Mathematics

Theory article

An Adomian decomposition method with some orthogonal polynomials to solve nonhomogeneous fractional differential equations (FDEs)

Mariam Al-Mazmumy[1], **Maryam Ahmed Alyami**[1], **Mona Alsulami**[1], **Asrar Saleh Alsulami**[1] and **Saleh S. Redhwan**[2,*]

[1] Department of Mathematics and Statistics, Faculty of Science, University of Jeddah, Jeddah 23218, Saudi Arabia

[2] Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China

* **Correspondence:** Email: salehredhwan@zjnu.edu.cn.

Abstract: The present study introduced modifications to the standard Adomian decomposition method (ADM) by combining the Taylor series with orthogonal polynomials, such as Legendre polynomials and the first and second kinds of Chebyshev polynomials. These improvements can be applied to solve fractional differential equations with initial-value problems in the Caputo sense. The approaches are based on the use of orthogonal polynomials, which are essential components in approximation theories. The study carefully analyzed their respective absolute error differences, highlighting the computational benefits of the proposed modifications, which offer improved accuracy and require fewer computational steps. The effectiveness and accuracy of the approach were validated through numerical examples, confirming its efficiency and reliability.

1. Introduction

Fractional calculus is a branch of mathematical analysis that generalizes integration and differentiation to arbitrary orders. Its origins can be traced back to the speculations of G. W. Leibniz in 1695 and L. Euler in 1730. Despite a long history, fractional calculus and the corresponding fractional differential equations (FDEs) are only gaining attention and popularity now. Different definitions of fractional derivatives exist, such as Riemann-Liouville, Caputo, Grunwald-Letnikov, Weyl, Marchaud, and Prabhakar. The history of this topic is available in [1–3]. Fractional calculus has several practical applications in various functional sciences, engineering, and technology fields.

These include economics [4], chaotic systems [5], viscoelastic materials dynamics [6], electrochemical processes [7], traffic models in fluid dynamics [8], and optics [9]. Fractional derivatives are also helpful in describing the history profile and heritable properties of various constituents and practices. Furthermore, several researchers have investigated the existence and uniqueness of solutions for FDEs [2,3]. Additionally, because most fractional differential equations lack exact analytical solutions, approximate and numerical techniques, including semi-analytical methods, are widely employed to address fractional differential models. These methods include the homotopy analysis method [10], the variational iteration technique [11], Adams methods [12], Multi-Step Methods [13], Haar wavelet method [14], as well as several others [15–18].

The Adomian decomposition method (ADM) [19] has seen extensive use over the past decades in solving a broad range of different types of equations. Originally introduced by George Adomian in the 1980s, the method has experienced numerous developments, modifications, and improvements. For further details regarding the convergence and stability of the ADM in its application to different types of equations, see [20–23]. The extended review by Duan et al. [24] covers the method and its applications to fractional differential equations; see also [25, 26].

In recent times, ADM has been extensively used to provide analytical approximate solutions for linear and nonlinear fractional differential equations [27–29]. The ADM has been improved in several ways to enhance its accuracy, speed, and computational efficiency. These modifications have also made it adaptable to different types of functional equations. The upgraded ADM has shown significant progress by rapidly converging to a series solution compared to the standard ADM. It has also proven to be computationally efficient in various models, which is crucial for researchers in applied science [30–32]. Modifications to the ADM have produced notable results and offer flexibility in solving complex higher-order equations. However, in specific cases, explicitly solving definite integrals with parameters in the domain, or even performing approximate numerical integrations for specific right-hand terms, is not possible. As a result, it becomes essential to approximate these right-hand terms using series expansions before applying the ADM. Orthogonal functions are highly esteemed in areas such as numerical methods and approximation theory.

However, in relation to their practical applications. Wazwaz et al. were the first to propose approximating right-hand terms using the Taylor series before applying the ADM [33]. Later, Hosseini [34] introduced the combination of ADM with the first kind of Chebyshev polynomials, demonstrating the effectiveness and reliability of this method for both linear and nonlinear equations. Following this, Liu [35] incorporated Legendre polynomials into the ADM and conducted a comparative analysis with the previously used Chebyshev polynomials. Furthermore, in [36], it was observed that using the second kind of Chebyshev polynomials with the ADM improved estimation accuracy, outperforming the first kind. Another approach for approximating right-hand terms, as suggested in [37], is through Gegenbauer and Jacobi polynomials. Finally, in [38], a study comparing Chebyshev, Legendre, and Laguerre polynomials found that Legendre polynomials were superior estimators to Chebyshev polynomials, contradicting earlier findings.

The present manuscript proposes several modification procedures for the standard ADM aimed at solving initial-value problems of fractional order. These procedures leverage the application of orthogonal polynomials, which play a crucial role in approximation theories. Specifically, we will explore the use of Legendre polynomials and Chebyshev polynomials of both the first and second kinds to develop enhanced modification methods for the standard ADM. Additionally, a comparative

analysis will be conducted to evaluate the accuracy and efficiency of the proposed approach. The primary advantage of the proposed modification lies in its high level of accuracy, along with fewer computational steps and reduced complexity in the calculations.

The organization of the manuscript is as follows: Section 2 recalls some basics on fractional calculus and formulation and basic features of the Taylor series and orthogonal polynomials, Section 3 outlines the modified approaches to the fractional differential equations, Section 4 applies the constructed scheme on certain test models, while Section 5 presents some concluding notes.

2. Main concepts and theorems

In the current section, we present some definitions and properties of the fractional calculus. Further, we discuss the formulation and basic features of the Taylor series and orthogonal polynomials.

2.1. Basic definitions and properties of the fractional calculus

Definition 2.1. *For a function $y(x) \in C([a, b])$ and $a < x < b$, the Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as [1, 2]*

$$I_a^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x y(t)(x - t)^{\alpha-1} dt. \tag{2.1}$$

Note from the above that for $\alpha = 0$, we have $I_a^0 y(x) = y(x)$, which is the identity operator. Additionally, when $\alpha \in \mathbb{N}$, then $I_a^\alpha y(x)$ coincides with that classical integral.

Definition 2.2. *For $y(x) \in C([a, b])$, the Caputo fractional derivative of order $\alpha > 0$ is defined by [1, 2]*

$$D_*^\alpha y(x) = \begin{cases} I^{m-\alpha} y^{(m)}(x), & m - 1 < \alpha < m, \\ \frac{d^m}{dt^m} y(t), & \alpha = m. \end{cases} \tag{2.2}$$

In the following, we will outline some important properties for the fractional differential and integral operators based on the aforementioned definitions [1, 2].

i)
$$I_a^\alpha I_a^\beta y(x) = I^{\alpha+\beta} y(x), \quad \alpha, \beta \geq 0. \tag{2.3}$$

ii)
$$I_a^\alpha D_*^\alpha y(x) = y(x) - \sum_{k=0}^{m-1} y^{(k)}(0^+) \frac{x^k}{\Gamma(k + 1)}, \tag{2.4}$$

where $m - 1 < \alpha \leq m$.

iii)
$$I_a^\alpha x^n = \frac{\Gamma(n + 1)}{\Gamma(\alpha + n + 1)} x^{n+\alpha}, \quad x > 0, \tag{2.5}$$

where $m - 1 < \alpha \leq m$, $n > -1$.

iv)
$$D_*^\alpha C = 0, \quad \text{where } C \text{ is a real constant.} \tag{2.6}$$

v).

$$D_*^\alpha x^n = \begin{cases} 0, & n < m - 1, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \geq m - 1. \end{cases} \tag{2.7}$$

In this study, we are solving fractional differential equations in the Caputo sense. We have chosen the Caputo definition to solve the fractional differential equations because it allows us to obtain a unique and exact solution by specifying additional initial conditions for the fractional equation. The Caputo definition is widely accepted as it enables defining initial conditions related to the integer derivatives of the derived functions in the considered models. It is worth noting that many studies have extensively explored the geometric interpretation of fractional derivatives.

2.2. The structure and fundamental characteristics of the Taylor series and orthogonal polynomials

In this study, we focus on the function $g(x)$, referred to as the "source term". Our goal is to approximate this term using various methods to reduce computational complexity and simplify the solution process. The significance of orthogonal functions in estimation theory and numerical methods is well known. Therefore, these functions and polynomials are utilized in this study to further enhance the accuracy of the standard Adomian decomposition method (ADM). We begin by applying the Taylor series expansion to the source term $g(x)$, for an arbitrary positive integer l, as follows:

$$g_{Taylor}(x) \approx \sum_{k=0}^{l-1} \frac{g^{(k)}(0)}{k!} x^k. \tag{2.8}$$

Thus, in the subsequent sections, we have developed a series of Adomian modification methods using orthogonal polynomials. In particular, we employed orthogonal polynomials such as Legendre and the first and second kinds of Chebyshev polynomials.

(1) Chebyshev's polynomials of the first kind
   The first kind of Chebyshev polynomial $T_k(x)$ expanding $g(x)$ is

$$gT(x) \approx \sum_{k=0}^{l-1} a_k T_k(x), \tag{2.9}$$

where

$$\begin{cases} T_0(x) = 1, \\ T_1(x) = x, \\ T_2(x) = 2x^2 - 1, \text{ and in general}, \\ T_{k+1}(x) = 2xT_k - T_{k-1}, k \geq 1, \end{cases} \tag{2.10}$$

and

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{g(0.5x + 0.5)}{\sqrt{1 - x^2}} T_0(x) dx, \tag{2.11}$$

$$a_i = \frac{2}{\pi} \int_{-1}^{1} \frac{g(0.5x + 0.5)}{\sqrt{1 - x^2}} T_i(x) dx. \tag{2.12}$$

(2) Chebyshev's polynomials of the second kind

Similarly, we employ Chebyshev polynomials of the second kind to approximate the source term $g(x)$, rather than using Chebyshev polynomials of the first kind, as shown below:

$$gU(x) \approx \sum_{k=0}^{l-1} b_k U_k(x), \tag{2.13}$$

where

$$\begin{cases} U_0(x) = 1, \\ U_1(x) = 2x, \\ U_2(x) = 4x^2 - 1, \text{ and in general,} \\ U_{k+1}(x) = 2xU_k - U_{k-1}, k \geq 1, \end{cases} \tag{2.14}$$

and

$$b_k = \frac{2}{\pi} \int_{-1}^{1} g(0.5x + 0.5) \sqrt{1 - x^2} U_k(x) dx. \tag{2.15}$$

Finally, to introduce a modification method utilizing Legendre polynomials, we represent the source term $g(x)$ as a series expansion of Legendre polynomials as follows.

(3) Legendre's polynomials

$$gP(x) \approx \sum_{k=0}^{l-1} c_k P_k(x), \tag{2.16}$$

where

$$\begin{cases} P_0(x) = 1, \\ P_1(x) = -x + 1, \\ P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \\ P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \\ \vdots \end{cases} \tag{2.17}$$

and

$$c_k = \frac{2k + 1}{\pi} \int_{-1}^{1} g(0.5x + 0.5) P_k(x) dx. \tag{2.18}$$

3. Modified Adomian decomposition method

This section introduces modified Adomian decomposition methods with some orthogonal polynomials for addressing initial-value problems (IVPs) of nonhomogeneous fractional differential equations.

We consider the initial-value problem (IVP) for a nonlinear fractional differential equation as shown below:

$$D_*^{\alpha} y(x) + Ry(x) + Ny(x) = g(x), \quad \alpha > 0. \tag{3.1}$$

$$y^k(0) = c_k, \quad k = 0, 1, 2, ..., m - 1, \quad m - 1 < \alpha < m.$$

In the above equation, $y(x)$ is the unknown function, D_*^{α} is the Caputo fractional derivative operator of high order α, which is easily invertible, N is the nonlinear operator, R is the remaining linear operator,

which might include other Caputo fractional derivative operators D_*^v ($v < \alpha$), and g is the given source function.

To solve the above initial-value problem (IVP), the approach involves applying the operator I^α on both sides of (3.1), which is indeed the inverse operator of D_*^α, and further make use of (2.4) to obtain

$$I^\alpha[D_*^\alpha y(x) + Ry(x) + Ny(x)] = I^\alpha[g(x)]. \tag{3.2}$$

$$y(x) = \sum_{k=0}^{m-1} c_k \frac{x^k}{k!} + [I^\alpha g(x)] - [I^\alpha Ry(x)] - [I^\alpha Ny(x)]. \tag{3.3}$$

Alternatively, by expressing the latter equation in accordance with the Adomian decomposition method, we arrive at the following formulation:

$$\sum_{n=0}^{\infty} y_n(x) = \sum_{k=0}^{m-1} c_k \frac{x^k}{k!} + [I^\alpha g(x)] - [I^\alpha R \sum_{n=0}^{\infty} y_n(x)] - [I^\alpha \sum_{n=0}^{\infty} A_n(x)], \tag{3.4}$$

where the A_n's are the Adomian polynomials, which are to be computed using the following compacted formula:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots.$$

The recurrence relation for $y(x)$ can be established by simply defining:

$$\begin{cases} y_0(x) = \sum_{k=0}^{m-1} c_k \frac{x^k}{k!} + I^\alpha[g(x)], \\ y_n(x) = -I^\alpha R[y_{n-1}(x)] - I^\alpha[A_{n-1}], \quad n \geq 1. \end{cases} \tag{3.5}$$

This approach follows the structure outlined below, resulting in the corresponding recursive solution.

(1) Modification method via Taylor's series

$$\begin{cases} y_0(x) = \sum_{k=0}^{m-1} c_k \frac{x^k}{k!} + I^\alpha[\sum_{k=0}^{l-1} \frac{g^{(k)}(0)}{k!} x^k], \\ y_n(x) = -I^\alpha R[y_{n-1}(x)] - I^\alpha[A_{n-1}], \quad n \geq 1. \end{cases} \tag{3.6}$$

(2) Modification method via the first kind of Chebyshev's polynomials

$$\begin{cases} y_0(x) = \sum_{k=0}^{m-1} c_k \frac{x^k}{k!} + I^\alpha[\sum_{k=0}^{l-1} a_k T_k(x)], \\ y_n(x) = -I^\alpha R[y_{n-1}(x)] - I^\alpha[A_{n-1}], \quad n \geq 1. \end{cases} \tag{3.7}$$

(3) Modification method via the seconed kind of Chebyshev's polynomials

$$\begin{cases} y_0(x) = \sum_{k=0}^{m-1} c_k \frac{x^k}{k!} + I^\alpha[\sum_{k=0}^{l-1} b_k U_k(x)], \\ y_n(x) = -I^\alpha R[y_{n-1}(x)] - I^\alpha[A_{n-1}], \quad n \geq 1. \end{cases} \tag{3.8}$$

(4) Modification method via Legendre's polynomials

$$\begin{cases} y_0(x) = \sum_{k=0}^{m-1} c_k \frac{x^k}{k!} + I^\alpha[\sum_{k=0}^{l-1} c_k P_k(x)], \\ y_n(x) = -I^\alpha R[y_{n-1}(x)] - I^\alpha[A_{n-1}], \quad n \geq 1. \end{cases} \tag{3.9}$$

Considering the rapid convergence of the decomposition series, it will be demonstrated through a representative example that only a small number of terms are needed to achieve an accurate solution. Subsequently, a comparison will be made between the approximate $y(x)$ and those derived from Taylor expansion, Chebyshev expansion, and Legendre expansion. This comparison aims to validate both the accuracy of the obtained solution and the efficacy of the proposed method.

Algorithm 1 Coding algorithm for the modification method of the ADM

1: **Input:** Define the problem and the source term $g(x)$.
2: **Step 1:** Apply the operator I^α on both sides of the problem.
3: **Step 2:** Expand $g(x)$ using a Taylor series or using Legendre and Chebyshev polynomials.
4: **Step 3:** Apply the Adomian decomposition method (ADM) for further modification.
5: **Step 4:** Derive the recursive solution based on the approximation.
6: **Step 5:** Compute the numerical results.
7: **Output:** Recursive solution to the problem.

---

4. Numerical applications

This section makes use of some test IVPs to ascertain the effectiveness of the proposed recursive schemes numerically for the classes of the fractional differential equations. All calculations are performed, and the plots are generated by using Maple 22 with 8 digits precision.

Example 4.1. *Consider the following inhomogeneous fractional differential equation [39]:*

$$D_*^\alpha y(x) + y(x) = \sin(x), \quad y(0) = 0. \tag{4.1}$$

$$0 < \alpha \le 1, \quad x \in [0, 1].$$

The exact solution of the above problem when $\alpha = 1$ is

$$y(x) = \frac{-1}{2} \cos(x) + \frac{1}{2} \sin(x) + \frac{1}{2} e^{-x}. \tag{4.2}$$

In the above equation, $y(x)$ is the unknown function, D_*^α is the Caputo fractional derivative operator of high order α, and $g(x) = \sin(x)$.

To solve the above IVP, we apply the operator I^α to both sides of the IVP to get:

$$I^\alpha[D_*^\alpha y(x)] + I^\alpha[y(x)] = I^\alpha[\sin(x)], \tag{4.3}$$

$$y(x) - y(0) + I^\alpha[y(x)] = I^\alpha[\sin(x)]. \tag{4.4}$$

Applying the standard Adomian decomposition method yields the following result:

$$\sum_{n=0}^{\infty} y_n(x) + I^\alpha[\sum_{n=0}^{\infty} y_n(x)] = I^\alpha[\sin(x)]. \tag{4.5}$$

The recurrence relation for $y(x)$ can be established by simply defining:

$$\begin{cases} y_0(x) = I^\alpha[\sin(x)], \\ y_n(x) = -I^\alpha[y_{n-1}(x)], \quad \forall n \ge 1. \end{cases} \tag{4.6}$$

It is essential to implement the suggested modifications while dealing with the complex source term associated with the $sin(x)$ term in computations. It is important to recognize that directly applying the ADM has certain disadvantages, such as a large computation size, extended computation time, and complicated computation of the remaining terms since it relies on the first term. Considering the preceding discussion, we propose the following formulations to relieve these issues.

- The expansion of $g(x) = \sin(x)$ using the Taylor series $l = 6$ is

$$\begin{cases} y_0(x) = I^\alpha[\sum_{k=0}^{5} \frac{g^{(k)}(0)}{k!} x^k] = I^\alpha[x - \frac{1}{6}x^3], \\ y_n(x) = -I^\alpha[y_{n-1}(x)], \quad \forall n \geq 1, \end{cases} \tag{4.7}$$

such that upon summing the above components yields the following series solution:

$$y_{Taylor}(x) = \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{x^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \dots \tag{4.8}$$

- The expansion of $g(x) = \sin(x)$ using the first kind of Chebyshev polynomials $l = 6$ is

$$\begin{cases} y_0(x) = I^\alpha[\sum_{k=0}^{5} a_k T_k(x)] \\ = I^\alpha[3.0908 \times 10^{-7} + 0.99997898x + 0.0023523x^2 - 0.16761545x^3 + 0.001633897x^4 \\ +0.0072373478x^5], \\ y_n(x) = -I^\alpha[y_{n-1}(x)], \quad \forall n \geq 1, \end{cases} \tag{4.9}$$

such that their summation results in

$$yT(x) = \frac{3.0908 \times 10^{-7} x^\alpha}{\Gamma(\alpha+1)} + \frac{0.99997898 x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{0.0047046 x^{\alpha+2}}{\Gamma(\alpha+3)} + \dots \tag{4.10}$$

- The expansion of $g(x) = \sin(x)$ using the second kind of Chebyshev polynomials $l = 6$ is

$$\begin{cases} y_0(x) = I^\alpha[\sum_{k=0}^{5} b_k U_k(x)] \\ = I^\alpha[1.00504 \times 10^{-6} + 0.99996781x + 0.00028067x^2 - 0.16767393x^3 + 0.001648541x^4 \\ +0.0072481459x^5], \\ y_n(x) = -I^\alpha[y_{n-1}(x)], \quad \forall n \geq 1, \end{cases} \tag{4.11}$$

which results in the subsequent series solution:

$$yU(x) = \frac{1.00504 \times 10^{-6} x^\alpha}{\Gamma(\alpha+1)} + \frac{0.99996781 x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{0.00056134 x^{\alpha+2}}{\Gamma(\alpha+3)} + \dots \tag{4.12}$$

- The expansion of $g(x) = \sin(x)$ using Legendre polynomials $l = 6$ is

$$\begin{cases} y_0(x) = I^\alpha[\sum_{k=0}^{5} c_k P_k(x)] \\ = I^\alpha[6.6871 \times 10^{-7} + 0.99997284x + 0.0002611x^2 - 0.16764828x^3 + 0.00164224x^4 \\ +0.007243161x^5], \\ y_n(x) = -I^\alpha[y_{n-1}(x)], \quad \forall n \geq 1, \end{cases} \tag{4.13}$$

that sums to the following:

$$yP(x) = \frac{6.6871 \times 10^{-7} x^\alpha}{\Gamma(\alpha+1)} + \frac{0.99997284 x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{0.0005222 x^{\alpha+2}}{\Gamma(\alpha+3)} + \dots \tag{4.14}$$

Additionally, in the following Table 1, we present an analysis of the simulated error difference for the orthogonal polynomial with $n = 5$, compared to the exact solution of the problem with $\alpha = 1$. In addition, we have visually represented the obtained solution for the IVP in the Figure 1(a)–(e), showcasing the graphical visualization for various values of α.

Table 1. Comparison of the proposed modified ADM's solution to the exact solution for Example (4.1).

x	$|exact - y_{Taylor}|$	$|exact - yP(x)|$	$|exact - yT(x)|$	$|exact - yU(x)|$
0	0	0	0	0
0.2	9.3×10^{-8}	1.9×10^{-8}	2.0×10^{-8}	1.3×10^{-8}
0.4	5.373×10^{-6}	9.0×10^{-9}	1.4×10^{-8}	8.0×10^{-9}
0.6	5.965×10^{-5}	4.3×10^{-7}	4.2×10^{-7}	4.2×10^{-7}
0.8	3.2628×10^{-4}	3.76×10^{-6}	3.77×10^{-6}	3.76×10^{-6}
1	1.21277×10^{-3}	2.2040×10^{-5}	2.2030×10^{-5}	2.201×10^{-5}
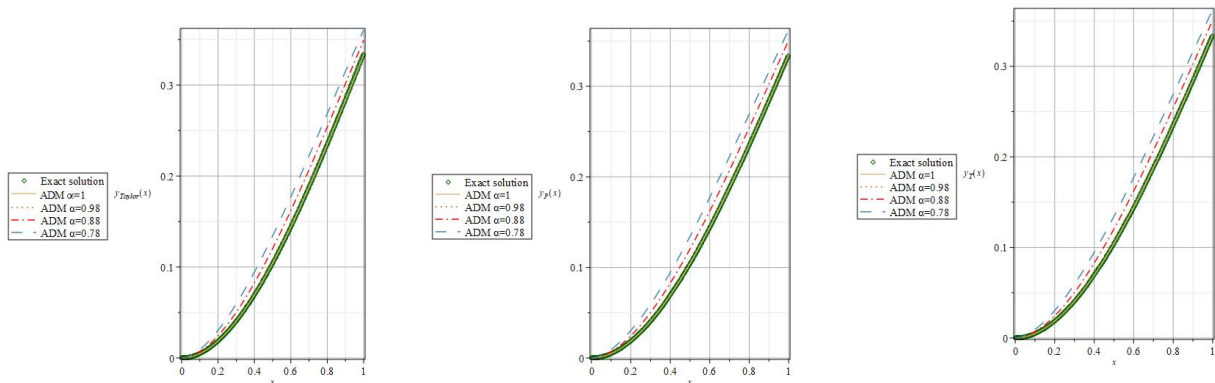
(a) Exact and approximate solution by the ADM with the Taylor series.

(b) Exact and approximate solution by the ADM with Legendre polynomials.

(c) Exact and approximate solution by the ADM with the first kind of Chebyshev polynomials.
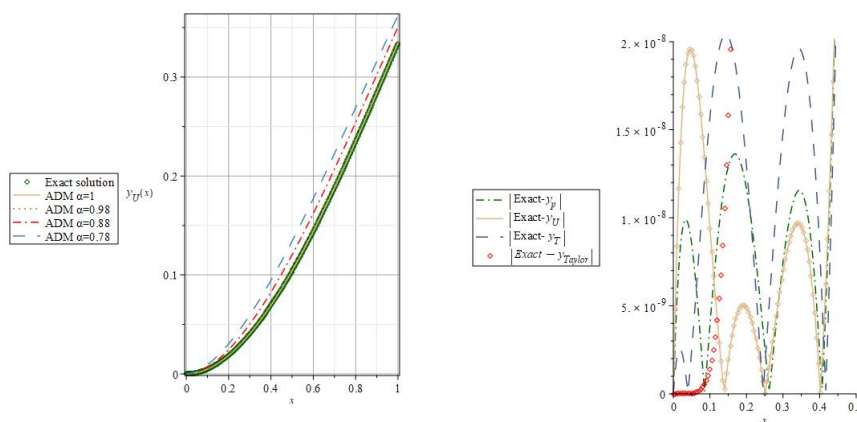
(d) Exact and approximate solution by the ADM with the second kind of Chebyshev polynomials.

(e) The absolute error between the exact solution and proposed modified ADM.

Figure 1. The exact solution and proposed modified ADM of Example (4.1) for different values of α.

Example 4.2. *Let us make consideration of the nonlinear inhomogeneous fractional IVP as follows [40]:*

$$D_*^\alpha y(x) + 2y(x) - y^3(x) = \sin(x)\cos^2(x), \quad y(0) = 0, y'(0) = 1. \tag{4.15}$$

$$0 < \alpha \le 2, \quad x \in [0, 1].$$

The exact solution of the above problem when $\alpha = 2$ is

$$y(x) = \sin(x). \tag{4.16}$$

In the above equation, $y(x)$ is the unknown function, D_*^α is the Caputo fractional derivative operator of high order α, $y^3(x)$ is a nonlinear term, and $g(x) = \sin(x)\cos^2(x)$.

To solve the above IVP, we apply the operator I^α to both sides of the IVP to get:

$$I^\alpha[D_*^\alpha y(x)] + I^\alpha[2y(x)] - I^\alpha[y^3(x)] = I^\alpha[\sin(x)\cos^2(x)], \tag{4.17}$$

$$y(x) - y(0) - y'(0)x + I^\alpha[2y(x)] - I^\alpha[y^3(x)] = I^\alpha[\sin(x)\cos^2(x)]. \tag{4.18}$$

Applying the standard Adomian decomposition method yields the following result:

$$\sum_{n=0}^{\infty} y_n(x) - x + I^{\alpha}[2\sum_{n=0}^{\infty} y_n(x)] - I^{\alpha}[\sum_{n=0}^{\infty} A_n] = I^{\alpha}[\sin(x)\cos^2(x)], \qquad (4.19)$$

where A_n are the Adomian polynomials for nonlinear term $y^3(x)$ which when computed iteratively takes the following form:

$$\begin{aligned}
A_0 &= y_0^3, \\
A_1 &= 3(y_0)^2 y_1, \\
A_2 &= 3(y_0)^2 y_2 + 3y_1^2 y_0, \\
A_3 &= 3(y_0)^2 y_3 + 6y_0 y_1 y_2 + y_1^3, \\
&\vdots
\end{aligned} \qquad (4.20)$$

The recurrence relation for $y(x)$ can be established by simply defining:

$$\begin{cases}
y_0(x) = x + I^{\alpha}[\sin(x)\cos^2(x)], \\
y_n(x) = -I^{\alpha}[2y_{n-1}(x)] + I^{\alpha}[A_{n-1}(x)], \quad \forall n \geq 1.
\end{cases} \qquad (4.21)$$

To simplify the complexity of $g(x) = \sin(x)\cos^2(x)$, we recommend using the following formulations:

- The expansion of $g(x) = \sin(x)\cos^2(x)$ using the Taylor series $l = 6$ is

$$\begin{cases}
y_0(x) = x + I^{\alpha}[\sum_{k=0}^{5} \frac{g^{(k)}(0)}{k!} x^k] = x + I^{\alpha}[x - \frac{7}{6}x^3], \\
y_n(x) = -I^{\alpha}[2y_{n-1}(x)] + I^{\alpha}[A_{n-1}(x)], \quad \forall n \geq 1,
\end{cases} \qquad (4.22)$$

leading to the following series solution:

$$y_{Taylor}(x) = \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{7x^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \dots \qquad (4.23)$$

- The expansion of $g(x) = \sin(x)\cos^2(x)$ using the first kind of Chebyshev polynomials $l = 6$ is

$$\begin{cases}
y_0(x) = x + I^{\alpha}[\sum_{k=0}^{5} a_k T_k(x)] \\
\quad = x + I^{\alpha}[0.0001117718 + 0.99203869x + 0.09116705x^2 - 1.5421702x^3 + 0.67021928x^4 \\
\quad + 0.034394566x^5], \\
y_n(x) = -I^{\alpha}[2y_{n-1}(x)] + I^{\alpha}[A_{n-1}(x)], \quad \forall n \geq 1,
\end{cases} \qquad (4.24)$$

that sums to the following:

$$yT(x) = \frac{0.0001117718x^{\alpha}}{\Gamma(\alpha+1)} + \frac{0.99203869x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{0.18233410x^{\alpha+2}}{\Gamma(\alpha+3)} + \dots \qquad (4.25)$$

- The expansion of $g(x) = \sin(x)\cos^2(x)$ using the second kind of Chebyshev polynomials $l = 6$ is

$$\begin{cases} y_0(x) = x + I^\alpha[\sum_{k=0}^{5} b_k U_k(x)] \\ = x + I^\alpha[0.0003936422 + 0.9875480x + 0.10989235x^2 - 1.570291x^3 + 0.68366181x^4 \\ +0.034844186x^5], \\ y_n(x) = -I^\alpha[2y_{n-1}(x)] + I^\alpha[A_{n-1}(x)], \quad \forall n \geq 1, \end{cases}$$

(4.26)

that leads to the following series solution:

$$yU(x) = \frac{0.0003936422x^\alpha}{\Gamma(\alpha + 1)} + \frac{0.9875480x^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{0.21978470x^{\alpha+2}}{\Gamma(\alpha + 3)} + \dots$$

(4.27)

- The expansion of $g(x) = \sin(x)\cos^2(x)$ using Legendre polynomials $l = 6$ is

$$\begin{cases} y_0(x) = x + I^\alpha[\sum_{k=0}^{5} c_k P_k(x)] \\ = x + I^\alpha[0.0002485684 + 0.98967633x + 0.1012779x^2 - 1.5574782x^3 + 0.6775396x^4 \\ +0.034636179x^5], \\ y_n(x) = -I^\alpha[2y_{n-1}(x)] + I^\alpha[A_{n-1}(x)], \quad \forall n \geq 1, \end{cases}$$

(4.28)

which totals to the following:

$$yP(x) = \frac{0.0002485684x^\alpha}{\Gamma(\alpha + 1)} + \frac{0.98967633x^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{0.2025558x^{\alpha+2}}{\Gamma(\alpha + 3)} + \dots$$

(4.29)

In the following Table 2, we present an analysis of the simulated error difference for the orthogonal polynomial with $n = 5$, compared to the exact solution of the problem with $\alpha = 2$. Further, we graphically visualize the obtained solution for the IVP in Figure 2(a)–(e), showcasing the various visualizations for different values of α.

Based on the provided illustration and the outcomes obtained through analyses and comparisons, we can conclude that incorporating orthogonal polynomials such as Chebyshev, Legendre, and Taylor series can significantly enhance the performance of the ADM. Comparison between the standard ADM and Taylor series with Legendre and Chebyshev polynomials shows that the latter provide more accurate estimations.

**Table 2.** Comparison of the proposed modified ADM's solution to the exact solution for Example (4.2).

x	$|exact - y_{Taylor}|$	$|exact - yP(x)|$	$|exact - yT(x)|$	$|exact - yU(x)|$
0	0	0	0	0
0.2	1.54×10^{-7}	2.447737×10^{-7}	9.41263×10^{-7}	7.65213×10^{-7}
0.4	1.93662×10^{-5}	1.517810×10^{-7}	2.74701×10^{-7}	1.62808×10^{-6}
0.6	3.21843×10^{-4}	1.600035×10^{-7}	5.02968×10^{-7}	2.27452×10^{-6}
0.8	2.32654×10^{-3}	3.882715×10^{-7}	1.81022×10^{-6}	2.5491×10^{-6}
1	1.06477×10^{-2}	7.837075×10^{-6}	9.06708×10^{-6}	4.60779×10^{-6}
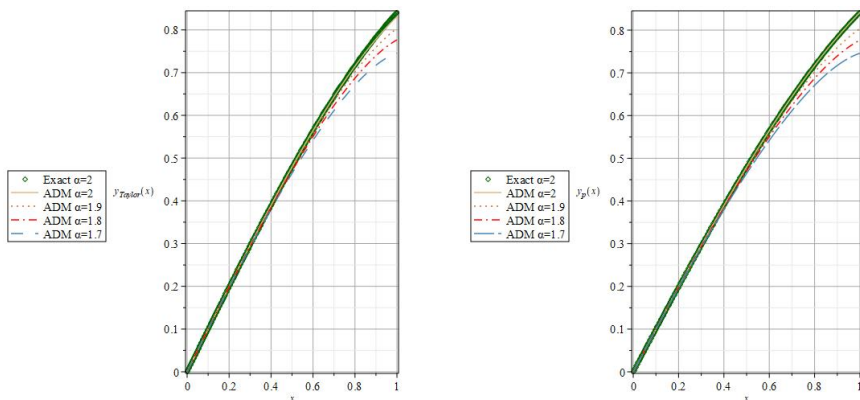
(a) Exact and approximate solution by the ADM with the Taylor series.

(b) Exact and approximate solution by the ADM with Legendre polynomials.
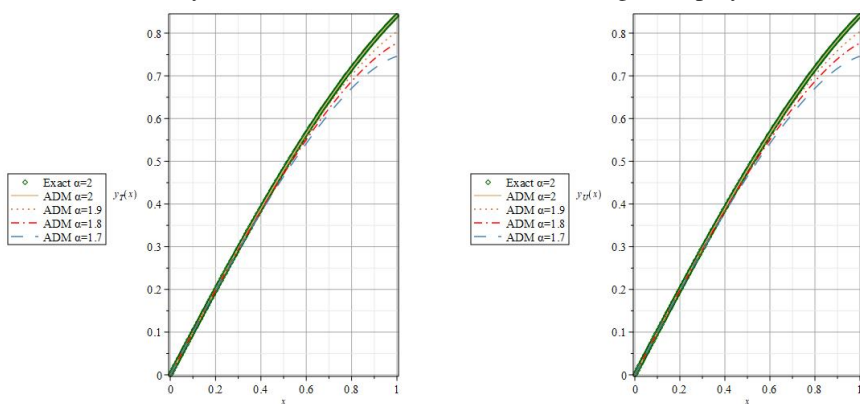
(c) Exact and approximate solution by the ADM with the first kind of Chebyshev polynomials.

(d) Exact and approximate solution by the ADM with the second kind of Chebyshev polynomials.
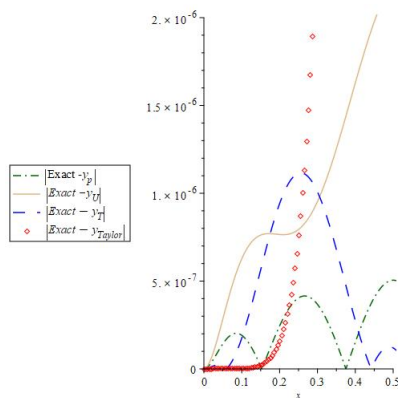
(e) The absolute error between the exact solution and proposed modified of ADM.

Figure 2. The exact solution and proposed modified ADM of Example (4.2) for different values of α.

4.1. Special application

Consider the fractional order forced Duffing-van der Pol oscillator equation as follows

$$D_*^\alpha y(x) - \mu(1 - y^2(x))y'(x) + ay(x) + by^3(x) = f\cos(\omega x), \quad y(0) = 1, \ y'(0) = 0. \qquad (4.30)$$

$$1 < \alpha \le 2, \quad x \in [0, 1].$$

- (Single-well $a > 0, b > 0$). $a = b = 0.5$, $\mu = 0.1$, $f = 0.5$, and $\omega = 0.79$.
- (Double-well $a < 0, b > 0$). $a = -0.5, b = 0.5$, $\mu = 0.1$, $f = 0.5$, and $\omega = 0.79$.
- (Double-hump $a > 0, b < 0$). $a = 0.5, b = -0.5$, $\mu = 0.1$, $f = 0.5$, and $\omega = 0.79$.

To solve Eq (4.30) using the modification method of the ADM, we apply the previously outlined algorithm as follows:

$$y(x) - 1 - I^\alpha[\mu y'(x)] + I^\alpha[y^2(x)y'(x)] + I^\alpha[ay(x)] + I^\alpha[by^3(x)] = I^\alpha[f\cos(\omega x)], \qquad (4.31)$$

such that when the ADM is deployed reveals

$$\sum_{n=0}^{\infty} y_n(x) - 1 - I^\alpha\left[\mu \sum_{n=0}^{\infty} y_n'(x)\right] - I^\alpha\left[\sum_{n=0}^{\infty} A_n(x)\right] + I^\alpha\left[a \sum_{n=0}^{\infty} y_n(x)\right] + I^\alpha\left[b \sum_{n=0}^{\infty} B_n(x)\right] = I^\alpha\left[f\cos(\omega x)\right], \quad (4.32)$$

where A_n and B_n are Adomian polynomials corresponding to the nonlinear terms $y^2(x)y'(x)$ and $y^3(x)$ respectively, defined as follows:

$$
\begin{aligned}
A_0 &= y_0(x)^2\left(\frac{d}{dx}y_0(x)\right), \\
A_1 &= 2y_0(x)\left(\frac{d}{dx}y_0(x)\right)y_1(x) + y_0(x)^2\left(\frac{d}{dx}y_1(x)\right), \\
A_2 &= y_1(x)^2\left(\frac{d}{dx}y_0(x)\right) + 2y_0(x)\left(\frac{d}{dx}y_1(x)\right)y_1(x) + 2y_0(x)\left(\frac{d}{dx}y_0(x)\right)y_2(x) + y_0(x)^2\left(\frac{d}{dx}y_2(x)\right), \\
&\vdots
\end{aligned}
\qquad (4.33)
$$

$$
\begin{aligned}
B_0 &= y_0(x)^3, \\
B_1 &= 3 \cdot y_0(x)^2 \cdot y_1(x), \\
B_2 &= 3 \cdot y_0(x) \cdot y_1(x)^2 + 3 \cdot y_0(x)^2 \cdot y_2(x), \\
&\vdots
\end{aligned}
\qquad (4.34)
$$

Moreover, the formal recursive relationship for the governing fractional IVP is obtained as follows:

$$
\begin{cases}
y_0(x) = 1 + I^\alpha\left[f\cos(\omega x)\right], \\
y_n(x) = I^\alpha\left[\mu y_{n-1}'(x)\right] - I^\alpha\left[A_{n-1}(x)\right] - I^\alpha\left[ay_{n-1}(x)\right] - I^\alpha\left[bB_{n-1}(x)\right], n \ge 1.
\end{cases}
\qquad (4.35)
$$

In the subsequent discussion, to avoid the complexity of calculating the source term $f\cos(\omega x)$, we will apply the proposed modification methods to solve the governing equation of the forced Duffing-van der Pol oscillator. We will first introduce the classical Taylor series approach, followed by the proposed schemes. The solution will be represented based on the respective modifications as follows: $y_{Taylor}(x)$ for the Taylor series expansion, $yP(x)$ for the Legendre series expansion, and $yT(x)$ and $yU(x)$ for the first and second kind of Chebyshev series expansion.

4.1.1. Single-well situation

The recursive formula for the fractional problem (4.30) in this case is given by:

$$\begin{cases} y_0(x) = 1 + I^\alpha\left[\frac{1}{2}\cos(0.79x)\right], \\ y_n(x) = I^\alpha\left[0.1y'_{n-1}(x)\right] - I^\alpha\left[A_{n-1}(x)\right] - I^\alpha\left[\frac{1}{2}y_{n-1}(x)\right] - I^\alpha\left[\frac{1}{2}B_{n-1}(x)\right], n \geq 1. \end{cases} \tag{4.36}$$

- Modification method via Taylor's series
  The source term $g(x) = \frac{1}{2}\cos(0.79x)$ will be expanded for $l = 6$ in the following manner:

$$g_{Taylor}(x) = \frac{1}{2} - 0.156025000x^2 + 0.00811460020x^4 + O\left(x^6\right). \tag{4.37}$$

This yields the subsequent iterative components.

$$y_0(x) = 1 + I^\alpha\left[\frac{1}{2} - 0.156025000x^2 + 0.00811460020x^4\right],$$

$$y_1(x) = I^\alpha\left[0.1y'_0(x)\right] - I^\alpha\left[A_0(x)\right] - I^\alpha\left[\frac{1}{2}y_0(x)\right] - I^\alpha\left[\frac{1}{2}B_0(x)\right], \tag{4.38}$$

$$\vdots$$

such that upon summing the above components yields the following series solution:

$$y_{Taylor} = 1 + \frac{x^\alpha}{2\Gamma(\alpha+1)} - \frac{0.312050000x^{2+\alpha}}{\Gamma(\alpha+3)} + \frac{0.1947504048x^{4+\alpha}}{\Gamma(\alpha+5)} + \ldots \tag{4.39}$$

- Modification method via Chebyshev's polynomials of the first kind
  We Proceed by expanding the source term $g(x)$ in the following manner:

$$gT(x) = 0.500000002 - 5.3 \times 10^{-8}x - 0.156023884x^2 - 7.0081 \times 10^{-6}x^3$$
$$+ 0.00813387558x^4 - 0.000025329921x^5 - 0.000154945607x^6. \tag{4.40}$$

Therefore, the solution components are as follows:

$$y_0(x) = 1 + I^\alpha\left[gT(x)\right],$$

$$y_1(x) = I^\alpha\left[0.1y'_0(x)\right] - I^\alpha\left[A_0(x)\right] - I^\alpha\left[\frac{1}{2}y_0(x)\right] - I^\alpha\left[\frac{1}{2}B_0(x)\right], \tag{4.41}$$

$$\vdots$$

which results in the subsequent series solution:

$$yT(x) = 1 + \frac{0.500000002x^\alpha}{\Gamma(\alpha+1)} - \frac{5.3 \times 10^{-8}x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{0.312047768x^{2+\alpha}}{\Gamma(3+\alpha)} + \ldots \tag{4.42}$$

- Modification method via Chebyshev's polynomials of the second kind

We begins by expressing the source term $g(x)$ in the following manner:

$$gU(x) = 0.500000004 - 1.30 \times 10^{-7}x - 0.156023427x^2 - 8.0266 \times 10^{-6}x^3$$
$$+ 0.00813475220x^4 - 0.000025461710x^5 - 0.000155053667x^6. \tag{4.43}$$

This gives the following iterative solutions:

$$y_0(x) = 1 + I^\alpha\left[gU(x)\right],$$

$$y_1(x) = I^\alpha\left[0.1y_0'(x)\right] - I^\alpha\left[A_0(x)\right] - I^\alpha\left[\frac{1}{2}y_0(x)\right] - I^\alpha\left[\frac{1}{2}B_0(x)\right], \tag{4.44}$$

$$\vdots$$

that sums to the following:

$$yU(x) = 1 + \frac{0.500000004x^\alpha}{\Gamma(\alpha+1)} - \frac{1.3 \times 10^{-7}x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{0.312046854x^{2+\alpha}}{\Gamma(3+\alpha)} + \ldots \tag{4.45}$$

- Modification method via Legendre's polynomials
  Now, the expansion of the source term $g(x)$ using Legendre polynomials gives:

$$gP(x) = 0.500000004 - 1.43 \times 10^{-7}x - 0.156023485x^2 - 7.7201 \times 10^{-6}x^3$$
$$+ 0.00813440004x^4 - 0.000025402532x^5 - 0.000155003233x^6. \tag{4.46}$$

This gives the following iterative solutions:

$$y_0(x) = 1 + I^\alpha\left[gP(x)\right],$$

$$y_1(x) = I^\alpha\left[0.1y_0'(x)\right] - I^\alpha\left[A_0(x)\right] - I^\alpha\left[\frac{1}{2}y_0(x)\right] - I^\alpha\left[\frac{1}{2}B_0(x)\right], \tag{4.47}$$

$$\vdots$$

that sums to the following:

$$yP(x) = 1 + \frac{0.500000004x^\alpha}{\Gamma(\alpha+1)} - \frac{1.43 \times 10^{-7}x^\alpha}{\Gamma(\alpha+1)} - \frac{0.31204696x^{2+\alpha}}{\Gamma(3+\alpha)} + \ldots \tag{4.48}$$

4.1.2. Double-well situation

The recursive formula for the fractional problem (4.30) in this case is given by:

$$\begin{cases} y_0(x) = 1 + I^\alpha\left[\frac{1}{2}\cos(0.79x)\right], \\ y_n(x) = I^\alpha\left[0.1y_{n-1}'(x)\right] - I^\alpha\left[A_{n-1}(x)\right] + I^\alpha\left[\frac{1}{2}y_{n-1}(x)\right] - I^\alpha\left[\frac{1}{2}B_{n-1}(x)\right], n \geq 1. \end{cases} \tag{4.49}$$

Therefore, without further details, we provide the corresponding solution by applying the proposed modification methods for $l = 6$, following the same procedure as in the single-well situation, as outlined below.

$$y_{Taylor} = 1 + \frac{x^\alpha}{2\Gamma(\alpha+1)} - \frac{0.312050000x^{2+\alpha}}{\Gamma(\alpha+3)} + \frac{0.1947504048x^{4+\alpha}}{\Gamma(\alpha+5)} - \frac{2.60389834 \times 10^{-14}x^{6+\alpha}}{\Gamma(7+\alpha)} + \ldots \tag{4.50}$$

$$yT(x) = 1 + \frac{0.500000002x^\alpha}{\Gamma(\alpha+1)} - \frac{5.3 \times 10^{-8}x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{0.312047768x^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{4.803621032 \times 10^{-10}x^{4+\alpha}}{\Gamma(5+\alpha)} \cdots \quad (4.51)$$

$$yU(x) = 1 + \frac{0.500000004x^\alpha}{\Gamma(\alpha+1)} - \frac{1.3 \times 10^{-7}x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{0.312046854x^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{4.806971329 \times 10^{-10}x^{4+\alpha}}{\Gamma(3+\alpha)} + \ldots \quad (4.52)$$

$$yP(x) = 1 + \frac{0.500000004x^\alpha}{\Gamma(\alpha+1)} - \frac{1.43 \times 10^{-7}x^\alpha}{\Gamma(\alpha+1)} - \frac{0.31204696x^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{4.805407763 \times 10^{-10}x^{4+\alpha}}{\Gamma(5+\alpha)} + \ldots \quad (4.53)$$

4.1.3. Double-hump situation

The recursive formula for the fractional problem (4.30) in this case is given by:

$$\begin{cases} y_0(x) = 1 + I^\alpha\left[\frac{1}{2}\cos(0.79x)\right], \\ y_n(x) = I^\alpha\left[0.1y'_{n-1}(x)\right] - I^\alpha\left[A_{n-1}(x)\right] - I^\alpha\left[\frac{1}{2}y_{n-1}(x)\right] + I^\alpha\left[\frac{1}{2}B_{n-1}(x)\right], n \geq 1. \end{cases} \quad (4.54)$$

Furthermore, without loss of generality, using the same approach as in the above situations, we present the corresponding solutions by applying the proposed modification methods for $l = 6$.

$$y_{Taylor} = 1 + \frac{x^\alpha}{2\Gamma(\alpha+1)} - \frac{0.312050000x^{2+\alpha}}{\Gamma(\alpha+3)} - \frac{8.249652602 \times 10^{-6}x^{3+\alpha}}{\Gamma(4+\alpha)}$$
$$+ \frac{0.1947504048x^{4+\alpha}}{\Gamma(\alpha+5)} - \frac{2.60389834 \times 10^{-14}x^{6+\alpha}}{\Gamma(7+\alpha)} + \ldots \quad (4.55)$$

$$yT(x) = 1 + \frac{0.500000002x^\alpha}{\Gamma(\alpha+1)} - \frac{5.3 \times 10^{-8}x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{0.312047768x^{1+\alpha}}{\Gamma(2+\alpha)} - \frac{0.001388888891x^{2+\alpha}}{\Gamma(3+\alpha)}$$
$$+ \frac{0.0008680555500x^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{4.803621032 \times 10^{-10}x^{4+\alpha}}{\Gamma(5+\alpha)} + \ldots \quad (4.56)$$

$$yU(x) = 1 + \frac{0.500000004x^\alpha}{\Gamma(\alpha+1)} - \frac{1.3 \times 10^{-8}x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{0.312047768x^{1+\alpha}}{\Gamma(2+\alpha)} - \frac{0.001388888891x^{2+\alpha}}{\Gamma(3+\alpha)}$$
$$+ \frac{0.0008680555792x^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{4.8036971329 \times 10^{-10}x^{4+\alpha}}{\Gamma(5+\alpha)} + \ldots \quad (4.57)$$

$$yP(x) = 1 + \frac{0.500000004x^\alpha}{\Gamma(\alpha+1)} - \frac{1.43 \times 10^{-7}x^\alpha}{\Gamma(\alpha+1)} - \frac{0.31204696x^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{0.001388888913x^{1+\alpha}}{\Gamma(\alpha+2)} +$$
$$+ \frac{0.0008680555792x^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{4.805407763 \times 10^{-10}x^{4+\alpha}}{\Gamma(5+\alpha)} + \ldots \quad (4.58)$$

The results obtained using the modification method of the ADM with $l = 6$ for three different scenarios (single-well, double-well, and double-hump) are presented in Tables 4, 6, and 8, respectively. In this analysis, we set the fractional order of the problem to $\alpha = 2$. The numerical results are compared with those from the variational iteration method (VIM) [41], the ultraspherical wavelets method (UWM) [42], the fourth-order Runge-Kutta (RK) method, and Bernoulli wavelet method (BWM) [43], see Tables 3, 5, and 7, respectively. In addition, CPU times of the presented method are listed in Tables 4, 6, and 8. Figure 3 shows the obtained numerical solutions by the presented modification method of the ADM.

Table 3. Single-well situation: Comparison of numerical solutions for $\alpha = 2$.

x	VIM	UWM	RK-4	BWM
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	0.99750286	0.99750276	0.99750272	0.99750285
0.2	0.99004534	0.99004513	0.99004504	0.99004516
0.3	0.97772778	0.97772579	0.97772567	0.97772579
0.4	0.96071284	0.96070262	0.96070236	0.96070247
0.5	0.93922114	0.93918360	0.93918299	0.93918309
0.6	0.91352389	0.91341578	0.91341497	0.913415063
0.7	0.88393496	0.88367502	0.88367344	0.88367350
0.8	0.85080112	0.85052195	0.85024907	0.85024912
0.9	0.81449135	0.81343957	0.81343631	0.81343635
CPU time	–	–	–	0.047

Table 4. Single-well situation: Comparison of the modification method for $\alpha = 2$.

x	y_{Taylor}	$yT(x)$	$yU(x)$	$yP(x)$
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	0.997511148	0.997511148	0.997511148	0.997511148
0.2	0.990111067	0.990111067	0.990111067	0.990111067
0.3	0.977945843	0.977945843	0.977945842	0.977945843
0.4	0.961217057	0.961217047	0.961217046	0.961217047
0.5	0.940171867	0.940171858	0.940171846	0.940171859
0.6	0.915091741	0.915091701	0.915091690	0.915091693
0.7	0.886280168	0.886280013	0.886279999	0.886280002
0.8	0.854049540	0.854049049	0.854049037	0.854049041
0.9	0.818704518	0.818703259	0.818703244	0.818703257
CPU time	0.0489	0.0696	0.06844	0.0781

Table 5. Double-well situation: Comparison of numerical solutions for $\alpha = 2$.

x	VIM	UWM	RK-4	BWM
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.00249660	1.00249669	1.00249670	1.00249660
0.2	1.00994530	1.00994542	1.00994545	1.00994536
0.3	1.0222113	1.02222174	1.02222179	1.02222170
0.4	1.03911114	1.03911446	1.03911459	1.03911450
0.5	1.06030866	1.06032214	1.06032231	1.06032223
0.6	1.08540584	1.08544887	1.08544906	1.08544898
0.7	1.11388470	1.11400072	1.11400108	1.11400100
0.8	1.14510669	1.14538393	1.14538468	1.14538460
0.9	1.17830101	1.17890570	1.17890664	1.17890656
CPU time	–	–	–	0.062

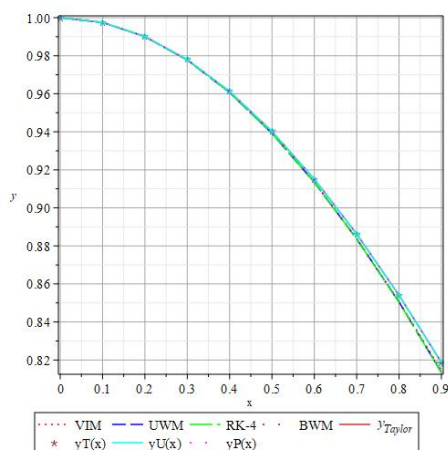**Table 6.** Double-well situation: Comparison of the modification method for $\alpha = 2$.

x	Y_{Taylor}	$yT(x)$	$yU(x)$	$yP(x)$
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.00248830	1.00248830	1.00248830	1.00248830
0.2	1.00987939	1.00987939	1.00987939	1.00987939
0.3	1.02200111	1.02200111	1.02200111	1.02200111
0.4	1.03859815	1.03859814	1.03859814	1.03859814
0.5	1.05932967	1.05932966	1.05932965	1.05932966
0.6	1.08376731	1.08376727	1.08376726	1.08376726
0.7	1.11139358	1.11139342	1.11139341	1.11139341
0.8	1.14160208	1.14160159	1.14160158	1.14160158
0.9	1.17369994	1.17369868	1.17369867	1.17369868
CPU time	0.0489	0.0696	0.06844	0.0781

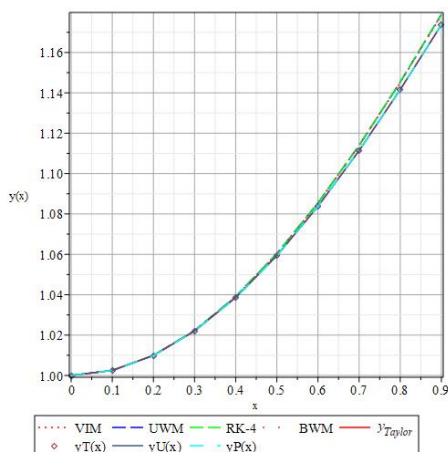**Table 7.** Double-hump situation: Comparison of numerical solutions for $\alpha = 2$.

x	VIM	UWM	RK-4	BWM
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.00250077	1.00250086	1.00250078	1.00250077
0.2	1.01001232	1.01001258	1.01001240	1.01001236
0.3	1.02256255	1.02256352	1.02256311	1.02256305
0.4	1.04019982	1.04020320	1.04020266	1.04020259
0.5	1.06299669	1.06300878	1.06300754	1.06300746
0.6	1.09105590	1.09109104	1.09108901	1.09108890
0.7	1.12451829	1.12460856	1.12460496	1.12460473
0.8	1.16357278	1.16377964	1.16377494	1.16377462
0.9	1.20846809	1.20890608	1.20890103	1.20890063
CPU time	–	–	–	0.171

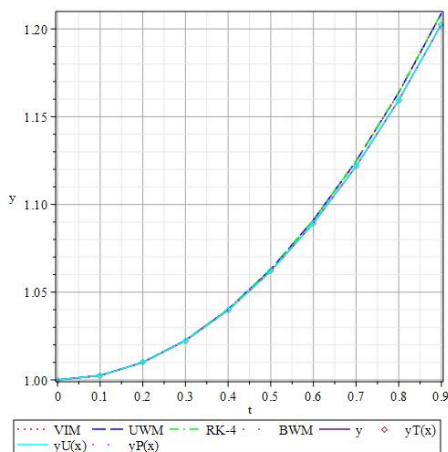**Table 8.** Double-hump situation: Comparison of the modification method for $\alpha = 2$.

x	Y_{Taylor}	$yT(x)$	$yU(x)$	$yP(x)$
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.00249245	1.00249245	1.00249245	1.00249245
0.2	1.00994585	1.00994585	1.00994585	1.00994585
0.3	1.02233836	1.02233836	1.02233836	1.02233836
0.4	1.03966877	1.03966876	1.03966876	1.03966876
0.5	1.06196079	1.06196078	1.06196077	1.06196078
0.6	1.08927009	1.08927005	1.08927004	1.08927004
0.7	1.12169449	1.12169433	1.12169432	1.12169432
0.8	1.15938728	1.15938678	1.15938677	1.15938677
0.9	1.20257483	1.20257354	1.20257353	1.20257354
CPU time	0.0489	0.0696	0.06844	0.0781

(a) Single-well situation: Comparison of numerical solutions for $\alpha = 2$.

(b) Double-well situation: Comparison of numerical solutions for $\alpha = 2$.

(c) Double-hump situation: Comparison of numerical solutions for $\alpha = 2$.

Figure 3. Comparison of numerical solutions for the fractional IVP (4.30).

5. Conclusions

This study investigates the application of the Taylor series, Legendre polynomials, and both the first and second kinds of Chebyshev polynomials within the Adomian decomposition method (ADM) for solving initial-value problems (IVPs) of fractional differential equations. The findings demonstrate the efficiency and accuracy of these approaches. Additionally, the study presents several modification methods for the ADM, highlighting the critical role of orthogonal polynomials in numerical analysis. Among the nonlinear inhomogeneous IVPs tested, the Legendre polynomial methods exhibited the lowest error rates. Given their computational benefits, these methods are recommended for fractional-order IVP models in scientific and engineering applications. Furthermore, the results obtained can be extended or generalized to various types of fractional differential equations and their associated operators.

Author contributions

Mariam Al-Mazmumy: Conceptualization, methodology, investigation, software, writing review & editing; Maryam Ahmed Alyami: Methodology, investigation, writing review & editing; Mona Alsulami: Methodology, investigation, writing review & editing; Asrar Saleh Alsulami: Formal analysis, methodology, investigation, writing original draft preparation, software, writing review & editing; Saleh S. Redhwan: Writing review & editing. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Willey & Sons, 1993.

2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, **204** (2006).

3. I. Podlubny, *Fractional differential equations*, New York: Academic Press, 1999.

4. R. T. Baillie, Long memory processes and fractional integration in econometrics, *J. Econometrics*, **73** (1996), 5–59. https://doi.org/10.1016/0304-4076(95)01732-1

5. G. C. Wu, Z. G. Deng, D. Baleanu, D. Q. Zeng, New variable-order fractional chaotic systems for fast image encryption, *Chaos Interd. J. Nonlinear Sci.*, **29** (2019). https://doi.org/10.1063/1.5110347

6. A. Bonfanti, J. L. Kaplan, G. Charras, A. Kabla, Fractional viscoelastic models for power-law materials, *Soft Matter*, **16** (2020), 6002–6020. https://doi.org/10.1039/D0SM00354A

7. J. F. G. Aguilar, J. E. E. Martínez, C. C. Ramón, L. J. M. Mendoza, M. B. Cruz, M. G. Lee, Equivalent circuits applied in electrochemical impedance spectroscopy and fractional derivatives with and without singular kernel, *Adv. Math. Phys.*, 2016. https://doi.org/10.1155/2016/9720181

8. N. Singh, K. Kumar, P. Goswami, H. Jafari, Analytical method to solve the local fractional vehicular traffic flow model, *Math. Method. Appl. Sci.*, **45** (2022), 3983–4001. https://doi.org/10.1002/mma.8027

9. H. Bulut, T. A. Sulaiman, H. M. Baskonus, H. Rezazadeh, M. Eslami, M. Mirzazadeh, Optical solitons and other solutions to the conformable space-time fractional Fokas-Lenells equation, *Optik*, **172** (2018), 20–27. https://doi.org/10.1016/j.ijleo.2018.06.108

10. S. R. Saratha, M. Bagyalakshmi, S. Sundara, G. Krishnan, Fractional generalised homotopy analysis method for solving nonlinear fractional differential equations, *Comput. Appl. Math.*, **39** (2020), 1–32. https://doi.org/10.1007/s40314-020-1133-9

11. B. R. Sontakke, A. S. Shelke, A. S. Shaikh, Solution of non-linear fractional differential equations by variational iteration method and applications, *Far East J. Math. Sci.*, **110** (2019), 113–129. http://dx.doi.org/10.17654/MS110010113

12. I. Ameen, P. Novati, The solution of fractional order epidemic model by implicit Adams methods, *Appl. Math. Model.*, **43** (2017), 78–84. https://doi.org/10.1016/j.apm.2016.10.054

13. R. Garrappa, Numerical solution of fractional differential equations: A survey and a software tutorial, *Mathematics*, **6** (2018), 16. https://doi.org/10.3390/math6020016

14. R. Amin, B. Alshahrani, M. Mahmoud, A. H. A. Aty, K. Shah, W. Deebani, Haar wavelet method for solution of distributed order time-fractional differential equations, *Alex. Eng. J.*, **60** (2021), 3295–3303. https://doi.org/10.1016/j.aej.2021.01.039

15. H. M. Ahmed, Enhanced shifted Jacobi operational matrices of integrals: Spectral algorithm for solving some types of ordinary and fractional differential equations, *Bound. Value Probl.*, **2024** (2024), 75. https://doi.org/10.1186/s13661-024-01880-0

16. H. M. Srivastava, W. Adel, M. Izadi, A. A. El-Sayed, Solving some physics problems involving fractional-order differential equations with the Morgan-Voyce polynomials, *Fractal Fract.*, **7** (2023), 301. https://doi.org/10.3390/fractalfract7040301

17. A. G. Atta, W. M. A. Elhameed, Y. H. Youssri, Shifted fifth-kind Chebyshev polynomials Galerkin-based procedure for treating fractional diffusion-wave equation, *Int. J. Mod. Phys. C*, **33** (2022), 2250102. https://doi.org/10.1142/S0129183122501029

18. H. M. Ahmed, A new first finite class of classical orthogonal polynomials operational matrices: An application for solving fractional differential equations, *Contemp. Math.*, 2023, 974–994. https://doi.org/10.37256/cm.4420232716

19. G. Adomian, *Nonlinear stochastic systems theory and applications to physics*, Springer Science & Business Media, **46** (1988).

20. K. Abbaoui, Y. Cherruault, Convergence of Adomian's method applied to differential equations, *Comput. Math. Appl.*, **28** (1994), 103–109. https://doi.org/10.1016/0898-1221(94)00144-8

21. M. M. Hosseini, H. Nasabzadeh, On the convergence of Adomian decomposition method, *Appl. Math. Comput.*, **182** (2006), 536–543. https://doi.org/10.1016/j.amc.2006.04.015

22. A. Aminataei, S. S. Hosseini, The comparison of the stability of Adomian decomposition method with numerical methods of equation solution, *Appl. Math. Comput.*, **186** (2007), 665–669. https://doi.org/10.1016/j.amc.2006.08.011

23. V. D. Gejji, H. Jafari, Adomian decomposition: A tool for solving a system of fractional differential equations, *J. Math. Anal. Appl.*, **301** (2005), 508–518. https://doi.org/10.1016/j.jmaa.2004.07.039

24. J. S. Duan, R. Rach, D. Baleanu, A. M. Wazwaz, A review of the Adomian decomposition method and its application to fractional differential equations, *Community Fract. Calculator*, **2** (2012), 73–99.

25. I. Sumiati, E. Rusyaman, S. Sukono, A. T. Bon, *A review of Adomian decomposition method and applied to differential equations*, Proceedings of the International Conference on Industrial Engineering and Operations Management, Pilsen, Czech Republic, 2019, 23–26.

26. M. Kumar, Umesh, Recent development of Adomian decomposition method for ordinary and partial differential equations, *Int. J. Appl. Comput. Math.*, **8** (2022), 81. https://doi.org/10.1007/s40819-022-01285-6

27. A. Sadeghinia, P. Kumar, One solution of multi-term fractional differential equations by Adomian decomposition method, *Int. J. Sci. Innov. Math. Res.*, **3** (2015).

28. P. Guo, The Adomian decomposition method for a type of fractional differential equations, *J. Appl. Math. Phys.*, **7** (2019), 2459–2466. https://doi.org/10.4236/jamp.2019.710166

29. A. Afreen, A. Raheem, Study of a nonlinear system of fractional differential equations with deviated arguments via Adomian decomposition method, *Int. J. Appl. Comput. Math.*, **8** (2022), 269. https://doi.org/10.1007/s40819-022-01464-5

30. M. Botros, E. A. A. Ziada, I. L. El-Kalla, Solutions of fractional differential equations with some modifications of Adomian Decomposition method, *Delta Univ. Sci. J.*, **6** (2023), 292–299. https://doi.org/10.21608/dusj.2023.291073

31. H. O. Bakodah, M. A. Mazmumy, S. O. Almuhalbedi, An efficient modification of the Adomian decomposition method for solving integro-differential equations, *Math. Sci. Lett.*, **21** (2017), 15–21. http://dx.doi.org/10.18576/msl/060103

32. J. Mulenga, P. A. Phiri, Solving different types of differential equations using modified and new modified Adomian decomposition methods, *J. Appl. Math. Phys.*, **11** (2023), 1656–1676. http://dx.doi.org/10.4236/jamp.2023.116108

33. A. M. Wazwaz, S. M. El-Sayed, A new modification of the Adomian decomposition method for linear and nonlinear operators, *Appl. Math. Comput.*, **122** (2001), 393–405. https://doi.org/10.1016/S0096-3003(00)00060-6

34. M. M. Hosseini, Adomian decomposition method with Chebyshev polynomials, *Appl. Math. Comput.*, **175** (2006), 1685–1693. https://doi.org/10.1016/j.amc.2005.09.014

35. Y. Liu, Adomian decomposition method with orthogonal polynomials: Legendre polynomials, *Math. Comput. Model.*, **49** (2009), 1268–1273. https://doi.org/10.1016/j.mcm.2008.06.020

36. Y. Liu, Adomian decomposition method with second kind Chebyshev polynomials, *Proc. Jangjeon Math. Soc.*, **12** (2009), 57–67.

37. Y. Çenesiz, A. Kurnaz, Adomian decomposition method by Gegenbauer and Jacobi polynomials, *Int. J. Comput. Math.*, **88** (2011), 3666–3676. https://doi.org/10.1080/00207160.2011.611503

38. Y. Xie, L. Li, M. Wang, Adomian decomposition method with orthogonal polynomials: Laguerre polynomials and the second kind of Chebyshev polynomials, *Mathematics*, **9** (2021), 1796. https://doi.org/10.3390/math9151796

39. N. Khodabakhshi, S. M. Vaezpour, D. Baleanu, Numerical solutions of the initial value problem for fractional differential equations by modification of the Adomian decomposition method, *Fract. Calc. Appl. Anal.*, **17** (2014)), 382–400. https://doi.org/10.2478/s13540-014-0176-2

40. Z. Odibat, On Legendre polynomial approximation with the VIM or HAM for numerical treatment of nonlinear fractional differential equations, *J. Comput. Appl. Math.*, **235** (2011), 2956–2968. https://doi.org/10.1016/j.cam.2010.12.013

41. V. Y. Shenas, Application of numerical and semi-analytical approach on van der Pol-duffing oscillators, *J. Adv. Res. Mech. Eng.*, **1** (2010).

42. Y. H. Youssri, W. M. Abd-Elhameed, E. H. Doha, Ultraspherical wavelets method for solving Lane-Emden type equations, *Rom. J. Phys.*, **60** (2015), 1298–1314.

43. P. Rahimkhani, R. Moeti, Numerical solution of the fractional order Duffing-van der Pol oscillator equation by using Bernoulli wavelets collocation method, *Int. J. Appl. Comput. Math.*, **4** (2018), 1–18. https://doi.org/10.1007/s40819-018-0494-x