i'(\alpha_1)n+2];$$

$$S(x_p^{(i)},\beta_1) = \{(\tau_i(\alpha_1)-1)n+1+2,\dots,\tau_i'(\alpha_1)n+2\} = [(\tau_i(\alpha_1)-1)n+3,\tau_i'(\alpha_1)n+2], 2\le p\le n-1.$$

Second, note that

$$d_{\widetilde{G}}(x_p^{(i)}) = d_H(x_p^{(i)}) + d_G(x_p^{(i)})n = \begin{cases} (\tau_i'(\alpha_1) - \tau_i(\alpha_1))n + n, H_i \cong \overline{K_n}; \\ (\tau_i'(\alpha_1) - \tau_i(\alpha_1))n + n + 1, H_i \cong P_n, and p = 1 \; or \; p = n; \\ (\tau_i'(\alpha_1) - \tau_i(\alpha_1))n + n + 2, H_i \cong P_n, and 2 \le p \le n-1. \end{cases}$$

Clearly, we have

$$\max S(x_p^{(i)}, \beta_1) - \min S(x_p^{(i)}, \beta_1) = d_{\widetilde{G}}(x_p^{(i)}) - 1, 1 \le p \le n.$$

This implies that β_1 is a proper edge coloring of $\widetilde{G}$.

Finally, we show that in the coloring β_1, all colors are used. Clearly, there exists an edge $x_{p_0}^{(i_0)} x_{p_0+1}^{(i_0)} \in E(\widetilde{G})$ such that $\beta_1(x_{p_0}^{(i_0)} x_{p_0+1}^{(i_0)}) = 1$, since in the coloring α_1 there exists a vertex u_{i_0} such that $\tau_{i_0}(\alpha_1) = 1$, and since $\beta_1(x_1^{(i_0)} x_2^{(i_0)}) = (\tau_{i_0}(\alpha_1) - 1)n + 1$ when $(n)_2 = 1$ and $\beta_1(x_2^{(i_0)} x_3^{(i_0)}) = (\tau_{i_0}(\alpha_1) - 1)n + 1$ when $(n)_2 = 0$. Similarly, there exists an edge $x_{p_1}^{(i_1)} x_{q_1}^{(j_1)} \in E(\widetilde{G})$ such that $\beta_1(x_{p_1}^{(i_1)} x_{q_1}^{(j_1)}) = w(G)n + 2$, since in the coloring α_1 there exists an edge $u_{i_1} u_{j_1} \in E(G)$ with $\alpha_1(u_{i_1} u_{j_1}) = w(G)$, and since $\beta_1(x_2^{(i_1)} x_n^{(j_1)}) = (\alpha_1(u_{i_1} u_{j_1}) - 1)n + (n + 2 - 3)_n + 3 = w(G)n + 2$ when $(n)_2 = 1$ and $\beta_1(x_1^{(i_1)} x_n^{(j_1)}) = (\alpha_1(u_{i_1} u_{j_1}) - 1)n + (n + 1 - 3)_n + 3 = w(G)n + 2$ when $(n)_2 = 0$.

Now, by Lemma 1.1, we have that β_1 is an interval $(w(G)n + 2)$-edge coloring of the graph $\widetilde{G}$.

Next, we prove that $W(\widetilde{G}) \ge (W(G) + 1)n - 1$. For this, we define an edge-coloring β_2 of the graph $\widetilde{G}$ in the following two steps:

Step 1. For every edge $x_p^{(i)} x_q^{(j)} \in E(G^{(1)})$, $1 \le p, q \le n$, let

$$\beta_2(x_p^{(i)} x_q^{(j)}) = \begin{cases} (\alpha_2(u_i u_j) - 1)n + p + q, & p + q \ne 2n; \\ \alpha_2(u_i u_j)n, & p + q = 2n. \end{cases}$$

Step 2. For every edge $x_p^{(i)} x_{p+1}^{(i)} \in E(G^{(2)})$, $1 \le p \le n - 1$, let

$$\beta_2(x_p^{(i)} x_{p+1}^{(i)}) = (\tau_i(\alpha_2) - 1)n + p, p = 1, \cdots, n - 1.$$

Let us prove that β_2 is an interval edge coloring of $\widetilde{G}$.

First, we prove that the set $S(x_p^{(i)}, \beta_2)$ is an interval for each vertex $x_p^{(i)} \in V(\widetilde{G})$, where $i = 1, \ldots, m$, $p = 1, \ldots, n$.

If H_i is isomorphic to a path, by the definition of β_2, we have

$$\begin{aligned} S(x_1^{(i)}, \beta_2) &= \{(\tau_i(\alpha_2) - 1)n + 1 + 1, \ldots, (\tau_i'(\alpha_2) - 1)n + n + 1\} \cup \{(\tau_i(\alpha_2) - 1)n + 1\} \\ &= [(\tau_i(\alpha_2) - 1)n + 1, \tau_i'(\alpha_2)n + 1]; \end{aligned}$$

$$\begin{aligned} S(x_p^{(i)}, \beta_2) &= \{(\tau_i(\alpha_2) - 1)n + p + 1, \ldots, (\tau_i'(\alpha_2) - 1)n + p + n\} \\ &\cup \{(\tau_i(\alpha_2) - 1)n + p - 1\} \cup \{(\tau_i(\alpha_2) - 1)n + p \\ &= [(\tau_i(\alpha_2) - 1)n + p - 1, \tau_i'(\alpha_2)n + p], 2 \le p \le n - 1; \end{aligned}$$

$$\begin{aligned} S(x_n^{(i)}, \beta_2) &= \{\tau_i(\alpha_2)n, \ldots, (\tau_i'(\alpha_2) - 1)n + n + n - 1\} \cup \{(\tau_i(\alpha_2) - 1)n + n - 1\} \\ &= [\tau_i(\alpha_2)n, (\tau_i'(\alpha_2) + 1)n - 1]; \end{aligned}$$

If $H_i \cong \overline{K_n}$, by the definition of β_2, we have

$$S(x_1^{(i)}, \beta_2) = \{(\tau_i - 1)n + 1 + 1, \dots, (\tau_i' - 1)n + n + 1\} = [(\tau_i - 1)n + 2, \tau_i' n + 1];$$

$$\begin{aligned} S(x_p^{(i)}, \beta_2) &= \{(\tau_i(\alpha_2) - 1)n + p + 1, \dots, (\tau_i'(\alpha_2) - 1)n + p + n\} \\ &= [(\tau_i(\alpha_2) - 1)n + p + 1, \tau_i'(\alpha_2)n + p], 2 \le p \le n - 1; \end{aligned}$$

$$S(x_n^{(i)}, \beta_2) = \{\tau_i(\alpha_2)n, \dots, (\tau_i'(\alpha_2) - 1)n + n + n - 1\} = [\tau_i(\alpha_2)n, (\tau_i'(\alpha_2) + 1)n - 1];$$

Second, note that

$$d_{\widetilde{G}}(x_p^{(i)}) = d_H(x_p^{(i)}) + d_G(x_p^{(i)})n = \begin{cases} (\tau_i'(\alpha_2) - \tau_i(\alpha_2))n + n, H_i \cong \overline{K_n}; \\ (\tau_i'(\alpha_2) - \tau_i(\alpha_2))n + n + 1, H_i \cong P_n, and\ p = 1\ or\ p = n; \\ (\tau_i'(\alpha_2) - \tau_i(\alpha_2))n + n + 2, H_i \cong P_n, and\ 2 \le p \le n - 1. \end{cases}$$

Clearly, we have

$$\max S(x_p^{(i)}, \beta_2) - \min S(x_p^{(i)}, \beta_2) = d_{\widetilde{G}}(x_p^{(i)}) - 1, 1 \le p \le n.$$

This implies that β_2 is a proper edge coloring of $\widetilde{G}$.

Finally, we show that in the coloring β_2, all colors are used. Clearly, there exists an edge $x_{p_0}^{(i_0)} x_{p_0+1}^{(i_0)} \in E(\widetilde{G})$ such that $\beta_2(x_{p_0}^{(i_0)} x_{p_0+1}^{(i_0)}) = 1$, since in the coloring α_2 there exists a vertex u_{i_0} such that $\tau_{i_0}(\alpha_2) = 1$, and since $\beta_2(x_1^{(i_0)} x_2^{(i_0)}) = (\tau_{i_0}(\alpha_2) - 1)n + 1$. Similarly, there exists an edge $x_{p_1}^{(i_1)} x_{q_1}^{(j_1)} \in E(\widetilde{G})$ such that $\beta_2(x_{p_1}^{(i_1)} x_{q_1}^{(j_1)}) = (W(G) + 1)n - 1$, since in the coloring α_2 there exists an edge $u_{i_1} u_{j_1} \in E(G)$ with $\alpha_2(u_{i_1} u_{j_1}) = W(G)$, and since $\beta_2(x_{n-1}^{(i_1)} x_n^{(j_1)}) = (\alpha_2(u_{i_1} u_{j_1}) - 1)n + 2n - 1 = (W(G) + 1)n - 1$.

Now, by Lemma 1.1, we have that β_2 is an interval $((W(G)n + 1)n - 1)$-edge coloring of the graph $\widetilde{G}$.

Case 2. $\widetilde{G} \in \mathcal{G}_1^2$.

Then Condition I holds and $\lambda = 2$. By Lemma 1.2, we can assume that $\alpha(u_{i_0} u_{j_0}) = 1$. It is easy to see that $w(G)n + 2(\lambda)_2 + (\lambda + 1)_2(n)_2 = w(G)n + (n)_2, (W(G) + 1)n - \lambda = (W(G) + 1)n - 2$.

Now, we prove that $w(\widetilde{G}) \le w(G)n + (n)_2$. For this, we define an edge-coloring β_3 of the graph $\widetilde{G}$ in the following two steps:

Step 1. For every edge $x_p^{(i)} x_q^{(j)} \in E(G^{(1)})$, $1 \le p, q \le n$, if $(n)_2 = 1$, then let

$$\beta_3(x_p^{(i)} x_q^{(j)}) = \begin{cases} (\alpha_1(u_i u_j) - 1)n + q, & p = 1; \\ (\alpha_1(u_i u_j) - 1)n + (p + q - 3)_n + 2, & 2 \le p \le n, \end{cases}$$

if $(n)_2 = 0$, then let

$$\beta_3(x_p^{(i)} x_q^{(j)}) = \begin{cases} (\alpha_1(u_i u_j) - 1)n + (p + q - 1)_n, & p + q \ne n + 1; \\ \alpha_1(u_i u_j)n, & p + q = n + 1. \end{cases}$$

Step 2. For every edge $x_p^{(i)} x_{p+1}^{(i)} \in E(G^{(2)})$, $1 \le p \le n - 1$, let

$$\beta_3(x_p^{(i)} x_{p+1}^{(i)}) = \begin{cases} (\tau_i(\alpha_1) - 1)n + (p + 1)_2, (n)_2 = 1; \\ (\tau_i(\alpha_1) - 1)n - (p + 1)_2, (n)_2 = 0. \end{cases}$$

Let us prove that β_3 is an interval $(w(G)n + (n)_2)$-edge coloring of the graph $\widetilde{G}$.

First, for any edge $x_p^{(i)}x_q^{(j)} \in E(\widetilde{G})$, by the definitions of colorings β_1 and β_3, if $(n)_2 = 1$, then $\beta_3(x_p^{(i)}x_q^{(j)}) = \beta_1(x_p^{(i)}x_q^{(j)}) - 1$, otherwise, $\beta_3(x_p^{(i)}x_q^{(j)}) = \beta_1(x_p^{(i)}x_q^{(j)}) - 2$. Since β_1 is an interval edge coloring of $\widetilde{G}$, the coloring β_3 is a proper edge coloring of $\widetilde{G}$ and the color set of each vertex forms an integer interval.

Next, we show that in the coloring β_3, all colors are used. Clearly, there exists an edge $x_{p_0}^{(i_0)}x_{q_0}^{(j_0)} \in E(\widetilde{G})$ such that $\beta_3(x_{p_0}^{(i_0)}x_{q_0}^{(j_0)}) = 1$, since in the coloring α_1 there exists a vertex u_{i_0} such that $\tau_{i_0}(\alpha_1) = 1$, and since $\beta_3(x_1^{(i_0)}x_1^{(j_0)}) = (\tau_{i_0}(\alpha_1) - 1)n + 1$. Similarly, there exists an edge $x_{p_1}^{(i_1)}x_{q_1}^{(j_1)} \in E(\widetilde{G})$ such that $\beta_3(x_{p_1}^{(i_1)}x_{q_1}^{(j_1)}) = w(G)n + (n)_2$, since in the coloring α_1 there exists an edge $u_{i_1}u_{j_1} \in E(G)$ with $\alpha_1(u_{i_1}u_{j_1}) = w(G)$, and since $\beta_3(x_2^{(i_1)}x_n^{(j_1)}) = (w(G) - 1)n + n + 1 = w(G)n + 1$ when $(n)_2 = 1$ and $\beta_3(x_1^{(i_1)}x_n^{(j_1)}) = \alpha_1(u_{i_1}u_{j_1})n = w(G)n$ when $(n)_2 = 0$.

Now, by Lemma 1.1, we have that β_3 is an interval $(w(G)n + (n)_2)$-edge coloring of the graph $\widetilde{G}$.

Next, we prove that $W(\widetilde{G}) \geq (W(G) + 1)n - 2$. For this, we define an edge-coloring β_4 of the graph $\widetilde{G}$ in the following two steps:

Step 1. For every edge $x_p^{(i)}x_q^{(j)} \in E(G^{(1)})$, $1 \leq p, q \leq n$, let

$$\beta_4(x_p^{(i)}x_q^{(j)}) = \begin{cases} (\alpha_2(u_iu_j) - 1)n + p + q - 1, & p + q \neq 2n; \\ \alpha_2(u_iu_j)n - 1, & p + q = 2n. \end{cases}$$

Step 2. For every edge $x_p^{(i)}x_{p+1}^{(i)} \in E(G^{(2)})$, $1 \leq p \leq n - 1$, let

$$\beta_4(x_p^{(i)}x_{p+1}^{(i)}) = (\tau_i(\alpha_2) - 1)n + p - 1.$$

Let us prove that β_4 is an interval $((W(G) + 1)n - 2)$-edge coloring of the graph $\widetilde{G}$.

First, for any edge $x_p^{(i)}x_q^{(j)} \in E(\widetilde{G})$, by the definitions of colorings β_2 and β_4, we have $\beta_4(x_p^{(i)}x_q^{(j)}) = \beta_2(x_p^{(i)}x_q^{(j)}) - 1$. Since β_2 is an interval edge coloring of $\widetilde{G}$, the coloring β_4 is a proper edge coloring of $\widetilde{G}$ and the color set of each vertex forms an integer interval.

Next, we show that in the coloring β_4, all colors are used. Clearly, there exists an edge $x_{p_0}^{(i_0)}x_{q_0}^{(j_0)} \in E(\widetilde{G})$ such that $\beta_4(x_{p_0}^{(i_0)}x_{q_0}^{(j_0)}) = 1$, since in the coloring α_2 there exists a vertex u_{i_0} such that $\tau_{i_0}(\alpha_2) = 1$, and since $\beta_4(x_1^{(i_0)}x_1^{(j_0)}) = (\tau_{i_0}(\alpha_2) - 1)n + 1$. Similarly, there exists an edge $x_{p_1}^{(i_1)}x_{q_1}^{(j_1)} \in E(\widetilde{G})$ such that $\beta_4(x_{p_1}^{(i_1)}x_{q_1}^{(j_1)}) = (W(G) + 1)n - 2$, since in the coloring α_2 there exists an edge $u_{i_1}u_{j_1} \in E(G)$ with $\alpha_2(u_{i_1}u_{j_1}) = W(G)$, and since $\beta_4(x_{n-1}^{(i_1)}x_n^{(j_1)}) = (\alpha_2(u_{i_1}u_{j_1}) - 1)n + 2n - 2 = (W(G) + 1)n - 2$.

Now, by Lemma 1.1, we have that β_4 is an interval $((W(G) + 1)n - 2)$-edge coloring of the graph $\widetilde{G}$.

□

From Theorem 2.1, one can easily see that if $(n)_2 = 0$ and $H_i \cong \overline{k_n}$, then Theorem 2.1 can obtain the same upper bound on $w(\widetilde{G})$ as Theorem 1.1.

We extend the result of Theorem 1.2 from the lexicographic product of graphs to the generalized lexicographic product of graphs and obtain the following result:

Theorem 2.2. *If $\widetilde{G} \in \mathcal{G}_2$, then $\widetilde{G} \in \mathfrak{N}$. Moreover, $w(\widetilde{G}) \leq w(G)n + w(H_{k_0})$, $W(\widetilde{G}) \geq W(G)n + w(H_{k_0})$.*

Proof. For the proof, let α be an interval t-coloring of G, and we consider the following two cases.

Case 1. $\widetilde{G} \in \mathcal{G}_2^1$.

Then Condition I holds, and by Lemma 1.2, we can assume that $\alpha(u_{i_0}u_{j_0}) = 1$.

Let

$$C_1 = \{w(H_{k_0}) + 1, \ldots, w(H_{k_0}) + tn\};\ C_2 = \{1, \ldots, w(H_{k_0})\};$$

$$C_3^{(i)} = \{w(H_{k_0}) - (w(H_i) - 1), \ldots, w(H_{k_0})\}, i \in \{1, \ldots, m\}\backslash\{k_0\}.$$

Obviously, $C_1 \cap C_2 = \varnothing$, $C_3^{(i)} \subseteq C_2$, and $|C_3^{(i)}| = w(H_i)$.

Now, we construct an edge coloring β of $\widetilde{G}$ as follows.

First, we define the following edge-coloring β_1 of the graph $G^{(1)}$ with nt colors in C_1. For every edge $x_p^{(i)}x_q^{(j)} \in E(G^{(1)})$, $1 \le p, q \le n$, let

$$\beta_1(x_p^{(i)}x_q^{(j)}) = \begin{cases} w(H_{k_0}) + (\alpha(u_iu_j) - 1)n + (p + q - 1)_n, & p + q \neq n + 1; \\ w(H_{k_0}) + \alpha(u_iu_j)n, & p + q = n + 1. \end{cases}$$

Second, we define the following edge-coloring β_2 of the graph $G^{(2)}$. If $H_i \cong H_{k_0}$, we color the edges of H_{k_0} with $w(H_{k_0})$ colors in C_2 such that the colors on the edges incident to any vertex are consecutive; if $H_i \ncong H_{k_0}$, we color the edges of H_i with $w(H_i)$ colors in $C_3^{(i)}$ such that the colors on the edges incident to any vertex are consecutive.

Finally, for every edge $e \in E(\widetilde{G})$, let

$$\beta(e) = \begin{cases} \beta_1(e), & e \in E(G^{(1)}); \\ \beta_2(e), & e \in E(G^{(2)}). \end{cases}$$

Let us prove that β is an interval $(tn + w(H_{k_0}))$-edge coloring of $\widetilde{G}$. It is easy to see that the set $S(x_p^{(i)}, \beta)$ is an interval for each vertex $x_p^{(i)} \in V(\widetilde{G})$, since both $S(x_p^{(i)}, \beta_1)$ and $S(x_p^{(i)}, \beta_2)$ are intervals, and since $\min S(x_p^{(i)}, \beta_1) = \max S(x_p^{(i)}, \beta_2) + 1$.

Next, we show that in the coloring β all colors are used. Clearly, there exists an edge $x_{p_0}^{(i_0)}x_{q_0}^{(j_0)} \in E(\widetilde{G})$ such that $\beta(x_{p_0}^{(i_0)}x_{q_0}^{(j_0)}) = 1$, since in the coloring α there exists an edge $u_{i_0}u_{j_0} \in E(G)$ with $\alpha(u_{i_0}u_{j_0}) = 1$, and since $\beta(x_1^{(i_0)}x_1^{(j_0)}) = (\alpha(u_{i_0}u_{j_0}) - 1)n + 1$. Similarly, there exists an edge $x_{p_1}^{(i_1)}x_{q_1}^{(j_1)} \in E(\widetilde{G})$ such that $\beta(x_{p_1}^{(i_1)}x_{q_1}^{(j_1)}) = tn + w(H_{k_0})$, since in the coloring α there exists an edge $u_{i_1}u_{j_1} \in E(G)$ with $\alpha(u_{i_1}u_{j_1}) = t$, and since $\beta(x_1^{(i_1)}x_n^{(j_1)}) = \alpha(u_{i_1}u_{j_1})n + w(H_{k_0}) = tn + w(H_{k_0})$.

Therefore, β is an interval $(tn + w(H_{k_0}))$-edge coloring of $\widetilde{G}$. By the definition of β, we have $w(\widetilde{G}) \le w(G)n + w(H_{k_0})$, $W(\widetilde{G}) \ge W(G)n + w(H_{k_0})$.

Case 2. $\widetilde{G} \in \mathcal{G}_2^2$.

Let

$$\mathcal{D}_1 = \{1, \ldots, tn\};\ \mathcal{D}_2 = \{tn + 1, \ldots, tn + w(H_{k_0})\};$$

$$\mathcal{D}_3^{(i)} = \{tn + w(H_{k_0}) - (w(H_i) - 1), \ldots, tn + w(H_{k_0})\}, i \in \{1, \ldots, m\}\backslash\{k_0\}.$$

Obviously, $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$, $\mathcal{D}_3^{(i)} \subseteq \mathcal{D}_2$, and $|\mathcal{D}_3^{(i)}| = w(H_i)$.

Since $G \in \mathfrak{N}$, there exists an interval t-coloring α of G. Now, we construct an edge coloring β' of $\widetilde{G}$ as follows.

First, we define the following edge-coloring β_3 of the graph $G^{(1)}$ with nt colors in $\mathcal{D}_1$. For every edge $x_p^{(i)}x_q^{(j)} \in E(G^{(1)})$, $1 \le p, q \le n$, let

$$\beta_3(x_p^{(i)}x_q^{(j)}) = \begin{cases} (\alpha(u_iu_j) - 1)n + (p + q - 1)_n, & p + q \neq n + 1; \\ \alpha(u_iu_j)n, & p + q = n + 1. \end{cases}$$

Second, we define the following edge-coloring β_4 of the graph $G^{(2)}$. If $H_i \cong H_{k_0}$, we color the edges of H_{k_0} with $w(H_{k_0})$ colors in $\mathcal{D}_2$ such that the colors on the edges incident to any vertex are consecutive; if $H_i \ncong H_{k_0}$, we color the edges of H_i with $w(H_i)$ colors in $\mathcal{D}_3^{(i)}$ such that on the edges incident to any vertex are consecutive.

Finally, for every edge $e \in E(\widetilde{G})$, let

$$\beta'(e) = \begin{cases} \beta_3(e), & e \in E(G^{(1)}); \\ \beta_4(e), & e \in E(G^{(2)}). \end{cases}$$

It is easy to see that β' is an interval $(tn + w(H_{k_0}))$-edge coloring of the graph $\widetilde{G}$. Its proof is similar to the proof of case 1. By the definition of β', we have $w(\widetilde{G}) \leq w(G)n + w(H_{k_0})$, $W(\widetilde{G}) \geq W(G)n + w(H_{k_0})$. □

It is not difficult to see that if H_i is a r-regular graph and $H_i \in \mathfrak{N}$ for any $i = 1, \ldots, m$, from Theorem 2.2, we can directly derive Theorem 1.2.

We extend Theorem 1.3 from the lexicographic product of graphs to the generalized lexicographic product of graphs, and obtained the following results:

Theorem 2.3. *If $\widetilde{G} \in \mathcal{G}_3$, then $\widetilde{G} \in \mathfrak{N}$. Moreover,*

(i) If $\widetilde{G} \in \mathcal{G}_3^1$, then $w(\widetilde{G}) \leq w(G)n + 2$, $W(\widetilde{G}) \geq W(G)n + \frac{n}{2} + 1$;

(ii) If $\widetilde{G} \in \mathcal{G}_3^2$, then $w(\widetilde{G}) \leq w(G)n$, $W(\widetilde{G}) \geq W(G)n + \frac{n}{2} - 1$.

Proof. For the proof, we consider the following two cases:

Case 1. $\widetilde{G} \in \mathcal{G}_3^1$.

Then Condition I holds, and by Lemma 1.2, we can assume that $\alpha(u_{i_0}u_{j_0}) = 1$. Now, we prove that $w(\widetilde{G}) \leq w(G)n + 2$. For this, we define an edge-coloring β_1 of the graph $\widetilde{G}$ in the following two steps:

Step 1. For every edge $x_p^{(i)} x_q^{(j)} \in E(G^{(1)})$, $1 \leq p, q \leq n$, let

$$\beta_1(x_p^{(i)} x_q^{(j)}) = \begin{cases} (\alpha_1(u_i u_j) - 1)n + (p + q - 1)_n + 2, & p + q \neq n + 1; \\ \alpha_1(u_i u_j)n + 2, & p + q = n + 1. \end{cases}$$

Step 2. For every edge $x_p^{(i)} x_{p+1}^{(i)} \in E(G^{(2)})$, $1 \leq p \leq n - 1$, let

$$\beta_1(x_p^{(i)} x_{p+1}^{(i)}) = (\tau_i(\alpha_1) - 1)n + (p)_2 + 1;$$

if H_i is isomorphic to a cycle, for the edge $(x_1^{(i)} x_n^{(i)}) \in E(G^{(2)})$, let

$$\beta_1(x_1^{(i)} x_n^{(i)}) = (\tau_i(\alpha_1) - 1)n + 1.$$

Let us prove that β_1 is an interval edge coloring of the graph $\widetilde{G}$.

First, we prove that the set $S(x_p^{(i)}, \beta_1)$ is an interval for each vertex $x_p^{(i)} \in V(\widetilde{G})$, where $i = 1, \ldots, m$, $p = 1, \ldots, n$.

If $H_i \cong \overline{K_n}$, by the definition of β_1, we have

$$S(x_p^{(i)}, \beta_1) = \{(\tau_i(\alpha_1) - 1)n + 3, \ldots, \tau_i'(\alpha_1)n + 2\}.$$

If H_i is isomorphic to a path or a cycle, by the definition of β_1, we have

$$\begin{aligned} S(x_p^{(i)},\beta_1) &= \{(\tau_i(\alpha_1)-1)n+3,\ldots,\tau_i'(\alpha_1)n+2\} \\ &\quad \cup \{(\tau_i(\alpha_1)-1)n+(p-1)_2+1\} \cup \{(\tau_i(\alpha_1)-1)n+(p)_2+1 \\ &= [(\tau_i(\alpha_1)-1)n+1, \tau_i'(\alpha_1)n+2], 2 \le p \le n-1; \end{aligned}$$

and when H_i is isomorphic to a path, we have

$$\begin{aligned} S(x_1^{(i)},\beta_1) = S(x_n^{(i)},\beta_1) &= \{(\tau_i(\alpha_1)-1)n+3,\ldots,\tau_i'(\alpha_1)n+2\} \cup \{(\tau_i(\alpha_1)-1)n+1+1\} \\ &= [(\tau_i(\alpha_1)-1)n+2, \tau_i'(\alpha_1)n+2], \end{aligned}$$

when H_i is isomorphic to a cycle, we have

$$\begin{aligned} S(x_1^{(i)},\beta_1) = S(x_n^{(i)},\beta_1) &= \{(\tau_i(\alpha_1)-1)n+3,\ldots,\tau_i'(\alpha_1)n+2\} \\ &\quad \cup \{(\tau_i(\alpha_1)-1)n+1\} \cup \{(\tau_i(\alpha_1)-1)n+1+1\} \\ &= [(\tau_i(\alpha_1)-1)n+1, \tau_i'(\alpha_1)n+2]. \end{aligned}$$

Second, note that

$$d_{\widetilde{G}}(x_p^{(i)}) = d_H(x_p^{(i)}) + d_G(x_p^{(i)})n = \begin{cases} (\tau_i'(\alpha_1)-\tau_i(\alpha_1))n+n, H_i \cong \overline{K_n}; \\ (\tau_i'(\alpha_1)-\tau_i(\alpha_1))n+n+1, H_i \cong P_n \textit{ and } p=1 \textit{ or } p=n; \\ (\tau_i'(\alpha_1)-\tau_i(\alpha_1))n+n+2, H_i \cong C_n, \textit{or } H_i \cong P_n \textit{ and } 2 \le p \le n-1. \end{cases}$$

Clearly, we have

$$\max S(x_p^{(i)},\beta_1) - \min S(x_p^{(i)},\beta_1) = d_{\widetilde{G}}(x_p^{(i)}) - 1, 1 \le p \le n.$$

This implies that β_1 is a proper edge coloring of $\widetilde{G}$.

Finally, we show that β_1 all colors are used. Clearly, there exists an edge $x_{p_0}^{(i_0)}x_{p_0+1}^{(i_0)} \in E(\widetilde{G})$ such that $\beta_1(x_{p_0}^{(i_0)}x_{p_0+1}^{(i_0)}) = 1$, since in the coloring α_1 there exists a vertex u_{i_0} such that $\tau_{i_0}(\alpha_1) = 1$, and since $\beta_1(x_1^{(i_0)}x_2^{(i_0)}) = (\tau_{i_0}(\alpha_1)-1)n+1 = 1$. Similarly, there exists an edge $x_{p_1}^{(i_1)}x_{q_1}^{(j_1)} \in E(\widetilde{G})$ such that $\beta_1(x_{p_1}^{(i_1)}x_{q_1}^{(j_1)}) = w(G)n+2$, since in the coloring α_1 there exists an edge $u_{i_1}u_{j_1} \in E(G)$ with $\alpha(u_{i_1}u_{j_1}) = w(G)$, and since $\beta_1(x_1^{(i_1)}x_n^{(j_1)}) = \alpha(u_{i_1}u_{j_1})n+2 = w(G)n+2$.

Now, by Lemma 1, we have that β_1 is an interval $(w(G)n+2)$-edge coloring of the graph $\widetilde{G}$.

Now, we prove that $W(\widetilde{G}) \ge W(G)n + \frac{n}{2} + 1$. For this, we define an edge-coloring β_2 of the graph $\widetilde{G}$ in the following two steps:

Step 1. For every edge $x_p^{(i)}x_q^{(j)} \in E(G^{(1)})$, let

$$\beta_2(x_p^{(i)}x_q^{(j)}) = \begin{cases} (\alpha_2(u_iu_j)-1)n+p+q+1, & 1 \le p \le \frac{n}{2} \textit{ and } 1 \le q \le \frac{n}{2}; \\ (\alpha_2(u_iu_j)+1)n+3-p-q, & \frac{n}{2}+1 \le p \le n \textit{ and } \frac{n}{2}+1 \le q \le n; \\ \alpha_2(u_iu_j)n+\frac{n}{2}+2-|p-q|, & \textit{otherwise}; \end{cases}$$

Step 2. For every edge $x_p^{(i)}x_{p+1}^{(i)} \in E(G^{(2)})$, $1 \le p \le n-1$, let

$$\beta_1(x_p^{(i)}x_{p+1}^{(i)}) = \beta_1(x_{n-p}^{(i)}x_{n-p+1}^{(i)}) = (\tau_i(\alpha_2)-1)n+p+1, 1 \le p \le \frac{n}{2};$$

if H_i is isomorphic to a cycle, for the edge $(x_1^{(i)}x_n^{(i)}) \in E(G^{(2)})$, let

$$\beta_1(x_1^{(i)}x_n^{(i)}) = (\tau_i(\alpha_2) - 1)n + 1.$$

Let us prove that β_2 is an interval $(W(G)n + \frac{n}{2} + 1)$-edge coloring of the graph $\widetilde{G}$.

First we prove that the set $S(x_p^{(i)}, \beta_2)$ is an interval for each vertex $x_p^{(i)} \in V(\widetilde{G})$, where $i = 1, \ldots, m$, $p = 1, \ldots, n$.

If H_i is isomorphic to a path or a cycle, by the definition of β_2, we have

$$\begin{aligned} S(x_p^{(i)}, \beta_1) = S(x_{n-p+1}^{(i)}, \beta_1) &= \{(\tau_i(\alpha_2) - 1)n + p + 2, \ldots, \tau_i'(\alpha_2)n + \frac{n}{2} + 2 - |p - \frac{n}{2} - 1|\} \\ &\cup \{(\tau_i(\alpha_2) - 1)n + p\} \cup \{(\tau_i(\alpha_2) - 1)n + p + 1\} \\ &= [(\tau_i(\alpha_2) - 1)n + p, \tau_i'(\alpha_2)n + p + 1], 2 \le p \le \frac{n}{2}; \end{aligned}$$

and when H_i is isomorphic to a path, we have

$$\begin{aligned} S(x_1^{(i)}, \beta_1) = S(x_n^{(i)}, \beta_1) &= \{(\tau_i(\alpha_2) - 1)n + 3, \ldots, \tau_i'(\alpha_2)n + 2\} \cup \{(\tau_i(\alpha_2) - 1)n + 2\} \\ &= [(\tau_i(\alpha_2) - 1)n + 2, \tau_i'(\alpha_2)n + 2]; \end{aligned}$$

when H_i is isomorphic to a cycle, we have

$$\begin{aligned} S(x_1^{(i)}, \beta_1) = S(x_n^{(i)}, \beta_1) &= \{(\tau_i(\alpha_2) - 1)n + 3, \ldots, \tau_i'(\alpha_2)n + 2\} \\ &\cup \{(\tau_i(\alpha_2) - 1)n + 2\} \cup \{(\tau_i(\alpha_2) - 1)n + 1\} \\ &= [(\tau_i(\alpha_2) - 1)n + 1, \tau_i'(\alpha_2)n + 2]; \end{aligned}$$

If $H_i \cong \overline{K_n}$, by the definition of β_2, we have

$$S(x_p^{(i)}, \beta_1) = S(x_{n-p+1}^{(i)}, \beta_1) = \{(\tau_i(\alpha_2) - 1)n + p + 2, \ldots, \tau_i'(\alpha_2)n + p + 1\}, 1 \le p \le \frac{n}{2};$$

Second, note that

$$d_{\widetilde{G}}(x_p^{(i)}) = d_H(x_p^{(i)}) + d_G(x_p^{(i)})n = \begin{cases} (\tau_i'(\alpha_2) - \tau_i(\alpha_2))n + n, H_i \cong \overline{K_n}; \\ (\tau_i'(\alpha_2) - \tau_i(\alpha_2))n + n + 1, H_i \cong P_n \textit{ and } p = 1 \textit{ or } p = n; \\ (\tau_i'(\alpha_2) - \tau_i(\alpha_2))n + n + 2, H_i \cong C_n, \textit{or } H_i \cong P_n \textit{ and } 2 \le p \le n - 1. \end{cases}$$

Clearly, we have

$$\max S(x_p^{(i)}, \beta_2) - \min S(x_p^{(i)}, \beta_2) = d_{\widetilde{G}}(x_p^{(i)}) - 1, 1 \le p \le n.$$

This implies that β_2 is a proper edge coloring of $\widetilde{G}$.

Finally, we show that in the coloring β_2, all colors are used. Clearly, there exists an edge $x_1^{(i_0)}x_n^{(i_0)} \in E(\widetilde{G})$ such that $\beta_1(x_1^{(i_0)}x_n^{(i_0)}) = 1$, since in the coloring α_2 there exists a vertex u_{i_0} such that $\tau_{i_0}(\alpha_2) = 1$, and since $\beta_2(x_1^{(i_0)}x_n^{(i_0)}) = (\tau_{i_0}(\alpha_2) - 1)n + 1$. Similarly, there exists an edge $x_{p_1}^{(i_1)}x_{q_1}^{(j_1)} \in E(\widetilde{G})$ such that $\beta_2(x_{p_1}^{(i_1)}x_{q_1}^{(j_1)}) = W(G)n + \frac{n}{2} + 1$, since in the coloring α_2 there exists an edge $u_{i_1}u_{j_1} \in E(G)$ with $\alpha_2(u_{i_1}u_{j_1}) = W(G)$, and since $\beta_2(x_{p_1}^{(i_1)}x_{q_1}^{(j_1)}) = (W(G) - 1)n + \frac{3n}{2} + 1 = W(G)n + \frac{n}{2} + 1$.

Now, by Lemma 1.1, we have that β_2 is an interval $(W(G)n + \frac{n}{2} + 1)$-edge coloring of the graph $\widetilde{G}$.

Case 2. $\widetilde{G} \in \mathcal{G}_3^2$**.**

Now, we prove that $w(\widetilde{G}) \leq w(G)n$. For this, we define an edge-coloring β_3 of the graph $\widetilde{G}$ in the following two steps:

Step 1. For every edge $x_p^{(i)}x_q^{(j)} \in E(G^{(1)})$, let

$$\beta_3(x_p^{(i)}x_q^{(j)}) = \begin{cases} (\alpha_1(u_iu_j) - 1)n + (p+q-1)_n, & p+q \neq n+1; \\ \alpha_1(u_iu_j)n, & p+q = n+1. \end{cases}$$

Step 2. For every edge $x_p^{(i)}x_{p+1}^{(i)} \in E(G^{(2)})$, $1 \leq p \leq n-1$, let

$$\beta_3(x_p^{(i)}x_{p+1}^{(i)}) = (\tau_i(\alpha_1) - 1)n - (p+1)_2;$$

if H_i is isomorphic to a cycle, for the edge $(x_1^{(i)}x_n^{(i)}) \in E(G^{(2)})$, let

$$\beta_3(x_1^{(i)}x_n^{(i)}) = (\tau_i(\alpha_1) - 1)n - 1.$$

Similar to the coloring β_1 in case 1 to discuss. It is easy to see that β_3 is an interval $w(G)n$-coloring of $\widetilde{G}$.

Now, we prove that $W(\widetilde{G}) \geq W(G)n + \frac{n}{2} - 1$. For this, we define an edge-coloring β_4 of the graph $\widetilde{G}$ in the following two steps:

Step 1. For every edge $x_p^{(i)}x_q^{(j)} \in E(G^{(1)})$, $1 \leq p, q \leq n$, let

$$\beta_4(x_p^{(i)}x_q^{(j)}) = \beta_2(x_p^{(i)}x_q^{(j)}) - 2;$$

Step 2. For every edge $x_p^{(i)}x_q^{(i)} \in E(G^{(2)})$, $1 \leq p, q \leq n$, let

$$\beta_4(x_p^{(i)}x_q^{(i)}) = \beta_2(x_p^{(i)}x_q^{(i)}).$$

It is easy to see that β_4 is an interval $(W(G)n + \frac{n}{2} - 1)$-coloring of $\widetilde{G}$. □

It is not difficult to see that if H_i is isomorphic to a cycle for any $i = 1, \ldots, m$, from Theorem 2.3, we can directly derive Theorem 1.3.

From Theorems 2.1–2.3, we can see that $\mathcal{G}_z \subset \mathfrak{N}, z = 1, 2, 3$.

3. Conclusions

In this paper, we studied the interval edge coloring of the generalized lexicographic product $\widetilde{G}$ of an interval colorable graph G with m vertices and a sequence of vertex-disjoint graphs $h_m = (H_i)_{i\in\{1,\ldots,m\}}$, where each graph in h_m has n vertices, and proved that $\widetilde{G}$ is interval colorable if and only if $h_m = (H_i)_{i\in\{1,\ldots,m\}}$ satisfies one of the following three conditions: (i) each graph H_i in h_m is isomorphic to a path or an empty graph; (ii) each graph H_i in h_m is isomorphic to an empty graph or an interval-colorable regular graph, but not all graphs in h_m are empty graphs; (iii) each graph H_i in h_m is isomorphic to a path or a cycle or an empty graph of even order, and H_{k_0} is a cycle. Moreover, we obtain the bounds on $w(\widetilde{G})$ and $W(\widetilde{G})$.

Author contributions

M. Jin: Writing-original draft preparation; M. Jin, P. Chen and S. Tian: Formal analysis; M. Jin, P. Chen and S. Tian: Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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