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Research article

A note on Kaliman's weak Jacobian Conjecture

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Abstract: We improve Kaliman's weak Jacobian Conjecture by the Hurwitz formula and resolution of singular curves. Furthermore, we give a more general form of Kaliman's weak Jacobian Conjecture.

Keywords: Jacobian Conjecture; Keller map; Hurwitz formula
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1. Introduction

Let $\mathbb{C}[X_1, \cdots, X_n]$ denote the polynomial ring in the variables $X_1, \cdots, X_n$ over $\mathbb{C}$. A *polynomial map* is a map $F = (F_1, \cdots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$(z_1, \cdots, z_n) \to (F_1(z_1, \cdots, z_n), \cdots, F_n(z_1, \cdots, z_n)),$$

where each F_i belongs to $\mathbb{C}[X_1, \cdots, X_n]$. Such a polynomial map is called *invertible* if there exists a polynomial map $G = (G_1, \cdots, G_n) : \mathbb{C}^n \to \mathbb{C}^n$ such that $X_i = G_i(F_1, \cdots, F_n)$ for all $1 \leq i \leq n$, i.e., G is the left inverse of F. It is easy to show that G is also a right inverse of F. So F is invertible, i.e., F is an isomorphism, in the sense of morphisms of algebraic varieties.

Consider a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$. How can we recognize if a polynomial map F is invertible?

Let $J(F) = (\partial F_i/\partial X_j)$ be the Jacobian matrix of F. Clearly, the invertibility of the matrix $J(F)$ is equivalent to $\det J(F) \in \mathbb{C}^\times$. It is easy to show that if $F : \mathbb{C}^n \to \mathbb{C}^n$ is invertible, then $\det J(F) \in \mathbb{C}^\times$. Conversely, there is the following famous conjecture.

Conjecture 1.1. *If* $\det J(F) \in \mathbb{C}^\times$, *then* F *is invertible.*

The Jacobian Conjecture was first formulated by O. H. Keller in 1939. Aside from the trivial case $n = 1$, this conjecture remains an open problem for all $n \geq 2$ up to now. The Jacobian Conjecture appeared as Problem 16 on a list of 18 famous open problems in the paper by Steve Smale [11].

The Jacobian Conjecture has been reduced to the case of degree 3 using the method of algebraic K-theory by Bass, Connell, and Wright [1]. The second author has achieved some results in algebraic K-theory [12, 13].

When $n = 2$, Kaliman proposed the weak Jacobian conjecture in [6].

Conjecture 1.2. *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ with $\det J(F) \in \mathbb{C}^\times$. Suppose that for every $c \in \mathbb{C}$ the fibre $V(F_1) := \{(x, y) \mid F_1(x, y) = c\}$ is irreducible. Then the map F is invertible.*

The fiber $V(F_1)$ is irreducible if and only if the polynomial $F_1(x, y) - c$ is irreducible. For a polynomial $f(x, y) \in \mathbb{C}[x, y]$ with the degree $\deg f(x, y) > 1$, in general the polynomial $f(x, y) - c$ is not always irreducible for each $c \in \mathbb{C}$. Hence, our main improvement is the following theorem (see Section 2, Thm.2.8)

Theorem 1.3. *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ with $\det J(F) \in \mathbb{C}^\times$. Suppose that there exist infinitely many $c \in \mathbb{C}$ such that the polynomial $F_1(x, y) - c$ is irreducible. Then the map F is invertible.*

Furthermore, we give a general form of the above theorem

Theorem 1.4. *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ with $\det J(F) \in \mathbb{C}^\times$. If there exist infinitely many points $(a, b, c) \in \mathbb{C}^3$ such that the polynomial $aF_1(x, y) + bF_2(x, y) + c$ is irreducible, then the map F is invertible.*

In the above theorem, the condition that $aF_1(x, y) + bF_2(x, y) + c$ is irreducible can be independent of the Jacobian conjecture. This leads us to propose the following conjecture.

Conjecture 1.5. *Let $F_1(x, y), F_2(x, y) \in \mathbb{C}[x, y]$ be algebraically independent polynomials. Then there exist infinitely many points $(a, b, c) \in \mathbb{C}^3$ such that the polynomial $aF_1(x, y) + bF_2(x, y) + c$ is irreducible.*

There are many works on the case $n = 2$. A good introduction about the classical results can be found in chapter 10 in [4]. Miyanishi [8] proved that the Jacobian conjecture holds true if a generalized Sard property holds true for the affine plane and an $\mathbb{A}^1$-fibration on $\mathbb{A}^2$. Jedrzejewicz and Zieliński in [5] give a survey of a new purely algebraic approach to the Jacobian Conjecture in terms of irreducible elements and square-free elements. A similar result has been achieved in [2, 3]. However, our methods are based on the Hurwitz formula and resolution of the singular curve.

2. Proof of theorem

Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map such that $\det J(F) \in \mathbb{C}^\times$. Denote $m = \max\{\deg F_1, \deg F_2\}$ the maximal degree of F_1 and F_2. Then we have a rational map of projective spaces

$$\bar{F} = (\bar{F}_1, \bar{F}_2, Z^m) : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}, \quad (x : y : z) \mapsto (\bar{F}_1 : \bar{F}_2 : z^m),$$

where $\bar{F}_i(x, y, z) = z^m F_i(\frac{x}{z}, \frac{y}{z})$ are the homogeneous polynomials. Let

$$L_\infty := \mathbb{P}^1_{\mathbb{C}} = \{(x : y : z) \mid z = 0\}.$$

Then $\mathbb{P}^2_{\mathbb{C}} = \mathbb{C}^2 \bigcup \mathbb{P}^1_{\mathbb{C}}$. Moreover, the restriction of $\bar{F}$ on $\mathbb{C}^2$ is F and

$$\bar{F}|_{\mathbb{P}^1_{\mathbb{C}}} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}, \quad (x : y : 0) \mapsto (\bar{F}_1 : \bar{F}_2 : 0).$$

Lemma 2.1. *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map such that* $\det J(F) \neq 0$ *in* $\mathbb{C}[x, y]$*. Then F_1, F_2 are algebraically independent over $\mathbb{C}$ and $\mathbb{C}(F_1, F_2) \subset \mathbb{C}(x, y)$ is a finite field extension.*

Proof. A proof can be found in [4, Prop.1.1.31]. □

Lemma 2.2. *For the polynomial map $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$, if* $\det J(F) \neq 0$ *in* $\mathbb{C}[x, y]$*, then the cardinality of the fibers of F is bounded by the degree*

$$\deg F := [\mathbb{C}(x, y) : \mathbb{C}(F_1, F_2)].$$

Proof. See [4, Thm.1.1.32]. □

Lemma 2.3. *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be the polynomial map such that* $\det J(F) \neq 0$ *in* $\mathbb{C}[x, y]$*. Then there exists a Zariski open set $U \subset \mathbb{C}^2$ such that*

$$\# F^{-1}(p) = [\mathbb{C}(x, y) : \mathbb{C}(F_1, F_2)], \quad \forall p \in U.$$

Proof. See [9, Prop.3.17]. The condition $\det J(F) \neq 0$ ensures that the map F is dominating map. □

Lemma 2.4. *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be the polynomial map such that* $\det J(F) \in \mathbb{C}^{\times}$*. Denote $V(F_1) = \{(x, y) \in \mathbb{C}^2 \mid F_1(x, y) = c,\ c \in \mathbb{C}\}$ and $L_c = \{(x, y) \in \mathbb{C}^2 \mid x = c\}$. Then the morphism*

$$F|_{V(F_1)} : V(F_1) \to L_c, \quad (x, y) \mapsto (c, F_2(x, y))$$

is étale morphism.

Proof. First, the morphism F is étale. Let $L_c = \{(x, y) \in \mathbb{C}^2 \mid x = c\}$. Then $V(F_1) = F^{-1}(L_c)$. Hence we have the fiber product

$$\begin{array}{ccc} V(F_1) & \xrightarrow{F|_{V(F_1)}} & L_c \\ \downarrow & & \downarrow \\ \mathbb{C}^2 & \xrightarrow{F} & \mathbb{C}^2. \end{array}$$

Because the étale map is stable under fibered products (see [7, Chap.4, Prop.3.22]), the map $F_2 : V(F_1) \to L_c$ is étale. □

Lemma 2.5. *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be the polynomial map such that* $\det J(F) \neq 0$*. Denote $V(F_1)$ and L_c as in* Lemma 2.4*. Consider*

$$F|_{V(F_1)} : V(F_1) \to L_c, \quad (x, y) \mapsto (c, F_2(x, y)).$$

Then there exists a Zariski open set $U \subset \mathbb{C}^2$ such that for almost all $c \in \mathbb{C}$,

$$\# F|_{V(F_1)}^{-1}(P) = \# F^{-1}(P) = [\mathbb{C}(x, y) : \mathbb{C}(F_1, F_2)], \quad \forall P \in U \bigcap L_c.$$

Proof. By Lemma 2.3, $\mathbb{C}^2 \setminus U$ contains at most finitely many lines L_c. Hence the lemma follows: □

Lemma 2.6. *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be the polynomial map such that* $\det J(F) \in \mathbb{C}^{\times}$*. Denote $m = \deg F_1$ and $\bar{F}_1(x, y, z) = z^m F_1(\frac{x}{z}, \frac{y}{z})$. Let $V(\bar{F}_1) := \{(x : y : z) \in \mathbb{P}^2_{\mathbb{C}} \mid \bar{F}_1(x, y, z) = 0\}$ and $L_\infty = \{(x : y : z) \in \mathbb{P}^2_{\mathbb{C}} \mid z = 0\}$. Then the curve $V(\bar{F}_1)$ is smooth at $V(\bar{F}_1) \setminus F^{-1}(L_\infty)$. The set $V(\bar{F}_1) \bigcap F^{-1}(L_\infty)$ may be singular points of $V(\bar{F}_1)$.*

Proof. This lemma is easy, because $V(\bar{F}_1) \setminus F^{-1}(L_\infty) = V(F_1)$ and $V(F_1)$ is smooth. □

Lemma 2.7. (Hurwitz) *Let $\phi : C_1 \to C_2$ be a morphism of Riemann surfaces of genera g_1 and g_2. Then*

$$2g_1 - 2 = \deg\phi(2g_2 - 2) + \sum_{P\in C_1}(e_\phi(P) - 1),$$

where $\deg\phi$ *is the degree of the map ϕ, $e_\phi(P)$ is the ramification index of ϕ at P.*

Proof. A proof can be found in [10, Thm.5.9]. □

Let $m = \deg F_1(x, y)$. Then $\bar{F}_1(x, y, z) = z^m F_1(\frac{x}{z}, \frac{y}{z})$ is an irreducible polynomial if and only if $F_1(x, y)$ is irreducible. Now we can prove the main theorems in this section.

Theorem 2.8. *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be the polynomial map such that* $\det J(F) \in \mathbb{C}^\times$. *Suppose that there exist infinitely many $c \in \mathbb{C}$ such that the polynomial $F_1(x, y) - c$ is irreducible. Then the two-dimensional Jacobian Conjecture holds.*

Proof. Let $\bar{F}_1(x, y, z) = z^m F_1(\frac{x}{z}, \frac{y}{z}) - cz^m$. The projective set $V(\bar{F}_1) \subset \mathbb{P}^2_{\mathbb{C}}$ is defined by $\bar{F}_1(x, y, z) = 0$ and $L_c \subset \mathbb{P}^2_{\mathbb{C}}$ defined by $x = c$. Consider the map

$$\phi_c = \bar{F}|_{V(\bar{F}_1)} : V(\bar{F}_1) \to L_c.$$

Since there exist infinitely many $c \in \mathbb{C}$ such that the polynomial $F_1(x, y) - c$ is irreducible, we can find some $c \in \mathbb{C}$ such that $V(\bar{F}_1)$ is irreducible and satisfying Lemma 2.5, that is, $\deg\phi_c = \deg F$. Further, ϕ_c is étale restricting on the affine curve $V(F_1)$ by Lemma 2.4, where $V(F_1)$ is the affine part of $V(\bar{F}_1)$.

If $V(\bar{F}_1)$ is singular at $V(\bar{F}_1) \setminus V(F_1) = \phi_c^{-1}(\infty)$, where $\infty = (c : 1 : 0) \in L_c$, then from resolution of singularity, we can find a smooth curve C such that the morphism

$$r : C \to V(\bar{F}_1)$$

satisfying that r is isomorphic on $W := r^{-1}(V(F_1))$ (see [9, Chp.7, P.128]). Then we have

$$\phi = \phi_c \circ r : C \to L_c$$

is étale on W.

Since the genus of L_c is 0, by Lemma 2.7,

$$2g - 2 = -2\deg\phi + \sum_{P\in C}(e_\phi(P) - 1),$$

where g is the genus of C. Since ϕ is étale on W, we have $e_\phi(P) = 1$ for $P \in W$. But $C \setminus W = \phi^{-1}(\infty)$, by Proposition 2.6 in [10], we have

$$\sum_{P\in\phi^{-1}(\infty)} e_\phi(P) = \deg\phi.$$

Hence,

$$2g - 2 = -2\deg\phi + \sum_{P\in\phi^{-1}(\infty)} e_\phi(P) - \#\phi^{-1}(\infty)$$

$$= -\deg\phi - \#\phi^{-1}(\infty).$$

Since $\deg\phi \geq 1$, $\#\phi^{-1}(\infty) \geq 1$, the right side in the above is negative. Then $2g - 2 < 0$, therefore we have $g = 0$. Furthermore,

$$\deg\phi = \deg\phi_c = \deg F = 1.$$

This implies that F is injective. By Theorem 4.1.1 in [4], F is isomorphic. □

Theorem 2.9. *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ with $\det J(F) \in \mathbb{C}^\times$. If there exist infinitely many points $(a, b, c) \in \mathbb{C}^3$ such that the polynomial $aF_1(x, y) + bF_2(x, y) + c$ is irreducible, then the map F is invertible.*

Proof. The proof of this theorem is similar to Theorem 2.8, because $ax + by + c = 0$ defines a line in $\mathbb{C}^2$, which is isomorphic to $\mathbb{C}^1$. □

3. Irreducibility of polynomials

Let $F(x, y) \in \mathbb{C}[x, y]$ such that $F(x, y) \notin \mathbb{C}[x]$ nor $F(x, y) \notin \mathbb{C}[y]$. We check Conjecture 1.5 when $\deg F = 2$.

Proposition 3.1. *Conjecture 1.5 holds true when $\deg F = 2$.*

Proof. Let $F(x, y) = ax^2 + bxy + cy^2 + dx + ey + f \in \mathbb{C}[x, y]$. Since $\deg F = 2$, at least one of a, b, c is not 0, we can assume $a \neq 0$. Consider $F(x, y)/a$. Then we can assume $a = 1$. Since $F(x, y) \notin \mathbb{C}[x], F(x, y) \notin \mathbb{C}[y]$, we discuss it in several cases.
Case 1. $b = c = 0, e \neq 0$. Then for each $z \in \mathbb{C}$, $F(x, y) + z$ is irreducible.
Case 2. At least one of b, c is not 0. Supposing for some $z \in \mathbb{C}$ there is

$$F(x, y) + z = (x + a_1y + a_2)(x + b_1y + b_2).$$

Comparing the homogeneous part of degree 2, we have

$$a_1 = \frac{b + \sqrt{b^2 - 4c}}{2}, \quad b_1 = \frac{b - \sqrt{b^2 - 4c}}{2}.$$

Comparing the homogeneous part of degree 1, we have

$$\begin{cases} a_2 + b_2 = d, \\ b_1a_2 + a_1b_2 = e. \end{cases}$$

If $a_1 - b_1 = \sqrt{b^2 - 4c} \neq 0$, then the above equation has a unique solution for a_2, b_2. Hence, there exists only one $z \in \mathbb{C}$ such that $F(x, y) + z$ is reducible.

If $a_1 - b_1 = \sqrt{b^2 - 4c} = 0$, then $a_1 \neq 0$; otherwise, we have $b = c = 0$, a contradiction. Then the above equation becomes

$$\begin{cases} a_2 + b_2 = d, \\ a_2 + b_2 = e/a_1. \end{cases}$$

If $d \neq e/a_1$, then there exists no solution for the above equation. Hence, for each $z \in \mathbb{C}$, $F(x, y) + z$ is irreducible. □

4. Conclusions

In this paper, we generalize Kaliman's weak Jacobian Conjecture utilizing the Hurwitz formula and resolution of singular curves. At the same time, we give a conjecture about the property of irreducibility of linear combination polynomials in two variables. Furthermore, we check this conjecture in the case of polynomials with degree 2.

Author contributions

Yan Tian: conceptualization, formal analysis, investigation, methodology, writing-original draft, writing-review and editing, funding acquisition; Chaochao Sun: supervision, methodology, formal analysis, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interest.

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