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Research article

On univalent spirallike log-harmonic mappings

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Abstract: This study investigates univalent log-harmonic mappings, a class of functions that map the unit disk into the complex plane. First, we establish the necessary and sufficient conditions for univalent log-harmonic mappings to map the unit disk onto a spirallike region. We then explore the relationship between starlike harmonic mappings and spirallike log-harmonic mappings, providing several examples to illustrate our results. Finally, under specific conditions, we present growth and covering theorems for univalent log-harmonic mappings and determine the radius of spirallikeness for starlike log-harmonic mappings.

Keywords: univalent log-harmonic mappings; spirallike log-harmonic mappings; starlike harmonic mappings
Mathematics Subject Classification: 30C35, 30C45, 35Q30

1. Introduction

Let $\mathcal{A}$ denote the space of analytic functions h defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

and let $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. The function $h \in \mathcal{S}$ is referred to as a starlike function. It maps the unit disk $\mathbb{D}$ relative to the origin into a starlike domain, i.e., when $z \in \mathbb{D} \setminus \{0\}$, it satisfies $\operatorname{Re}\{zh'(z)/h(z)\} > 0$. Similarly, if the analytic function $h \in \mathcal{S}$ maps the unit disk $\mathbb{D}$ onto a convex domain, i.e., if it satisfies $\operatorname{Re}\{1 + zh''(z)/h'(z)\} > 0$, then it is referred to as a convex function. If a function $h \in \mathcal{S}$, and for all z in the unit disk $\mathbb{D}$ except the origin, it satisfies $\operatorname{Re}\left\{e^{-i\lambda} zh'(z)/h(z)\right\} > 0$, then h is called λ-spirallike. To learn more about these classes, please see details in [1–4].

A log-harmonic mapping f is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\bar{z}}(z)}}{\overline{f(z)}} = \mu(z)\frac{f_z(z)}{f(z)}, \tag{1.1}$$

where $\mu(z) \in \mathcal{A}$ and $|\mu(z)| < 1$ for all $z \in \mathbb{D}$, which is known as the second complex dilatation of f. Note that the Jacobian J_f of f is given by

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |f_z|^2(1 - |\mu|^2).$$

If J_f is positive, then all non-constant log-harmonic mappings are sense-preserving in the unit disk $\mathbb{D}$. A log-harmonic mapping f vanishes at $z = 0$ if and only if f has the form

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)},$$

where $\mathrm{Re}\beta > -1/2$, $h, g \in \mathcal{A}$, $h(0) \neq 0$ and $g(0) = 1$ (cf. [5, 6]). For simplicity, we set $\beta = 0$ and let $\mathcal{S}_{LH}$ be the class of all univalent sense-preserving log-harmonic functions $f(z) = zh(z)\overline{g(z)}$ in $\mathbb{D}$ with

$$h(z) = \exp\left(\sum_{n=1}^{\infty} a_n z^n\right) \quad \text{and} \quad g(z) = \exp\left(\sum_{n=1}^{\infty} b_n z^n\right), \tag{1.2}$$

the dilatation of the function $f \in \mathcal{S}_{LH}$ is satisfied

$$\mu(z) = \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)}. \tag{1.3}$$

If the function f is first-order continuously differentiable on the complex plane, then the differential operator D of $f \in C^1(\mathbb{D})$ is defined as

$$Df(z) = zf_z(z) - \bar{z}f_{\bar{z}}(z).$$

As is known to all, the properties convexity and starlikeness of a univalent analytic function are hereditary, i.e., if the analytic function f is convex or starlike, then for $0 < r < 1$, $f(\mathbb{D}_r)$ is convex or starlike domain, respectively, where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. Clunie and Sheil–Small [7] succeeded in extending the theory of starlikeness, convexity, and close-to-convex from analytic to harmonic functions. The classic results on spirallike functions can be extended to log-harmonic mappings. In [8, 9], the authors introduce a concept called hereditarily spirallikeness of log-harmonic mappings in the unit disk $\mathbb{D}$. For the real λ, if $|\lambda| < \frac{\pi}{2}$, then the log-harmonic mapping $f(z) = zh(z)\overline{g(z)} \in \mathcal{S}_{LH}$ is called the λ-spirallike log-harmonic mapping if f is univalent on $\mathbb{D}$ and for every $r < 1$, $f(\mathbb{D}_r)$ is a λ-spirallike domain.

Definition 1. Let $\lambda \in (-\pi/2, \pi/2)$. A univalent function $f \in C^1(\mathbb{D})$ with $f(0) = 0$ is called λ-spirallike, denoted by $\mathcal{SP}_{LH}(\lambda)$, if

$$\mathrm{Re}\left(e^{-i\lambda}\frac{Df(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \setminus \{0\}.$$

To facilitate the following discussion, we will employ the symbols defined below:

(1) $\mathcal{S}^*$ is the set of all $f \in \mathcal{S}$, and $f(\mathbb{D})$ is a starlike domain;

(2) $\mathcal{SP}(\lambda)$ is the set of all $f \in \mathcal{S}$ and $f(\mathbb{D})$ is a λ-spirallike domain;
(3) $\mathcal{S}_H$ is the set of all univalent harmonic mappings $f(z) = h(z) + \overline{g(z)}$ defined on $\mathbb{D}$ such that $f(0) = h(0) = g(0) = h'(0) - 1 = 0$;
(4) $\mathcal{SP}_H(\lambda)$ is the set of all $f(z) \in \mathcal{S}_H$, and $f(\mathbb{D})$ is a λ-spirallike domain;
(5) $\mathcal{SP}_{LH}(\lambda)$ is the set of all $f(z) \in \mathcal{S}_{LH}$ and $f(\mathbb{D})$ is a λ-spirallike domain;
(6) $\mathcal{S}^*_{LH} = \mathcal{SP}_{LH}(0)$ and $\mathcal{S}^*_H = \mathcal{SP}_H(0)$, for which $f(\mathbb{D})$ is starlike.

The main content of this paper is as follows: In Section 2, we obtain coefficient determination conditions for λ-spirallike log-harmonic mappings. In Section 3, we investigate the relationship between $\mathcal{S}^*_{H'}$ and $\mathcal{SP}_{LH}(\lambda)$. Finally, in Section 4, we propose modulus estimates and covering theorems for univalent log-harmonic mappings satisfying certain conditions. Based on the coefficient estimation of starlike log-harmonic mappings, we derive the radius of λ-spirallikeness for starlike log-harmonic mappings.

2. Discriminant conditions for spirallike log-harmonic mappings

Currently, a wealth of conclusions has been drawn for the class of mappings $\mathcal{S}^*_{LH}$, $\mathcal{S}^*_H$, and $\mathcal{S}^*$. In this section, we obtain several results for λ-spirallike log-harmonic mappings. Firstly, we present the sufficient and necessary conditions for the $\mathcal{SP}_{LH}(\lambda)$.

Theorem 2.1. *Let $\lambda \in (-\pi/2, \pi/2)$ and $f(z) = zh(z)\overline{g(z)}$ be a sense-preserving log-harmonic mapping on $\mathbb{D}$ such that $f(z) = 0$ only for $z = 0$. Then $f \in \mathcal{SP}_{LH}(\lambda)$ if and only if*

$$\cos\lambda > \mathrm{Re}\left\{e^{-i\lambda}\left(\frac{zg'(z)}{g(z)} - \frac{zh'(z)}{h(z)}\right)\right\}. \tag{2.1}$$

Proof. Since $f \in \mathcal{SP}_{LH}(\lambda)$, by Definiton 1, it follows that

$$\begin{aligned}\mathrm{Re}\left(e^{-i\lambda}\frac{Df(z)}{f(z)}\right) &= \mathrm{Re}\left(e^{-i\lambda}\frac{zh(z)\overline{g(z)} + z^2h'(z)\overline{g(z)} - |z|^2h(z)\overline{g'(z)}}{zh(z)\overline{g(z)}}\right)\\ &= \mathrm{Re}\left(e^{-i\lambda}\left(1 + \frac{zh'(z)}{h(z)} - \overline{\frac{zg'(z)}{g(z)}}\right)\right)\\ &= \cos\lambda + \mathrm{Re}\left(e^{-i\lambda}\left(\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right)\right) > 0.\end{aligned}$$

Since $\lambda \in (-\pi/2, \pi/2)$, it follows that $\cos\lambda > 0$. Thus, we get (2.1).

Conversely, through simple calculations, by the condition (2.1) and $\lambda \in (-\pi/2, \pi/2)$ yield $\mathrm{Re}\left(e^{-i\lambda}\frac{Df(z)}{f(z)}\right) > 0$. According to Definition 1, it follows that $f \in \mathcal{SP}_{LH}(\lambda)$. □

Remark 1. We claim that the right-half plane log-harmonic mapping

$$f_1(z) = zh_1(z)\overline{g_1(z)} = \frac{z}{1-z}\exp\left(\mathrm{Re}\left(\frac{2z}{1-z}\right)\right)$$

is not $\frac{\pi}{6}$-spirallike region, where $g_1(z) = \exp\left(\frac{z}{1-z}\right), h_1(z) = \frac{1}{1-z}\exp\left(\frac{z}{1-z}\right)$ and $z \in \mathbb{D}$. Because for $\lambda = \frac{\pi}{6}$ and $z = \frac{1}{2}$ the inequality (2.1) becomes $\frac{\sqrt{3}}{2} > \frac{\sqrt{3}}{2}$. According to Theorem 2.1, it is known that $f_1 \notin \mathcal{SP}_{LH}(\frac{\pi}{6})$.

In the realm of studying spirallike log-harmonic mappings, direct assessments frequently pose challenges. Consequently, there arises a necessity to transpose numerous research findings pertaining to spirallike analytic functions into the realm of spirallike log-harmonic mappings. Theorem 2.2 forges a bridge between the families of mappings, specifically linking $\mathcal{SP}(\lambda)$ and $\mathcal{SP}_{LH}(\lambda)$.

Theorem 2.2. *Let $\lambda \in (-\pi/2, \pi/2)$ and $f(z) = zh(z)\overline{g(z)} = \varphi(z)|g(z)|^2$ be a log-harmonic mapping on $\mathbb{D}$, where $\varphi(z) = \frac{zh(z)}{g(z)}$, $g(z) \in \mathcal{A}$ such that $\varphi(z)$ and $g(z)$ are non-vanishing in $\mathbb{D} \setminus \{0\}$. Then $f \in \mathcal{SP}_{LH}(\lambda)$ if and only if $\varphi \in \mathcal{SP}(\lambda)$.*

Proof. After a simple calculation shows that

$$D\varphi(z) = z\varphi'(z) = z\frac{(h(z) + zh'(z))g(z) - g'(z)zh(z)}{(g(z))^2},$$

and

$$Df(z) = zf_z(z) - \overline{z}f_{\overline{z}}(z) = zh(z)\overline{g(z)} + z^2h'(z)\overline{g(z)} - |z|^2\left(zh(z)\overline{g'(z)}\right).$$

Which implies that

$$\text{Re}\left(e^{-i\lambda}\frac{Df(z)}{f(z)}\right) = \text{Re}\left(e^{-i\lambda}\left(1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right)\right) = \text{Re}\left(e^{-i\lambda}\frac{D\varphi(z)}{\varphi(z)}\right).$$

□

Theorem 2.2 establishes the direct equivalence between $\varphi \in \mathcal{SP}(\lambda)$ and $f \in \mathcal{SP}_{LH}(\lambda)$. By imposing certain restrictions on φ, the conditions for $\mathcal{SP}(\lambda)$ can be relaxed to $\mathcal{S}^*$, thereby establishing the relationship between $\mathcal{S}^*$ and $\mathcal{SP}_{LH}(\lambda)$.

Corollary 2.3. *Let $f(z) = zh(z)\overline{g(z)}$, and $\varphi(z) = \frac{zh(z)}{g(z)} \in \mathcal{S}^*$. If*

(1) $\lambda \in (0, \pi/2)$ and $\text{Im}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) \geq 0$, or
(2) $\lambda \in (-\pi/2, 0)$ and $\text{Im}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) \leq 0$.

Then $f(z) = zh(z)\overline{g(z)} \in \mathcal{SP}_{LH}(\lambda)$.

Proof. According to Theorem 2.2, we have

$$\text{Re}\left(e^{-i\lambda}\frac{Df(z)}{f(z)}\right) = \text{Re}\left(e^{-i\lambda}\frac{z\varphi'(z)}{\varphi(z)}\right) = \cos\lambda\,\text{Re}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) + \sin\lambda\,\text{Im}\left(\frac{z\varphi'(z)}{\varphi(z)}\right).$$

As $\varphi(z) \in \mathcal{S}^*$, $\text{Im}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) \geq 0$ and $\lambda \in (0, \pi/2)$, it follows that

$$\cos\lambda\,\text{Re}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) + \sin\lambda\,\text{Im}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) > 0.$$

As $\varphi(z) \in \mathcal{S}^*$, $\text{Im}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) \leq 0$ and $\lambda \in (-\pi/2, 0)$. Then $f(z) = zh(z)\overline{g(z)} \in \mathcal{SP}_{LH}(\lambda)$. The proof process is the same as the previous text. □

Now we present a few results for $f \in \mathcal{SP}_{LH}(\lambda)$ in the unit disk. Before we do that, let us jot down a symbol that we will use throughout the paper.

$$B := \left|1 + e^{-i\lambda}\right| - \left|1 - e^{-i\lambda}\right|. \tag{2.2}$$

Theorem 2.4. *Let $\lambda \in (-\pi/2, \pi/2)$ and $f(z) = zh(z)\overline{g(z)}$ be of the form* (1.2) *such that*

$$\sum_{n=1}^{\infty} \frac{n}{B}(|a_n| + |b_n|) \leq \frac{1}{2}, \tag{2.3}$$

where B is given by (2.2). *Then $f(z) = zh(z)\overline{g(z)} \in \mathcal{SP}_{LH}(\lambda)$.*

Proof. Note that

$$\frac{B}{2} = \frac{|1 + e^{-i\lambda}| - |1 - e^{-i\lambda}|}{2} \leq \frac{|1 + e^{-i\lambda}|}{2} \leq 1.$$

First, to prove that f is sense-preserving and locally univalent in $\mathbb{D}$, we need only show that $|\mu| < 1$. By (1.3) and (2.3), we have

$$\begin{aligned} \left|1 + z\frac{h'(z)}{h(z)}\right| - \left|z\frac{g'(z)}{g(z)}\right| &= \left|1 + \sum_{n=1}^{\infty} na_n z^n\right| - \left|\sum_{n=1}^{\infty} nb_n z^n\right| \\ &> 1 - \sum_{n=1}^{\infty} n|a_n| - \sum_{n=1}^{\infty} n|b_n| \\ &\geq 1 - \frac{B}{2} \\ &\geq 0. \end{aligned}$$

Next, in order to prove that f is λ-spirallike, it is sufficient to show that

$$\operatorname{Re}\left(e^{-i\lambda}\frac{Df(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \setminus \{0\},$$

which is equivalent to

$$\left|1 + e^{-i\lambda}\frac{Df(z)}{f(z)}\right| > \left|1 - e^{-i\lambda}\frac{Df(z)}{f(z)}\right|,$$

or equivalent to

$$M := \left|1 + e^{-i\lambda}\left(1 + \sum_{n=1}^{\infty} n(a_n - b_n)z^n\right)\right| - \left|1 - e^{-i\lambda}\left(1 + \sum_{n=1}^{\infty} n(a_n - b_n)z^n\right)\right| > 0.$$

Now

$$\begin{aligned} M &= \left|(1 + e^{-i\lambda}) + e^{-i\lambda}\sum_{n=1}^{\infty} n(a_n - b_n)z^n\right| - \left|(1 - e^{-i\lambda}) - e^{-i\lambda}\sum_{n=1}^{\infty} n(a_n - b_n)z^n\right| \\ &\geq |1 + e^{-i\lambda}| - \sum_{n=1}^{\infty} n(|a_n| + |b_n|)|z|^n - |1 - e^{-i\lambda}| - \sum_{n=1}^{\infty} n(|a_n| + |b_n|)|z|^n \\ &> |1 + e^{-i\lambda}| - |1 - e^{-i\lambda}| - 2\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \\ &= B\left(1 - 2\sum_{n=1}^{\infty} \frac{n}{B}(|a_n| + |b_n|)\right). \end{aligned}$$

Thus from (2.3), for $z \in \mathbb{D} \setminus \{0\}$, we have

$$M > B\left(1 - 2\sum_{n=1}^{\infty} \frac{n}{B}(|a_n| + |b_n|)\right) \geq 0.$$

Then $f(z) = zh(z)\overline{g(z)} \in \mathcal{SP}_{LH}(\lambda)$. This completes the proof. □

Example 1. We claim that $f_2(z) = zh(z)\overline{g(z)} = z\overline{\exp\left(\frac{Bz^k}{2k}\right)} \in \mathcal{SP}_{LH}(\lambda)$, where $k \in \mathbb{N}^+$, $\lambda \in (-\pi/2, \pi/2)$ and B is given by (2.2).

Here $h(z) = 1$ and $g(z) = \exp\left(\frac{Bz^k}{2k}\right)$. Since

$$|\mu_2(z)| = \left|\frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)}\right| = \left|\frac{Bz^k}{2}\right| < \left|\frac{B}{2}\right| \leq 1.$$

So $f_2 \in \mathcal{S}_{LH}$. Next, we provide the coefficients of the series terms for $f_2(z)$,

$$a_n = 0, \qquad b_n = \begin{cases} \frac{B}{2n}, & k = n, \\ 0, & k \neq n. \end{cases}$$

Finally, we see that

$$\sum_{n=1}^{\infty} \frac{n}{B}(|a_n| + |b_n|) = \frac{n}{B}\left(0 + \frac{B}{2n}\right) = \frac{1}{2}.$$

According to Theorem 2.4, we can deduce that $f_2 \in \mathcal{SP}_{LH}(\lambda)$. The images of $\mathbb{D}$ under f_2 for certain values of k and λ are shown in Figure 1. As shown in Figure 1, it can be observed that the value of k determines the number of cusps of $f_2(\mathbb{D})$ (see Figure 1(a,c) for details), while the value of λ determines the smoothness of the image at the cusps (compare Figures 1(a–d)).

Let $\lambda = 0$ in Theorem 2.4, and we obtain the following result.

Corollary 2.5. *Let $f(z) = zh(z)\overline{g(z)}$ be of the form* (1.2) *such that*

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq 1. \tag{2.4}$$

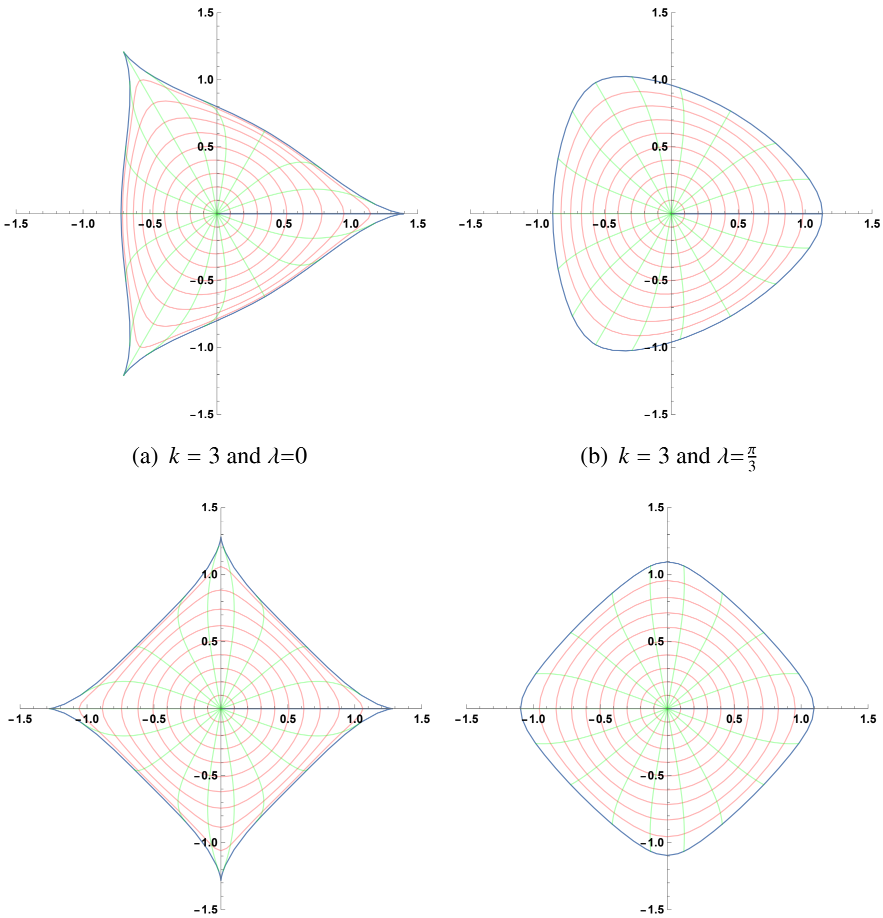
Then $f(z) = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}$.

Our next result provides the necessary conditions for a log-harmonic mapping $f \in \mathcal{SP}_{LH}(\lambda)$.

Theorem 2.6. *Let $\lambda \in (-\pi/2, \pi/2)$ and $f(z) = zh(z)\overline{g(z)} \in \mathcal{SP}_{LH}(\lambda)$ be of the form* (1.2). *Then*

$$\sum_{n=1}^{\infty} \frac{B}{2} n(|a_n| + |b_n|) \leq 1, \tag{2.5}$$

where B is given by (2.2).



(a) $k = 3$ and λ=0 (b) $k = 3$ and λ=$\frac{\pi}{3}$

(c) $k = 4$ and λ=0 (d) $k = 4$ and λ=$\frac{\pi}{3}$

Figure 1. $f_2(\mathbb{D})$ for certain values of k and λ.

Proof. Let $f \in \mathcal{SP}_{LH}(\lambda)$, then f is univalent and satisfies

$$\mathrm{Re}\left(e^{-i\lambda}\frac{Df(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \setminus \{0\}.$$

For $\lim_{z\to 1} z = 1$, we obtain

$$\begin{aligned}\mathrm{Re}\left(e^{-i\lambda}\frac{Df(z)}{f(z)}\right) &= \cos\lambda \mathrm{Re}\left(1 + \sum_{n=1}^{\infty} n(a_n - b_n)z^n\right)\\ &\geq \cos\lambda\left(1 - \sum_{n=1}^{\infty} n(|a_n| + |b_n|)\right) > 0.\end{aligned}$$

Since $\frac{B}{2} \leq 1$, it follows that

$$\sum_{n=1}^{\infty}\frac{B}{2}n(|a_n| + |b_n|) \leq \sum_{n=1}^{\infty} n(|a_n| + |b_n|) < 1.$$

This completes the proof. □

Remark 2. Let $f(z) = zh(z)\overline{g(z)}$. Then $f \in \mathcal{S}^*_{LH}$ if and only if f satisfies the coefficient inequality

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq 1.$$

3. The relationship between $\mathcal{S}^*_{H'}$ and $\mathcal{SP}_{LH}(\lambda)$

In this section, we will discuss some commonalities in the judgment conditions between the spirallike harmonic mappings and the λ-spirallike log-harmonic mappings. According to some theorems, a family of spirallike log-harmonic mappings can be generated through the starlike harmonic family. Before describing the following conclusions, we need a brief explanation of harmonic mappings. A continuous twice differentiable complex-valued function $f = u + iv$ is called a harmonic mapping in the complex domain Ω if both u and v are real harmonic on Ω. In any simply connected domain Ω, each harmonic mapping f can be expressed as $f = h + \overline{g}$, where h and g are analytic in Ω [4]. Let $\mathcal{H}$ be the class of all sense-preserving and univalent harmonic functions $f = h + \overline{g}$ in $\mathbb{D}$ with

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{3.1}$$

and $\mathcal{H}'$ be the class of harmonic functions $F = H + \overline{G}$ in $\mathcal{H}$ with the representation

$$H(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad G(z) = \sum_{n=1}^{\infty} |b_n| z^n. \tag{3.2}$$

If a harmonic mapping $F \in \mathcal{S}_{H'}$ and $F(\mathbb{D})$ is a starlike region, then this subclass of mappings is denoted by $\mathcal{S}^*_{H'}$. We can extend the conclusion to the relationship between $\mathcal{S}^*_{H'}$ and $\mathcal{S}^*_{LH}(\lambda)$. Before this, we need the following lemma.

Lemma 3.1. ([10, Theorem 2]) *For $F(z) = H(z) + \overline{G(z)} \in \mathcal{H}'$ be of the form* (3.2) *if and only if it satisfies the coefficient inequality*

$$\sum_{n=1}^{\infty} n\left(|a_n| + |b_n|\right) \leq 2. \tag{3.3}$$

Theorem 3.2. $F(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \overline{\sum_{n=1}^{\infty} |b_n| z^n} \in \mathcal{S}^*_{H'}$ *if and only if*

$$f(z) = z \exp\left(\sum_{n=2}^{\infty} a_n z^n\right) \overline{\exp\left(\sum_{n=1}^{\infty} b_n z^n\right)} \in \mathcal{S}^*_{LH}.$$

Proof. Since $F(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \overline{\sum_{n=1}^{\infty} |b_n| z^n} \in \mathcal{S}^*_{H'}$, by Lemma 3.1, we obtain

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1.$$

For the log-harmonic mapping $f(z) = z\exp\left(\sum_{n=2}^{\infty} a_n z^n\right)\overline{\exp\left(\sum_{n=1}^{\infty} b_n z^n\right)}$, $a_1 = 0$. Then we have

$$\sum_{n=1}^{\infty} n\left(|a_n| + |b_n|\right) = \sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1.$$

By Corrollary 2.5, we have

$$f(z) = z\exp\left(\sum_{n=2}^{\infty} a_n z^n\right)\overline{\exp\left(\sum_{n=1}^{\infty} b_n z^n\right)} \in \mathcal{S}^*_{LH}.$$

Conversely, if

$$f(z) = z\exp\left(\sum_{n=2}^{\infty} a_n z^n\right)\overline{\exp\left(\sum_{n=1}^{\infty} b_n z^n\right)} \in \mathcal{S}^*_{LH},$$

it follows from Remark 2 that $\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq 1$. After simple calculations, it can be deduced that

$$F(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \overline{\sum_{n=1}^{\infty} |b_n| z^n} \in \mathcal{S}^*_{H'}.$$

□

The following establishes the relationship between starlike harmonic mappings and spirallike log-harmonic mappings.

Theorem 3.3. *Let* $F(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \overline{\sum_{n=1}^{\infty} |b_n| z^n} \in \mathcal{S}^*_{H'}$. *Then*

$$f(z) = z\exp\left(\sum_{n=2}^{\infty} d_n a_n z^n\right)\overline{\exp\left(\sum_{n=1}^{\infty} d_n b_n z^n\right)} \in \mathcal{SP}_{LH}(\lambda),$$

where $\{d_n\}$ *is a sequence such that* $|d_n| \leq B/2$ *for* $n \geq 1$ *and* B *is given by* (2.2).

Proof. Since $F(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \overline{\sum_{n=1}^{\infty} |b_n| z^n}$ be starllike harmonic mapping, by Lemma 3.1, the coefficient condition (3.3) holds. Since $\{d_n\}$ is a sequence of complex numbers with $|d_n| \leq B/2$. Then

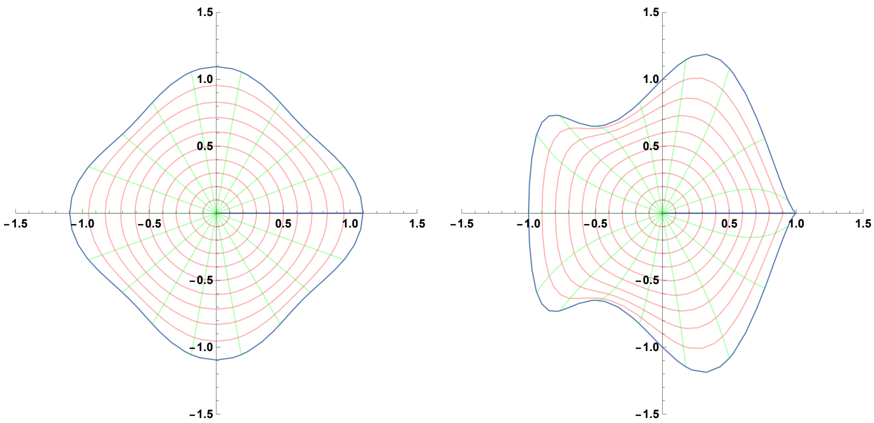
$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2n}{B}\left(|d_n a_n| + |d_n b_n|\right) &= \sum_{n=2}^{\infty} \frac{2n}{B}|d_n||a_n| + \sum_{n=1}^{\infty} \frac{2n}{B}|d_n||b_n| \\ &\leq \sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \\ &\leq 1.\end{aligned}$$

Hence, by Theorem 2.4, we have

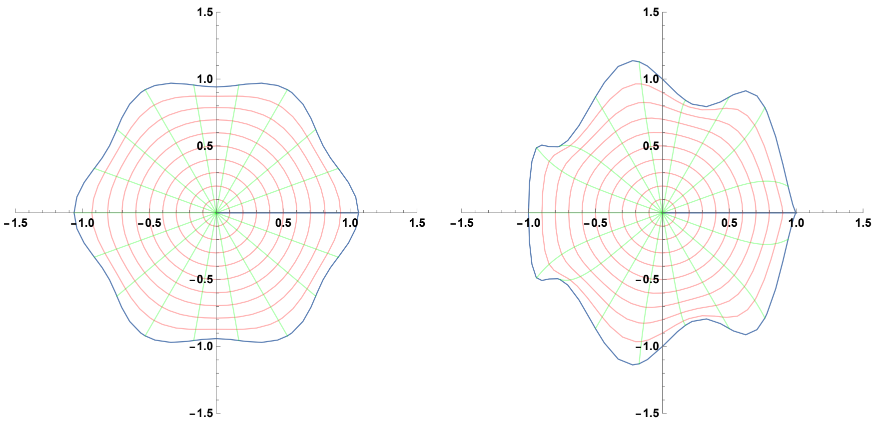
$$f(z) = z\exp\left(\sum_{n=2}^{\infty} d_n a_n z^n\right)\overline{\exp\left(\sum_{n=1}^{\infty} d_n b_n z^n\right)} \in \mathcal{SP}_{LH}(\lambda).$$

□

For positive integers n greater than 2, $f_3(z) = z\exp\left(\frac{B}{4n}z^n\right)\overline{\exp\left(\frac{B}{4n}z^n\right)} \in \mathcal{SP}_{LH}(\lambda)$ as from Lemma 3.1 the mapping $F(z) = z - \frac{1}{2n}z^n + \frac{1}{2n}\overline{z}^n \in \mathcal{S}^*_{H'}$. Here, the value of d_n is $B/2$. Figure 2 illustrates the images of $\mathbb{D}$ under $f_3(z)$ and $F(z)$ for certain values of n and λ. From the images, it can be concluded that $F(\mathbb{D})$ has at least one real axis as its axis of symmetry, and $f_3(\mathbb{D})$ has at least n axes of symmetry, including the real axis.



(a) $f_3(\mathbb{D})$ for $n = 4$ and $\lambda = \frac{\pi}{3}$ (b) $F(\mathbb{D})$ for $n = 4$



(c) $f_3(\mathbb{D})$ for $n = 6$ and $\lambda = \frac{\pi}{3}$ (d) $F(\mathbb{D})$ for $n = 6$

Figure 2. $f_3(\mathbb{D})$, $F(\mathbb{D})$ for certain values of n and λ.

4. Growth estimate and $\lambda-$spirallike radius

Next, we present a growth estimate for the modulus of sense-preserving log-harmonic λ-spirallike mappings.

Theorem 4.1. *Let $f(z) = zh(z)\overline{g(z)}$ be a log-harmonic mapping of the form* (1.2) *and satisfies the coefficient inequality* (2.3)*. Then the sharp inequality*

$$r\exp\left(-\frac{B}{2}r\right) < |f(z)| < r\exp\left(\frac{B}{2}r\right) \quad \textit{for} \quad |z| = r$$

holds and

$$\left\{w \in \mathbb{C} : |w| < \exp\left(-\frac{B}{2}\right)\right\} \subset f(\mathbb{D}).$$

where B is given by (2.2).

Proof. Let $|z| = r < 1$. As the log-harmonic mapping $f(z) = z\exp\left(\sum_{n=2}^{\infty} a_n z^n\right)\overline{\exp\left(\sum_{n=1}^{\infty} b_n z^n\right)}$ satisfies the condition (2.3), it follows that

$$\begin{aligned}|f(z)| &= r\exp\left(\mathrm{Re}\left(\sum_{n=1}^{\infty} a_n z^n + \overline{b_n z^n}\right)\right)\\ &\le r\exp\left(\sum_{n=1}^{\infty}(|a_n| + |b_n|)r\right)\\ &< r\exp\left(\sum_{n=1}^{\infty} n(|a_n| + |b_n|)r\right)\\ &< r\exp\left(\frac{B}{2}r\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}|f(z)| &= r\exp\left(\mathrm{Re}\left(\sum_{n=1}^{\infty} a_n z^n + \overline{b_n z^n}\right)\right)\\ &\ge r\exp\left(\sum_{n=1}^{\infty} -(|a_n| + |b_n|)r\right)\\ &> r\exp\left(\sum_{n=1}^{\infty} -n(|a_n| + |b_n|)r\right)\\ &> r\exp\left(-\frac{B}{2}r\right).\end{aligned}$$

When we approach r to 1 from the left side, we obtain

$$\left\{w \in \mathbb{C} : |w| < \exp\left(-\frac{B}{2}\right)\right\} \subset f(\mathbb{D}).$$

□

Similar to the Bieberbach conjecture for analytic and harmonic mappings, the following Bieberbach conjecture for log-harmonic mappings can be naturally proposed with the help of the Koebe log-harmonic mapping. The case where $f(z) = h(z)\overline{g(z)} \in \mathcal{S}^*_{LH}$ has already been confirmed by the author in ([11], Theorem 3.2) and ([12], Theorem 3.3):

Lemma 4.2. ([11, Theorem 3.2]) *Let* $f(z) = zh(z)\overline{g(z)}$ *belong to* $\mathcal{S}^*_{LH}$, *where* $h(z)$ *and* $g(z)$ *are given by* (1.2). *Then for all* $n \ge 1$,

$$|a_n - b_n| \le \frac{2}{n}. \tag{4.1}$$

Moreover

(1) $|a_n| \le 2 + \frac{1}{n}$*;*
(2) $|b_n| \le 2 - \frac{1}{n}$*.*

Equality holds if

$$f(z) = \varphi(z)|g(z)|^2 = \frac{z}{(1-z)^2}\left|(1-z)\exp\left(\frac{2z}{1-z}\right)\right|^2,$$

or one of its rotations.

Next, we will derive the λ-spirallike radius for log-harmonic mappings $f \in \mathcal{S}^*_{LH}$.

Theorem 4.3. *Let* $f(z) = zh(z)\overline{g(z)}$ *be a log-harmonic mapping of the form* (1.2)*, and* $f \in \mathcal{S}^*_{LH}$*. Then the* λ*-spirallike radius of* $f(z)$ *is* $r_s(\lambda)$*, where* $\lambda \in (-\pi/2, \pi/2)$*, and*

$$r_s(\lambda) = \frac{\cos\lambda}{\cos\lambda + 2} \in (0, \frac{1}{3}). \tag{4.2}$$

Proof. According to Theorem 2.1 in this paper, to prove $f \in \mathcal{SP}_{LH}(\lambda)$, it suffices to obtain

$$\cos\lambda > \mathrm{Re}\left(e^{-i\lambda}\left(\frac{zg'(z)}{g(z)} - \frac{zh'(z)}{h(z)}\right)\right).$$

By using the coefficient estimates $|a_n|$ and $|b_n|$ (Lemma 4.2), we have

$$\begin{aligned}
&\cos\lambda - \mathrm{Re}\left(e^{-i\lambda}\left(\frac{zg'(z)}{g(z)} - \frac{zh'(z)}{h(z)}\right)\right)\\
&= \cos\lambda - \mathrm{Re}\left(e^{-i\lambda}\left(\sum_{n=1}^{\infty} n(b_n - a_n)z^n\right)\right)\\
&\ge \cos\lambda - \left(\sum_{n=1}^{\infty} n(|a_n - b_n|)r^n\right)\\
&\ge \cos\lambda - \left(\sum_{n=1}^{\infty} 2r^n\right)\\
&= \cos\lambda - \frac{2r}{1-r}.
\end{aligned}$$

Therefore,

$$f \in \mathcal{SP}_{LH}(\lambda) \text{ if } \cos\lambda - \frac{2r}{1-r} > 0,$$

that is, if

$$\cos\lambda - (\cos\lambda + 2)r > 0,$$

through simple calculations, it can be obtained that

$$r_s(\lambda) = r < \frac{\cos\lambda}{\cos\lambda + 2}.$$

Therefore, for all $\lambda \in (-\pi/2, \pi/2)$, $r_s(\lambda) \in (0, \frac{1}{3})$. Consequently, when $r \leq r_s(\lambda)$, we have $f \in \mathcal{SP}_{LH}(\lambda)$. To prove the accuracy of $r_s(\lambda)$, it is necessary to consider the cases where equality holds during the proof. For the Koebe log-harmonic mapping, we have

$$\begin{aligned} g(z) &= (1-z)\exp\left(\frac{2z}{1-z}\right) = \exp\left(\sum_{n=1}^{\infty}(2+\frac{1}{n})z^n\right), \\ h(z) &= \frac{1}{1-z}\exp\left(\frac{2z}{1-z}\right) = \exp\left(\sum_{n=1}^{\infty}(2-\frac{1}{n})z^n\right), \\ f(z) &= \frac{z}{(1-z)^2}\left|(1-z)\exp\left(\frac{2z}{1-z}\right)\right|^2, \end{aligned}$$

when $z = \dfrac{\cos\lambda}{\cos\lambda + 2}$ satisfies

$$\begin{aligned} \mathrm{Re}\left(e^{-i\lambda}\frac{Df(z)}{f(z)}\right) &= \mathrm{Re}\left(e^{-i\lambda}\frac{1+z}{1-z}\right) \\ &= \mathrm{Re}\left(e^{-i\lambda}(1+\cos\lambda)\right) \\ &= \cos\lambda(1+\cos\lambda) > 0. \end{aligned}$$

When λ approaches $-\pi/2$ or $\pi/2$, $\mathrm{Re}\left(e^{-i\lambda}\frac{Df(z)}{f(z)}\right)$ approaches 0. Therefore, $r_s(\lambda)$ is sharp. □

Author contributions

Chuang, Wang: Investigation, Writing-original draft, Writing-review, Functions; Junzhe Mo: Validation; Zhihong Liu: Supervision, Resources, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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