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Research article

An L_∞ performance control for time-delay systems with time-varying delays: delay-independent approach via ellipsoidal $\mathcal{D}$-invariance

Hyung Tae Choi[1] **and Jung Hoon Kim**[1,2,*]

[1] Department of Electrical Engineering, Pohang University of Science and Technology (POSTECH), Pohang 37673, Republic of Korea

[2] Institute for Convergence Research and Education in Advanced Technology, Yonsei University, Incheon 21983, Republic of Korea

* **Correspondence:** Email: junghoonkim@postech.ac.kr; Tel: +82542792230; Fax: +82542792903.

Abstract: This paper is concerned with a delay-independent output-feedback controller synthesis suppressing the L_∞-gain of linear time-delay systems with time-varying delays. We first proposed a continuous-time version of the existing discrete-time ellipsoidal $\mathcal{D}$-invariant set and established its existence condition in terms of some linear matrix inequalities (LMIs). This existence condition was further extended to characterizing the L_∞-gain of linear time-delay systems with time-varying delays. Because of the delay-independent property of the proposed $\mathcal{D}$-invariant set, the L_∞-gain analysis does not depend on the choice of delays including their magnitudes and time derivatives. Based on this analysis method, we also constructed an output-feedback controller synthesis for ensuring the L_∞-gain of time-delay systems bounded by a performance level ρ. In an equivalent fashion to the L_∞-gain analysis method, this controller synthesis is independent of the delays in the sense that the obtained controller coefficients do not depend on the delay characteristics. Finally, numerical results were given to demonstrate the effectiveness and validity of the proposed results.

Keywords: L_∞-gain; time-varying delay; ellipsoidal $\mathcal{D}$-invariance
Mathematics Subject Classification: 34H05, 93C23, 93C43

1. Introduction

Based on the fact that the L_∞-gain of dynamical systems means the maximum L_∞ norm of the output signal for the input signal with the unit L_∞ norm, the issues on computing and/or minimizing the L_∞-gain have been deeply discussed in [1]. Here, it would be worthwhile to note that the L_∞-gain problem has been also called the L_1 problem because the L_∞-gain coincides with the L_1 norm of the impulse response in the case of single-input single-output (SISO) linear time-invariant (LTI) systems. The L_1 problem has attracted much interest in the system and control community because of the practical importance of the L_∞-gain. More precisely, some early results on the L_∞-gain problem [1–3] have shown that the L_∞-gain of continuous-time LTI systems can be exactly obtained by using the arguments on linear programming (LP) [4] and operator theory [5]. Motivated by the successful results in those studies, the L_∞-gain has been widely applied to various practical systems such as robotic manipulators [6, 7], bipedal robots [8, 9], and power systems [10].

In a similar fashion, the L_∞-gain problem has been actively explored for advanced dynamical systems such as sampled-data systems [11–14], positive systems [15–17], event-triggered control systems [18,19], and nonlinear systems [20–23]. To simply put it, the L_∞-gain of a sampled-data system can be computed in [11–14] by finding an (approximate) equivalent discrete-time linear system, which are concerned with the piecewise constant and/or linear approximations of input/output signals. For positive systems, it is shown in [15] that the L_∞-gain of linear positive systems can be equivalently characterized by the arguments on LP and this idea is also extended to positive delay systems [16] and positive fuzzy systems [17]. Some linear matrix inequalities (LMIs) together with piecewise linear models are introduced in [18, 19] to characterize the L_∞-gains of event-triggered systems. The L_∞-gain of nonlinear systems is also discussed in [20–23] by employing the arguments on set-invariance principles [24–26].

On the other hand, it is also meaningful to tackle the L_∞-gain for another type of dynamical system of time-delay systems [27], in which delayed (or retarded) terms are considered for the differential/difference equations. This is because time-delay systems have been used for describing various fields of population dynamics [28], epidemic models [29], communication network systems [30], time-delay-based robot control [31], and so on. Despite this broad applicability, there are only a few studies that deal with the L_∞-gain of general time-delay systems, while there have been a number of studies on the asymptotic stability [32–35] and the H_∞ performance [36–39] of time-delay systems. With respect to this, the Lyapunov-Krasovski functional (LKF) approach is taken in [40–42] to derive invariant sets for time-delay systems, by which the L_∞-gain of such systems can be characterized. These LKF-based arguments lead to controller synthesis procedures for bounding the L_∞-gain of time-delay systems in terms of some LMIs. However, the results in [40–42] do depend on characteristics of delays, and thus a resulting controller should be re-determined if the magnitudes and/or time derivatives of the considered delays are changed. This is because the LKFs in [40–42] are intrinsically dependent on the delays. Furthermore, there are other limitations of the arguments in those studies. The delays in [40,41] are confined to constant values with sufficiently small sizes. Even though time-varying delays can be considered in [42], the bound for L_∞-gain in that study is determined after the controller synthesis is completed. Thus, we can only lead to a conservative bound for the L_∞-gain of time-delay systems by employing the arguments in [42], and this feature does not fit into suppressing the L_∞-gains in a prescribed level.

To alleviate the aforementioned limitations in [40–42], it would be required to develop a delay-independent approach to deal with the L_∞-gain of time-delay systems with time-varying delays. Regarding this, the so-called $\mathcal{D}$-invariance was introduced recently in [43–45] to derive set-invariance properties for discrete-time systems without depending on the choice of delays. More precisely, it is shown in [43] that a polyhedral $\mathcal{D}$-invariance set can be formulated and such a set can be computed in terms of the LP. In a parallel line, an ellipsoidal $\mathcal{D}$-invariance set and its LMI-based characterizations were also introduced in [44, 45]. However, it should be remarked for the results in [43–45] that they were limited to the case of discrete-time systems and no discussion on the L_∞-gain was obtained.

Motivated by the previous works [40–45], this paper aims to establish a delay-independent output-feedback controller synthesis ensuring that the L_∞-gain of the resulting time-delay closed-loop system is bounded by a prescribed level ρ regardless of the choice of delays. To do this, we first propose a continuous-time version of the ellipsoidal $\mathcal{D}$-invariance, by which a given ellipsoidal set becomes forward invariant regardless of the choice of delays. We then derive the existence condition of an ellipsoidal $\mathcal{D}$-invariant set in terms of LMIs. This result allows us to obtain a condition for bounding the L_∞-gain of time-delay systems with time-varying delays in terms of LMIs. On the basis of the L_∞-gain analysis, an output-feedback controller leading to the L_∞-gain of the resulting closed-loop system to be bounded by a prescribed level ρ is obtained. Because both the analysis and synthesis results on the L_∞-gain of time-delay systems with time-varying delays are established based on the ellipsoidal $\mathcal{D}$-invariance, they do not depend on characteristics of delays such as magnitudes and time derivatives. The validity and effectiveness of the proposed arguments are verified by some comparative numerical results. Finally, the overall contributions of this paper compared to the existing studies [40–42] are summarized in the following and are also described in Table 1.

- The continuous-time version of the ellipsoidal $\mathcal{D}$-invariance is introduced for time-delay systems with time-varying delays.
- A delay-independent condition for the L_∞-gain analysis of time-delay systems with time-varying delays is developed.
- An L_∞ performance controller ensuring that the L_∞-gain for the resulting closed-loop system is bounded by a prescribed level ρ is established.
- The coefficient matrices involved in the above L_∞ performance controller are obtained through the LMI-based approach, in which the relevant constraints do not depend on the delay characteristics.
- Thus, once such an L_∞ performance controller is obtained by the aforementioned method, it can be employed regardless of the modifications of the sizes and/or rates of the considered delays.

Table 1. The contributions of the proposed methods compared to the existing results in [40–42].

	This paper	[40, 41]	[42]
Delay	**Time-varying**	Constant	**Time-varying**
Approach	**Ellipsoidal $\mathcal{D}$-invariance**	LKFs	LKFs
Controller coefficients	**Delay-independent**	Delay-dependent	Delay-dependent
Bound for L_∞-gain	**Prescribed**	**Prescribed**	Determined after synthesis

This paper is organized as follows. The problem definition is introduced in Section 2. The notion of an ellipsoidal $\mathcal{D}$-invariance and the L_∞-gain analysis are discussed in Section 3. We then provide the synthesis method of an L_∞ performance controller in Section 4. Numerical results are provided in Section 5. Finally, the concluding remarks are given in Section 6.

The notations used in this paper can be summarized as follows. The notations $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the set of n-dimensional real vectors and the set of $n \times m$-dimensional real matrices, respectively. For any symmetric matrix A, the notation $A \preceq 0$ (or $A \prec 0$) implies that A is negative semi-definite (or negative definite). The notation $\star$ is used to represent the symmetric part of the matrix. The notations

$|\cdot|$ and $\|\cdot\|_\infty$ are used to imply the Euclidean norm of a vector and the L_∞-norm of a continuous signal, respectively, i.e.,

$$|v| := \Big(\sum_{i=1}^{n} v_i^2\Big)^{1/2}, \quad \|f\|_\infty := \operatorname*{ess\,sup}_{0\le t<\infty} |f(t)|.$$

For a non-empty set S in $\mathbb{R}^n$, the notations $\mathrm{Int}(S)$, ∂S, and $\overline{S}$ are used to denote the sets of interior points of S, boundary points of S, and closure points of S, respectively. For a continuously differentiable function $f:\mathbb{R}^n\to\mathbb{R}$, ∇f means the gradient of f defined as

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n}\end{bmatrix}^T.$$

2. Problem formulation

Let us consider the continuous-time plant P and the output-feedback controller C with the time-varying delay $d(t)$ described by

$$P : \begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d(t)) + Bw(t) + B_u u(t) \\ z(t) = C_1 x(t) + D_1 w(t) \\ y(t) = C_2 x(t) + C_{2d} x(t-d(t)) + D_2 w(t) \end{cases}, \tag{2.1}$$

$$C : \begin{cases} \dot{x}_c(t) = A_c x_c(t) + A_{cd} x_c(t-d(t)) + B_c y(t) \\ u(t) = C_c x_c(t) + C_{cd} x_c(t-d(t)) + D_c y(t) \end{cases}, \tag{2.2}$$

where $x(t)\in\mathbb{R}^n$, $w(t)\in\mathbb{R}^p$, $u(t)\in\mathbb{R}^m$, $z(t)\in\mathbb{R}^q$, and $y(t)\in\mathbb{R}^r$ in P imply the state, the external disturbance, the control input, the regulated output, and the measurement output, respectively, and $x_c(t)\in\mathbb{R}^{n_c}$ is the state of the controller C.

Here, it should be noted that the delay $d(t)$ considered in (2.1) and (2.2) is assumed to be not necessarily bounded, i.e.,

$$0 < d(t) \le d_M \le \infty. \tag{2.3}$$

To put it another way, this paper considers the case when the upper bound d_M is allowed to be ∞ and $d(t)$ has a fast changing rate for a wider applicability of the associated arguments. For a generality of the considered plant, the measurement output $y(t)$ is assumed to consist of both the current state $x(t)$ and delayed state $x(t-d(t))$ and the external disturbance $w(t)$ is related with not only $\dot{x}(t)$ but also $z(t)$ and $y(t)$ in (2.1). To effectively tackle the disturbance rejection problem relevant to this generalized plant P, the delayed state value $x(t-d(t))$ is also employed in designing the output-feedback controller C as in (2.2), with which the closed-loop system Σ consisting of P and C as shown in Figure 1 is described by

$$\Sigma : \begin{cases} \dot{\xi}(t) = A_{cl}\xi(t) + A_{d,cl}\xi(t-d(t)) + B_{cl}w(t) \\ z(t) = C_{cl}\xi(t) + D_{cl}w(t) \end{cases}, \tag{2.4}$$

where $\xi(t) = \begin{bmatrix} x^T(t) & x_c^T(t)\end{bmatrix}^T \in \mathbb{R}^{n+n_c}$ and the matrices are given by

$$A_{cl} = \begin{bmatrix} A + B_u D_c C_2 & B_u C_c \\ B_c C_2 & A_c \end{bmatrix}, \quad A_{d,cl} = \begin{bmatrix} A_d + B_u D_c C_{2d} & B_u C_{cd} \\ B_c C_{2d} & A_{cd} \end{bmatrix},$$

$$B_{cl} = \begin{bmatrix} B + B_u D_c D_2 \\ B_c D_2 \end{bmatrix}, \quad C_{cl} = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad D_{cl} = D_1. \tag{2.5}$$

Here, these expressions of A_{cl} and $A_{d,cl}$ allow us to derive tractable assertions on the L_∞-gain analysis and the L_∞ performance controller synthesis, and the details will be discussed in the following sections.

Based on this representation of Σ, the L_∞-gain (or the L_∞ performance) of Σ can be defined as

$$||\Sigma||_{\infty/\infty} := \sup_{w\neq 0,\ \varphi\equiv 0} \frac{||z||_\infty}{||w||_\infty} = \sup_{||w||_\infty \leq 1,\ \varphi\equiv 0} ||z||_\infty, \tag{2.6}$$

where

$$\varphi(t) := \xi(t), \quad t \in [-d_M, 0]. \tag{2.7}$$

This implies the maximum ratio from $||w||_\infty$ to $||z||_\infty$ when the initial condition of Σ is given by zero. With this notion in mind, let us consider the following synthesis problem for an L_∞ performance controller.

Problem 1. *For a given $\rho > 0$, design a controller C such that the following assertions are established for the resulting closed-loop system Σ with an arbitrary $d(t)$ given by (2.3).*

(i) The zero solution of the differential equation in Σ is asymptotically stable when $w(t) \equiv 0$.*
(ii) The L_∞-gain of Σ is bounded by ρ, i.e., $||\Sigma||_{\infty/\infty} \leq \rho$.

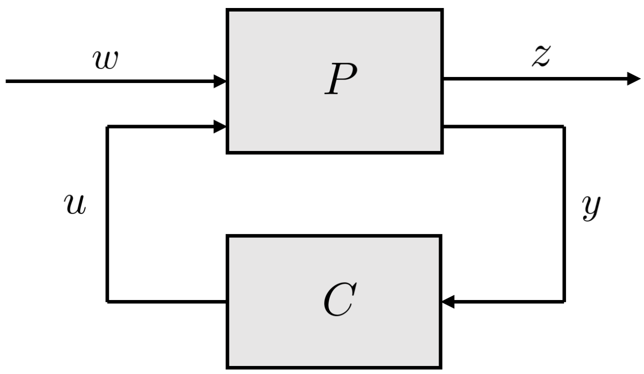

Figure 1. The feedback connection Σ between plant P and controller C.

This problem definition is for ensuring the L_∞-gain of Σ to be bounded by ρ regardless of the choice of delay $d(t)$, and the following section is devoted to developing a method of the L_∞-gain analysis for Σ as a preliminary step to obtaining a relevant controller synthesis procedure.

3. L_∞-gain analysis for Σ via ellipsoidal $\mathcal{D}$-invariance

Assuming that a controller C is fixed, this section considers the L_∞-gain of the closed-loop system Σ. To this end, we first define the continuous-time $\mathcal{D}$-invariant set for Σ as follows.

*The zero solution is said to be asymptotically stable [27] with respect to Σ if it is stable and there exists a $\delta > 0$ such that $\lim_{t\to\infty} \xi(t) = 0$ for any initial condition φ with $\sup_{-d_M \leq t \leq 0} |\varphi(t)| < \delta$.

Definition 1. *For a closed set $K \in \mathbb{R}^{n+n_c}$, it is said to be a (continuous-time) $\mathcal{D}$-invariant set of Σ if*

$$\xi(t) \in K, \quad \forall t \geq 0 \tag{3.1}$$

holds for any delay $d(t)$ given by (2.3) and any initial condition φ with $\varphi(t) \in K, \forall t \in [-d_M, 0]$.

This is a generalized concept of the conventional forward invariant set [46], in the sense that the invariance of K for (2.4) does not depend on the choice of the delay $d(t)$. This could also be regarded as a continuous-time version of the existing discrete-time $\mathcal{D}$-invariance set [43–45].

To obtain a condition ensuring that a given set becomes a $\mathcal{D}$-invariant set, let us consider a scalar-valued function $B : \mathbb{R}^{n+n_c} \to \mathbb{R}$ and its 0-sublevel set C described by

$$C := \{\xi \in \mathbb{R}^{n+n_c} \mid B(x) \leq 0\}. \tag{3.2}$$

With this C, we are led to the following lemma associated with characterizing the aforementioned condition.

Lemma 1. *Let $B : \mathbb{R}^{n+n_c} \to \mathbb{R}$ be a continuously differentiable function and C be the sublevel set defined as (3.2). Suppose that the following assertion holds.*

$$\nabla B(\xi_1)(A_{cl}\xi_1 + A_{d,cl}\xi_2 + B_{cl}w) \leq 0, \quad \forall \xi_1 \in \mathbb{R}^{n+n_c} \setminus C,\ \forall \xi_2 \in C, \forall w \text{ s.t. } |w| \leq 1. \tag{3.3}$$

Then, C is a $\mathcal{D}$-invariant set of Σ.

Proof. Assume that C is not a $\mathcal{D}$-invariant set of Σ. Then, there exist $t_0, \delta > 0$ such that

$$\xi(t_0) \in \partial C, \quad \xi(t) \in \mathbb{R}^{n+n_c} \setminus C, \quad \xi(t - d(t)) \in C, \quad \forall t \in (t_0, t_0 + \delta] \tag{3.4}$$

for a solution ξ of the differential equation in Σ with an initial condition $\varphi(t) \in C,\ \forall t \in [-d_M, 0]$. By the definition of C, it follows that

$$B(\xi(t_0)) = 0, \quad B(\xi(t)) > 0, \quad B(\xi(t - d(t))) \leq 0, \quad \forall t \in (t_0, t_0 + \delta]. \tag{3.5}$$

Then, we can obtain that

$$\begin{aligned} 0 < B(\xi(t_0 + \delta)) &= B(\xi(t_0 + \delta)) - B(\xi(t_0)) \\ &= \int_{t_0}^{t_0+\delta} \frac{d}{dt} B(\xi(t))dt \\ &= \int_{t_0}^{t_0+\delta} \nabla B(\xi(t)) \cdot \dot{\xi}(t)dt \\ &= \int_{t_0}^{t_0+\delta} \nabla B(\xi(t)) \cdot (A_{cl}\xi(t) + A_{d,cl}\xi(t - d(t)) + B_{cl}w(t))dt \\ &\leq 0, \end{aligned} \tag{3.6}$$

where the last inequality is obtained by letting $\xi_1 = \xi(t)$, $\xi_2 = \xi(t - d(t))$, and $w = w(t)$ in (3.3). This contradicts the assumption that C is not a $\mathcal{D}$-invariant set. Thus, the proof is established. □

It would be also worthwhile to note that this lemma can be regarded as an extended version of the existing arguments on the forward invariance and the barrier function [26], in which we are not concerned with delay term $d(t)$.

In connection with establishing an applicability of the $\mathcal{D}$-invariance property to the L_∞-gain analysis for Σ, we next introduce the notion of an ellipsoidal $\mathcal{D}$-invariant set. For a $P > 0$, the ellipsoidal set Ω_P described by

$$\Omega_P := \{\xi \in \mathbb{R}^{n+n_c} \mid \xi^T P\xi \leq 1\} \tag{3.7}$$

is called an *ellipsoidal $\mathcal{D}$-invariant set* if it is a $\mathcal{D}$-invariant set, and its schematic diagram is shown in Figure 2.

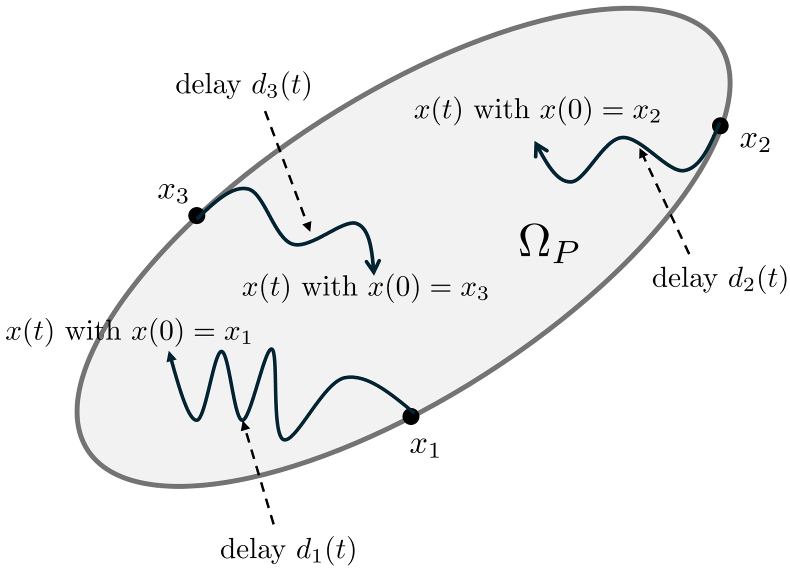

Figure 2. The ellipsoidal $\mathcal{D}$-invariant set Ω_P and the trajectories within Ω_P regardless of the choice of delays.

Before proceeding to apply Lemma 1 to this Ω_P, let us note that Ω_P is a 0-sublevel set of the quadratic function B defined as

$$B(\xi) = \xi^T P\xi - 1 \tag{3.8}$$

and the corresponding gradient is given by

$$\nabla B(\xi) = 2P\xi. \tag{3.9}$$

This fact together with Lemma 1 leads to the following proposition establishing the existence condition of an ellipsoidal $\mathcal{D}$-invariant in terms of linear matrix inequalities (LMIs).

Proposition 1. *There exists an ellipsoidal $\mathcal{D}$-invariant set if the following LMI with the decision variable $P > 0$ is feasible for given positive constants α_i $(i = 1, 2)$.*

$$\mathcal{M} = \begin{bmatrix} PA_{cl} + A_{cl}^T P + (\alpha_1 + \alpha_2)P & PA_{d,cl} & PB_{cl} \\ A_{d,cl}^T P & -\alpha_1 P & 0 \\ B_{cl}^T P & 0 & -\alpha_2 I \end{bmatrix} \leq 0. \tag{3.10}$$

Proof. It readily follows from (3.10) that

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ w \end{bmatrix}^T \mathcal{M} \begin{bmatrix} \xi_1 \\ \xi_2 \\ w \end{bmatrix} \leq 0. \tag{3.11}$$

Here, the left-hand side (LHS) of (3.11) admits the representation

$$2\xi_1^T P(A_{cl}\xi_1 + A_{d,cl}\xi_2 + B_{cl}w) + \alpha_1(\xi_1^T P\xi_1 - \xi_2^T P\xi_2) + \alpha_2(\xi_1^T P\xi_1 - w^T w) \leq 0. \tag{3.12}$$

This turns out that the inequality

$$\xi_1^T P(A_{cl}\xi_1 + A_{d,cl}\xi_2 + B_{cl}w) \leq 0 \tag{3.13}$$

is established for any ξ_i $(i = 1, 2)$ and w such that

$$\xi_1^T P\xi_1 \geq 1, \quad \xi_2^T P\xi_2 \leq 1, \quad w^T w \leq 1. \tag{3.14}$$

Because the assertions in (3.14) are equivalent to

$$\xi_1 \in \mathbb{R}^{n+n_c} \setminus \Omega_P, \quad \xi_2 \in \Omega_P, \quad w^T w \leq 1, \tag{3.15}$$

it follows from Lemma 1 that $C = \Omega_P$ is a $\mathcal{D}$-invariant set. This completes the proof. □

It is obvious that the LMI given by (3.10) is independent of the delay $d(t)$. Hence, Proposition 1 can be regarded as an extension of the existing Lyapunov-Krasovski functional (LKF)-based approach [41], in which a delay-dependent argument is established for characterizing an invariant set. The feasibility of this LMI corresponds to the asymptotic stability of Σ because the first 2×2 submatrix of $\mathcal{M}$ in (3.10) can be represented by [47]

$$\begin{bmatrix} PA_{cl} + A_{cl}^T P + \alpha_1 P & PA_{d,cl} \\ A_{d,cl}^T P & -\alpha_1 P \end{bmatrix} \prec 0 \tag{3.16}$$

and this leads to the asymptotic stability of Σ. Thus, we can see that the first assertion of Problem 1 is solved if the LMI given by (3.10) is feasible. The scalars α_i $(i = 1, 2)$ in Proposition 1 are associated with the so-called $\mathcal{S}$-procedure [47], by which a constrained optimization can be transformed to a non-constrained one.

To characterize the relationship between an ellipsoidal $\mathcal{D}$-invariant set and the L_∞-gains of Σ, we next define a set $\Xi(\rho)$ as follows.

$$\Xi(\rho) := \{\xi \in \mathbb{R}^{n+n_c} \mid |C_{cl}\xi + D_{cl}w| \leq \rho, \quad \forall w \text{ s.t. } |w| \leq 1\}. \tag{3.17}$$

This is the set of all states ξ of Σ such that the norm of regulated output is bounded by ρ for any disturbances with the unit L_∞ norm. With this set in mind, let us introduce the following lemma.

Lemma 2. *The L_∞-gain of Σ is bounded by ρ, i.e., $\|\Sigma\|_{\infty/\infty} \leq \rho$, if there exists an ellipsoidal $\mathcal{D}$-invariant set Ω_P such that*

$$(0 \in)\ \Omega_P \subseteq \Xi(\rho). \tag{3.18}$$

We omit the proof of this lemma since it is obvious and similar to the relevant arguments in the conventional studies [41, 48]. For a practical application of Lemma 2, we next give the following proposition associate with replacing the assertion of (3.18) with an LMI condition.

Proposition 2. *The relation $\Omega_P \subseteq \Xi(\rho)$ is established if the following LMI with the decision variable $P(>0)$ is feasible for a given constant $\lambda(>0)$.*

$$\mathcal{M}_o = \begin{bmatrix} -\lambda P & 0 & C_{cl}^T \\ 0 & -(\rho^2-\lambda)I & D_{cl}^T \\ C_{cl} & D_{cl} & -I \end{bmatrix} \leq 0. \tag{3.19}$$

Proof. By taking the Schur complement of $\mathcal{M}_o$ with respect to the (3,3)-component, (3.19) is equivalent to

$$\begin{bmatrix} \xi \\ w \end{bmatrix}^T \begin{bmatrix} C_{cl}^T C_{cl} - \lambda P & C_{cl}^T D_{cl} \\ D_{cl}^T C_{cl} & D_{cl}^T D_{cl} - (\rho^2-\lambda)I \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \leq 0, \quad \forall \xi \in \mathbb{R}^{n+n_c},\ \forall w \in \mathbb{R}^p. \tag{3.20}$$

This turns out that

$$(C_{cl}\xi + D_{cl}w)^T(C_{cl}\xi + D_{cl}w) - \rho^2 w^T w - \lambda(\xi^T P \xi - w^T w) \leq 0. \tag{3.21}$$

Then, it immediately follows from (3.21) that

$$\sup_{|w|\leq 1}(C_{cl}\xi + D_{cl}w)^T(C_{cl}\xi + D_{cl}w) \leq \rho^2 - \lambda(1 - \xi^T P \xi). \tag{3.22}$$

This clearly implies that

$$\sup_{|w|\leq 1}(C_{cl}\xi + D_{cl}w)^T(C_{cl}\xi + D_{cl}w) \leq \rho^2, \quad \forall \xi \in \Omega_p \tag{3.23}$$

and thus $\Omega_P \subseteq \Xi(\rho)$. This completes the proof. □

Note that λ should be taken as $0 < \lambda \leq \rho^2$ for the LMI of (3.19) to be feasible. By combining Propositions 1 and 2 with Lemma 2, we are led to the following theorem relevant to the L_∞-gain analysis for Σ through an LMI condition.

Theorem 1. *Both the assertions in Problem 1 are established if the LMIs given by* (3.10) *and* (3.19) *with the common decision variable $P(>0)$ are feasible for given positive constants α_i $(i = 1, 2)$ and λ.*

It should be stressed that all the LMIs in this theorem do not depend on the delay $d(t)$. In other words, once the LMIs given by (3.10) and (3.19) are shown to be feasible, then it can be ensured that the L_∞-gain of Σ is bounded by ρ for any choice of the delay $d(t)$. Thus, this delay-independent result can be regarded as an effective alternative to the conventional results [40–42] dependent on the properties of delays such as their magnitudes and time derivatives, especially when their detailed values cannot be obtained.

4. L_∞ performance controller synthesis via ellipsoidal $\mathcal{D}$-invariance

This section aims at establishing a controller synthesis procedure with respect to solving Problem 1. In other words, we would determine the coefficient matrices in C ensuring that the assertions in Theorem 1 are satisfied with the resulting closed-loop system Σ. Because (3.10) and (3.19) do not correspond to LMIs in terms of the control coefficients, it should be required to consider adequate variable transformations.

In connection with this, let us represent the matrix P in Theorem 1 and its inverse, respectively, by

$$P = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \overline{X} & \overline{Y} \\ \overline{Y}^T & \overline{Z} \end{bmatrix}. \tag{4.1}$$

Then, it immediately follows from the Schur complement [49] that

$$P > 0 \quad \Leftrightarrow \quad \begin{bmatrix} X & I \\ I & \overline{X} \end{bmatrix} > 0. \tag{4.2}$$

With this representation, we introduce the following new decision variables to replace the elements in $\mathcal{M}$ and $\mathcal{M}_o$ given by (3.10) and (3.19), respectively, as follows [37].

$$\begin{cases} A_k := XA\overline{X} + (YA_c + XB_uC_c)\overline{Y}^T + (YB_c + XB_uD_c)C_2\overline{X} \\ \tilde{A}_k := XA_d\overline{X} + (YA_{cd} + XB_uC_{cd})\overline{Y}^T + (YB_c + XB_uD_c)C_{2d}\overline{X} \\ B_k := YB_c + XB_uD_c \\ C_k := C_c\overline{Y}^T + D_cC_2\overline{X} \\ \tilde{C}_k := C_{cd}\overline{Y}^T + D_cC_{2d}\overline{X} \end{cases}. \tag{4.3}$$

Next, we are led to the following theorem replacing the assertions in Theorem 1 with LMI-based conditions with the decision variables X, $\overline{X}$ and those in (4.3), by which a synthesis procedure of the L_∞ performance controller discussed in Problem 1 is obtained.

Theorem 2. *Assume that $n = n_c$ and positive constants ρ, λ, α_i $(i = 1, 2)$ are given. The LMIs given by* (3.10) *and* (3.19) *are feasible with the decision variable $P > 0$ if and only if the LMIs given by*

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & \tilde{A}_k & XB + B_kD_2 \\ \star & (2,2) & (2,3) & A_d\overline{X} + B_u\tilde{C}_k & B + B_uD_cD_2 \\ \star & \star & -\alpha_1X & -\alpha_1I & 0 \\ \star & \star & \star & -\alpha_1\overline{X} & 0 \\ \star & \star & \star & \star & -\alpha_2I \end{bmatrix} \leq 0, \tag{4.4}$$

$$\begin{bmatrix} -\lambda X & -\lambda I & 0 & C_1^T \\ \star & -\lambda\overline{X} & 0 & \overline{X}C_1^T \\ \star & \star & -(\sigma^2 - \lambda)I & D^T \\ \star & \star & \star & -I \end{bmatrix} \leq 0, \tag{4.5}$$

$$\begin{bmatrix} X & I \\ I & \overline{X} \end{bmatrix} > 0 \tag{4.6}$$

are feasible with the decision variables X*,* $\overline{X}$*,* A_k*,* $\tilde{A}_k$*,* B_k*,* C_k*,* $\tilde{C}_k$*, and* D_c*, where*

$$\begin{cases} (1,1) = XA + B_kC_2 + A^TX + C_2^TB_k^T + (\alpha_1+\alpha_2)X \\ (1,2) = A_k + A^T + C_2^TD_c^TB_u^T + (\alpha_1+\alpha_2)I \\ (1,3) = XA_d + B_kC_{2d} \\ (2,2) = A\overline{X} + \overline{X}A^T + B_uC_k + C_k^TB_u^T + (\alpha_1+\alpha_2)\overline{X} \\ (2,3) = A_d + B_uD_cC_{2d} \end{cases}. \tag{4.7}$$

Furthermore, if the LMIs (4.4)–(4.6) are feasible, then an L_∞ *performance controller* C *solving Problem 1 can be determined by*

$$\begin{cases} C_c = (C_k - D_cC_2\overline{X})(\overline{Y}^T)^{-1} \\ C_{cd} = (\tilde{C}_k - D_cC_{2d}\overline{X})(\overline{Y}^T)^{-1} \\ B_c = Y^{-1}(B_k - XB_uD_c) \\ A_c = Y^{-1}\{(A_k - XA\overline{X} - B_kC_2\overline{X})(\overline{Y}^T)^{-1} - XB_uC_c\} \\ A_{cd} = Y^{-1}\{(\tilde{A}_k - XA_d\overline{X} - B_kC_{2d}\overline{X})(\overline{Y}^T)^{-1} - XB_uC_{cd}\} \end{cases}. \tag{4.8}$$

Proof. Suppose that the LMIs given by (4.4)–(4.6) are feasible. From $n = n_n$ and applying the Schur complement to (4.6), we can see that $X - \bar{X}^{-1}$ is positive definite and its spectral decomposition can be described by

$$X - \overline{X}^{-1} = YZ^{-1}Y^T > 0, \tag{4.9}$$

where Y and Z are invertible matrices. With these matrices, a positive definite matrix P as in (4.1) is obtained, and thus $\overline{Y}$ and $\overline{Z}$ involved in a positive definite matrix P^{-1} are also determined. Because $\overline{Y}$ can be also given by $\overline{Y} = -X^{-1}Y\overline{Z}$ and $\overline{Z}$ is an invertible matrix, $\overline{Y}$ is ensured to be an invertible matrix. From this $\overline{X}$ together with $\overline{Y}$ in P^{-1}, we take the (nonsingular) transformation matrix described by

$$T = \begin{bmatrix} I & \overline{X} \\ 0 & \overline{Y}^T \end{bmatrix} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}. \tag{4.10}$$

With this T, we note that the matrices dealt with in Theorem 1 can be described by

$$\mathcal{M} = T_1^T \text{ (LHS of (4.4)) } T_1, \tag{4.11}$$

$$\mathcal{M}_o = T_2^T \text{ (LHS of (4.5)) } T_2, \tag{4.12}$$

$$P = T_3^T \text{ (LHS of (4.6)) } T_3, \tag{4.13}$$

where T_i $(i = 1, 2, 3)$ are defined as

$$\begin{cases} T_1 = \text{diag}(T^{-1}, T^{-1}, I) \\ T_2 = \text{diag}(T^{-1}, I, I) \\ T_3 = T^{-1} \end{cases}. \tag{4.14}$$

This clearly implies that the LMIs given by (3.10), (3.19), and $P > 0$ are established. The converse direction also holds because the above process is reversible. Finally, the last assertion associated with determining an L_∞ performance controller holds from taking the inverse computations of (4.3). □

Algorithm 1 Algorithm for the controller synthesis.

Require: Initialized constants α_i $(i = 1, 2)$ and λ
Require: Set $T = 0$
  while $T = 0$ **do**
    solve (4.4)–(4.6) with respect to X, $\overline{X}$, A_k, $\tilde{A}_k$, B_k, C_k, $\tilde{C}_k$, and D_c
    if (4.4)–(4.6) are feasible **then**
      determine Y and Z with respect to (4.9)
      determine P as described in (4.1)
      determine $\overline{Y}$ by taking the inverse of P
      compute A_c, A_{cd}, B_c, C_c, C_{cd}, and D_c with respect to (4.8)
      set $T = 1$
    end if
    update $\alpha_i \leftarrow \alpha_i + \Delta\alpha_i$ $(i = 1, 2)$ and $\lambda \leftarrow \lambda + \Delta\lambda$
  end while

The details for employing the arguments in this theorem to obtain an L_∞ performance controller are summarized in Algorithm 1. In an equivalent fashion to the L_∞-gain analysis for Σ discussed in the preceding section, the control parameters obtained in Theorem 2 do not depend on the delay $d(t)$. Hence, this synthesis procedure can be also regarded as a delay-independent extension of the existing works [40–42], in which the relevant analysis and synthesis are confined to delay-dependent arguments. With respect to theoretical and practical characteristics of the arguments in Theorem 2, it would be also worthwhile to discuss the following remarks.

Remark 1. *The computational cost required for solving the LMI-based condition in Theorem 2 (i.e., (4.4)–(4.6)) can be regarded as similar to or smaller than those in the conventional results [40–42]. This is because the sizes of the matrices in (4.4)–(4.6) and the number of tuning parameters such as* α_i $(i = 1, 2)$ *are not larger than the counterparts in those studies.*

Remark 2. *The LMIs in Theorem 2 could be (empirically) shown to be feasible if the pairs* (A, B_u) *and* (C_2, A) *are stabilizable and detectable, respectively. Because these conditions are necessary for the nominal system without considering the effects of time delays to be stabilizable, the feasibility of (4.4)–(4.6) does not restrict the practical applicability of Theorem 2 and the relevant numerical verification will be provided in Section 5.*

Remark 3. *The assumption of* $n = n_c$ *in Theorem 2 is for ensuring the invertibility of the component matrices* Y *and* Z *in the proof of this theorem. In connection with this, it would be meaningful to obtain a reduced-order controller (i.e.,* $n_c < n$*) as in [50, 51]. This might be achieved by applying Finsler's lemma (or an elimination lemma) [50, 51] to the proof of Theorem 2, but such an application is not straightforward and the relevant issue is left for an important future study.*

5. Numerical results

This section is devoted to verifying the feasibility of the developed methods (i.e., Theorems 1 and 2) and their effectiveness compared to the existing studies [40–42] through some numerical comparisons.

Let us consider the plant P given by [42]:

$$A = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}, \quad A_d = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = B_u = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_{2d} = D_1 = D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.1}$$

In the perspective of the arguments in [30], we note that this plant P can be regarded as a system with an internal network delay $d(t)$. For this system, we would like to verify that the LMIs given by (3.10) and (3.19) discussed in Theorem 1 are feasible without control, i.e., $u = 0$. To this end, we take the relevant parameters by $\alpha_1 = \alpha_2 = 1$, $\lambda = 28$, and we solve the LMIs (3.10) and (3.19) by using MATLAB's optimization library called *mosek*. As a result, we can see that both the LMIs are feasible for $\rho \geq 5.3$, and the resulting solution matrix $P \in \mathbb{R}^{2\times 2}$ with $\rho = 5.3$ is given by

$$P = \begin{bmatrix} 1.3027 & -2.5511 \\ -2.5511 & 6.0793 \end{bmatrix}. \tag{5.2}$$

Thus, the L_∞-gain of the above P (without control) is bounded by $\rho = 5.3$ from Theorem 1.

Regarding a verification of this result, we consider the three different disturbances $w_i(t)$ and delays $d_i(t)$ $(i = 1, 2, 3)$ which are given by

$$\begin{cases} w_1(t) = \begin{bmatrix} \text{square}(t) & 0 \end{bmatrix}^T \\ w_2(t) = \begin{bmatrix} 0.9 + 0.1 \cdot \text{square}(t) & 0 \end{bmatrix}^T \\ w_3(t) = \begin{bmatrix} 0.5 + 0.5 \cdot \sin(t) & 0 \end{bmatrix}^T \end{cases}, \quad \begin{cases} d_1(t) = 2 \\ d_2(t) = 3 + \text{square}(t) \\ d_3(t) = 5 + 2 \cdot \text{square}(5t) \end{cases}, \tag{5.3}$$

where square : $\mathbb{R}_+ \to \mathbb{R}$ is defined by

$$\text{square}(t) := \begin{cases} 1 & \text{if } t \in [2k\pi, (2k+1)\pi), k = 0, 1, 2, \cdots \\ -1 & \text{if } t \in [(2k+1)\pi, (2k+2)\pi), k = 0, 1, 2, \cdots \end{cases}. \tag{5.4}$$

We then obtain the corresponding simulation results of z as shown in Figure 3, and we can observe from this figure that all the vector 2-norms of z are bounded by the 5.3. This undoubtedly demonstrates the theoretical validity of the L_∞-gain analysis obtained in Theorem 1 for the considered three different cases of w, regardless of the choice of any (possibly time-varying) delays.

Beyond the effectiveness of the proposed methods in the L_∞-gain analysis, we next consider the dynamic behavior of this system in terms of obtaining an ellipsoidal $\mathcal{D}$-invariant set. In this regard, we obtain some simulation results for three different cases of the state $x(t)$ on the ellipsoidal $\mathcal{D}$-invariant set Ω_P with P given by (5.2), as shown in Figure 4, where the initial conditions v_i, disturbances $w_i(t)$, and delays $d_i(t)$ $(i = 1, 2, 3)$ are described as follows.

$$\begin{cases} v_1 = \begin{bmatrix} -1.8814 & -0.9607 \end{bmatrix}^T \\ v_2 = \begin{bmatrix} 1.9785 & 0.7077 \end{bmatrix}^T \\ v_3 = \begin{bmatrix} -0.7515 & 0.0627 \end{bmatrix}^T \end{cases}, \quad \begin{cases} \tilde{w}_1(t) = \begin{bmatrix} -0.9 + 0.1 \cdot \text{square}(t) & 0 \end{bmatrix}^T \\ \tilde{w}_2(t) = \begin{bmatrix} 0.9 + 0.1 \cdot \text{square}(t) & 0 \end{bmatrix}^T \\ \tilde{w}_3(t) = \begin{bmatrix} \text{square}(t) & 0 \end{bmatrix}^T \end{cases}, \quad \begin{cases} \tilde{d}_1(t) = 2 \\ \tilde{d}_2(t) = 3 + \text{square}(t) \\ \tilde{d}_3(t) = 3 + 2 \cdot \text{square}(5t) \end{cases}. \tag{5.5}$$

We could observe from Figure 4 that all the cases of the state do not escape from the ellipsoidal $\mathcal{D}$-invariant set Ω_P even if all the initial conditions v_i are taken from the boundary of Ω_P, regardless of the choices for disturbances and delays. This verifies that this Ω_P becomes an ellipsoidal $\mathcal{D}$-invariant set for P, and the state should be limited inside Ω_P, regardless of the choice of delay $d(t)$.

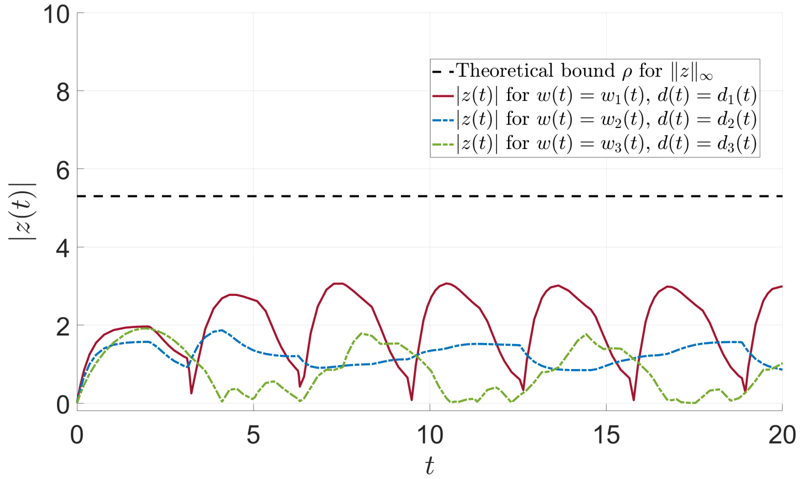

Figure 3. The vector 2-norms of z for the plant in (5.1) (without control) with three distinct pairs for disturbances and delays.

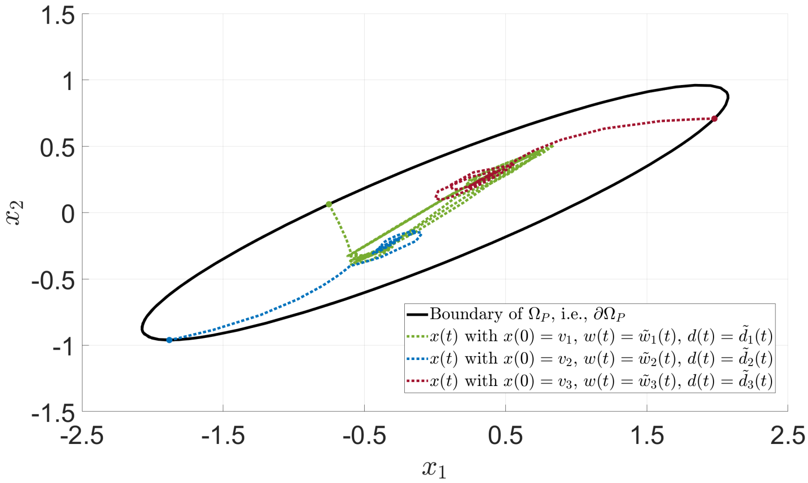

Figure 4. The obtained ellipsoidal $\mathcal{D}$-invariant set for the plant (5.1) and state trajectories with three distinct tuples of initial condition, disturbance, and delays.

Next, we would like to verify the effectiveness of the proposed controller synthesis discussed in Theorem 2. By following Algorithm 1, we can obtain the tuning parameters $\alpha_1 = 1$, $\alpha_2 = 5$ and $\lambda = 0.202$, by which the LMIs in (4.4)–(4.6) are ensured to be feasible for $\rho \geq 0.45$. To achieve the L_∞-gain of the resulting closed-loop system as small as possible, we determine the decision variables X, $\overline{X}$, A_k, $\tilde{A}_k$, B_k, C_k, $\tilde{C}_k$, and D_c with taking $\rho = 0.45$. Substituting these variables into (4.8) derives the

L_∞ performance controller C given by

$$A_c = 10^4 \times \begin{bmatrix} -0.5574 & -8.4477 \\ -0.2377 & -3.7758 \end{bmatrix}, \quad A_{cd} = \begin{bmatrix} 0.0481 & 0.1345 \\ -0.0079 & 0.3440 \end{bmatrix},$$
$$B_c = \begin{bmatrix} -16.7534 & -591.0538 \\ -777.9141 & -474.9024 \end{bmatrix}, \quad C_c = 10^3 \times \begin{bmatrix} 0.0506 & 0.7828 \\ -0.1206 & -1.8136 \end{bmatrix},$$
$$C_{cd} = \begin{bmatrix} -0.0053 & -0.0138 \\ 0.0434 & -0.0375 \end{bmatrix}, \quad D_c = \begin{bmatrix} 0.9521 & -0.7404 \\ 3.9557 & -9.5838 \end{bmatrix}. \tag{5.6}$$

Because the arguments in Theorem 2 are independent of the delay $d(t)$, the L_∞-gain of the closed-loop system obtained by connecting P and C is not larger than 0.45, regardless of the choice of the delay. To verify this fact, we obtain the simulation results for the three different cases of $w_i(t), d_i(t)$ ($i = 1, 2, 3$) described by (5.3), as shown in Figure 5. We can observe from this figure that all the vector 2-norms of z are not larger than $\rho = 0.45$, regardless of the different choices of $d(t)$. It would be also worthwhile to note that the vector 2-norms in Figure 5 associated with the L_∞ performance controller synthesis are much smaller than those in Figure 3 relevant to the plant P without any controller.

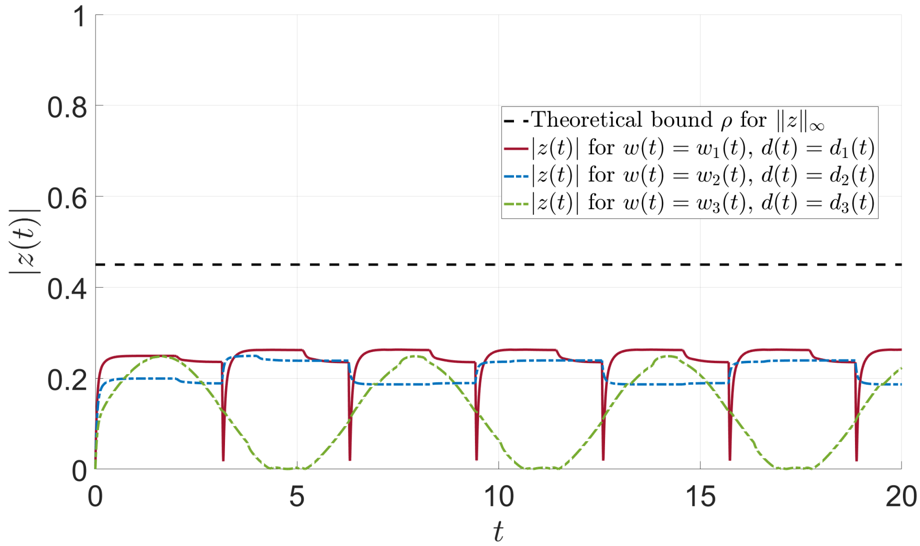

Figure 5. The vector 2-norms of z for the closed-loop system obtained by connecting (5.1) with the controller (5.6) with three distinct tuples of disturbances and delays.

On the other hand, we are in a position to evaluate the proposed methods in the presence of model uncertainties and measurement noises. To this end, we consider the model uncertainty ΔP and the measurement noise n described as follows.

$$\Delta P : \begin{cases} \Delta A := -0.1A \\ \Delta A_d := -0.1A_d \\ \Delta B := 0.1B \\ \Delta C := 0.1C \end{cases}, \quad n(t) := 0.01\sin(15t). \tag{5.7}$$

In other words, P is assumed to be changed into $P + \delta P$, by which we mean that A, A_d, B, B_u, and C_1 are replaced by $A + \Delta A$, $A_d + \Delta A_d$, $B + \Delta B$, $B_u + \Delta B$, and $C_1 + \Delta C$, respectively, and the measurement y is also affected by the noise n, i.e., y is also supposed to be changed into $y + n$. With the disturbance and delay described by (5.3) and the model uncertainty and the noise given by (5.7), we deal with the following cases.

- The case of $w_1(t)$, $d_1(t)$, and ΔP without $n(t)$.
- The case of $w_2(t)$, $d_3(t)$, and $n(t)$ without ΔP.
- The case of $w_3(t)$, $d_3(t)$, ΔP, and $n(t)$.

The corresponding simulation results are shown in Figure 6. We can observe from Figure 6 that the vector 2-norms of z are always smaller than the prescribed theoretical bound $\rho = 0.45$ in the presence of the model uncertainty ΔP and/or the measurement noise $n(t)$. This observation implies that the closed-loop system Σ obtained through the proposed L_∞ performance controller is not too sensitive with respect to such uncertain elements, and clearly demonstrates the possibility of extending the arguments in Theorems 1 and 2 to an L_∞ robust controller synthesis for uncertain time-delay systems.

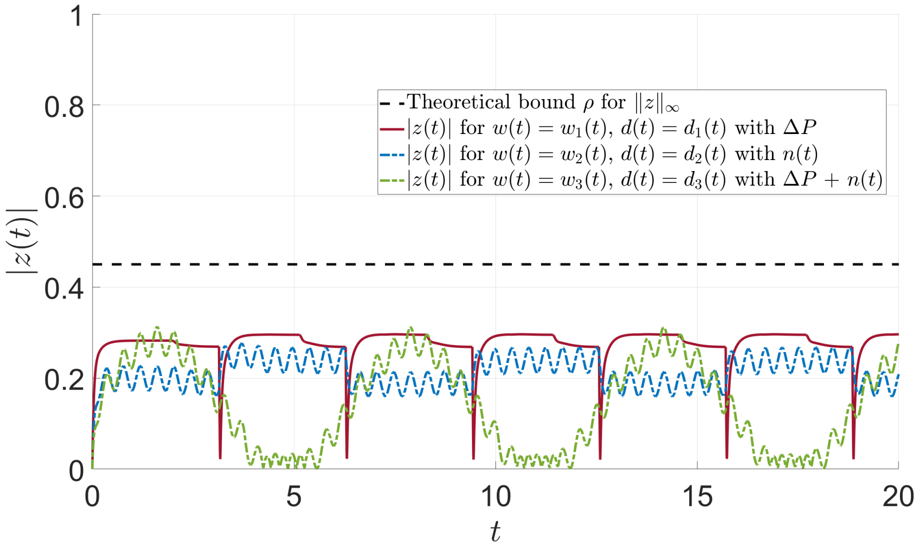

Figure 6. The vector 2-norms of z for the closed-loop system obtained by connecting (5.1) with the controller (5.6) with three distinct tuples of disturbances, delays, model uncertainties, and measurement noises.

With respect to comparing the proposed results to the conventional studies [40,41], in which a delay-dependent controller synthesis is provided for suppressing the L_∞-gain of linear systems with constant delays (i.e., $d(t) \equiv d$), we would like to note that the LMI-based synthesis procedure described in [40] is feasible for the cases of $d(t) \equiv 0.1$ and $d(t) \equiv 0.2$ for $\rho = 0.45$, but the resulting controllers are different from each other. More precisely, the controller for the case of $d(t) \equiv 0.1$ and $\rho = 0.45$ obtained by the arguments in [40, 41] can be described by

$$A_c = \begin{bmatrix} -0.1599 & 0.0299 \\ -0.0718 & -4.3722 \end{bmatrix}, \quad B_c = \begin{bmatrix} -0.0216 & -0.1242 \\ -1.0892 & 0.1895 \end{bmatrix}, \quad C_c = \begin{bmatrix} -5.0255 & 8.2399 \\ 10.0813 & -31.0444 \end{bmatrix}, \tag{5.8}$$

where the other coefficients in (2.2) (e.g., A_{cd}) are taken by the zero, and that for the case of $d(t) \equiv 0.2$ and $\rho = 0.45$ is given by

$$A_c = \begin{bmatrix} -0.1695 & 0.0143 \\ -0.0027 & -4.6346 \end{bmatrix}, \quad B_c = \begin{bmatrix} -0.0278 & -0.1346 \\ -1.2190 & 0.2489 \end{bmatrix}, \quad C_c = \begin{bmatrix} -4.7963 & 8.0529 \\ 9.8212 & -27.9233 \end{bmatrix}. \tag{5.9}$$

To put it another way, the synthesis procedure in [40, 41] does depend on the delay $d(t)$ and thus the associated controller should be re-designed when we switch the characteristics of the delay. In connection with this, it should be noted that the LMI-based synthesis procedure in [40, 41] becomes infeasible when we take $d(t) \equiv d \geq 0.5$. To simply put it, this is because all the eigenvalues of dA_d should be located in the unit disk to ensure the feasibility of the LMIs in [40, 41] (see Theorem 1 in [40]). Hence, we can see that the results in [40, 41] are only applicable to constant delays with sufficiently small sizes.

To summarize, the aforementioned observations clearly validate the fact that the proposed L_∞ performance controller can be an effective alternative to the existing controllers introduced in [40, 41] for the following two practical aspects. First of all, it is not required to modify/change a controller depending on the characteristics of time delays once it is obtained through the results in Theorem 2, while a trial-and-error process should be taken for different delays when we employ the results in [40, 41]. Furthermore, the scope of delays taken in the proposed controller synthesis is broader than that of the conventional methods [40, 41] confined to constant delays. These features are summarized in Table 2.

Table 2. Feasibility of controller synthesis procedures via the proposed method and the existing method [40, 41] for $\rho = 0.45$.

	$d(t) = 0.1$	$d(t) = 0.2$	$d(t) \geq 0.5$
Proposed method		feasible with (5.6)	
[40, 41]	feasible with (5.8)	feasible with (5.9)	**infeasible**

Finally, we compare the proposed method with the other existing method [42], in which a static controller ensuring a finite L_∞-gain of linear systems with time-varying delays is introduced. Even though the allowable scope of the delays in [42] is broader than that of [40, 41], the synthesis procedure in the former study is also dependent on the delays. More precisely, the controller synthesis arguments in [42] are feasible for $d(t) \equiv 0.1$, $d(t) \equiv 0.3$, $d(t) \equiv 0.7$, and $d(t) \equiv 1.5$ with the resulting control parameters given, respectively, by

$$C_c = \begin{bmatrix} -4.5831 & -2.0647 \\ -0.8747 & -2.4454 \end{bmatrix}, \quad D_c = \begin{bmatrix} 0.0763 & 0.9995 \\ 1.8695 & -2.0883 \end{bmatrix}, \tag{5.10}$$

$$C_c = \begin{bmatrix} -4.5526 & -2.0392 \\ -0.8522 & -2.4607 \end{bmatrix}, \quad D_c = \begin{bmatrix} 0.0760 & 0.9997 \\ 1.8696 & -2.0880 \end{bmatrix}, \tag{5.11}$$

$$C_c = \begin{bmatrix} -4.4125 & -1.9194 \\ -0.7451 & -2.4948 \end{bmatrix}, \quad D_c = \begin{bmatrix} 0.0748 & 1.0009 \\ 1.8698 & -2.0865 \end{bmatrix}, \tag{5.12}$$

$$C_c = \begin{bmatrix} -3.9759 & -1.5102 \\ -0.3649 & -2.6734 \end{bmatrix}, \quad D_c = \begin{bmatrix} 0.0709 & 1.0040 \\ 1.8713 & -2.0816 \end{bmatrix}. \tag{5.13}$$

Here, it should be remarked that the arguments in [42] could be interpreted as providing a quite conservative L_∞-gain compared to our proposed method through simulation results. In fact, the resulting L_∞-gains for the cases of $d(t) \equiv 0.1$, $d(t) \equiv 0.3$, $d(t) \equiv 0.7$, and $d(t) \equiv 1.5$ with respect to the controllers of (5.10)–(5.13) are obtained, respectively, by

$$\rho = 12.5276, \quad \rho = 13.5607, \quad \rho = 17.1909, \quad \rho = 29.8269. \tag{5.14}$$

These results obviously validate the fact that the methods developed in this paper are more effective than the existing methods in [42] for reducing the L_∞-gain of time-delay systems since the L_∞-gains given in (5.14) are much larger than $\rho = 0.45$ obtained through the developed methods and $\rho = 5.3$ for the case of $u = 0$. This conservatism in [42] might have arisen from the fact that the arguments in [42] are only for ensuring a finite L_∞-gain and no performance level ρ is employed in the controller synthesis procedure. These features are summarized in Table 3.

Table 3. The ensured L_∞-gains obtained through the proposed method and the existing method [42].

	$d(t) \equiv 0.1$	$d(t) \equiv 0.3$	$d(t) \equiv 0.7$	$d(t) \geq 1.5$
Proposed method		**0.45**		
[42]	12.5276	13.5607	17.1909	≥ 29.8269

6. Conclusions

This paper was concerned with establishing an extensive controller synthesis for linear time-delay systems with time-varying delays. In other words, a delay-independent output-feedback controller, so-called an L_∞ performance controller, was obtained, by which the L_∞-gain of such systems is bounded by a predetermined performance level ρ. To this end, we first introduced a continuous-time version of the conventional ellipsoidal $\mathcal{D}$-invariance set. We then characterized the existence condition of this continuous-time set in terms of some linear matrix inequalities (LMIs). On the basis of the condition, we could derive not only an analysis method of the L_∞-gain of the aforementioned time-delay systems but also an L_∞ controller synthesis procedure through the LMI approach. Because of the delay-independent property of the continuous-time ellipsoidal $\mathcal{D}$-invariance set, both the analysis and controller synthesis methods do not depend on the choice of delays including their magnitudes and time derivatives. The effectiveness and validity of the arguments developed in this paper were also demonstrated through some numerical results.

Finally, we would like to discuss some possible future studies relevant to this paper. Motivated by the fact that the overall arguments in this paper are related to an output-feedback L_∞ performance controller, one might consider an extension of this controller to an observer-based form as in [52, 53]. This observer-based control is expected to allow us to not only reduce the corresponding L_∞ performance but also obtain an improved state estimation. As mentioned in Remark 3, it is also quite practically meaningful to modify the LMIs in Theorem 1, tailored to a reduced-order controller synthesis, as in [50, 51]. Furthermore, constructing a theoretical framework for the robust controller synthesis relevant to measurement noises and model uncertainties is left for an interesting future study, although some empirical discussions are provided in Section 5.

Author contributions

Hyung Tae Choi: Writing original draft; Jung Hoon Kim: Supervision, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that there is no conflict of interest.

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