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Research article

Weak Hardy spaces associated with para-accretive functions and their applications

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Abstract: In this paper, we introduced a new class of weak Hardy spaces, denoted by $H_b^{p,\infty}$, and provided an analysis of their atomic decomposition. As an application, we established the boundedness of Calderón-Zygmund operators (CZOs) from H^p to $H_b^{p,\infty}$ including cases at the critical exponent

$$p = \frac{n}{n+\delta},$$

where δ represents the regularity index of the distributional kernel. Moreover, the boundedness of CZOs from $H^{p,\infty}$ to $H_b^{p,\infty}$ was demonstrated for

$$\frac{n}{n+\delta} < p \leq 1.$$

Keywords: weak Hardy space; para-accretive function; atomic decomposition; Calderón-Zygmund operator
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1. Introduction

Hardy spaces are well-recognized as effective substitutes for Lebesgue spaces in analyzing the boundedness of singular integral operators for $0 < p \leq 1$. We consider T in Calderón-Zygmund operators (CZOs) with regularity δ (where $0 < \delta \leq 1$) and satisfying

$$T^*(1) = 0$$

with T^* denoting the adjoint operator of T. It is known that T is bounded on $H^p(\mathbb{R}^n)$ for

$$\frac{n}{n+\delta} < p \leq 1,$$

although T may fail to be bounded on $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$. In this context, the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$ was introduced in [1] and it was demonstrated that certain T in CZOs is bounded from $H^{1,\infty}(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Liu [2] showed that T in CZOs is bounded from $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$ to $H^{\frac{n}{n+\delta},\infty}(\mathbb{R}^n)$ by atomic decomposition. A novel Littlewood-Paley characterization of weak Hardy spaces was introduced by He in [3], along with a new inequality that parallels the Fefferman-Stein vector-valued inequality. This development has contributed notably to the theoretical study of weak Hardy spaces. Furthermore, Yan et al. [4] developed variable weak Hardy spaces via radial grand maximal functions, characterized these spaces with atoms and the Littlewood-Paley theory, and established the boundedness of CZOs. Additionally, for some discussions on generalized weak Hardy spaces, see [5–8].

On the other hand, Calderón and Zygmund extended the Hilbert and Riesz transforms by introducing a broader class of singular integral operators, specifically those of convolution type. For these operators, the L^2-boundedness is established through the application of Plancherel's Theorem. Nevertheless, many significant singular integral operators, such as Calderón commutators and layer potential operators, are not of convolution type, rendering Plancherel's Theorem inapplicable to them. To address this limitation, the $T1$ theorem was established by the authors in [9], providing a general criterion for the L^2-boundedness of singular integral operators.

However, there are still cases where the $T1$ theorem is not applicable, such as with the Cauchy integral on Lipschitz curves. To extend such results to this case, a Tb theorem was established by McIntosh and Meyer in [10] by replacing the function 1 with an accretive function b. Building on this, David et al. [11] developed a more general Tb theorem using a para-accretive function b. Additionally, Han et al. [12] introduced a class of Hardy spaces associated with para-accretive functions, denoted as $H^p_b(\mathbb{R}^n)$, and provided the necessary and sufficient conditions for the boundedness of CZOs in these new Hardy spaces, specifically for

$$\frac{n}{n+\delta} < p \leq 1.$$

Further work has extended these results to Besov and Triebel-Lizorkin spaces (see [13–15]) and to the variable index setting (see [16]).

It is natural to investigate whether the results in [4] can be extended to the weak Hardy spaces associated with para-accretive functions. The aim of this paper is to develop the theory of weak Hardy spaces and to investigate the boundedness of the operators. More precisely, we first introduce the weak Hardy spaces associated with para-accretive functions, denoted as $H^{p,\infty}_b(\mathbb{R}^n)$, and provide their atomic decomposition. Building on this foundation, we establish the boundedness of CZOs from $H^p(\mathbb{R}^n)$ to $H^{p,\infty}_b(\mathbb{R}^n)$ for

$$\frac{n}{n+\delta} \leq p \leq 1$$

with $\delta \leq \varepsilon$ (where ε is the exponent in the approximation to the identity) and additionally show that these operators are bounded from $H^{p,\infty}(\mathbb{R}^n)$ to $H^{p,\infty}_b(\mathbb{R}^n)$ for

$$\frac{n}{n+\delta} < p \leq 1.$$

This paper is organized as follows. In Section 2, we introduce the weak Hardy spaces associated with para-accretive functions and show that such spaces are well defined. Next we give the atom decomposition of these spaces in Section 3. In Section 4, we demonstrate that CZOs are bounded on the weak Hardy spaces associated with para-accretive functions.

2. Weak Hardy spaces associated with para-accretive functions

We introduce the new class of function spaces and show that these spaces are well defined in this section. To further analyze the underlying structure of these spaces, it is necessary to review the definitions of para-accretive functions and their associated approximations to the identity, as these concepts are crucial for the definition and analysis of weak Hardy spaces.

Definition 2.1. *[11] A bounded function b: $\mathbb{R}^n \to \mathbb{C}$ is called a para-accretive function if there exists $\gamma > 0$ such that for every cube $Q \subset \mathbb{R}^n$,*

$$\frac{1}{|Q|}\Big|\int_{Q'} b(x)\,dx\Big| \gtrsim 1,$$

where $Q^{'}$ is a subcube of Q satisfying $\gamma|Q| \leq |Q^{'}|$.

From this point forward, any function denoted by b in this paper will refer to para-accretive functions, unless explicitly stated otherwise.

Definition 2.2. *[11] Let $\varepsilon \in (0,1]$. A sequence of operators $\{S_l\}_{l\in\mathbb{Z}}$ is called an approximation to the identity associated with para-accretive functions b with regularity index ε if for all $l \in \mathbb{Z}$ and all $x, x', y, y' \in \mathbb{R}^n$, the kernel $S_l(x,y)$ of S_l satisfies the following conditions:*

(i) $|S_l(x,y)| \lesssim \frac{2^{-l\varepsilon}}{(2^{-l}+|x-y|)^{n+\varepsilon}}$;

(ii) $|S_l(x,y) - S_l(x,y')| \lesssim \left(\frac{|y-y'|}{2^{-l}+|x-y|}\right)^{\varepsilon} \frac{2^{-l\varepsilon}}{(2^{-l}+|x-y|)^{n+\varepsilon}}$ *for* $|y-y'| \leq (2^{-l}+|x-y|)/2$;

(iii) $|S_l(x,y) - S_l(x',y)| \lesssim \left(\frac{|x-x'|}{2^{-l}+|x-y|}\right)^{\varepsilon} \frac{2^{-l\varepsilon}}{(2^{-l}+|x-y|)^{n+\varepsilon}}$ *for* $|x-x'| \leq (2^{-l}+|x-y|)/2$;

(iv) $|(S_l(x,y) - S_l(x',y)) - (S_l(x,y') - S_l(x',y'))| \lesssim \left(\frac{|x-x'|}{2^{-l}+|x-y|}\right)^{\varepsilon}\left(\frac{|y-y'|}{2^{-l}+|x-y|}\right)^{\varepsilon} \frac{2^{-l\varepsilon}}{(2^{-l}+|x-y|)^{n+\varepsilon}}$ *for* $|x-x'| \leq (2^{-l} + |x-y|)/2$ *and* $|x-x'| \leq (2^{-l}+|x-y|)/2$;

(v) $\int_{\mathbb{R}^n} S_l(x,y)b(y)dy = 1$;

(vi) $\int_{\mathbb{R}^n} S_l(x,y)b(x)dx = 1$.

The class of all sequences defined above is denoted by $\mathrm{AIP}(\varepsilon, b)$. Here is the definition of the test function on $\mathbb{R}^n$ associated with para-accretive functions.

Definition 2.3. *[17] Let $0 < \beta \leq 1$ and $\gamma > 0$ be two fixed exponents. A function f: $\mathbb{R}^n \to \mathbb{C}$ is called a test function of type (β, γ), centered at $x_0 \in \mathbb{R}^n$ with width $r > 0$, if there exists a constant $C > 0$ such that*

$$|f(x)| \leq C\frac{r^{\gamma}}{(r+|x-x_0|)^{1+\gamma}}, \tag{2.1}$$

$$|f(x)-f(y)| \leq C\Big(\frac{|x-y|}{r+|x-x_0|}\Big)^{\beta}\frac{r^{\gamma}}{(r+|x-x_0|)^{1+\gamma}} \tag{2.2}$$

for

$$|x-y| \leq \frac{1}{2}(r+|x-x_0|)$$

and

$$\int_{\mathbb{R}^n} f(x)b(x)dx = 0.$$

Moreover, the class of all such functions is denoted by $\mathcal{M}(x_0, r, \beta, \gamma)$, and the norm is defined as

$$\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} = \inf\{C : (2.1) \ \ and \ \ (2.2) \ \ hold\}.$$

The class $\mathcal{M}(\beta, \gamma)$ is defined as the set of functions $f \in \mathcal{M}(x_0, 1, \beta, \gamma)$ for some fixed x_0 and r. This space $\mathcal{M}(\beta, \gamma)$ is a Banach space with respect to its norm. Importantly, for any $x_1 \in \mathbb{R}^n$ and $r > 0$, the spaces $\mathcal{M}(x_1, r, \beta, \gamma)$ and $\mathcal{M}(\beta, \gamma)$ are equivalent, meaning that they have the same structural properties and equivalent norms. For a para-accretive function b, the space $b\mathcal{M}(\beta, \gamma)$ is defined as

$$b\mathcal{M}(\beta, \gamma) = \{bg : g \in \mathcal{M}(\beta, \gamma)\}.$$

The norm of $f = bg$ in $b\mathcal{M}(\beta, \gamma)$ is given by

$$\|f\|_{b\mathcal{M}(\beta,\gamma)} = \|g\|_{\mathcal{M}(\beta,\gamma)}.$$

The dual space $(b\mathcal{M}(\beta, \gamma))'$ consists of all linear functionals £ defined on $b\mathcal{M}(\beta, \gamma)$ that satisfy

$$|£(f)| \lesssim \|f\|_{\mathcal{M}(\beta,\gamma)} \ \text{ for all } \ f \in b\mathcal{M}(\beta, \gamma).$$

Given $\varepsilon \in (0, 1]$, let $\widetilde{\mathcal{M}}(\beta, \gamma)$ denote the completion of $\mathcal{M}(\varepsilon, \varepsilon)$ in the space $\mathcal{M}(\beta, \gamma)$, where $0 < \beta, \gamma \leq \varepsilon$.

Now we can introduce the desired space.

Definition 2.4. *Let $\{S_l\}_{l\in\mathbb{Z}}$ be in AIP(ε, b) and*

$$D_l = S_l - S_{l-1}$$

for $l \in \mathbb{Z}$. For $0 < \beta, \gamma < \varepsilon$, and $0 < p \leq 1$, the weak Hardy space $H_b^{p,\infty}(\mathbb{R}^n)$ is defined as the set of all functions $f \in (b\widetilde{\mathcal{M}}(\beta, \gamma))'$ such that the function $g_b(f)(x)$ given by

$$g_b(f)(x) = \left\{ \sum_{l\in\mathbb{Z}} |D_l b f(x)|^2 \right\}^{\frac{1}{2}}$$

satisfies the condition

$$\|f\|_{H_b^{p,\infty}} = \|g_b(f)\|_{L^{p,\infty}} < \infty.$$

It is essential to establish that the weak Hardy space $H_b^{p,\infty}(\mathbb{R}^n)$ is well defined, meaning its definition does not depend on the specific choice of the approximation to the identity. To demonstrate this, we must prove the following result.

Theorem 2.1. *Suppose that $0 < p \leq 1$, and $\{P_j\}_{j\in\mathbb{Z}}$ and $\{S_k\}_{k\in\mathbb{Z}}$ are in AIP(ε,b). Let*

$$D_j = P_j - P_{j-1}$$

and

$$E_k = S_k - S_{k-1}$$

for all $k, j \in \mathbb{Z}$. *Then*

$$\left\| \left\{ \sum_j \sum_{Q_j} \Big(\sup_{x \in Q_j} |D_j bf(x)| \Big)^2 \chi_{Q_j} \right\}^{\frac{1}{2}} \right\|_{L^{p,\infty}} \approx \left\| \left\{ \sum_k \sum_{Q_k} \Big(\inf_{x \in Q_k} |E_k bf(x)| \Big)^2 \chi_{Q_k} \right\}^{\frac{1}{2}} \right\|_{L^{p,\infty}}$$

for all $f \in \Big(b\widetilde{\mathcal{M}}(\beta, \gamma) \Big)'$.

To establish Theorem 2.1, it is necessary to review the following vector-valued inequality.

Lemma 2.1. *[3, 18] Let* $1 < p, q < \infty$, *and then for any sequence of functions* $\{f_j\}_j \subset \mathbb{X}$, *we have*

$$\left\| \Big(\sum_j [M(f_j)]^q \Big)^{1/q} \right\|_{\mathbb{X}} \lesssim \left\| \Big(\sum_j |f_j|^q \Big)^{1/q} \right\|_{\mathbb{X}},$$

where $\mathbb{X}$ *can be either* $L^p(\mathbb{R}^n)$ *or* $L^{p,\infty}(\mathbb{R}^n)$ *and* M *is the Hardy-Littlewood maximal operator.*

Let Q_l be the collection of all dyadic cubes with side length 2^{-l-N}, where $l \in \mathbb{Z}$ and N is a fixed large positive integer. For each dyadic cube Q_l, x_{Q_l} denotes an arbitrary fixed point within Q_l. With this notion of dyadic cubes, the following Calderón reproducing formula is also required.

Lemma 2.2. *[12] Suppose that all notations are as defined in Definition 2.4. Then there exists a family of operators* $\{\widetilde{\widetilde{D}}_l\}_{l \in \mathbb{Z}}$ *such that*

$$f(y) = \sum_l \sum_{Q_l} D_l bf(x_{Q_l}) \int_{Q_l} \widetilde{\widetilde{D}}_l(x, y) b(x) dx$$

for $f \in \Big(b\widetilde{\mathcal{M}}(\beta, \gamma) \Big)'$ *with* $0 < \beta, \gamma < \varepsilon$, *where the series converge in the sense of distribution. Moreover, the kernel* $\widetilde{\widetilde{D}}_l(x, y)$ *of* $\widetilde{\widetilde{D}}_l$ *satisfies: for* $0 < \varepsilon' < \varepsilon$,

$$|\widetilde{\widetilde{D}}_l(x, y)| \lesssim \frac{2^{-l\varepsilon'}}{(2^{-l} + |x - y|)^{n+\varepsilon'}},$$

$$|\widetilde{\widetilde{D}}_l(x, y) - \widetilde{\widetilde{D}}_l(x, y')| \lesssim \Big(\frac{|y - y'|}{2^{-l} + |x - y|} \Big)^{\varepsilon'} \frac{2^{-l\varepsilon'}}{(2^{-l} + |x - y|)^{n+\varepsilon'}}$$

for

$$|y - y'| \leq \frac{1}{2}(2^{-l} + |x - y|)$$

and

$$\int_{\mathbb{R}^n} \widetilde{\widetilde{D}}_l(x, y) b(y) dy = \int_{\mathbb{R}^n} \widetilde{\widetilde{D}}_l(x, y) b(x) dx = 0.$$

With the necessary groundwork laid, we turn to demonstrating Theorem 2.1.

Proof of Theorem 2.1. For $0 < \varepsilon'' < \varepsilon' < \varepsilon$, we have

$$|E_k b\widetilde{\widetilde{D}}_j(x,y)| \lesssim 2^{-|j-k|\varepsilon''} \frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)} + |x-y|)^{n+\varepsilon'}},$$

which follows from Lemma 2.2 and the almost orthogonal estimate (see [19]). Let $f \in \left(b\widetilde{\mathcal{M}}(\beta,\gamma)\right)'$ and fix $y \in Q_k$, and then

$$\begin{aligned}
|E_k b(f)(y)| &\leq \sum_j \sum_{Q_j} |D_j bf(x_{Q_j})| \int_{Q_j} |E_k b\widetilde{\widetilde{D}}_j(x,y)| dx \\
&\lesssim \sum_j \sum_{Q_j} 2^{-|j-k|\varepsilon''} |D_j bf(x_{Q_j})| \int_{Q_j} \frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)} + |x-y|)^{n+\varepsilon'}} dx \\
&\lesssim \sum_j \sum_{Q_j} 2^{-|j-k|\varepsilon''} 2^{-jn} \frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)} + |y-x_{Q_j}|)^{n+\varepsilon'}} |D_j bf(x_{Q_j})|.
\end{aligned}$$

By an estimate in [20],

$$\sum_{Q_j} \frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)} + |y-x_{Q_j}|)^{n+\varepsilon'}} |D_j(bf)(x_{Q_j})| \lesssim 2^{(j\wedge k)n} 2^{[j-(j\wedge k)]n/r} M_r\left(\sum_{Q_j} |D_j(bf)(x_{Q_j})| \chi_{Q_j}\right)(y),$$

where

$$M_r(f)(x) := [M(|f|^r)(x)]^{1/r}$$

and

$$\frac{n}{n+\varepsilon''} < r < p.$$

This result, combined with Hölder's inequality and

$$\begin{aligned}
&\sup_k \sum_k 2^{-|j-k|\varepsilon''} 2^{-jn} 2^{(j\wedge k)n} 2^{[j-(j\wedge k)]n/r} < \infty, \\
&\sup_k \sum_j 2^{-|j-k|\varepsilon''} 2^{-jn} 2^{(j\wedge k)n} 2^{[j-(j\wedge k)]n/r} < \infty,
\end{aligned}$$

leads to the result that

$$\begin{aligned}
&\left\{ \sum_k \sum_{Q_k} \sup_{y\in Q_k} |E_k bf(y)|^2 \chi_{Q_k}(y) \right\}^{\frac{1}{2}} \\
&\lesssim \left\{ \sum_k \sum_{Q_k} \left(\sum_j \sum_{Q_j} 2^{-jn} 2^{-|j-k|\varepsilon''} \frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)} + |y-x_{Q_k}|)^{n+\varepsilon'}} |D_j(bf)(x_{Q_j})| \chi_{Q_k}(y) \right)^2 \right\}^{\frac{1}{2}} \\
&\lesssim \left\{ \sum_k \sum_{Q_k} \left[\sum_j 2^{-jn} 2^{-|j-k|\varepsilon''} 2^{(j\wedge k)n} 2^{[j-(j\wedge k)]n/r} M_r\left(\sum_{Q_j} |D_j(bf)(x_{Q_j})| \chi_{Q_j}(x) \right) \chi_{Q_k}(y) \right]^2 \right\}^{\frac{1}{2}}
\end{aligned}$$

$$\lesssim \left\{ \sum_k \sum_j 2^{-jn} 2^{-|j-k|\varepsilon''} 2^{(j\wedge k)n} 2^{[j-(k\wedge j)]n/r} \left[M_r\left(\sum_{Q_j} |D_j(bf)(x_{Q_j})| \chi_{Q_j} \right)(y) \right]^2 \right\}^{\frac{1}{2}}$$

$$\lesssim \left\{ \sum_j \left[M_r\left(\sum_{Q_j} |D_j(bf)(x_{Q_j})| \chi_{Q_j} \right)(y) \right]^2 \right\}^{\frac{1}{2}}.$$

Since x_{Q_j} is any fixed point in Q_j, then

$$\left\{ \sum_k \sum_{Q_k} \sup_{y\in Q_k} |E_k bf(y)|^2 \chi_{Q_k}(y) \right\}^{\frac{1}{2}} \lesssim \left\{ \sum_k \left[M_r\left(\sum_{Q_j} \inf_{x\in Q_j} |D_j(bf)(x)| \chi_{Q_j} \right)(y) \right]^2 \right\}^{\frac{1}{2}}.$$

From Lemma 2.1 and $0 < r < p$, we obtain

$$\left\| \left\{ \sum_k \sum_{Q_k} \left(\sup_{y\in Q_k} |E_k bf(y)| \right)^2 \chi_{Q_k} \right\}^{\frac{1}{2}} \right\|_{L^{p,\infty}} \lesssim \left\| \left\{ \sum_j \sum_{Q_j} \left(\inf_{x\in Q_j} |D_j bf(x)| \right)^2 \chi_{Q_j} \right\}^{\frac{1}{2}} \right\|_{L^{p,\infty}}.$$

By an analogue argument, we can obtain the converse inequality, and thus we get the desired result. □

The next result is necessary, with the proof omitted due to its similarity to [12, Theorem 3.3].

Proposition 2.1. *For*

$$\frac{n}{n+\varepsilon} \leq p \leq 1$$

and $f \in \left(b\mathcal{M}(\beta,\gamma)\right)'$, *define*

$$S_b(f)(x) = \left(\sum_l \|D_l b(f) \chi_{\overline{B_l}}\|_{L^2}^2 \right)^{\frac{1}{2}}$$

with

$$B_l = B(x, 2^{-l}),$$

and then

$$\|S_b(f)\|_{L^{p,\infty}} \approx \|g_b(f)\|_{L^{p,\infty}}.$$

3. Atomic decomposition of $H_b^{p,\infty}(\mathbb{R}^n)$

Let b be a para-accretive function. We first introduce the notion of the (p,q,b) atom, which plays a fundamental role in the atomic decomposition of $H_b^{p,\infty}(\mathbb{R}^n)$.

Definition 3.1. *Let* $0 < p \leq 1 < q \leq \infty$. *A function a is called a* (p,q,b) *atom centered at a ball B if it satisfies:*

(i) supp $a \subset B$;

(ii) $\|a\|_{L^q} \leq |B|^{\frac{1}{q}-\frac{1}{p}}$;

(iii) $\int_{\mathbb{R}^n} a(x)b(x)dx = 0.$

Next, we present the definition of $H^{p,\infty}_{b,atom}(\mathbb{R}^n)$, which is closely tied to the structure of para-accretive functions. The class of atom spaces serves as a fundamental setting for analyzing the boundedness of CZOs and other significant operators in harmonic analysis.

Definition 3.2. *Let $0 < p \le 1 < q \le \infty$. A function $f \in H^{p,\infty}_{b,atom}(\mathbb{R}^n)$ if there exists a sequence of (p, q, b) atoms $\{a_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ and a sequence of coefficients $\{\lambda_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ such that $f \in (b\widetilde{\mathcal{M}}(\beta,\gamma))'$ can be expressed as*

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}} \lambda_{j,k} a_{j,k}, \tag{3.1}$$

where the above series hold in $(b\widetilde{\mathcal{M}}(\beta,\gamma))'$. Moreover,

$$\lambda_{j,k} := C2^j |B_{j,k}|^{\frac{1}{p}}$$

for all $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, where $B_{j,k}$ is the ball associated with the atom $a_{j,k}$ and satisfies

$$\sum_{k\in\mathbb{N}} \chi_{cB_{j,k}}(x) \le C$$

for all $j \in \mathbb{Z}$. The norm defined as

$$\|f\|_{H^{p,\infty}_{b,atom}} = \inf\left\{\sup_{j\in\mathbb{Z}}\left\|\left\{\sum_{k\in\mathbb{N}}\left(\frac{\lambda_{j,k}\chi_{B_{j,k}}}{|B_{j,k}|^{\frac{1}{p}}}\right)^p\right\}^{\frac{1}{p}}\right\|_{L^p}\right\},$$

where the infimum is taken over all decomposition of f as in (3.1).

Before presenting the major conclusion of this section, a useful and simple result is provided, which corresponds to a special case of [4, Remark 2.5].

Lemma 3.1. *Let $1 \le \beta < \infty$ and $0 < r < p < \infty$, and then we have*

$$\left\|\sum_k \chi_{\beta B_k}\right\|_{L^p} \le C\beta^{n/r}\left\|\sum_k \chi_{B_k}\right\|_{L^p},$$

since Lemma 2.1 and the inequality

$$\chi_{\beta B} \le \beta^{n/r}[M(\chi_B)]^{1/r}$$

hold for any ball $B \subset \mathbb{R}^n$.

The following conclusion can be drawn, with the details omitted for brevity.

Remark 3.1. *From the Definition 3.2 and Lemma 3.1, it can be showed that*

$$\begin{aligned} \|f\|_{H^{p,\infty}_{b,atom}} &\approx \sup_{j\in\mathbb{Z}} 2^j\left\|\left(\sum_{k\in\mathbb{N}}\chi_{B_{j,k}}\right)^{\frac{1}{p}}\right\|_{L^p} \approx \sup_{j\in\mathbb{Z}} 2^j\left\|\left(\sum_{k\in\mathbb{N}}\chi_{cB_{j,k}}\right)^{\frac{1}{p}}\right\|_{L^p} \\ &\approx \sup_{j\in\mathbb{Z}} 2^j\left\|\sum_{k\in\mathbb{N}}\chi_{cB_{j,k}}\right\|_{L^p} \approx \sup_{j\in\mathbb{Z}} 2^j\left\|\sum_{k\in\mathbb{N}}\chi_{B_{j,k}}\right\|_{L^p}, \end{aligned} \tag{3.2}$$

where the implicit equivalent positive constants are independent of f.

The following useful result, known as the generalized Grafakos-Kalton lemma, is a special case of [4, Lemma 4.5] and is also presented here.

Lemma 3.2. *Let* $0 < r < p \leq 1$ *and* $1 < q < \infty$*, and there exist two sequences of numbers* $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ *and functions* $\{a_k\}_{k\in\mathbb{N}}$ *such that*

$$\left\|\left(\sum_k |\lambda_k a_k|^r\right)^{1/r}\right\|_{L^p} \lesssim \left\|\left(\sum_k |\lambda_k \chi_{B_k}|^r\right)^{1/r}\right\|_{L^p},$$

where supp $a_k \subset B_k$ *and*

$$\|a_k\|_{L^q} \leq |B_k|^{1/q}.$$

The lemma originated from Grafakos and Kalton's study of multilinear singular integral operators on Hardy spaces, which was later generalized by Sawano in [21], Cruz-Uribe et al. in [22], and Tan in [23]. This lemma plays a crucial role in establishing atomic decomposition and proving boundedness on general Hardy spaces, especially in situations where a single atom cannot be handled effectively. Next, the relationship between the two function spaces introduced above will be explored.

Theorem 3.1. *Let*

$$\frac{n}{n+\varepsilon} < p \leq 1 < q \leq \infty,$$

and then

$$H_b^{p,\infty}(\mathbb{R}^n) = H_{b,atom}^{p,\infty}(\mathbb{R}^n)$$

with equivalent quasi-norms.

Proof. The proof starts by establishing that

$$H_{b,atom}^{p,\infty}(\mathbb{R}^n) \subset H_b^{p,\infty}(\mathbb{R}^n).$$

Suppose that $f \in H_{b,atom}^{p,\infty}(\mathbb{R}^n)$, and it is suffice to prove that

$$\left\|\chi_{\{x:|g_b(f)(x)|>\alpha\}}\right\|_{L^p} \lesssim \alpha^{-1}\|f\|_{H_{b,atom}^{p,\infty}}.$$

From Definition 3.2 and Remark 3.1, there exist two sequences of (p, q, b) atoms $\{a_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$, associated with balls $\{B_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$, and coefficients $\{\lambda_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}} \subset \mathbb{C}$ such that the decomposition (3.1) holds in $(b\widetilde{\mathcal{M}}(\beta,\gamma))'$ and

$$\|f\|_{H_{b,atom}^{p,\infty}} \approx \sup_{j\in\mathbb{Z}} 2^j \left\|\sum_{k\in\mathbb{N}} \chi_{B_{j,k}}\right\|_{L^p}. \tag{3.3}$$

To proceed, it will be convenient to break the sum (3.1) into two parts:

$$f = \sum_{j=-\infty}^{j_0-1}\sum_{k\in\mathbb{N}} \lambda_{j,k}a_{j,k} + \sum_{j=j_0}^{\infty}\sum_{k\in\mathbb{N}} \lambda_{j,k}a_{j,k} =: f_1 + f_2.$$

Choose $j_0 \in \mathbb{Z}$ such that

$$2^{j_0} \leq \alpha < 2^{j_0+1}$$

for any given $\alpha \in (0,\infty)$. Therefore, it follows that

$$\begin{aligned}\left\|\chi_{\{x: g_b(f)(x)>\alpha\}}\right\|_{L^p} &\lesssim \left\|\chi_{\{x: g_b(f_1)(x)>\frac{\alpha}{2}\}}\right\|_{L^p} + \left\|\chi_{\{x\in A^{j_0}: g_b(f_2)(x)>\frac{\alpha}{2}\}}\right\|_{L^p} + \left\|\chi_{\{x\in (A^{j_0})^c: g_b(f_2)(x)>\frac{\alpha}{2}\}}\right\|_{L^p}\\ &=: I_1 + I_2 + I_3,\end{aligned}$$

where

$$A^{j_0} = \bigcup_{j=j_0}^{\infty}\bigcup_{k\in\mathbb{N}}(2B_{j,k}).$$

We rewrite I_1 as

$$\begin{aligned}I_1 &\le \left\|\chi_{\left\{x: \sum_{j=-\infty}^{j_0-1}\sum_{k\in\mathbb{N}}\lambda_{j,k}g_b(a_{j,k})(x)\chi_{2B_{j,k}}(x)>\frac{\alpha}{4}\right\}}\right\|_{L^p} + \left\|\chi_{\left\{x: \sum_{j=-\infty}^{j_0-1}\sum_{k\in\mathbb{N}}\lambda_{j,k}g_b(a_{j,k})(x)\chi_{(2B_{j,k})^c}(x)>\frac{\alpha}{4}\right\}}\right\|_{L^p}\\ &=: I_{1,1} + I_{1,2}.\end{aligned}$$

Let $s \in (0,p)$, $t \in \left(1, \min\{q, \frac{1}{s}\}\right)$, and $a \in (0, 1-\frac{1}{t})$, and according to Hölder's inequality, we have

$$\begin{aligned}&\sum_{j=-\infty}^{j_0-1}\sum_{k\in\mathbb{N}}\lambda_{j,k}g_b(a_{j,k})(x)\chi_{2B_{j,k}}(x)\\ &\le \left(\sum_{j=-\infty}^{j_0-1}2^{jat'}\right)^{\frac{1}{t'}}\left(\sum_{j=-\infty}^{j_0-1}2^{-jat}\left(\sum_{k\in\mathbb{N}}\lambda_{j,k}g_b(a_{j,k})(x)\chi_{2B_{j,k}}(x)\right)^t\right)^{\frac{1}{t}}\\ &= \frac{2^{j_0a}}{(2^{at'}-1)^{\frac{1}{t'}}}\left(\sum_{j=-\infty}^{j_0-1}2^{-jat}\left(\sum_{k\in\mathbb{N}}\lambda_{j,k}g_b(a_{j,k})(x)\chi_{2B_{j,k}}(x)\right)^t\right)^{\frac{1}{t}},\end{aligned}$$

where t and t' are conjugate exponents.

Since $1 < t' < \infty$, then

$$0 < \left(2^{at'}-1\right)^{\frac{1}{t'}} < \infty.$$

Set

$$C(a,t) = 2^{-2t}\left(2^{at'}-1\right)^{\frac{1}{t'}},$$

and observe that $ts < 1$ and $p/s > 1$. Then we obtain

$$\begin{aligned}I_{1,1} &\le \left\|\chi_{\left\{x: \frac{2^{j_0a}}{(2^{at'}-1)^{1/t'}}\left(\sum_{j=-\infty}^{j_0-1}2^{-jat}\left(\sum_{k\in\mathbb{N}}\lambda_{j,k}g_b(a_{j,k})\chi_{2B_{j,k}}\right)^t\right)^{\frac{1}{t}}>2^{j_0-2}\right\}}\right\|_{L^p}\\ &\le C(a,t)2^{-j_0t(1-a)}\left\|\sum_{j=-\infty}^{j_0-1}2^{-jat}\left(\sum_{k\in\mathbb{N}}\lambda_{j,k}g_b(a_{j,k})\chi_{2B_{j,k}}\right)^t\right\|_{L^p}\\ &\lesssim 2^{-j_0t(1-a)}\left\|\sum_{j=-\infty}^{j_0-1}2^{(1-a)jts}\sum_{k\in\mathbb{N}}\left(\|\chi_{B_{j,k}}\|_{L^p}g_b(a_{j,k})\chi_{2B_{j,k}}\right)^{ts}\right\|_{L^{p/s}}^{\frac{1}{s}}\end{aligned}$$

$$\lesssim 2^{-j_0t(1-a)}\Big(\sum_{j=-\infty}^{j_0-1}2^{(1-a)jts}\Big\|\Big(\sum_{k\in\mathbb{N}}\Big(\|\chi_{B_{j,k}}\|_{L^p}g_b(a_{j,k})\chi_{2B_{j,k}}\Big)^{ts}\Big)^{\frac{1}{s}}\Big\|_{L^p}^s\Big)^{\frac{1}{s}}.$$

Let

$$r=\frac{q}{t},$$

and then $1<r<\infty$. For $j\in\mathbb{Z}$ and $k\in\mathbb{N}$, using the Littlewood-Paley characterization of the Lebesgue spaces by g_b, we have

$$\begin{aligned}\Big\|\Big(\|\chi_{B_{j,k}}\|_{L^p}g_b(a_{j,k})\chi_{2B_{j,k}}\Big)^t\Big\|_{L^r}&\le\|\chi_{B_{j,k}}\|_{L^p}^t\|g_b(a_{j,k})\|_{L^q}^t\\&\lesssim\|\chi_{B_{j,k}}\|_{L^p}^t\|a_{j,k}\|_{L^q}^t\\&\lesssim|B_{j,k}|^{\frac{1}{r}}.\end{aligned}$$

Thus, Lemmas 3.1 and 3.2, and (3.3) yield that

$$\begin{aligned}I_{1,1}&\lesssim2^{-j_0t(1-a)}\Big(\sum_{j=-\infty}^{j_0-1}2^{(1-a)jts}\Big\|\Big(\sum_{k\in\mathbb{N}}\chi_{2B_{j,k}}\Big)^{\frac{1}{s}}\Big\|_{L^p}^s\Big)^{\frac{1}{s}}\\&\lesssim2^{-j_0t(1-a)}\Big(\sum_{j=-\infty}^{j_0-1}2^{(1-a)jts}\Big\|\Big(\sum_{k\in\mathbb{N}}\chi_{cB_{j,k}}\Big)^{\frac{1}{s}}\Big\|_{L^p}^s\Big)^{\frac{1}{s}}\\&\lesssim2^{-j_0t(1-a)}\Big(\sum_{j=-\infty}^{j_0-1}2^{[(1-a)t-1]js}\Big)^{\frac{1}{s}}\sup_{j\in\mathbb{Z}}2^j\Big\|\sum_{k\in\mathbb{N}}\chi_{B_{j,k}}\Big\|_{L^p}\\&\lesssim\alpha^{-1}\|f\|_{H^{p,\infty}_{b,atom}}.\end{aligned}\tag{3.4}$$

To deal with $I_{1,2}$, we need some estimates on $g_b(a_{j,k})$. Let

$$B_{j,k}:=B(x_{j,k},r_{j,k})$$

for all $j\in\mathbb{Z},\ k\in\mathbb{N}$. If $j\in\mathbb{Z}\cap[j_0,\infty)$ and $x\in(2B_{j,k})^c$, then using the cancellation condition of $a_{j,k}$ and Hölder's inequality, we have

$$\begin{aligned}|D_lba(x)|&=\Big|\int_{B_{j,k}}b(y)a_{j,k}(y)(D_l(x,y)-D_l(x,x_{j,k}))dy\\&\lesssim\int_{B_{j,k}}|a_{j,k}(y)|\Big(\frac{|y-x_{j,k}|}{2^{-l}+|x-x_{j,k}|}\Big)^{\varepsilon}\frac{2^{-l\varepsilon}}{(2^{-l}+|x-x_{j,k}|)^{n+\varepsilon}}dy\\&\lesssim|B_{j,k}|^{\frac{\varepsilon}{n}}\frac{2^{-l\varepsilon}}{(2^{-l}+|x-x_{j,k}|)^{n+2\varepsilon}}\int_{B_{j,k}}|a_{j,k}(y)|dy\\&\lesssim|B_{j,k}|^{\frac{\varepsilon}{n}}\frac{2^{-l\varepsilon}}{(2^{-l}+|x-x_{j,k}|)^{n+2\varepsilon}}\Big(\int_{2B_{j,k}}|a(y)|^qdy\Big)^{\frac{1}{q}}\Big(\int_{2B_{j,k}}1dy\Big)^{\frac{1}{q'}}\\&\lesssim\frac{2^{-l\varepsilon}}{(2^{-l}+|x-x_{j,k}|)^{n+2\varepsilon}}|B_{j,k}|^{1-\frac{1}{p}+\frac{\varepsilon}{n}},\end{aligned}$$

which implies that

$$\begin{aligned} g_b(a_{j,k})(x) =& \Big(\sum_l |D_l b(a_{j,k})(x)|^2 \Big)^{\frac{1}{2}} \\ \lesssim& |B_{j,k}|^{1-\frac{1}{p}+\frac{\varepsilon}{n}} \Bigg(\sum_l \frac{2^{-2l\varepsilon}}{(2^{-l}+|x-x_{j,k}|)^{2n+4\varepsilon}} \Bigg)^{\frac{1}{2}} \\ \lesssim& |B_{j,k}|^{1-\frac{1}{p}+\frac{\varepsilon}{n}} \Bigg(\sum_{2^{-l}\le|x-x_{j,k}|} \frac{2^{-2l\varepsilon}}{|x-x_{j,k}|^{2n+4\varepsilon}} + \sum_{2^{-l}\ge|x-x_{j,k}|} \frac{2^{-2l\varepsilon}}{2^{-l(2n+4\varepsilon)}} \Bigg)^{\frac{1}{2}} \\ \lesssim& |B_{j,k}|^{1-\frac{1}{p}+\frac{\varepsilon}{n}} |x-x_{j,k}|^{-n-\varepsilon} \\ \lesssim& |B_{j,k}|^{-\frac{1}{p}} [M(\chi_{B_{j,k}})(x)]^{\frac{n+\varepsilon}{n}}. \end{aligned} \tag{3.5}$$

Choose $u \in (0, \frac{n}{n+\varepsilon})$, $v \in (\frac{n}{(n+\varepsilon)u}, \frac{1}{u})$, and $w \in (0, 1-\frac{1}{v})$. By Hölder's inequality, it is available that

$$\sum_{j=-\infty}^{j_0-1} \sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(a_{j,k})(x) \chi_{(2B_{j,k})^c}(x) \le \frac{2^{j_0 w}}{(2^{wv'}-1)^{\frac{1}{v'}}} \Bigg(\sum_{j=-\infty}^{j_0-1} 2^{-jwv} \Bigg(\sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(a_{j,k})(x) \chi_{(2B_{j,k})^c}(x) \Bigg)^v \Bigg)^{\frac{1}{v}}, \tag{3.6}$$

where v and v' are conjugate indices.

Combining estimates (3.5) and (3.6) with Lemma 2.1, we get

$$\begin{aligned} I_{1,2} \le& \Bigg\| \chi_{\Big\{x: \frac{2^{j_0 w}}{(2^{wv'}-1)^{1/v'}} \Big(\sum_{j=-\infty}^{j_0-1} 2^{-jwv} \Big(\sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(a_{j,k})(x) \chi_{(2B_{j,k})^c}(x) \Big)^v \Big)^{\frac{1}{v}} > 2^{j_0-2} \Big\}} \Bigg\|_{L^p} \\ \lesssim& 2^{-j_0 v(1-w)} \Bigg\| \sum_{j=-\infty}^{j_0-1} 2^{-jwv} \Bigg(\sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(a_{j,k})(x) \chi_{(2B_{j,k})^c} \Bigg)^v \Bigg\|_{L^p} \\ \lesssim& 2^{-j_0 v(1-w)} \Bigg(\sum_{j=-\infty}^{j_0-1} 2^{(1-w)jvu} \Bigg\| \sum_{k\in\mathbb{N}} [M(\chi_{B_{j,k}})]^{\frac{(n+\varepsilon)vu}{n}} \Bigg\|_{L^{\frac{p}{u}}} \Bigg)^{\frac{1}{u}} \\ \lesssim& 2^{-j_0 v(1-w)} \Bigg(\sum_{j=-\infty}^{j_0-1} 2^{(1-w)jvu} \Bigg\| \sum_{k\in\mathbb{N}} \chi_{B_{j,k}} \Bigg\|_{L^{\frac{p}{u}}} \Bigg)^{\frac{1}{u}} \\ \lesssim& 2^{-j_0 v(1-w)} \Bigg(\sum_{j=-\infty}^{j_0-1} 2^{[(1-w)v-1]} 2^{ju} \Bigg\| \Bigg(\sum_{k\in\mathbb{N}} \chi_{cB_{j,k}} \Bigg)^{\frac{1}{u}} \Bigg\|_{L^p}^u \Bigg)^{\frac{1}{u}} \\ \lesssim& \alpha^{-1} \sup_{j\in\mathbb{Z}} 2^j \Bigg\| \sum_{k\in\mathbb{N}} \chi_{B_{j,k}} \Bigg\|_{L^p} \\ \lesssim& \alpha^{-1} \parallel f \parallel_{H^{p,\infty}_{b,atom}}. \end{aligned}$$

Thus, we can conclude that

$$I_1 \lesssim \alpha^{-1} \|f\|_{H^{p,\infty}_{b,atom}}.$$

For I_2, we choose $r_1 \in [\frac{1}{p}, \infty)$, and from Lemma 3.1 and (3.2), we get

$$\begin{aligned}
I_2 &\le \left\| \sum_{j=j_0}^{\infty} \sum_{k\in\mathbb{N}} \chi_{2B_{j,k}} \right\|_{L^p} \\
&\lesssim \left\| \sum_{j=j_0}^{\infty} \sum_{k\in\mathbb{N}} \chi_{B_{j,k}} \right\|_{L^p} \\
&\lesssim \left(\sum_{j=j_0}^{\infty} \left\| \sum_{k\in\mathbb{N}} \chi_{B_{j,k}} \right\|_{L^p(\mathbb{R}^n)}^{\frac{1}{r_1}} \right)^{r_1} \\
&\lesssim \left(\sum_{j=j_0}^{\infty} 2^{-\frac{j}{r_1}} \left(2^j \left\| \sum_{k\in\mathbb{N}} \chi_{B_{j,k}} \right\|_{L^p} \right)^{\frac{1}{r_1}} \right)^{r_1} \\
&\lesssim \sup_{j\in\mathbb{Z}} 2^j \left\| \sum_{k\in\mathbb{N}} \chi_{B_{j,k}} \right\|_{L^p} \left(\sum_{j=j_0}^{\infty} 2^{-\frac{j}{r_1}} \right)^{r_1} \\
&\lesssim \alpha^{-1} \parallel f \parallel_{H^{p,\infty}_{b,atom}} .
\end{aligned}$$

Let

$$\frac{n}{p(n+\varepsilon)} < r_2 < 1.$$

The value of $\lambda_{j,k}$, Lemma 2.1, and (3.5) imply that

$$\begin{aligned}
I_3 &\le \left\| \chi_{\left\{x\in(A_{j_0})^c:\, \sum_{j=j_0}^{\infty} \sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(a_{j,k})(x) > \alpha/2\right\}} \right\|_{L^p} \\
&\lesssim \alpha^{-r_2} \left\| \sum_{j=j_0}^{\infty} \sum_{k\in\mathbb{N}} [\lambda_{j,k} g_b(a_{j,k})]^{r_2} \chi_{(A_{j_0})^c} \right\|_{L^p} \\
&\lesssim \alpha^{-r_2} \left(\sum_{j=j_0}^{\infty} \left\| \left(\sum_{k\in\mathbb{N}} [\lambda_{j,k} g_b(a_{j,k})]^{r_2} \chi_{(A_{j_0})^c} \right)^{\frac{n}{r_2(n+\varepsilon)}} \right\|_{L^{\frac{r_2(n+\varepsilon)p}{n}}} \right)^{\frac{r_2(n+\varepsilon)}{n}} \\
&\lesssim \alpha^{-r_2} \left(\sum_{j=j_0}^{\infty} 2^{\frac{jn}{n+\varepsilon}} \left\| \left(\sum_{k\in\mathbb{N}} [M(\chi_{B_{j,k}})]^{\frac{r_2(n+\varepsilon)}{n}} \right)^{\frac{n}{r_2(n+\varepsilon)}} \right\|_{L^{\frac{r_2(n+\varepsilon)p}{n}}} \right)^{\frac{r_2(n+\varepsilon)}{n}} \\
&\lesssim \alpha^{-r_2} \left(\sum_{j=j_0}^{\infty} 2^{\frac{jn}{n+\varepsilon}} \left\| \sum_{k\in\mathbb{N}} \chi_{B_{j,k}} \right\|_{L^p}^{\frac{n}{r_2(n+\varepsilon)}} \right)^{\frac{r_2(n+\varepsilon)}{n}} \\
&\lesssim \sup_{j\in\mathbb{Z}} 2^j \left\| \sum_{k\in\mathbb{N}} \chi_{B_{j,k}} \right\|_{L^p} \alpha^{-r_2} \left(\sum_{j=j_0}^{\infty} 2^{\frac{jn}{n+\varepsilon}} 2^{-\frac{jn}{(n+\varepsilon)r_2}} \right)^{\frac{r_2(n+\varepsilon)}{n}} \\
&\lesssim \alpha^{-1} \|f\|_{H^{p,\infty}_{b,atom}}.
\end{aligned}$$

Together with the estimates of I_1–I_3, we obtain

$$\|f\|_{H^{p,\infty}_b} = \sup_{\alpha\in(0,\infty)} \alpha \left\| \chi_{\{x: g_b(f)(x)>\alpha\}} \right\|_{L^p} \lesssim \|f\|_{H^{p,\infty}_{b,atom}},$$

which implies that $f \in H_b^{p,\infty}(\mathbb{R}^n)$.

The converse result is now proved, showing that

$$H_b^{p,\infty}(\mathbb{R}^n) \subset H_{b,\mathrm{atom}}^{p,\infty}(\mathbb{R}^n).$$

For $f \in \left(b\widetilde{\mathcal{M}}(\beta,\gamma)\right)'$, [24, Theorem 4] provides that

$$f(x) = \sum_l \sum_{Q_l} |Q_l| D_l(x, x_{Q_l}) \widetilde{\widetilde{D}}_l(bf)(x_{Q_l}) b_{Q_l},$$

where

$$b_{Q_l} = \frac{1}{|Q_l|} \int_{Q_l} b(x)dx$$

and the series converges in the sense of distribution. Denote

$$\Omega := \bigcup_l \bigcup_{Q \in Q_l} Q.$$

For $j \in \mathbb{Z}$, let

$$\Omega_j = \{x \in \mathbb{R}^n : S_b(f)(x) > 2^j\} \tag{3.7}$$

and

$$\mathcal{R}_j = \left\{Q \in \Omega_j : |Q \cap \Omega_j| \geq \frac{|Q|}{2} \text{ and } |Q \cap \Omega_{j+1}| < \frac{|Q|}{2}\right\}.$$

Denote the maximal dyadic cubes in $\mathcal{R}_j$ by $\{Q_{j,k}\}_k$ for any $j \in \mathbb{Z}$. Thus, we can rewrite

$$\begin{aligned} f(x) &= \sum_j \sum_k \sum_{Q \in \mathcal{R}_j, Q \subset Q_{j,k}} |Q| D_Q(x, x_Q) \widetilde{\widetilde{D}}_Q(bf)(x_Q) b_Q \\ &=: \sum_j \sum_k \lambda_{j,k} a_{j,k}(x), \end{aligned}$$

where

$$D_Q := D_k$$

if

$$l(Q) = 2^{-k-N},$$

with a similar notation used for $\widetilde{\widetilde{D}}_Q$. Additionally,

$$\lambda_{j,k} = 2^j \|\chi_{cQ_{j,k}}\|_{L^p}$$

and

$$a_{j,k}(x) = \frac{1}{\lambda_{j,k}} \sum_{Q \in \mathcal{R}_j, Q \subset Q_{j,k}} |Q| D_Q(x, x_Q) \widetilde{\widetilde{D}}_Q(f)(x_Q) b_Q.$$

Next, it will be shown that $a_{j,k}$ is a multiple of a (p, q, b) atom corresponding to $cQ_{j,k}$. By definitions of $a_{j,k}$ and D_l, it is clear that supp $a_{j,k} \subset cQ_{j,k}$ and

$$\int a_{j,k}(x)b(x)dx = 0.$$

Let

$$\overline{\Omega}_j = \{x \in \mathbb{R}^n : M(\chi_{\Omega_j})(x) \geq \frac{1}{2}\}.$$

Since

$$|Q \cap \Omega_j| \geq |Q|/2$$

for any $Q \in \mathcal{R}_j$, it follows that $Q \subset \overline{\Omega}_j$. Given this fact, along with

$$|Q \cap \Omega_{j+1}| \leq |Q|/2,$$

then

$$M\Big(\chi_{Q\cap(\overline{\Omega}_j\setminus\Omega_{j+1})}\Big)(x) \geq \frac{\chi_{Q(x)}}{2}.$$

Moreover, we have

$$4M^2\Big(\chi_{Q\cap(\overline{\Omega}_j\setminus\Omega_{j+1})}\Big)(x) \geq \chi_{Q(x)}$$

for $x \in Q$.

Now we estimate $\|a_{j,k}\|_{L^q}$. Since $1 < q < \infty$, from Hölder's inequality and Lemma 2.1, we can deduce that

$$\begin{aligned}
\|a\|_{L^q} &= \frac{1}{\lambda_{j,k}} \sup_{\|h\|_{L^{q'}}\leq 1} \left| \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{R}_j, Q\subset Q_{j,k}} |Q| D_Q(x, x_Q) \widetilde{\widetilde{D}}_Q(bf)(x_Q) h(x) b_Q dx \right| \\
&= \frac{1}{\lambda_{j,k}} \sup_{\|h\|_{L^{q'}}\leq 1} \left| \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{R}_j, Q\subset Q_{j,k}} \widetilde{\widetilde{D}}_l(bf)(x_Q) D_Q^*(h)(x_Q) b_Q \chi_Q(x) dx \right| \\
&\leq \frac{1}{\lambda_{j,k}} \left\| \left\{ \sum_{Q\in\mathcal{R}_j, Q\subset Q_{j,k}} |\widetilde{\widetilde{D}}_Q(bf)(x_Q)|^2 \chi_Q(x) \right\}^{1/2} \right\|_{L^q} \\
&\lesssim \frac{1}{\lambda_{j,k}} \left\| \left\{ \sum_{Q\in\mathcal{R}_j, Q\subset Q_{j,k}} |\widetilde{\widetilde{D}}_Q(bf)(x_Q)|^2 M^2\Big(\chi_{Q\cap(\overline{\Omega}_j\setminus\Omega_{j+1})}\Big) \right\}^{1/2} \right\|_{L^q} \\
&\lesssim \frac{1}{\lambda_{j,k}} \left\| \left\{ \sum_{Q\in\mathcal{R}_j, Q\subset Q_{j,k}} |\widetilde{\widetilde{D}}_Q(bf)(x_Q)|^2 \chi_{Q\cap(\overline{\Omega}_j\setminus\Omega_{j+1})} \right\}^{1/2} \right\|_{L^q} \\
&\lesssim \frac{1}{\lambda_{j,k}} \left\| \left\{ \sum_{Q\in\mathcal{R}_j, Q\subset Q_{j,k}} |\widetilde{\widetilde{D}}_Q(bf)(x_Q)|^2 \right\}^{1/2} \chi_{Q_{j,k}\cap(\overline{\Omega}_j\setminus\Omega_{j+1})} \right\|_{L^q}
\end{aligned}$$

$$\lesssim \frac{1}{\lambda_{j,k}} \|2^j \chi_{Q_{j,k}}\|_{L^q}$$
$$\lesssim |Q_{j,k}|^{1/q-1/p}.$$

Finally, by Definition 3.2 and (3.7), it can be immediately obtained that

$$\|f\|_{H^{p,\infty}_{b,atom}} \lesssim \sup_j 2^j \Big\| \sum_j \chi_{Q_{j,k}} \Big\|_{L^p} \lesssim \sup_j 2^j \Big\| \chi_{\Omega_j} \Big\|_{L^p} \lesssim \|S_b(f)\|_{L^{p,\infty}} \lesssim \|f\|_{H^{p,\infty}_b},$$

which shows that $H^{p,\infty}_b \subset H^{p,\infty}_{b,atom}$ and thereby concludes Theorem 3.1. □

4. Boundedness of CZOs

Recall the definition of Calderón-Zygmund operators, as initated in [25], which are a class of singular integral operators fundamental to harmonic analysis. Generally, the space

$$C^\eta_0(\mathbb{R}^n) = \{f \in C_0(\mathbb{R}^n) : \exists C > 0,\ \forall x, y \in \mathbb{R}^n,\ \ |f(x) - f(y)| \le C|x-y|^\eta\},$$

and its norm

$$\|f\|_{C^\eta_0(\mathbb{R}^n)} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\eta} < \infty.$$

Suppose that b_1 and b_2 are both para-accretive functions. We define $b_1 C^\eta_0(\mathbb{R}^n)$ as the set $\{g : g = b_1 f, f \in C^\eta_0\}$ and $(b_2 C^\eta_0(\mathbb{R}^n))'$ as the dual space of $b_2 C^\eta_0(\mathbb{R}^n)$.

Definition 4.1. *[11] Let T be a continuous linear operator defined from $b_1 C^\eta_0(\mathbb{R}^n)$ to $(b_2 C^\eta_0(\mathbb{R}^n))'$ for all $\eta > 0$. Then T is called a generalized singular integral operator if it can be written in the form*

$$\langle T b_1 f, b_2 g\rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) b_2(x) g(x) b_1(y) f(y) dy dx,$$

for $f, g \in C^\eta_0(\mathbb{R}^n)$ with

$$supp(f) \cap supp(g) = \emptyset,$$

and the kernel $K(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{x = y : x, y \in \mathbb{R}^n\}$ satisfies:

$$|K(x,y)| \lesssim \frac{1}{|x-y|^n}, \tag{4.1}$$

$$|K(x,y) - K(x',y)| \lesssim \frac{|x-x'|^\delta}{|x-y|^{n+\delta}} \quad for\ \ |x-x'| \le \frac{|x-y|}{2}, \tag{4.2}$$

$$|K(x,y) - K(x,y')| \lesssim \frac{|y-y'|^\delta}{|x-y|^{n+\delta}} \quad for\ \ |y-y'| \le \frac{|x-y|}{2}, \tag{4.3}$$

where $0 < \delta < 1$. A generalized singular integral operator T that is bounded on $L^2(\mathbb{R}^n)$ is referred to as a CZO.

We say that $T^*(b) = 0$ if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y)b(x)\phi(y)\mathrm{d}y \, \mathrm{d}x = 0$$

for any $\phi \in \mathcal{D}_1$, where

$$\mathcal{D}_1 (\mathbb{R}^n) := \left\{ \phi \in \mathcal{D}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \phi(x)b(x)\mathrm{d}x = 0 \right\}.$$

The following results present two theorems that examine the behavior of CZOs with respect to weak Hardy spaces. We provide detailed proofs for these theorems to illustrate their validity and significance.

Theorem 4.1. *Let T be a CZO with regularity exponent δ. Suppose that*

$$\frac{n}{n+\delta} \leq p \leq 1$$

and

$$T^*(b) = 0,$$

and then T is bounded from $H^p(\mathbb{R}^n)$ to $H_b^{p,\infty}(\mathbb{R}^n)$.

Proof. To prove Theorem 4.1, observe that

$$\overline{H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)} = H^p(\mathbb{R}^n),$$

and thus we only need to verify

$$\|Tf\|_{H_b^{p,\infty}} \lesssim \|f\|_{H^p} \tag{4.4}$$

for

$$f \in H^p(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n).$$

Let

$$f \in H^p(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n),$$

and there exists a sequence of $(p, 2)$ atoms $\{a_k\}_{k\in\mathbb{N}}$, each corresponding to a ball B_k and coefficients $\{\lambda_k\}_{k\in\mathbb{N}}$, such that

$$f = \sum_{k\in\mathbb{N}} \lambda_k a_k$$

holds in $L^2(\mathbb{R}^n)$ and

$$\|f\|_{H^p} = \inf\left\{ \Big(\sum_{k\in\mathbb{N}} |\lambda_k|^p \Big)^{\frac{1}{p}} : f = \sum_{k\in\mathbb{N}} \lambda_k a_k, \text{where } a_k \text{ is a } (p,2) \text{ atom and } \sum_{k\in\mathbb{N}} |\lambda_k|^p < \infty \right\}.$$

To prove (4.4), it is sufficient to show that

$$\sup_{0<\alpha<\infty} \alpha^p \left| \left\{ x \in \mathbb{R}^n : g_b(Ta)(x) > \alpha \right\} \right| \lesssim 1, \tag{4.5}$$

where a is a $(p,2)$ atom and supp $a \subset B$. In fact, if

$$f \in H^p(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n),$$

then we have

$$\begin{aligned}\alpha^p\left|\left\{x \in \mathbb{R}^n : g_b(Tf)(x) > \alpha\right\}\right| &\leq \alpha^p\left|\left\{x \in \mathbb{R}^n : \sum_{k\in\mathbb{N}} g_b(T\lambda_k a_k)(x) > \alpha\right\}\right| \\ &\leq \alpha^p \sum_{k\in\mathbb{N}} \left|\left\{x \in \mathbb{R}^n : g_b(T\lambda_k a_k)(x) > \alpha\right\}\right| \\ &\leq \sum_{k\in\mathbb{N}} |\lambda_k|^p,\end{aligned}$$

where the second inequality arises from [26, Theorem 6.1] (see also in [27, Lemma 1.8]). Therefore, we conclude (4.4).

Denote

$$B := B(x_0, r)$$

and

$$4B := B(x_0, 4r).$$

We write

$$\begin{aligned}\alpha^p\left|\left\{x \in \mathbb{R}^n : g_b(Ta)(x) > \alpha\right\}\right| &\leq \alpha^p\left|\left\{x \in 4B : g_b(Ta)(x) > \alpha\right\}\right| + \alpha^p\left|\left\{x \in (4B)^c : g_b(Ta)(x) > \alpha\right\}\right| \\ &=: J_1 + J_2.\end{aligned}$$

We first consider J_1. The application of the Hölder inequality and the fact that T is bounded on $L^2(\mathbb{R}^n)$ result in

$$J_1 \leq \int_{4B} [g_b(Ta)(x)]^p dx \lesssim \|g_b(Ta)\|_{L^2}^p |B|^{1-\frac{p}{2}} \lesssim 1.$$

In order to deal with J_2, we begin by presenting some estimates of $D_l bTa(x)$. Here we fix ε as the regularity exponent of D_l for all $l \in \mathbb{Z}$. If $z \in B$ and $x \in (4B)^c$, then

$$|x_0 - z| \leq \frac{1}{2}|x - z|.$$

Choose η and t such that $0 < \eta < \delta \leq \varepsilon$ and $t > \eta$, and then the properties of atom a and [28, (2.4)] imply that

$$\begin{aligned}|D_l bTa(x)| &= \left|\int_{|y-x_0|\geq\frac{|x-x_0|}{2}} b(y) \int_{\mathbb{R}^n} K(y,z)a(z)dz D_l(x,y)dy\right| \\ &\leq \int_{\mathbb{R}^n} |a(z)| \left|\int_{\mathbb{R}^n} D_l(x,y)b(y)[K(y,z) - K(y,x_0)]dy\right| dz \\ &\lesssim \int_{\mathbb{R}^n} |a(z)| \left(\frac{|z-x_0|}{|x-z|}\right)^\delta \frac{1}{(2^{-l} + |x-z|)^n} dz\end{aligned}$$

$$\lesssim \frac{r^\eta}{|x-x_0|^\eta} \frac{2^{-l(t-\eta)}}{(2^{-l}+|x-x_0|)^{n+t-\eta}} \int_{\mathbb{R}^n} |a(z)|dz$$
$$\lesssim r^n |B|^{-\frac{1}{p}} \frac{r^\eta}{|x-x_0|^\eta} \frac{2^{-l(t-\eta)}}{(2^{-l}+|x-x_0|)^{n+t-\eta}}.$$

Thus, for $x \in (4B)^c$, then

$$g_b(Ta)(x) = \left(\sum_l |D_l b(Ta)(x)|^2\right)^{\frac{1}{2}} \lesssim \frac{r^{n+\eta}}{|x-x_0|^{n+\eta}} |B|^{-\frac{1}{p}},$$

and taking the infimum over $0 < \eta < \delta$ for the above estimates, we have

$$g_b(Ta)(x) \lesssim \frac{r^{n+\delta}}{|x-x_0|^{n+\delta}} |B|^{-\frac{1}{p}} \lesssim \left(M(\chi_B)(x)\right)^{\frac{n+\delta}{n}} |B|^{-\frac{1}{p}}. \tag{4.6}$$

Note that

$$\frac{p(n+\delta)}{n} \geq 1,$$

and using the fact that the maximal operator is weak $(\frac{p(n+\delta)}{n}, \frac{p(n+\delta)}{n})$-type bounded, we get that

$$\begin{aligned} J_2 &\lesssim \alpha^p \left|\left\{x \in (4B)^c : g_b(Ta)(x) > \alpha\right\}\right| \\ &\lesssim \alpha^p \left|\left\{x \in (4B)^c : \left(M(\chi_B)(x)\right)^{\frac{n+\delta}{n}} |B|^{-\frac{1}{p}} > \alpha\right\}\right| \\ &\lesssim \alpha^p \left|\left\{x \in (4B)^c : M(\chi_B)(x) > \left(\alpha |B|^{\frac{1}{p}}\right)^{\frac{n}{n+\delta}}\right\}\right| \\ &\lesssim \alpha^p \left(\left(\alpha |B|^{\frac{1}{p}}\right)^{-\frac{n}{n+\delta}} \|\chi_B\|_{L^{\frac{p(n+\delta)}{n}}}\right)^{\frac{p(n+\delta)}{n}} \\ &= C. \end{aligned}$$

Combining these majorizations yields (4.4) and we conclude the proof of Theorem 4.1. □

Theorem 4.2. *Under the hypothesis of Theorem 4.1, T is bounded from $H^{p,\infty}(\mathbb{R}^n)$ to $H_b^{p,\infty}(\mathbb{R}^n)$ if*

$$\frac{n}{n+\delta} < p \leq 1.$$

Proof. The goal is to prove that

$$\left|\left\{x : |g_b(Tf)(x)| > \alpha\right\}\right| \leq \left(\frac{\|f\|_{H^{p,\infty}}}{\alpha}\right)^p.$$

From [4, Definition 4.2, Remark 4.3 and Theorem 4.4] with replacing $p(x)$ with p, there exist two sequences of (p,q) atoms $\{a_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$, each corresponding to a ball $B_{j,k}$ and coefficients $\{\lambda_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ such that (3.1) holds in the sense of distribution and

$$\|f\|_{H_b^{p,\infty}} \approx \sup_{j\in\mathbb{Z}} 2^i \left\| \sum_{k\in\mathbb{N}} \chi_{B_{j,k}} \right\|_{L^p}.$$

Choose $j_0 \in \mathbb{Z}$ such that $2^{j_0} \leq \alpha < 2^{j_0+1}$ for any given $\alpha \in (0, \infty)$. We write

$$f = \sum_{j=-\infty}^{j_0-1} \sum_{k\in\mathbb{N}} \lambda_{j,k} a_{j,k} + \sum_{j=j_0}^{\infty} \sum_{k\in\mathbb{N}} \lambda_{j,k} a_{j,k}$$
$$=: f_1 + f_2.$$

Therefore, we have

$$\begin{aligned}&\left|\left\{x : |g_b(Tf)(x)| > \alpha\right\}\right| \\ &\leq \left|\left\{x : g_b(Tf_1)(x) > \frac{\alpha}{2}\right\}\right| + \left|\left\{x \in B^{j_0} : g_b(Tf_2)(x) > \frac{\alpha}{2}\right\}\right| + \left|\left\{x \in (B^{j_0})^c : g_b(Tf_2)(x) > \frac{\alpha}{2}\right\}\right| \\ &=: H_1 + H_2 + H_3,\end{aligned}$$

where

$$B^{j_0} = \bigcup_{j=j_0}^{\infty} \bigcup_{k\in\mathbb{N}} (4B_{j,k}).$$

We split H_1 into two parts,

$$\begin{aligned}H_1 &\lesssim \left|\left\{x : \sum_{j=-\infty}^{j_0-1} \sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(Ta_{j,k})(x)\chi_{4B_{j,k}}(x) > \frac{\alpha}{4}\right\}\right| + \left|\left\{x : \sum_{j=-\infty}^{j_0-1} \sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(Ta_{j,k})(x)\chi_{(4B_{j,k})^c}(x) > \frac{\alpha}{4}\right\}\right| \\ &=: H_{1,1} + H_{1,2}.\end{aligned}$$

Let

$$\frac{1}{t} + \frac{1}{t'} = 1.$$

Choose $s \in (0, p)$, $t \in \left(1, \min\{q, \frac{1}{s}\}\right)$, and $a \in (0, 1 - \frac{1}{t})$ such that

$$\sum_{j=-\infty}^{j_0-1} \sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(Ta_{j,k})(x)\chi_{4B_{j,k}}(x) \leq \frac{2^{j_0 a}}{(2^{at'} - 1)^{\frac{1}{t'}}} \left(\sum_{j=-\infty}^{j_0-1} 2^{-jat} \left(\sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(Ta_{j,k})(x)\chi_{4B_{j,k}}(x) \right)^t \right)^{\frac{1}{t}}, \tag{4.7}$$

where we use Hölder's inequality. Notice that $ts < 1$, $p/s > 1$, and

$$0 < \left(2^{at'} - 1\right)^{\frac{1}{t'}} < \infty$$

for $1 < t' < \infty$. These facts and (4.7) imply that

$$\begin{aligned}H_{1,1} &\leq \left|\left\{x : \frac{2^{j_0 a}}{(2^{at'} - 1)^{1/t'}} \left(\sum_{j=-\infty}^{j_0-1} 2^{-jat} \left(\sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(Ta_{j,k})(x)\chi_{4B_{j,k}}(x) \right)^t \right)^{\frac{1}{t}} > 2^{j_0-2}\right\}\right| \\ &\lesssim 2^{-j_0 t(1-a)p} \left\| \sum_{j=-\infty}^{j_0-1} 2^{-jat} \left(\sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(Ta_{j,k})(x)\chi_{4B_{j,k}} \right)^t \right\|_{L^p}^p \\ &\lesssim 2^{-j_0 t(1-a)p} \left\| \sum_{j=-\infty}^{j_0-1} 2^{(1-a)jts} \sum_{k\in\mathbb{N}} \left(\|\chi_{B_{j,k}}\|_{L^p} g_b(Ta_{j,k})\chi_{4B_{j,k}} \right)^{ts} \right\|_{L^{p/s}}^{\frac{p}{s}}\end{aligned}$$

$$\lesssim 2^{-j_0t(1-a)p}\bigg(\sum_{j=-\infty}^{j_0-1}2^{(1-a)jts}\bigg\|\bigg(\sum_{k\in\mathbb{N}}\Big(\|\chi_{B_{j,k}}\|_{L^p}g_b(Ta_{j,k})\chi_{4B_{j,k}}\Big)^{ts}\bigg)^{\frac{p}{s}}\bigg\|_{L^p}^{s}\bigg)^{\frac{p}{s}}.$$

Let

$$r=\frac{q}{t},$$

and then $1<r<\infty$. For $j\in\mathbb{Z}$ and $k\in\mathbb{N}$, in view of the Littlewood-Paley characterization of Lebesgue spaces and the size condition of atom a, we have

$$\begin{aligned}\Big\|\Big(\|\chi_{B_{j,k}}\|_{L^p}g_b(Ta_{j,k})\chi_{4B_{j,k}}\Big)^t\Big\|_{L^r}&\lesssim\|\chi_{B_{j,k}}\|_{L^p}^t\Big\|g_b(Ta_{j,k})\Big\|_{L^q}^t\\&\lesssim\|\chi_{B_{j,k}}\|_{L^p}^t\Big\|Ta_{j,k}\Big\|_{L^q}^t\\&\lesssim\|\chi_{B_{j,k}}\|_{L^p}^t\Big\|a_{j,k}\Big\|_{L^q}^t\\&\lesssim|B_{j,k}|^{\frac{1}{r}}.\end{aligned}$$

From Lemmas 2.1 and 3.1, and (3.2), it follows that

$$\begin{aligned}H_{1,1}&\lesssim 2^{-j_0t(1-a)p}\bigg(\sum_{j=-\infty}^{j_0-1}2^{(1-a)jts}\bigg\|\bigg(\sum_{k\in\mathbb{N}}\chi_{4B_{j,k}}\bigg)^{\frac{1}{s}}\bigg\|_{L^p}^{s}\bigg)^{\frac{p}{s}}\\&\lesssim 2^{-j_0t(1-a)p}\bigg(\sum_{j=-\infty}^{j_0-1}2^{(1-a)jts}\bigg\|\bigg(\sum_{k\in\mathbb{N}}\chi_{cB_{j,k}}\bigg)^{\frac{1}{s}}\bigg\|_{L^p}^{s}\bigg)^{\frac{p}{s}}\\&\lesssim 2^{-j_0t(1-a)p}\bigg(\sum_{j=-\infty}^{j_0-1}2^{[(1-a)t-1]js}\sup_{j\in\mathbb{Z}}2^j\bigg\|\sum_{k\in\mathbb{N}}\chi_{B_{j,k}}\bigg\|_{L^p}\bigg)^{\frac{p}{s}}\\&\lesssim\alpha^{-p}\|f\|_{H_b^{p,\infty}}^p.\end{aligned}$$

According to Lemmas 2.1 and 3.1, and (4.6), we can conclude

$$\begin{aligned}H_{1,2}&\le\bigg|\bigg\{x:\frac{2^{j_0a}}{(2^{at'}-1)^{1/t'}}\bigg(\sum_{j=-\infty}^{j_0-1}2^{-jat}\bigg(\sum_{k\in\mathbb{N}}\lambda_{j,k}g_b(Ta_{j,k})(x)\chi_{(4B_{j,k})^c}(x)\bigg)^t\bigg)^{\frac{1}{t}}>2^{j_0-2}\bigg\}\bigg|\\&\lesssim 2^{-j_0t(1-a)p}\bigg\|\sum_{j=-\infty}^{j_0-1}2^{-jat}\bigg(\sum_{k\in\mathbb{N}}\lambda_{j,k}g_b(Ta_{j,k})(x)\chi_{(4B_{j,k})^c}\bigg)^t\bigg\|_{L^p}^p\\&\lesssim 2^{-j_0t(1-a)q}\bigg(\sum_{j=-\infty}^{j_0-1}2^{(1-a)jts}\bigg\|\sum_{k\in\mathbb{N}}[M(\chi_{B_{j,k}})]^{\frac{(n+\delta)ts}{n}}\bigg\|_{L^{\frac{p}{s}}}\bigg)^{\frac{p}{s}}\\&\lesssim 2^{-j_0t(1-a)p}\bigg(\sum_{j=-\infty}^{j_0-1}2^{(1-a)jts}\bigg\|\sum_{k\in\mathbb{N}}\chi_{B_{j,k}}\bigg\|_{L^{\frac{p}{s}}}\bigg)^{\frac{p}{s}}\\&\lesssim 2^{-j_0t(1-a)p}\bigg(\sum_{j=-\infty}^{j_0-1}2^{[(1-a)t-1]}2^{js}\bigg\|\bigg(\sum_{k\in\mathbb{N}}\chi_{cB_{j,k}}\bigg)^{\frac{1}{s}}\bigg\|_{L^p}^{s}\bigg)^{\frac{p}{s}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim a^{-p}\Big(\sup_{j\in\mathbb{Z}} 2^j \Big\| \sum_{k\in\mathbb{N}} \chi_{B_{j,k}} \Big\|_{L^p}\Big)^p \\
&\lesssim a^{-p}\|f\|^p_{H^{p,\infty}}.
\end{aligned}
$$

By an analogue argument to I_2 in the previous section, we have

$$
H_2 \lesssim \alpha^{-p}\|f\|^p_{H^{p,\infty}}.
$$

The value of $\lambda_{j,k}$, Lemma 2.1, and (4.6) deduce that

$$
\begin{aligned}
H_3 &\leq \Big|\Big\{x\in (A_{j_0})^c : \sum_{j=j_0}^{\infty}\sum_{k\in\mathbb{N}} \lambda_{j,k} g_b(Ta_{j,k})(x) > \alpha/2\Big\}\Big| \\
&\lesssim \alpha^{-r_2 p}\Big\| \sum_{j=j_0}^{\infty}\sum_{k\in\mathbb{N}} [\lambda_{j,k} g_b(Ta_{j,k})]^{r_2}\chi_{(A_{j_0})^c}\Big\|^p_{L^p} \\
&\lesssim \alpha^{-r_2 p}\Big(\sum_{j=j_0}^{\infty} \Big\|\Big(\sum_{k\in\mathbb{N}} [\lambda_{j,k} g_b(Ta_{j,k})]^{r_2}\chi_{(A_{j_0})^c}\Big)^{\frac{n}{r_2(n+\delta)}}\Big\|_{L^{\frac{r_2(n+\delta)p}{n}}}\Big)^{\frac{r_2(n+\varepsilon)p}{n}} \\
&\lesssim \alpha^{-r_2 p}\Big(\sum_{j=j_0}^{\infty} 2^{\frac{jn}{n+\delta}} \Big\|\Big(\sum_{k\in\mathbb{N}} [M(\chi_{B_{j,k}})]^{\frac{r_2(n+\delta)}{n}}\Big)^{\frac{n}{r_2(n+\delta)}}\Big\|_{L^{\frac{r_2(n+\delta)p}{n}}}\Big)^{\frac{r_2(n+\delta)p}{n}} \\
&\lesssim \alpha^{-r_2 p}\Big(\sum_{j=j_0}^{\infty} 2^{\frac{jn}{n+\delta}} \Big\|\sum_{j\in\mathbb{Z}} \chi_{B_{j,k}}\Big\|^{\frac{n}{r_2(n+\delta)}}_{L^p}\Big)^{\frac{r_2(n+\delta)p}{n}} \\
&\lesssim \alpha^{-r_2 p}\Big(\sup_{j\in\mathbb{Z}} 2^j \Big\|\sum_{k\in\mathbb{N}} \chi_{B_{j,k}}\Big\|_{L^p}\Big)^p \Big(\sum_{j=j_0}^{\infty} 2^{\frac{jn}{n+\delta}} 2^{-\frac{jn}{(n+\delta)r_2}}\Big)^{\frac{r_2(n+\delta)p}{n}} \\
&\lesssim \alpha^{-p}\|f\|^p_{H^{p,\infty}},
\end{aligned}
$$

where

$$
\frac{n}{p(n+\delta)} < r_2 < 1.
$$

Combining with the estimates of H_1–H_3, we have

$$
\Big|\Big\{x : |g_b(Tf)(x)| > \alpha\Big\}\Big| \lesssim \left(\frac{\|f\|_{H^{p,\infty}}}{\alpha}\right)^p,
$$

which completes the proof of Theorem 4.2. □

5. Conclusions

We developed the theory of weak Hardy spaces by introducing a new class of function spaces, $H^{p,\infty}_b(\mathbb{R}^n)$, associated with para-accretive functions. Additionally, we derived their atomic decomposition and established two general criterions for the boundedness of Calderón-Zygmund singular integral operators.

Author contributions

Yan Wang: methodology, writing–review and editing; Xintian Dong: writing–original draft, formal analysis; Fanghui Liao: conceptualization, methodology, writing–review and editing, supervision, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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