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Research article

Asymptotics on a heriditary recursion

Yong-Guo Shi[1], **Xiaoyu Luo**[2,*] **and Zhi-jie Jiang**[2]

[1] Data Recovery Key Laboratory of Sichuan Province, College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641100, China

[2] College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong 64300, China

* **Correspondence:** Email: 323070108111@stu.suse.edu.cn.

Abstract: The asymptotic behavior for a heriditary recursion

$$x_1 > a \text{ and } x_{n+1} = \frac{1}{n^s}\sum_{k=1}^{n} f\left(\frac{x_k}{k}\right) \text{ for every } n \geq 1$$

is studied, where f is decreasing, continuous on (a, ∞) $(a < 0)$, and twice differentiable at 0. The result has been known for the case $s = 1$. This paper analyzes the case $s > 1$. We obtain an asymptotic sequence that is quite different from the case $s = 1$. Some examples and applications are provided.

Keywords: heriditary recursion; asymptotic expansion; Euler–Maclaurin formula
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1. Introduction

To the evaluation of sequences, which may be divergent, asymptotic expansions provide a way to compute sequences with arbitrarily high accuracy [4, 11, 12, 17]. Many researchers have studied asymptotics of partial sums and related inequalities. One of the most famous examples is the harmonic sum [10, 15], which has an asymptotic estimate

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + \varepsilon_n,$$

where $\gamma = \lim_{n\to\infty}(H_n - \ln n) = 0.577\ldots$ is the constant of Euler and $\varepsilon_n \to 0$. Zhu [20] calculated the asymptotic expansion of the finite sum of some sequences

$$S_n = \sum_{k=1}^{n} \left(n^2 + k\right)^{-1}.$$

Other well-known examples of asymptotic formulas include the Euler–Maclaurin formula [12], the Euler–Boole type summation formula [6] and the prime number theorem [1]. Grünberg [7] applied the Euler–Maclaurin formula to obtain the asymptotic expansions of the sums,

$$\sum_{k=1}^{n} \frac{(\log k)^p}{k^q}, \sum_{k=1}^{n} k^q (\log k)^p, \sum_{k=1}^{n} \frac{(\log k)^p}{(n-k)^q}, \sum_{k=1}^{n} \frac{1}{k^q (\log k)^p}$$

in closed form to arbitrary order ($p, q \in \mathbb{N}$). Wang and Wong [16] carried out the asymptotic estimation of the partial sum $\sum_{k=0}^{n} f_n(k) q^{g_n(k)}$. Xu [19] provided an estimate for the partial sum

$$\gamma_n(z) = \sum_{k=1}^{n} z^{k-1} \left(\frac{1}{k} - \ln\frac{k+1}{k}\right), \text{ where } 0 < z < 1.$$

Blagouchine and Moreau [2] derived the complete asymptotic expansion of the finite sum

$$S_n(\varphi, a) \equiv \sum_{l=1}^{n-1} \csc\left(\varphi + \frac{a\pi l}{n}\right), \quad n \in \mathbb{N} \setminus \{1\}, \quad \varphi + \frac{a\pi l}{n} \neq \pi k, \quad k \in \mathbb{Z}.$$

Some researchers have also given asymptotic estimates for some recurrences in combinatorial mathematics and algorithms. For example, Xu [18, 19] studied the asymptotic series of the generalized Somos recurrence. Hwang, Janson, and Tsai [9] gave exact and asymptotic solutions of a divide-and-conquer recurrence. Heuberger, Krenn, and Lipnik [8] presented some asymptotic analysis of q-recursive sequences.

However, due to computational complexity and lack of tools or methods, very few papers have investigated asymptotic expansions of heriditary recursions (refer to [5, Section 6.3, p. 291]). Recently, Popa [13] investigated a heriditary recursion

$$x_1 > a \text{ and } x_{n+1} = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{x_k}{k}\right) \quad \text{for every } n \geq 1,$$

where $f : (a, \infty) \to (0, \infty)$ and $a < 0$. He gave the first five terms of the asymptotic expansion of $(x_n)_{n\geq 1}$. The aim of this paper is to study the generalized form

$$x_1 > a \text{ and } x_{n+1} = \frac{1}{n^s} \sum_{k=1}^{n} f\left(\frac{x_k}{k}\right) \text{ for every } n \geq 1, \tag{1.1}$$

where $s > 1$. Using only elementary techniques, we establish an asymptotic estimate of $(x_n)_{n\geq 1}$ in (1.1). We obtain an asymptotic sequence for the case $s > 1$ that is quite different from the case $s = 1$. Some examples and applications are provided.

2. Preliminaries

In this section we first introduce the Euler–Maclaurin formula (see [3,6,14]), from which asymptotic expansions of many sequences and sums can be derived.

Lemma 2.1 (Euler–Maclaurin formula)**.** *Suppose f is k-times continuously differentiable on the interval $[a,b]$ with $a<b, a,b\in\mathbb{Z}$. Then*

$$\sum_{a<n\le b} f(n) = \int_a^b \left\{ f(x) - \frac{(-1)^k}{k!}\psi_k(x) f^{(k)}(x) \right\} dx + \sum_{\ell=1}^{k} \frac{(-1)^\ell}{\ell!} \left(f^{(\ell-1)}(b) - f^{(\ell-1)}(a) \right) B_\ell.$$

Suppose f and all its derivatives go to zero as $x\to\infty$. Then we obtain by letting $b\to\infty$ (and adding $f(a)$ to both sides),

$$\sum_{n=a}^{\infty} f(n) = \int_a^\infty f(x)dx + \frac{1}{2}f(a) - \sum_{\ell=2}^{k} \frac{(-1)^\ell}{\ell!} f^{(\ell-1)}(a) B_\ell - \frac{(-1)^k}{k!}\int_a^\infty f^{(k)}(x)\psi_k(x)dx,$$

where $B_\ell(x)$'s are Bernoulli polynomials, $B_\ell = B_\ell(0)$'s are Bernoulli numbers, and $\psi_k(x) = B_k(\{x\})$.

For convenience, let $(k-1)_i$ be Pochhammer's symbol defined as

$$(k-1)_0 := 1, \quad (k-1)_i := (k-1)k(k+1)(k+2)\cdots(k+i-2), \text{ for } i \ge 1.$$

Put $c_i(\alpha) = \dfrac{(\alpha-1)_i}{i!}$.

For $\alpha>1$, let $\zeta(\alpha)$ denote the Riemann zeta function, namely,

$$\zeta(\alpha) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha}.$$

In order to compute the asymptotic expansion, we need the following asymptotic estimate of sums and sequences.

Lemma 2.2. *Let $\alpha>1$. Then*

(i)

$$\sum_{k=n}^{\infty} \frac{1}{k^\alpha} = \frac{1}{(\alpha-1)n^{\alpha-1}} + \frac{1}{2n^\alpha} + o\Big(\frac{1}{n^\alpha}\Big), \quad n\to\infty.$$

(ii)

$$\sum_{k=1}^{n-1} \frac{1}{k^\alpha} = \zeta(\alpha) - \frac{1}{(\alpha-1)n^{\alpha-1}} - \frac{1}{2n^\alpha} + o\Big(\frac{1}{n^\alpha}\Big), \quad n\to\infty.$$

(iii)

$$\frac{1}{(n-1)^{\alpha-1}} - \frac{1}{n^{\alpha-1}} = \sum_{i=0}^{\infty} \frac{c_{i+1}(\alpha)}{n^{\alpha+i}},$$

where $c_i(\alpha) = \dfrac{(\alpha-1)_i}{i!}$.

Proof. (i) By the Euler–Maclaurin formula, $f(k) = \dfrac{1}{k^\alpha}$, $\alpha > 1$, we obtain

$$\begin{aligned}\sum_{k=n}^{\infty} \frac{1}{k^\alpha} &= \int_n^\infty \frac{1}{y^\alpha} dy + \frac{1}{2n^\alpha} + \sum_{i=2}^{\infty} \frac{B_i}{i!} \frac{(\alpha)_{i-1}}{n^{\alpha+i-1}} \\ &= \frac{1}{(\alpha-1)n^{\alpha-1}} + \frac{1}{2n^\alpha} + o\Big(\frac{1}{n^\alpha}\Big).\end{aligned}$$

(ii) It is clear that

$$\sum_{k=1}^{n-1} \frac{1}{k^\alpha} = \sum_{k=1}^{\infty} \frac{1}{k^\alpha} - \sum_{k=n}^{\infty} \frac{1}{k^\alpha}.$$

Then the result (ii) immediately follows from (i).

(iii) Using the Maclaurin series $(1+x)^\beta = 1 + \binom{\beta}{1}x + \binom{\beta}{2}x^2 + o(x^2)$, we have

$$\begin{aligned}\frac{1}{(n-1)^{\alpha-1}} - \frac{1}{n^{\alpha-1}} &= \frac{1}{n^{\alpha-1}} \left(\frac{1}{(1-\frac{1}{n})^{\alpha-1}} - 1 \right) \\ &= \frac{1}{n^{\alpha-1}} \sum_{i=1}^{\infty} \frac{\frac{(\alpha-1)_i}{i!}}{n^i} \\ &= \sum_{i=0}^{\infty} \frac{c_{i+1}(\alpha)}{n^{\alpha+i}}.\end{aligned}$$

□

3. Main results

We first give some asymptotic estimates of the sequence $(x_n)_{n\geq1}$, which is defined by (1.1).

Lemma 3.1. *Suppose that the sequence* $(x_n)_{n\geq1}$ *is defined by (1.1),* f *is decreasing, continuous on* (a,∞)*,* $a < 0$*, and twice differentiable at* 0 *with a Taylor series* $f(x) = f(0) + f'(0)x + f''(0)x^2 + o(x^2)$*. Then the following results hold:*

(i) $\lim_{n\to\infty} x_n = 0$,

(ii) $\lim_{n\to\infty} n^{s-1} x_n = f(0)$,

(iii) *the finite sum*

$$\sum_{k=1}^{n-1} \Big(f(\frac{x_k}{k}) - f(0) \Big) = f(0)f'(0)\Big(\zeta(s) - \frac{1}{(s-1)n^{s-1}} - \frac{1}{2n^s}\Big) + o\Big(\frac{1}{n^s}\Big), \; n \to \infty.$$

Proof. (i) Since f takes positive real numbers, we get $x_n > 0$ for all $n \geq 2$. Since the inequality $\frac{x_k}{k} > 0$ for all $k \geq 2$ and f is decreasing, we get $f\Big(\frac{x_k}{k}\Big) \leq f(0)$, and

$$\sum_{k=1}^{n} f\Big(\frac{x_k}{k}\Big) = f(x_1) + \sum_{k=2}^{n} f\Big(\frac{x_k}{k}\Big) \leq f(x_1) + (n-1)f(0).$$

We deduce that for every $n \geq 1$

$$0 < x_{n+1} = \frac{1}{n^s}\sum_{k=1}^{n} f\Big(\frac{x_k}{k}\Big) \leq \frac{f(x_1) + (n-1)f(0)}{n^s}. \tag{3.1}$$

Since $s > 1$, it follows from the squeeze theorem that $\lim_{n\to\infty} x_n = 0$.

(ii) Moreover, by (i), we have $\lim_{n\to\infty} x_n/n = 0$. From the Stolz–Cesàro theorem, and the continuity at 0, we obtain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} f\Big(\frac{x_k}{k}\Big) = \lim_{n\to\infty} f\Big(\frac{x_n}{n}\Big) = f(0).$$

Let $n \geq 2$. It follows from (1.1) that $x_n = \frac{1}{(n-1)^s}\sum_{k=1}^{n-1} f\Big(\frac{x_k}{k}\Big)$. Then

$$\begin{aligned}
\lim_{n\to\infty} n^{s-1}x_n &= \lim_{n\to\infty} \frac{x_n}{\frac{1}{n^{s-1}}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{(n-1)^s}\sum_{k=1}^{n-1} f(\frac{x_k}{k})}{\frac{1}{n^{s-1}}} \\
&= \lim_{n\to\infty} \frac{\frac{n}{(n-1)^s}\cdot\frac{1}{n}\cdot\sum_{k=1}^{n-1} f(\frac{x_k}{k})}{\frac{1}{n^{s-1}}} = f(0).
\end{aligned}$$

(iii) Since f is differentiable at 0, we have

$$\lim_{k\to\infty} \frac{f(\frac{x_k}{k}) - f(0)}{\frac{x_k}{k}} = f'(0).$$

It follows from (ii) and $f(x) = f(0) + f'(0)x + f''(0)x^2 + o(x^2)$ that

$$\begin{aligned}
f(\frac{x_k}{k}) - f(0) &= f'(0)\frac{x_k}{k} + f''(0)(\frac{x_k}{k})^2 + o\Big((\frac{x_k}{k})^2\Big) \\
&= \frac{f(0)f'(0)}{k^s} + \frac{f^2(0)f''(0)}{k^{2s}} + o\Big(\frac{1}{k^{2s}}\Big).
\end{aligned}$$

By Lemma 2.2 (ii), we have

$$\begin{aligned}
\sum_{k=1}^{n-1}\Big(f(\frac{x_k}{k}) - f(0)\Big) =& \sum_{k=1}^{n-1}\left(\frac{f(0)f'(0)}{k^s} + \frac{f^2(0)f''(0)}{k^{2s}} + o\Big(\frac{1}{k^{2s}}\Big)\right) \\
=& f(0)f'(0)\Big(\zeta(s) - \frac{1}{(s-1)n^{s-1}} - \frac{1}{2n^s}\Big) + o\Big(\frac{1}{n^s}\Big), \ n \to \infty.
\end{aligned}$$

□

The following theorem gives the first three terms of the asymptotic expansion of x_n.

Theorem 3.1. *Suppose that the sequence $(x_n)_{n\ge 1}$ is defined by (1.1), f is decreasing, continuous on (a,∞), $a<0$, and twice differentiable at 0 with a Taylor series $f(x)=f(0)+f'(0)x+f''(0)x^2+o(x^2)$. Then there exists a constant $C\in\mathbb{R}$ such that*

$$C=\lim_{n\to\infty} n^s\Big(x_n-\frac{f(0)}{n^{s-1}}\Big)=\zeta(s)+(s-1)f(0).$$

Moreover, the sequence $(x_n)_{n\ge 1}$ of (1.1) has the following asymptotic expansion:

$$x_n=\frac{f(0)}{n^{s-1}}+\frac{C}{n^s}+o\Big(\frac{1}{n^s}\Big).$$

Proof. For every $n\ge 2$, we have

$$x_n-\frac{f(0)}{n^{s-1}}=\frac{1}{(n-1)^s}\sum_{k=1}^{n-1}\Big(f(\frac{x_k}{k})-f(0)\Big)+\Big(\frac{1}{(n-1)^{s-1}}-\frac{1}{n^{s-1}}\Big)f(0).$$

From Lemma 2.2 (iii) and Lemma 3.1 (iii), we can obtain

$$\begin{aligned} x_n-\frac{f(0)}{n^{s-1}} &= \frac{1}{(n-1)^s}\sum_{k=1}^{n-1}\Big(f(\frac{x_k}{k})-f(0)\Big)+\Big(\frac{1}{(n-1)^{s-1}}-\frac{1}{n^{s-1}}\Big)f(0)\\ &\sim \frac{\zeta(s)+c_1(s)f(0)}{n^s}. \end{aligned}$$

It follows that

$$n^s\left(x_n-\frac{f(0)}{n^{s-1}}\right)\to C:=\zeta(s)+c_1(s)f(0)=\zeta(s)+(s-1)f(0).\quad as\quad n\to\infty.$$

This completes the proof. □

The following theorem gives the first four terms of the asymptotic expansion.

Theorem 3.2. *Suppose that the sequence $(x_n)_{n\ge 1}$ is defined by (1.1), f is decreasing, continuous on (a,∞), $a<0$, and twice differentiable at 0 with a Taylor series $f(x)=f(0)+f'(0)x+f''(0)x^2+o(x^2)$. Let C be defined by Theorem 3.1. Then the sequence $(x_n)_{n\ge 1}$ of (1.1) has the following asymptotic expansion:*

(i) *If $1<s<2$, then*

$$x_n=\frac{f(0)}{n^{s-1}}+\frac{C}{n^s}-\frac{1}{s-1}\cdot\frac{1}{n^{2s-1}}+o\Big(\frac{1}{n^{2s-1}}\Big).$$

(ii) *If $s=2$, then*

$$x_n=\frac{f(0)}{n^{s-1}}+\frac{C}{n^s}+\frac{(s-1)sf(0)-2}{2(s-1)n^{s+1}}+o\Big(\frac{1}{n^{s+1}}\Big).$$

(iii) *If $s>2$, then*

$$x_n=\frac{f(0)}{n^{s-1}}+\frac{C}{n^s}+\frac{(s-1)sf(0)}{2n^{s+1}}+o\Big(\frac{1}{n^{s+1}}\Big).$$

Proof. From Lemma 2.2 (iii) and Lemma 3.1 (iii), we deduce that

$$\begin{aligned} x_n - \frac{f(0)}{n^{s-1}} - \frac{C}{n^s} &= \frac{1}{(n-1)^s}\Big(\sum_{k=1}^{n-1} f\Big(\frac{x_k}{k}\Big) - f(0)\Big) + \Big(\frac{1}{(n-1)^{s-1}} - \frac{1}{n^{s-1}}\Big)f(0) - \frac{C}{n^s} \\ &= \frac{1}{n^s}\Big(C_1 - \frac{1}{(s-1)n^{s-1}} - \frac{1}{2n^s}\Big) + \Big(\frac{c_1(s)}{n^s} + \frac{c_2(s)}{n^{s+1}}\Big)f(0) - \frac{C}{n^s} + o\Big(\frac{1}{n^{2s}}\Big) \\ &= -\frac{1}{(s-1)n^{2s-1}} + \frac{c_2(s)f(0)}{n^{s+1}} - \frac{1}{2n^{2s}} + o\Big(\frac{1}{n^{2s}}\Big). \end{aligned}$$

Case 1. Let $1 < s < 2$. Then $2s-1 < s+1$, and

$$x_n - \frac{f(0)}{n^{s-1}} - \frac{C}{n^s} \sim -\frac{1}{(s-1)n^{2s-1}}.$$

Thus

$$x_n = \frac{f(0)}{n^{s-1}} + \frac{C}{n^s} - \frac{1}{s-1}\cdot\frac{1}{n^{2s-1}} + o\Big(\frac{1}{n^{2s-1}}\Big).$$

Case 2. Let $s = 2$. Then $2s-1 = s+1 < 2s$, and

$$x_n - \frac{f(0)}{n^{s-1}} - \frac{C}{n^s} = -\frac{1}{(s-1)n^{2s-1}} + \frac{c_2 f(0)}{n^{s+1}} + o\Big(\frac{1}{n^{s+1}}\Big) \sim \frac{D}{n^{s+1}},$$

where $D = c_2 f(0) - \frac{1}{(s-1)}$. Thus

$$x_n = \frac{f(0)}{n^{s-1}} + \frac{C}{n^s} + \frac{D}{n^{s+1}} + o\Big(\frac{1}{n^{s+1}}\Big).$$

Case 3. Let $s > 2$. Then $s+1 < 2s-1 < 2s$. Hence, we have

$$x_n = \frac{f(0)}{n^{s-1}} + \frac{C}{n^s} + \frac{c_2 f(0)}{n^{s+1}} + o\Big(\frac{1}{n^{s+1}}\Big).$$

□

Below is our main theorem, which gives the first five terms of the asymptotic expansion.

Theorem 3.3. *Suppose that the sequence $(x_n)_{n\geq 1}$ is defined by (1.1), f is decreasing, continuous on (a,∞), $a < 0$, and twice differentiable at 0 with a Taylor series $f(x) = f(0) + f'(0)x + f''(0)x^2 + o(x^2)$. Let C be defined by Theorem 3.1. Then the sequence $(x_n)_{n\geq 1}$ of (1.1) has the following asymptotic expansion:*

(i) *If $1 < s < 2$, then*

$$x_n = \frac{f(0)}{n^{s-1}} + \frac{C}{n^s} - \frac{1}{s-1}\cdot\frac{1}{n^{2s-1}} + \frac{(s-1)sf(0)}{2n^{s+1}} + o\Big(\frac{1}{n^{s+1}}\Big).$$

(ii) *If $s = 2$, then*

$$x_n = \frac{f(0)}{n^{s-1}} + \frac{C}{n^s} + \frac{(s-1)sf(0) - 2}{2(s-1)n^{s+1}} + \frac{(s-1)s(s+1)f(0) - 3}{6n^{s+2}} + o\Big(\frac{1}{n^{s+2}}\Big).$$

(iii) *If* $2 < s < 3$, *then*

$$x_n = \frac{f(0)}{n^{s-1}} + \frac{C}{n^s} + \frac{(s-1)sf(0)}{2n^{s+1}} - \frac{1}{s-1} \cdot \frac{1}{n^{2s-1}} + o\Big(\frac{1}{n^{2s-1}}\Big).$$

(iv) *If* $s = 3$, *then*

$$x_n = \frac{f(0)}{n^{s-1}} + \frac{C}{n^s} + \frac{(s-1)sf(0)}{2n^{s+1}} + \frac{(s-1)s(s+1)f(0) - 3}{6n^{s+2}} + o\Big(\frac{1}{n^{2s-1}}\Big).$$

(v) *If* $s > 3$, *then*

$$x_n = \frac{f(0)}{n^{s-1}} + \frac{C}{n^s} + \frac{(s-1)sf(0)}{2n^{s+1}} + \frac{(s-1)s(s+1)f(0)}{6n^{s+2}} + o\Big(\frac{1}{n^{2s-1}}\Big).$$

Proof. Case 1. Let $1 < s < 2$. Then $2s - 1 < s + 1 < 2s$. By Lemma 2.2 and Lemma 3.1, we have

$$\begin{aligned}
& x_n - \frac{f(0)}{n^{s-1}} - \frac{C}{n^s} + \frac{1}{(s-1)}\frac{1}{n^{2s-1}} \\
= \; & \frac{1}{(n-1)^s}\Big(\sum_{k=1}^{n-1} f\Big(\frac{x_k}{k}\Big) - f(0)\Big) + \Big(\frac{f(0)}{(n-1)^{s-1}} - \frac{f(0)}{n^{s-1}}\Big) - \frac{C}{n^s} + \frac{1}{(s-1)}\frac{1}{n^{2s-1}} \\
= \; & \frac{1}{n^s}\Big(\zeta(s) - \frac{1}{(s-1)n^{s-1}} - \frac{1}{2n^s}\Big) + \Big(\frac{c_1}{n^s} + \frac{c_2}{n^{s+1}} + \frac{c_3}{n^{s+2}}\Big)f(0) \\
& - \frac{C}{n^s} + \frac{1}{(s-1)}\frac{1}{n^{2s-1}} + o\Big(\frac{1}{n^{2s}}\Big) \\
= \; & \frac{c_2 f(0)}{n^{s+1}} - \frac{1}{2n^{2s}} + \frac{c_3 f(0)}{n^{s+2}} + o\Big(\frac{1}{n^{2s}}\Big) \sim \frac{c_2 f(0)}{n^{s+1}}.
\end{aligned}$$

Thus

$$x_n = \frac{f(0)}{n^{s-1}} + \frac{C}{n^s} - \frac{1}{s-1} \cdot \frac{1}{n^{2s-1}} + \frac{c_2 f(0)}{n^{s+1}} + o\Big(\frac{1}{n^{s+1}}\Big).$$

Case 2. Let $s = 2$. Then $s + 2 = 2s$, and

$$x_n - \frac{f(0)}{n^{s-1}} - \frac{C}{n^s} - \frac{D}{n^{s+1}} = -\frac{1}{2n^{2s}} + \frac{c_3 f(0)}{n^{s+2}} + o\Big(\frac{1}{n^{2s}}\Big) \sim \frac{E}{n^{s+2}},$$

where

$$D = c_2 f(0) - \frac{1}{(s-1)}, \quad E = c_3 f(0) - \frac{1}{2}.$$

Case 3. Let $2 < s < 3$. Then $s + 1 < 2s - 1 < s + 2 < 2s$, and

$$x_n - \frac{f(0)}{n^{s-1}} - \frac{C}{n^s} - \frac{c_2 f(0)}{n^{s+1}} \sim -\frac{1}{(s-1)}\frac{1}{n^{2s-1}}.$$

Case 4. Let $s = 3$. Then $2s - 1 = s + 2 < 2s$, and

$$x_n - \frac{f(0)}{n^{s-1}} - \frac{C}{n^s} - \frac{c_2 f(0)}{n^{s+1}} = -\frac{1}{(s-1)}\frac{1}{n^{2s-1}} + \frac{c_3 f(0)}{n^{s+2}} + o\Big(\frac{1}{n^{2s}}\Big) \sim \frac{F}{n^{s+2}},$$

where

$$F = c_3 f(0) - \frac{1}{(s-1)}.$$

Case 5. Let $s > 3$. Then $s + 2 < 2s - 1 < 2s$, and

$$x_n - \frac{f(0)}{n^{s-1}} - \frac{C}{n^s} - \frac{c_2 f(0)}{n^{s+1}} \sim \frac{c_3 f(0)}{n^{s+2}}.$$

□

4. Some examples

In this section, we give some applications of our results to several specific examples and a comparison with Popa's results.

Example 4.1. *Let $(x_n)_{n\geq 1}$ be a sequence of the real numbers defined by*

$$x_{n+1} = \frac{1}{n^s}\sum_{k=1}^{n} e^{-\frac{x_k}{k}}, \quad x_1 \in \mathbb{R}, \quad \forall n \geq 1.$$

Note that $f(x) = e^{-x}$ and $f(0) = 1$.

When $s = \frac{3}{2}$, it follows from Theorem 3.3 (i) that $\lim\limits_{n\to\infty} x_n = 0$. Moreover,

$$\sqrt{n}x_n = 1 + \frac{C}{n} - \frac{1}{n^{\frac{3}{2}}} + \frac{3}{8n^2} + o\left(\frac{1}{n^2}\right).$$

When $s = 1$, it follows from in [13, Theorem 5] that

$$x_n = 1 - \frac{\ln n}{n} + \frac{A}{n} - \frac{2\ln n}{n^2} + \frac{2A-1}{n^2} + o\left(\frac{1}{n^2}\right),$$

since $f'(0) = -1$.

This example corrects some printing errors in [13, Corollary 6]. And it is easy to see that when $s > 1$, the asymptotic sequence for the case $s > 1$ is completely different from the case $s = 1$.

Example 4.2. *Let $(x_n)_{n\geq 1}$ be a sequence of the real numbers defined by $x_1 > 1 - e^2$ and*

$$x_{n+1} = \frac{1}{n^s}\sum_{k=1}^{n} \frac{1}{\ln(e^2 + \frac{x_k}{k})}, \quad \forall n \geq 1.$$

Note that $f(x) = \frac{1}{\ln(e^2+x)}$, and $f(0) = \dfrac{1}{2}$.

When $s = \frac{3}{2}$, by Theorem 3.3 (i), we have $\lim\limits_{n\to\infty} x_n = 0$, and

$$2\sqrt{n}x_n = 1 + \frac{2C}{n} - \frac{4}{n^{3/2}} + \frac{3}{8n^2} + o\left(\frac{1}{n^2}\right).$$

When $s = 4$, by Theorem 3.3 (v), we have $\lim\limits_{n\to\infty} x_n = 0$, and

$$2n^3 x_n = 1 + \frac{2C}{n} + \frac{6}{n^2} + \frac{10}{n^3} + o\left(\frac{1}{n^3}\right).$$

5. Conclusions

Our analysis shows that this heriditary recursion can be further expanded according to the residual terms. By comparing asymptotic sequences, depending on the value of s, more and more terms of the asymptotic expansion are obtained.

Our results show that, for the case $s > 1$, Eq (1.1) has the asymptotic expansion of the form

$$x_n = \sum_{j=0}^{k} \frac{a_j}{n^j} + o\left(\frac{1}{n^k}\right), \quad \text{as } n \to \infty,$$

while, for the case $s = 1$, Eq (1.1) has the asymptotic expansion of the form

$$x_n = \sum_{j=1}^{k} a_l \frac{\ln^{p_j} n}{n^{q_j}} + o\left(\frac{\ln^{p_k} n}{n^{q_k}}\right),$$

where $q_j \in [0, \infty)$, $p_j \in \mathbb{R}$ and where some of the constants a_j may depend on the initial values.

Author contributions

Yong-Guo Shi: writing original draft, methodology, proof of conclusions; Xiaoyu Luo: writing original draft, methodology, proof of conclusions; Zhi-jie Jiang: validation, writing review, editing, proof of conclusions. The authors contributed equally to this work. All the authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflicts of interest.

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