Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(11): 30364–30383.
DOI: 10.3934/math.20241465
Received: 05 September 2024
Revised: 08 October 2024
Accepted: 14 October 2024
Published: 25 October 2024

Research article

Certain results on tangent bundle endowed with generalized Tanaka Webster connection (GTWC) on Kenmotsu manifolds

Rajesh Kumar[1], **Sameh Shenawy**[2,*], **Lalnunenga Colney**[1,3] **and Nasser Bin Turki**[4]

[1] Department of Mathematics, Pachhunga University College (Mizoram University), 796001, India
[2] Basic Science Department, Modern Academy for Engineering and Technology, Maadi, Egypt
[3] Department of Mathematics and Computer Science, Mizoram University, 796004, India
[4] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: drssshenawy@eng.modern-academy.edu.eg.

Abstract: This work studies the complete lifts of Kenmotsu manifolds associated with the generalized Tanaka-Webster connection (GTWC) in the tangent bundle. Using the GTWC, this study explores the complete lifts of various curvature tensors and geometric structures from Kenmotsu manifolds to their tangent bundles. Specifically, it examines the complete lifts of Ricci semi-symmetry, the projective curvature tensor, Φ-projectively semi-symmetric structures, the conharmonic curvature tensor, the concircular curvature tensor, and the Weyl conformal curvature tensor. Additionally, the research delves into the complete lifts of Ricci solitons on Kenmotsu manifolds with the GTWC within the tangent bundle framework, providing new insights into their geometric properties and symmetries in the lifted space. The data on the complete lifts of the Ricci soliton in Kenmotsu manifolds associated with the GTWC in the tangent bundle are also investigated. An example of the complete lifts of a 5-dimensional Kenmotsu manifold is also included.

Keywords: Kenmotsu manifolds; generalized Tanaka-Webster connection; Einstein manifolds; curvature tensors; vertical and complete lifts; Ricci soliton; tangent bundle; partial differential equation; mathematical operators
Mathematics Subject Classification: 53C05, 53C25, 53D15, 53E20, 58A30

1. Introduction

The study of tangent bundles in differential geometry has long been a focal point, posing new challenges for modern exploration. The use of complete lifts conveniently extends differentiable structures from any manifold to its tangent bundle. Yano and Ishihara [25] developed the theory of lifts, encompassing vertical, complete, and horizontal aspects, enabling the extension of various geometric structures and multiple types of connections from a manifold to its tangent bundle. Notable geometers such as Yano and Kobayashi [26], Yano and Ishihara [25], Tani [20], and Khan [7–10] have extensively delved into tangent bundle geometry. Several manifolds associated with distinct connections in the tangent bundle were extensively investigated in [12–14]. Recently, Kumar et al. [15] carried out a comprehensive investigation into the lifts of the semi-symmetric metric connection from Sasakian statistical manifolds, geometric structures combining Sasakian geometry and statistical manifolds, to the tangent bundle.

The canonical affine connection known as the Tanaka-Webster connection was established on a nondegenerate pseudo-Hermitian CR-manifold in [23, 24]. Tanno [22] initially explored the generalized Tanaka-Webster connection (GTWC) for contact metric manifolds by utilizing the canonical connection. This GTWC aligns with the Tanaka-Webster connection when the associated CR-structure is integrable. Several geometers have investigated certain characteristics of real hypersurfaces in complex space forms using the GTWC [21]. Kenmotsu manifolds, introduced by Kenmotsu in 1971 [6], have recently been the focus of numerous studies on the GTWC connection by various authors [3, 17, 18]. Ricci solitons, introduced by Hamilton [4], represent natural extensions of Einstein metrics and are defined on a Riemannian manifold (M, g).

A Ricci soliton denoted as (g, V_0, γ) is defined on a Riemannian manifold (M, g) by the equation

$$(\ddot{L}_{V_0} g)(X_0, Y_0) + 2\ddot{S}(X_0, Y_0) + 2\gamma g(X_0, Y_0) = 0, \tag{1.1}$$

where $\ddot{L}_{V_0}$ represents the Lie derivative, which measures how a tensor field changes along the direction of a vector field, of g along a vector field V_0, capturing how g changes along the flow generated by V_0. Here, γ is a constant, and X_0, Y_0 are arbitrary vector fields on M. The classification of a Ricci soliton as shrinking, steady, or expanding depends on whether γ can take on negative, zero, or positive values, respectively. Extensive research on Ricci solitons has been conducted in the context of contact geometry, as discussed in [5, 16, 19] and related references.

This paper is organized as follows: Section 1 is devoted to the introduction, and Section 2 concerns the foundational concepts or background information. In Section 3, we investigate the complete lifts of the curvature properties of Kenmotsu manifolds associated with GTWC in the tangent bundle, and several curvature properties and theorems are proved. Subsequently, we investigate the complete lifts of the Ricci soliton of Kenmotsu manifolds associated with the GTWC in the tangent bundle in Section 4. Lastly, in Section 5, we provide an example of the complete lifts of a 5-dimensional Kenmotsu manifold in the tangent bundle followed by a conclusion section of our proposed paper in Section 6.

2. Preliminaries

Consider M as a manifold of dimension $(2n + 1)$ equipped with an almost contact metric structure (Φ, ξ, η, g). This structure consists of a $(1, 1)$ tensor field Φ representing a specific linear transformation on the tangent spaces, a vector field ξ, a 1-form η, and Riemannian metric g on M satisfying [13]

$$\eta(\xi) = 1, \quad \Phi\xi = 0, \quad \eta\Big(\Phi(X_0)\Big) = 0, \quad g(X_0, \xi) = \eta(X_0), \tag{2.1}$$

$$\Phi^2(X_0) = -X_0 + \eta(X_0)\xi, \quad g(X_0, \Phi Y_0) = -g(\Phi X_0, Y_0), \tag{2.2}$$

$$g(\Phi X_0, \Phi Y_0) = g(X_0, Y_0) - \eta(X_0)\eta(Y_0). \tag{2.3}$$

A manifold with an almost contact metric structure (Φ, ξ, η, g) is classified as a Kenmotsu manifold if and only if it satisfies the following condition: [1]

$$(\ddot{\nabla}_{X_0}\Phi)Y_0 = g(\Phi X_0, Y_0)\xi - \eta(Y_0)\Phi X_0. \tag{2.4}$$

Using the above relations, we have some properties as given below [13]:

$$\ddot{\nabla}_{X_0}\xi = X_0 - \eta(X_0)\xi, \tag{2.5}$$
$$(\ddot{\nabla}_{X_0}\eta)Y_0 = g(X_0, Y_0) - \eta(X_0)\eta(Y_0) = g(\Phi X_0, \Phi Y_0), \tag{2.6}$$
$$\ddot{R}(X_0, Y_0)\xi = \eta(X_0)Y_0 - \eta(Y_0)X_0, \tag{2.7}$$
$$\ddot{R}(\xi, X_0)Y_0 = \eta(Y_0)X_0 - g(X_0, Y_0)\xi, \tag{2.8}$$
$$\ddot{R}(\xi, X_0)\xi = X_0 - \eta(X_0)\xi, \tag{2.9}$$
$$\eta(\ddot{R}(X_0, Y_0)Z_0) = g(X_0, Z_0)\eta(Y_0) - g(Y_0, Z_0)\eta(X_0), \tag{2.10}$$
$$\ddot{S}(\Phi X_0, \Phi Y_0) = \ddot{S}(X_0, Y_0) + 2n\eta(X_0)\eta(Y_0), \tag{2.11}$$
$$\ddot{S}(X_0, \xi) = -2n\eta(X_0), \tag{2.12}$$
$$\ddot{S}(X_0, Y_0) = g(\ddot{Q}, Y_0), \tag{2.13}$$

where $\ddot{R}, \ddot{S}$, and $\ddot{Q}$ refer to the curvature tensor, the Ricci tensor, and the Ricci operator, respectively. These entities are derived from the Levi-Civita connection, which is the unique connection that preserves the metric and is torsion-free.

2.1. Complete lifts of Kenmotsu manifolds to its tangent bundle

Suppose T_0M is the tangent bundle and $X_0 = X_0^i \frac{\partial}{\partial x^i}$ is a local vector field on M. Then its vertical and complete lifts in terms of partial differential equations are

$$X_0^v = X_0^i \frac{\partial}{\partial y^i}, \tag{2.14}$$
$$X_0^c = X_0^i \frac{\partial}{\partial x^i} + \frac{\partial X_0^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}. \tag{2.15}$$

Let T_0M denote the tangent bundle on the Kenmotsu manifolds M. Then, applying the complete lifts of the mathematical operators from Eqs (2.1)–(2.13), we obtain [13]

$$\eta^c(\xi^c) = 1, \;\; (\Phi\xi)^c = 0, \;\; \eta^c\Big(\big(\Phi(X_0)\big)^c\Big) = 0, \;\; g^c(X_0^c, \xi^c) = \eta^c(X_0^c), \tag{2.16}$$
$$(\Phi^2(X_0))^c = -X_0^c + \eta^c(X_0^c)\xi^v + \eta^v(X_0^c)\xi^c, \tag{2.17}$$
$$g^c\Big(X_0^c, (\Phi Y_0)^c\Big) = -g^c\Big((\Phi X_0)^c, Y_0^c\Big), \tag{2.18}$$
$$g^c\Big((\Phi X_0)^c, (\Phi Y_0)^c\Big) = g^c(X_0^c, Y_0^c) - \eta^c(X_0^c)\eta^v(Y_0^c) - \eta^v(X_0^c)\eta^c(Y_0^c), \tag{2.19}$$

$$\begin{aligned}(\ddot{\nabla}^c_{X_0^c}\Phi^c)Y_0^c = {}& g^c\Big((\Phi X_0)^c, Y_0^c\Big)\xi^v + g^c\Big((\Phi X_0)^v, Y_0^c\Big)\xi^c - \eta^c(Y_0^c)(\Phi X_0)^v \\ & - \eta^v(Y_0^c)(\Phi X_0)^c,\end{aligned} \tag{2.20}$$
$$\ddot{\nabla}^c_{X_0^c}\xi^c = X_0^c - \eta^c(X_0^c)\xi^v - \eta^v(X_0^c)\xi^c, \tag{2.21}$$

$$(\ddot{\nabla}^c_{X_0^c}\eta^c)Y_0^c = g^c(X_0^c, Y_0^c) - \eta^c(X_0^c)\eta^v(Y_0^c) - \eta^v(X_0^c)\eta^c(Y_0^c) = g^c\Big((\Phi X_0)^c, (\Phi Y_0)^c\Big), \tag{2.22}$$

$$\ddot{R}^c(X_0^c, Y_0^c)\xi^c = \eta^c(X_0^c)Y_0^v + \eta^v(X_0^c)Y_0^c - \eta^c(Y_0^c)X_0^v - \eta^v(Y_0^c)X_0^c, \tag{2.23}$$

$$\ddot{R}^c(\xi^c, X_0^c)Y_0^c = \eta^c(Y_0^c)X_0^v + \eta^v(Y_0^c)X_0^c - g^c(X_0^c, Y_0^c)\xi^v - g^c(X_0^v, Y_0^c)\xi^c, \tag{2.24}$$

$$\ddot{R}^c(\xi^c, X_0^c)\xi^c = X_0^c - \eta^c(X_0^c)\xi^v - \eta^v(X_0^c)\xi^c, \tag{2.25}$$

$$\eta^c(\ddot{R}^c(X_0^c, Y_0^c)Z_0^c) = g^c(X_0^c, Z_0^c)\eta^v(Y_0^c) + g^c(X_0^v, Z_0^c)\eta^c(Y_0^c) - g^c(Y_0^c, Z_0^c)\eta^v(X_0) - g^c(Y_0^v, Z_0^c)\eta^c(X_0), \tag{2.26}$$

$$\ddot{S}^c\Big((\Phi X_0)^c, (\Phi Y_0)^c\Big) = \ddot{S}^c(X_0^c, Y_0^c) + 2n\eta^c(X_0^c)\eta^v(Y_0^c) + 2n\eta^v(X_0^c)\eta^c(Y_0^c), \tag{2.27}$$

$$\ddot{S}^c(X_0^c, \xi^c) = -2n\eta^c(X_0^c), \tag{2.28}$$

$$\ddot{S}^c(X_0^c, Y_0^c) = g^c\Big((\ddot{Q}X_0)^c, Y_0^c\Big). \tag{2.29}$$

The notations η^c, η^v, g^c, g^v, Φ^c, Φ^v, $\ddot{\nabla}^c$, $\ddot{\nabla}^v$, $\ddot{R}^c$, $\ddot{R}^v$, $\ddot{S}^c$, $\ddot{S}^v$ are the complete and vertical lifts of η, g, Φ, $\ddot{\nabla}$, $\ddot{R}$, and $\ddot{S}$, respectively.

3. Complete lifts of the curvature properties of the Kenmotsu manifolds endowed with the GTWC to its tangent bundle

In a Kenmotsu manifolds $M^{(2n+1)}$, we will use the GTWC $\breve{\nabla}$ given by [3, 11]

$$\breve{\nabla}_{X_0}Y_0 = \ddot{\nabla}_{X_0}Y_0 - \eta(Y_0)\ddot{\nabla}_{X_0}\xi + (\ddot{\nabla}_{X_0}\eta)(Y_0)\xi - \eta(X_0)\Phi(Y_0), \tag{3.1}$$

where $\ddot{\nabla}$ is the Levi-Civita connection and X_0, Y_0 are vector fields on $M^{(2n+1)}$. Taking the complete lifts of the above equation by using mathematical operators, we get

$$\breve{\nabla}^c_{X_0^c}Y_0^c = \ddot{\nabla}^c_{X_0^c}Y_0^c - \eta^c(Y_0^c)(\ddot{\nabla}_{X_0}\xi)^v - \eta^v(Y_0^c)(\ddot{\nabla}_{X_0}\xi)^c + (\ddot{\nabla}_{X_0}\eta)^c Y_0^c\xi^v + (\ddot{\nabla}_{X_0}\eta)^c Y_0^v\xi^c + (\ddot{\nabla}_{X_0}\eta)^v Y_0^c\xi^c - \eta^c(X_0^c)(\Phi Y_0)^v - \eta^v(X_0^c)(\Phi Y_0)^c. \tag{3.2}$$

Employing Eqs (2.21) and (2.22) in the above equation, we get

$$\breve{\nabla}^c_{X_0^c}Y_0^c = \ddot{\nabla}^c{}_{X_0^c}Y_0^c + g^c(X_0^c, Y_0^c)\xi^v + g^c(X_0^v, Y_0^c)\xi^c - \eta^c(Y_0^c)X_0^v - \eta^v(Y_0^c)X_0^c - \eta^c(X_0^c)(\Phi Y_0)^v - \eta^v(X_0^c)(\Phi Y_0)^c. \tag{3.3}$$

By setting $Y_0 = \xi$ in the above equation and employing Eq (2.21), we get

$$\breve{\nabla}_{X_0}\xi = 0. \tag{3.4}$$

The complete lifts of the Riemannian curvature tensor in the tangent bundle are given by

$$\breve{R}^c(X_0^c, Y_0^c)Z_0^c = \breve{\nabla}^c_{X_0^c}\breve{\nabla}^c_{Y_0^c}Z_0^c + \breve{\nabla}^c_{Y_0^c}\breve{\nabla}^c_{X_0^c}Z_0^c - \breve{\nabla}_{[X_0^c, Y_0^c]}Z_0^c. \tag{3.5}$$

By employing Eq (3.3), we get

$$\breve{R}^c(X_0^c, Y_0^c)Z_0^c = \ddot{R}^c(X_0^c, Y_0^c)Z_0^c + g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c \\ - g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c. \tag{3.6}$$

By setting $Z_0 = \xi$ in the above equation and using Eq (2.23), we get

$$\breve{R}^c(X_0^c, Y_0^c)\xi^c = 0. \tag{3.7}$$

Contracting Eq (3.6), we get

$$\breve{S}^c(Y_0^c, Z_0^c) = \ddot{S}^c(Y_0^c, Z_0^c) + 2ng^c(Y_0^c, Z_0^c), \tag{3.8}$$

where $\breve{S}^c$ and $\ddot{S}^c$ are the complete lifts of the Ricci tensor associated with the GTWC and Levi-Civita connections, respectively. The complete lifts of the Ricci operator $\breve{Q}^c$ associated with the GTWC are obtained by

$$\breve{Q}^c Y_0^c = \ddot{Q}^c Y_0^c + 2nY_0^c, \tag{3.9}$$

where $\ddot{Q}^c$ is the complete lift of the Ricci operator associated with the Levi-Civita connection.

Again, contracting Eq (3.8), we get

$$\breve{r}^c = \ddot{r}^c + 2n(2n + 1). \tag{3.10}$$

where $\breve{r}^c$ and $\ddot{r}^c$ are the complete lifts of the scalar curvature associated with the GTWC and Levi-Civita connections, respectively.

Theorem 3.1. *The complete lifts of the GTWC of Kenmotsu manifolds in the tangent bundle are the only affine connection, which are metric and its complete lifts of the torsion are given by*

$$\breve{T}^c(X_0^c, Y_0^c) = \eta^c(X_0^c)Y_0^v + \eta^v(X_0^c)Y_0^c - \eta^c(Y_0^c)X_0^v - \eta^v(Y_0^c)X_0^c - \eta^c(X_0^c)(\Phi Y_0)^v \\ - \eta^v(X_0^c)(\Phi Y_0)^c + \eta^c(Y_0^c)(\Phi X_0)^v + \eta^v(Y_0^c)(\Phi X_0)^c. \tag{3.11}$$

Proof. We have

$$\left((\breve{\nabla}_{X_0}\eta)Y_0\right)^c = \left(\breve{\nabla}_{X_0}\eta(Y_0)\right)^c - \left(\eta(\breve{\nabla}_{X_0}Y_0)\right)^c. \tag{3.12}$$

Employing Eq (3.3) in the above equation, we get

$$\left((\breve{\nabla}_{X_0}\eta)Y_0\right)^c = \left(\ddot{\nabla}_{X_0}\eta(Y_0)\right)^c - g^c(X_0^c, Y_0^c) + \eta^c(Y_0^c)\eta^v(X_0^c) + \eta^v(Y_0^c)\eta^c(X_0^c). \tag{3.13}$$

Employing Eq (2.22) in the above equation, we get

$$\left((\breve{\nabla}_{X_0}\eta)Y_0\right)^c = 0. \tag{3.14}$$

Again,

$$\left((\breve{\nabla}_{X_0}g)(Y_0, Z_0)\right)^c = \left(\breve{\nabla}_{X_0}g(Y_0, Z_0)\right)^c - \left(g(\breve{\nabla}_{X_0}Y_0, Z_0)\right)^c - \left(g(Y_0, \breve{\nabla}_{X_0}Z_0)\right)^c. \tag{3.15}$$

Employing Eq (3.3) in the above equation, we get

$$\left((\breve{\nabla}_{X_0}g)(Y_0, Z_0)\right)^c = 0. \tag{3.16}$$

This shows that the GTWC is a metric connection. The expression of the complete lifts of the torsion tensor in the tangent bundle are given by

$$\breve{T}^c(X_0^c, Y_0^c) = \breve{\nabla}^c_{X_0^c} Y_0^c + \breve{\nabla}^c_{Y_0^c} X_0^c. \tag{3.17}$$

Employing Eq (3.3) in the above equation, we get

$$\begin{aligned}\breve{T}^c(X_0^c, Y_0^c) = \eta^c(X_0^c)Y_0^v + \eta^v(X_0^c)Y_0^c - \eta^c(Y_0^c)X_0^v - \eta^v(Y_0^c)X_0^c - \eta^c(X_0^c)(\Phi Y_0)^v \\ - \eta^v(X_0^c)(\Phi Y_0)^c + \eta^c(Y_0^c)(\Phi X_0)^v + \eta^v(Y_0^c)(\Phi X_0)^c.\end{aligned} \tag{3.18}$$

Any complete lifts of metric connection can be put in the expression with the help of $\breve{T}^c$ as

$$\begin{aligned}\Big(g(\breve{\nabla}_{X_0} Y_0, Z_0)\Big)^c = \Big(g(\ddot{\nabla}_{X_0} Y_0, Z_0)\Big)^c + \frac{1}{2}\Big[\Big(g(\breve{T}(X_0, Y_0), Z_0)\Big)^c \\ - \Big(g(\breve{T}(X_0, Z_0), Y_0)\Big)^c - \Big(g(\breve{T}(Y_0, Z_0), X_0)\Big)^c\Big].\end{aligned} \tag{3.19}$$

Employing Eq (3.18) in the above equation, we get

$$\begin{aligned}\Big(g(\breve{\nabla}_{X_0} Y_0, Z_0)\Big)^c = &\Big(g(\ddot{\nabla}_{X_0} Y_0, Z_0)\Big)^c + \eta^c(Z_0^c)g^c(X_0^v, Y_0^c) + \eta^v(Z_0^c)g^c(X_0^c, Y_0^c) \\ &- \eta^c(Y_0^c)g^c(X_0^v, Z_0^c) - \eta^v(Y_0^c)g^c(X_0^c, Z_0^c) - \eta^c(X_0^c)g^c((\Phi Y_0)^v, Z_0^c) \\ &- \eta^v(X_0^c)g^c((\Phi Y_0)^c, Z_0^c).\end{aligned} \tag{3.20}$$

Contracting the above equation with Z_0, we get

$$\begin{aligned}\breve{\nabla}^c_{X_0^c} Y_0^c = \ddot{\nabla}^c{}_{X_0^c} Y_0^c + g^c(X_0^c, Y_0^c)\xi^v + g^c(X_0^v, Y_0^c)\xi^c - \eta^c(Y_0^c)X_0^v \\ - \eta^v(Y_0^c)X_0^c - \eta^c(X_0^c)(\Phi Y_0)^v - \eta^v(X_0^c)(\Phi Y_0)^c.\end{aligned} \tag{3.21}$$

□

Proposition 3.1. *In the complete lifts of Kenmotsu manifolds associated with the GTWC, ξ^c, η^c, and g^c are parallel in the tangent bundle.*

Proposition 3.2. *The complete lifts of the Kenmotsu manifolds associated with GTWC in the tangent bundle are a metric connections.*

Proposition 3.3. *In the complete lifts of Kenmotsu manifolds associated with the GTWC, the complete lifts of the integral curves of the vector field ξ^c are geodesic in the tangent bundle.*

3.1. Complete lifts of Ricci semi-symmetry Kenmotsu manifolds endowed with the GTWC to its tangent bundle

The Kenmotsu manifolds associated with the GTWC are said to be Ricci semi-symmetry if [3]

$$(\breve{R}(X_0, Y_0).\breve{S})(V_0, U_0) = -\breve{S}(\breve{R}(X_0, Y_0)V_0, U_0) - \breve{S}(V_0, \breve{R}(X_0, Y_0)U_0). \tag{3.22}$$

Obtaining the complete lifts of the above equation by using mathematical operators, we get

$$\Big((\breve{R}(X_0, Y_0).\breve{S})(V_0, U_0)\Big)^c = -\Big(\breve{S}(\breve{R}(X_0, Y_0)V_0, U_0)\Big)^c - \Big(\breve{S}(V_0, \breve{R}(X_0, Y_0)U_0)\Big)^c. \tag{3.23}$$

Theorem 3.2. *The complete lifts of the Ricci semi-symmetric Kenmotsu manifolds associated with the GTWC and Levi-Civita connections are equal if and only if the manifold is an Einstein manifold, which is associated with the Levi-Civita connection in the tangent bundle.*

Proof. Employing Eq (3.6) in the above equation, we get

$$\begin{aligned}\Big((\breve{R}(X_0^c, Y_0).\breve{S})(V_0, U_0)\Big)^c = {}& \Big((\ddot{R}(X_0^c, Y_0).\ddot{S})(V_0, U_0)\Big)^c + g^c(X_0^c, V_0^c)\ddot{S}^v(Y_0^c, U_0^c) \\ & + g^c(X_0^v, V_0^c)\ddot{S}^c(Y_0^c, U_0^c) + g^c(X_0^c, U_0^c)\ddot{S}^v(V_0^c, Y_0^c) \\ & + g^c(X_0^v, U_0^c)\ddot{S}^c(V_0^c, Y_0^c) - g^c(Y_0^c, V_0^c)\ddot{S}^v(X_0^c, U_0^c) \\ & - g^c(Y_0^v, V_0^c)\ddot{S}^c(X_0^c, U_0^c) - g^c(Y_0^c, U_0^c)\ddot{S}^v(V_0^c, X_0^c) \\ & - g^c(Y_0^v, U_0^c)\ddot{S}^c(V_0^c, X_0^c).\end{aligned} \tag{3.24}$$

Suppose

$$\Big((\breve{R}(X_0^c, Y_0).\breve{S})(V_0, U_0)\Big)^c = \Big((\ddot{R}(X_0^c, Y_0).\ddot{S})(V_0, U_0)\Big)^c. \tag{3.25}$$

Then, Eq (3.24) becomes

$$\begin{aligned}& g^c(X_0^c, V_0^c)\ddot{S}^v(Y_0^c, U_0^c) + g^c(X_0^v, V_0^c)\ddot{S}^c(Y_0^c, U_0^c) + g^c(X_0^c, U_0^c)\ddot{S}^v(V_0^c, Y_0^c) \\ & + g^c(X_0^v, U_0^c)\ddot{S}^c(V_0^c, Y_0^c) - g^c(Y_0^c, V_0^c)\ddot{S}^v(X_0^c, U_0^c) - g^c(Y_0^v, V_0^c)\ddot{S}^c(X_0^c, U_0^c) \\ & - g^c(Y_0^c, U_0^c)\ddot{S}^v(V_0^c, X_0^c) - g^c(Y_0^v, U_0^c)\ddot{S}^c(V_0^c, X_0^c) \\ & = 0.\end{aligned} \tag{3.26}$$

By setting $X_0 = V_0 = e_i$, $1 \leq i \leq (2n + 1)$ in the above equation, we get

$$\ddot{S}^c(Y_0^c, U_0^c) = \frac{\ddot{r}^c}{(2n+1)} g^c(Y_0^c, U_0^c). \tag{3.27}$$

Again if

$$\ddot{S}^c(Y_0^c, U_0^c) = \frac{\ddot{r}^c}{(2n+1)} g^c(Y_0^c, U_0^c). \tag{3.28}$$

Then, from Eq (3.24), we get

$$\Big((\breve{R}(X_0^c, Y_0).\breve{S})(V_0, U_0)\Big)^c = \Big((\ddot{R}(X_0^c, Y_0).\ddot{S})(V_0, U_0)\Big)^c. \tag{3.29}$$

□

3.2. Complete lifts of projective curvature tensor of a Kenmotsu manifold endowed with GTWC to its tangent bundle

The complete lifts of the projective curvature tensor of the Kenmotsu manifolds associated with the GTWC in the tangent bundle are given by

$$\begin{aligned}\breve{P}^c(X_0^c, Y_0^c)Z_0^c = {}& \breve{R}^c(X_0^c, Y_0^c)Z_0^c - \frac{1}{2n}\Big[\breve{S}^c(Y_0^c, Z_0^c)X_0^v + \breve{S}^v(Y_0^c, Z_0^c)X_0^c \\ & - \breve{S}^c(X_0^c, Z_0^c)Y_0^v - \breve{S}^v(X_0^c, Z_0^c)Y_0^c\Big].\end{aligned} \tag{3.30}$$

Theorem 3.3. *If the complete lifts of the projective curvature tensor of a Kenmotsu manifold $M^{(2n+1)}$ associated with the GTWC to its tangent bundle vanish, then the complete lifts of the curvature tensor of Kenmotsu manifolds associated with the GTWC are also found to vanish to its tangent bundle.*

Proof. By setting $\breve{P}^c = 0$ in Eq (3.30), we get

$$\begin{aligned}\breve{R}^c(X_0^c, Y_0^c)Z_0^c = \frac{1}{2n}\Big[&\breve{S}^c(Y_0^c, Z_0^c)X_0^v + \breve{S}^v(Y_0^c, Z_0^c)X_0^c - \breve{S}^c(X_0^c, Z_0^c)Y_0^v \\ &- \breve{S}^v(X_0^c, Z_0^c)Y_0^c\Big].\end{aligned} \tag{3.31}$$

Employing Eqs (3.6) and (3.8) in the above equation, we get

$$\begin{aligned}&g^c\Big(\ddot{R}^c(X_0^c, Y_0^c)Z_0^c, U_0^c\Big) + g^c(Y_0^c, Z_0^c)g^c(X_0^v, U_0^c) + g^c(Y_0^v, Z_0^c)g^c(X_0^c, U_0^c) \\ &- g^c(X_0^c, Z_0^c)g^c(Y_0^v, U_0^c) - g^c(X_0^v, Z_0^c)g^c(Y_0^c, U_0^c) \\ &= \frac{1}{2n}\Big[\ddot{S}^c(Y_0^c, Z_0^c)g^c(X_0^v, U_0^c) + \ddot{S}^v(Y_0^c, Z_0^c)g^c(X_0^c, U_0^c) \\ &+ 2ng^c(Y_0^c, Z_0^c)g^c(X_0^v, U_0^c) + 2ng^c(Y_0^v, Z_0^c)g^c(X_0^c, U_0^c) \\ &- \ddot{S}^c(X_0^c, Z_0^c)g^c(Y_0^v, U_0^c) - \ddot{S}^v(X_0^c, Z_0^c)g^c(Y_0^c, U_0^c)\Big].\end{aligned} \tag{3.32}$$

By setting $U_0 = \xi$ in the above equation, we get

$$\begin{aligned}&\ddot{S}^v(Y_0^c, Z_0^c)\eta^c(X_0^c) + \ddot{S}^c(Y_0^c, Z_0^c)\eta^v(X_0^c) - \ddot{S}^v(X_0^c, Z_0^c)\eta^c(Y_0^c) - \ddot{S}^c(X_0^c, Z_0^c)\eta^v(Y_0^c) \\ &= 2n\Big[g^c(X_0^c, Z_0^c)\eta^v(Y_0^c) + g^c(X_0^v, Z_0^c)\eta^c(Y_0^c) - g^c(Y_0^c, Z_0^c)\eta^v(X_0^c) - g^c(Y_0^v, Z_0^c)\eta^c(X_0^c)\Big].\end{aligned} \tag{3.33}$$

Also, by setting $X_0 = \xi$ in the above equation, we get

$$\ddot{S}^c(Y_0^c, Z_0^c) = -2ng^c(Y_0^c, Z_0^c). \tag{3.34}$$

which gives

$$\ddot{r}^c = -2n(2n+1). \tag{3.35}$$

Employing Eq (3.34) in Eq (3.31), we get

$$\ddot{R}^c = 0, \tag{3.36}$$

which shows that the complete lifts of the curvature tensor of Kenmotsu manifolds associated with the GTWC vanish. □

Theorem 3.4. *In the complete lifts of the Kenmotsu manifolds $M^{(2n+1)}$ with GTWC to its tangent bundle, the following properties hold:*

(i) The complete lifts of the projective curvature tensor are skew-symmetric.

(ii) The complete lifts of the projective curvature tensor are cyclic.

Proof. For (i), we interchanged X_0 and Y_0 in Eq (3.30) as

$$\begin{aligned}\breve{P}^c(Y_0^c, X_0^c)Z_0^c = \breve{R}^c(Y_0^c, X_0^c)Z_0^c - \frac{1}{2n}\Big[\breve{S}^c(X_0^c, Z_0^c)Y_0^v + \breve{S}^v(X_0^c, Z_0^c)Y_0^c \\ - \breve{S}^c(Y_0^c, Z_0^c)X_0^v - \breve{S}^v(Y_0^c, Z_0^c)X_0^c\Big].\end{aligned} \tag{3.37}$$

On adding the above equation to Eq (3.30) and employing $\ddot{R}^c(X_0^c, Y_0^c)Z_0^c + \ddot{R}^c(Y_0^c, X_0^c)Z_0^c = 0$, we get

$$\breve{P}^c(X_0^c, Y_0^c)Z_0^c + \breve{P}^c(Y_0^c, X_0^c)Z_0^c = 0. \tag{3.38}$$

For (ii), employing Eqs (3.6) and (3.30) and the complete lifts of the first Bianchi identity $\ddot{R}^c(X_0^c, Y_0^c)Z_0^c + \ddot{R}^c(Y_0^c, Z_0^c)X_0^c + \ddot{R}^c(Z_0^c, X_0^c)Y_0^c = 0$, we get

$$\breve{P}^c(X_0^c, Y_0^c)Z_0^c + \breve{P}^c(Y_0^c, Z_0^c)X_0^c + \breve{P}^c(Z_0^c, X_0^c)Y_0^c = 0. \tag{3.39}$$

□

Definition 3.1. *The Kenmotsu manifold $M^{(2n+1)}$ associated with the GTWC is said to be ξ-projectively flat if [11]*

$$\breve{P}(X_0, Y_0)\xi = 0,$$

for any vector field X_0, Y_0 on $M^{(2n+1)}$.

Theorem 3.5. *The complete lifts of a Kenmotsu manifold $M^{(2n+1)}$ associated with GTWC to its tangent bundle are ξ-projectively flat.*

Proof. By setting $Z_0 = \xi$ in Eq (3.30) and employing Eqs (3.7) and (3.8), we get

$$\breve{P}^c(X_0^c, Y_0^c)\xi^c = 0. \tag{3.40}$$

□

3.3. *Complete lifts of Φ-projectively semi-symmetric Kenmotsu manifolds endowed with GTWC to its tangent bundle*

Definition 3.2. *A Kenmotsu manifolds $M^{(2n+1)}$ associated with the GTWC is said to be Φ-projectively semi-symmetric if [11]*

$$\breve{P}(X_0, Y_0).\Phi = 0. \tag{3.41}$$

for any vector field X_0, Y_0 on $M^{(2n+1)}$.

Taking a complete lift of Eq (3.41) by using mathematical operators,

$$\Big((\breve{P}(X_0, Y_0).\Phi)Z_0\Big)^c = 0. \tag{3.42}$$

The above equation becomes

$$\Big(\breve{P}(X_0, Y_0)\Phi Z_0\Big)^c - \Big(\Phi\breve{P}(X_0, Y_0)Z_0\Big)^c = 0. \tag{3.43}$$

Employing Eqs (3.30), (3.6), and (3.8) in the above equation, we get

$$
\begin{aligned}
&\left(\ddot{R}(X_0, Y_0)\Phi Z_0\right)^c - \left(\Phi\ddot{R}(X_0, Y_0)Z_0\right)^c - \frac{1}{2n}\Big[\ddot{S}^c(Y_0^c, (\Phi Z_0)^c)X_0^v \\
&+ \ddot{S}^v(Y_0^c, (\Phi Z_0)^c)X_0^c - \ddot{S}^c(X_0^c, (\Phi Z_0)^c)Y_0^v - \ddot{S}^v(X_0^c, (\Phi Z_0)^c)Y_0^c \\
&+ \ddot{S}^c(X_0^c, Z_0^c)(\Phi Y_0)^v + \ddot{S}^v(X_0^c, Z_0^c)(\Phi Y_0)^c - \ddot{S}^c(Y_0^c, Z_0^c)(\Phi X_0)^v \\
&- \ddot{S}^v(Y_0^c, Z_0^c)(\Phi X_0)^c\Big] \\
&= 0.
\end{aligned} \tag{3.44}
$$

Interchanging Y_0 by ξ in the above equation and employing Eqs (2.23) and (2.28), we get

$$
\begin{aligned}
&\ddot{S}^c(X_0^c, (\Phi Z_0)^c)\xi^v + \ddot{S}^v(X_0^c, (\Phi Z_0)^c)\xi^c \\
&\quad = -2n\Big[g^c(X_0^c, (\Phi Z_0)^c)\xi^v + g^c(X_0^v, (\Phi Z_0)^c)\xi^c\Big].
\end{aligned} \tag{3.45}
$$

Operating an inner product with ξ, interchanging X_0 by ΦX_0, and employing Eqs (2.19) and (2.27) in the above equation, we get

$$\ddot{S}^c(X_0^c, Z_0^c) = -2ng^c(X_0^c, Z_0^c), \textit{and} \tag{3.46}$$

$$\ddot{r}^c = -2n(2n+1). \tag{3.47}$$

Thus, by substituting Eq (3.46) in Eq (3.30), we get

$$
\begin{aligned}
\ddot{P}^c(X_0^c, Y_0^c)Z_0^c = \ddot{R}^c(X_0^c, Y_0^c)Z_0^c + \Big[&g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c \\
&- g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c\Big].
\end{aligned} \tag{3.48}
$$

Hence, we can claim the following theorem.

Theorem 3.6. *The complete lifts of a Kenmostsu manifold $M^{(2n+1)}$ associated with the GTWC are said to be Φ-projectively semi-symmetric if and only if $\ddot{S}^c(X_0^c, Z_0^c) = -2ng^c(X_0^c, Z_0^c)$ in the tangent bundle. Further, if $\breve{P}^c = 0$, then the complete lifts of the manifold are said to be a hyperbolic space $\ddot{H}^{(2n+1)}(-1)$ in the tangent bundle.*

By setting $\left[(\breve{P}(X_0, Y_0).\breve{S})(Z_0, U_0)\right]^c = 0$ on a Kenmotsu manifold $M^{(2n+1)}$, we get

$$\breve{S}^c(\breve{P}(X_0, Y_0)Z_0, U_0)^c + \breve{S}^c(U_0, \breve{P}(X_0, Y_0)U_0)^c = 0. \tag{3.49}$$

By setting $X_0 = \xi$ in the above equation, we get

$$\breve{S}^c(\breve{P}(\xi, Y_0)Z_0, U_0)^c + \breve{S}^c(U_0, \breve{P}(\xi, Y_0)U_0)^c = 0. \tag{3.50}$$

Employing Eq (3.30) in the above equation, we get

$$\breve{S}^c(Y_0^c, Z_0^c)\eta^v(U_0^c) + \breve{S}^v(Y_0^c, Z_0^c)\eta^c(U_0^c) + \breve{S}^c(Y_0^c, U_0^c)\eta^v(Z_0^c) + \breve{S}^v(Y_0^c, U_0^c)\eta^c(Z_0^c) = 0. \tag{3.51}$$

Again employing Eq (3.8) in the above equation, we get

$$
\begin{aligned}
&\ddot{S}^c(Y_0^c, Z_0^c)\eta^v(U_0^c) + \ddot{S}^v(Y_0^c, Z_0^c)\eta^c(U_0^c) + \ddot{S}^c(Y_0^c, U_0^c)\eta^v(Z_0^c) \\
&+ \ddot{S}^v(Y_0^c, U_0^c)\eta^c(Z_0^c) + 2n\Big[g^c(Y_0^c, Z_0^c)\eta^v(U_0^c) + g^c(Y_0^v, Z_0^c)\eta^c(U_0^c) \\
&+ g^c(Y_0^c, U_0^c)\eta^v(Z_0^c) + g^c(Y_0^v, U_0^c)\eta^c(Z_0^c)\Big] \\
&= 0.
\end{aligned} \tag{3.52}
$$

By setting $U_0 = \xi$ in the above equation and contracting it with Y_0 and Z_0, we get

$$\ddot{S}^c(Y_0^c, Z_0^c) = -2ng^c(Y_0^c, Z_0^c), \tag{3.53}$$

and

$$\ddot{r}^c = -2n(2n+1). \tag{3.54}$$

Thus, by substituting Eq (3.53) in Eq (3.30), we get

$$\begin{aligned} \breve{P}^c(X_0^c, Y_0^c)Z_0^c = \ddot{R}^c(X_0^c, Y_0^c)Z_0^c + \Big[& g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c \\ & - g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c \Big]. \end{aligned} \tag{3.55}$$

Hence, we can claim the following theorem.

Theorem 3.7. *The complete lifts of a Kenmostsu manifold $M^{(2n+1)}$ associated with the GTWC satisfy $\breve{P}^c.\breve{S}^c = 0$ if and only if $\ddot{S}^c(Y_0^c, Z_0^c) = -2ng^c(Y_0^c, Z_0^c)$ in the tangent bundle. Further, if $\breve{P}^c = 0$, then the complete lifts of the manifold are said to be a hyperbolic space $\ddot{H}^{(2n+1)}(-1)$ in the tangent bundle.*

3.4. Complete lifts of conharmonic curvature tensor of a Kenmotsu manifold endowed with GTWC to its tangent bundle

The complete lifts of the conharmonic curvature tensor associated with the GTWC in the tangent bundle are given as

$$\begin{aligned} \breve{K}^c(X_0^c, Y_0^c)Z_0^c = \breve{R}^c(X_0^c, Y_0^c)Z_0^c - \frac{1}{2n-1}\Big[& \breve{S}^c(Y_0^c, Z_0^c)X_0^v + \breve{S}^v(Y_0^c, Z_0^c)X_0^c \\ & - \breve{S}^c(X_0^c, Z_0^c)Y_0^v - \breve{S}^v(X_0^c, Z_0^c)Y_0^c + g^c(Y_0^c, Z_0^c)(\breve{Q}X_0)^v \\ & + g^c(Y_0^v, Z_0^c)(\breve{Q}X_0)^c - g^c(X_0^c, Z_0^c)(\breve{Q}Y_0)^v - g^c(X_0^v, Z_0^c)(\breve{Q}Y_0)^c \Big]. \end{aligned} \tag{3.56}$$

Theorem 3.8. *If the complete lifts of the conharmonic curvature tensor of a Kenmotsu manifolds $M^{(2n+1)}$ with GTWC to its tangent bundle vanishes, then the complete lifts of the curvature tensor of a Kenmotsu manifold with the GTWC are also found to vanish to its tangent bundle.*

Proof. By setting $\breve{K}(X_0, Y_0)Z_0 = 0$, Eq (3.56) becomes

$$\begin{aligned} \breve{R}^c(X_0^c, Y_0^c)Z_0^c = \frac{1}{2n-1}\Big[& \breve{S}^c(Y_0^c, Z_0^c)X_0^v + \breve{S}^v(Y_0^c, Z_0^c)X_0^c - \breve{S}^c(X_0^c, Z_0^c)Y_0^v \\ & - \breve{S}^v(X_0^c, Z_0^c)Y_0^c + g^c(Y_0^c, Z_0^c)(\breve{Q}X_0)^v + g^c(Y_0^v, Z_0^c)(\breve{Q}X_0)^c \\ & - g^c(X_0^c, Z_0^c)(\breve{Q}Y_0)^v - g^c(X_0^v, Z_0^c)(\breve{Q}Y_0)^c \Big]. \end{aligned} \tag{3.57}$$

Employing Eqs (3.6), (3.8), and (3.9) in the above equation, we get

$$\begin{aligned}
&\Big(g(\ddot{R}(X_0, Y_0)Z_0, U_0)\Big)^c + g^c(Y_0^c, Z_0^c)g^c(X_0^v, U_0^c) + g^c(Y_0^v, Z_0^c)g^c(X_0^c, U_0^c) \\
&- g^c(X_0^c, Z_0^c)g^c(Y_0^v, U_0^c) - g^c(X_0^v, Z_0^c)g^c(Y_0^c, U_0^c) \\
&= \frac{1}{2n-1}\Big[\ddot{S}^c(Y_0^c, Z_0^c)g^c(X_0^v, U_0^c)\ddot{S}^c(Y_0^c, Z_0^c)g^c(X_0^v, U_0^c) + \ddot{S}^v(Y_0^c, Z_0^c)g^c(X_0^c, U_0^c) \\
&+ 4ng^c(Y_0^c, Z_0^c)g^c(X_0^v, U_0^c) + 4ng^c(Y_0^v, Z_0^c)g^c(X_0^c, U_0^c) - \ddot{S}^c(X_0^c, Z_0^c)g^c(Y_0^v, U_0^c) \\
&- \ddot{S}^v(X_0^c, Z_0^c)g^c(Y_0^c, U_0^c) - 4ng^c(X_0^c, Z_0^c)g^c(Y_0^v, U_0^c) - 4ng^c(X_0^v, Z_0^c)g^c(Y_0^c, U_0^c) \\
&+ \ddot{S}^c(X_0^c, U_0^c)g^c(Y_0^v, Z_0^c) + \ddot{S}^v(X_0^c, U_0^c)g^c(Y_0^c, Z_0^c) - \ddot{S}^c(Y_0^c, U_0^c)g^c(X_0^v, Z_0^c) \\
&- \ddot{S}^v(Y_0^c, U_0^c)g^c(X_0^c, Z_0^c)\Big].
\end{aligned} \tag{3.58}$$

By setting $U_0 = \xi$ in the above equation, we get

$$\begin{aligned}
&\ddot{S}^c(Y_0^c, Z_0^c)\eta^v(X_0^c) + \ddot{S}^v(Y_0^c, Z_0^c)\eta^c(X_0^c) - \ddot{S}^c(X_0^c, Z_0^c)\eta^v(Y_0^c) \\
&- \ddot{S}^v(X_0^c, Z_0^c)\eta^c(Y_0^c) - 2n\Big[g^c(X_0^c, Z_0^c)\eta^v(Y_0^c) + g^c(X_0^v, Z_0^c)\eta^c(Y_0^c) \\
&- g^c(Y_0^c, Z_0^c)\eta^v(X_0^c) - g^c(Y_0^v, Z_0^c)\eta^c(X_0^c)\Big] \\
&= 0.
\end{aligned} \tag{3.59}$$

By setting $X_0 = \xi$ in the above equation, we get

$$\ddot{S}^c(Y_0^c, Z_0^c) = -2ng^c(Y_0^c, Z_0^c). \tag{3.60}$$

Contracting the above equation gives

$$\ddot{r}^c = -2n(2n+1). \tag{3.61}$$

Employing Eq (3.60) in (3.57), we get

$$\breve{R}^c(X_0^c, Y_0^c)Z_0^c = 0. \tag{3.62}$$

□

3.5. Complete lifts of concircular curvature tensor of a Kenmotsu manifold endowed with GTWC to its tangent bundle

The complete lifts of the concircular curvature tensor of the Riemannian manifold in the tangent bundle are

$$\begin{aligned}
\ddot{K}_0{}^c(X_0^c, Y_0^C)Z_0^c = \ddot{R}^c(X_0^c, Y_0^c)Z_0^c - \frac{\ddot{r}^c}{2n(2n+1)}\Big(&g^c(Y_0^c, Z_0^c)X_0^v \\
&+ g^c(Y_0^v, Z_0^c)X_0^c - g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c\Big),
\end{aligned} \tag{3.63}$$

where X_0, Y_0, Z_0 are vector fields on the Riemannian manifold, $\ddot{R}$ is the Riemmanian curvature tensor, and $\ddot{r}$ is the scalar curvature associated with the Levi-Civita connection.

Similarly, the complete lifts of the concircular curvature tensor of Kenmotsu manifolds in the tangent bundle are given as

$$\breve{K_0}^c(X_0^c, Y_0^C)Z_0^c = \breve{R}^c(X_0^c, Y_0^c)Z_0^c - \frac{\breve{r}^c}{2n(2n+1)}\Big(g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c - g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c\Big), \tag{3.64}$$

where X_0, Y_0, Z_0 are vector fields on Kenmotsu manifolds, $\breve{R}$ is the Riemmanian curvature tensor, and $\breve{r}$ is the scalar curvature associated with the GTWC.

Theorem 3.9. *The complete lifts of concircular curvature tensors of Kenmotsu manifolds associated with the GTWC and Levi-Civita connections are equal in the tangent bundle.*

Proof. Employing Eqs (3.6) and (3.10) in the above equation, we get

$$\breve{K_0}^c(X_0^c, Y_0^c)Z_0^c = \ddot{K_0}^c(X_0^c, Y_0^c)Z_0^c. \tag{3.65}$$

□

Theorem 3.10. *If the complete lifts of the Kenmotsu manifolds endowed with GTWC to its tangent bundle satisfy the condition $\breve{K_0}^c(X_0^c, Y_0^c).\breve{S}^c = 0$, then the manifold is classified as an Einstein manifold with respect to the Levi-Civita connection in the tangent bundle.*

Proof. By setting $\breve{K_0}^c(X_0^c, Y_0^c).\breve{S}^c = 0$, we have

$$\Big(\breve{S}(\breve{K}_0(X_0, Y_0)V_0, U_0)\Big)^c + \Big(\breve{S}(V_0, \breve{K}_0(X_0, Y_0)U_0)\Big)^c = 0, \tag{3.66}$$

for all $X_0, Y_0, V_0, U_0 \in \chi(M^{(2n+1)})$. Replacing X_0 with ξ in the above equation, we get

$$\Big(\breve{S}(\breve{K}_0(\xi, Y_0)V_0, U_0)\Big)^c + \Big(\breve{S}(V_0, \breve{K}_0(\xi, Y_0)U_0)\Big)^c = 0. \tag{3.67}$$

Employing Eqs (3.6), (3.10), and (3.64) in the above equation, we get

$$\eta^c(V_0^c)\breve{S}^v(Y_0^c, U_0^c) + \eta^v(V_0^c)\breve{S}^c(Y_0^c, U_0^c) - \eta^c(U_0^c)\breve{S}^v(V_0^c, Y_0^c) - \eta^v(U_0^c)\breve{S}^c(V_0^c, Y_0^c) = 0. \tag{3.68}$$

By setting $V_0 = \xi$ in the above equation, we get $\breve{S}^c(Y_0^c, U_0^c) = 0$. Then, from Eq (3.8) it follows that

$$\ddot{S}^c(Y_0^c, U_0^c) = -2ng^c(Y_0^c, U_0^c). \tag{3.69}$$

□

3.6. Complete lifts of Weyl conformal curvature tensor of a Kenmotsu manifold endowed with GTWC to its tangent bundle

In a Riemannian manifold, the complete lifts of the Weyl conformal curvature tensor associated with the GTWC in the tangent bundle is

$$\begin{aligned}\breve{C}^c(X_0^c, Y_0^c)Z_0^c = {} & \breve{R}^c(X_0^c, Y_0^c)Z_0^c - \frac{1}{2n-1}\Big[\breve{S}^c(Y_0^c, Z_0^c)X_0^v + \breve{S}^v(Y_0^c, Z_0^c)X_0^c \\ & + g^c(Y_0^c, Z_0^c)(\breve{Q}X_0)^v + g^c(Y_0^v, Z_0^c)(\breve{Q}X_0)^c - g^c(X_0^c, Z_0^c)(\breve{Q}Y_0)^v \\ & - g^c(X_0^v, Z_0^c)(\breve{Q}Y_0)^c\Big] + \frac{\breve{r}^c}{2n(2n-1)}\Big[g^c(Y_0^c, Z_0^c)X_0^v \\ & + g^c(Y_0^v, Z_0^c)X_0^c - g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c\Big].\end{aligned} \tag{3.70}$$

Theorem 3.11. *In a Kenmotsu manifolds $M^{(2n+1)}$ with the GTWC, the complete lifts of the Weyl conformal curvature tensor with the Levi-Civita connection and the complete lifts of the Weyl conformal curvature tensor with the GTWC are equivalent in the tangent bundle.*

Proof. Employing Eqs (3.6), (3.8)–(3.10) in Eq (3.70), we get

$$\breve{C}^c(X_0^c, Y_0^c)Z_0^c = \ddot{C}^c(X_0^c, Y_0^c)Z_0^c. \tag{3.71}$$

for all X_0, Y_0, Z_0 on $M^{(2n+1)}$. □

Definition 3.3. *A Kenmotsu manifold associated with the GTWC is called recurrent if its curvature tensor $\breve{R}$ satisfies the condition [11]*

$$(\breve{\nabla}_{U_0}\breve{R})(X_0, Y_0)Z_0 = D(U_0)\breve{R}(X_0, Y_0)Z_0, \tag{3.72}$$

where $\breve{R}$ is the curvature tensor associated with the GTWC and D is a 1-form associated with vector field ρ such that $D(X_0) = g(X_0, \rho)$.

Taking the complete lifts of Eq (3.72) by mathematical operators, we get

$$\Big((\breve{\nabla}_{U_0}\breve{R})(X_0, Y_0)Z_0\Big)^c = \Big(D(U_0)\breve{R}(X_0, Y_0)Z_0\Big)^c, \tag{3.73}$$

We can write the above equation as

$$\begin{aligned}&\Big(\breve{\nabla}_{U_0}\breve{R}(X_0, Y_0)Z_0\Big)^c - \Big(\breve{R}(\breve{\nabla}_{U_0}X_0, Y_0)Z_0\Big)^c - \Big(\breve{R}(X_0, \breve{\nabla}_{U_0}Y_0)Z_0\Big)^c - \Big(\breve{R}(X_0, Y_0)\breve{\nabla}_{U_0}Z_0\Big)^c\\ &\quad= \Big(D(U_0)\breve{R}(X_0, Y_0)Z_0\Big)^c.\end{aligned} \tag{3.74}$$

Employing Eqs (3.3), (3.6), and (3.8) in the above equation, we get

$$\begin{aligned}&g^c(U_0^c, \ddot{R}^c(X_0^c, Y_0^c)Z_0^c)\xi^v + g^c(U_0^v, \ddot{R}^c(X_0^c, Y_0^c)Z_0^c)\xi^c - \Big(g(U_0^c, X_0^c)\ddot{R}(\xi, Y_0)Z_0\Big)^c\\ &- \Big(g(U_0, Y_0)\ddot{R}(X_0, \xi)Z_0\Big)^c - \Big(g(U_0, Z_0)\ddot{R}(X_0, Y_0)\xi\Big)^c - \Big(\eta(\ddot{R}(X_0, Y_0)Z_0)U_0\Big)^c\\ &+ \Big(\eta(X_0)\ddot{R}(U_0, Y_0)Z_0\Big)^c + \Big(\eta(Y_0)\ddot{R}(X_0, U_0)Z_0\Big)^c + \Big(\eta(Z_0)\ddot{R}(X_0, Y_0)U_0\Big)^c\\ &- \Big[\eta(U_0)\Big(\Phi\ddot{R}(X_0, Y_0)Z_0 - \ddot{R}(\Phi X_0, Y_0)Z_0 - \ddot{R}(X_0, \Phi Y_0)Z_0 - \ddot{R}(X_0, Y_0)\Phi Z_0\Big)\Big]^c\\ &= D^c(U_0^c)\Big(g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c - g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c\Big).\end{aligned} \tag{3.75}$$

Interchanging Z_0 by ξ in the above equation and employing Eqs (2.16), (2.17), (2.23), (2.24), and (2.26) in the above equation, we get

$$\begin{aligned}&D_0^c\Big(\eta^c(Y_0^c)X_0^v + \eta^v(Y_0^c)X_0^c - \eta^c(X_0^c)Y_0^v - \eta^v(X_0^c)Y_0^c\Big)\\ &= g^c(U_0^c, Y_0^c)X_0^v + g^c(U_0^v, Y_0^c)X_0^c - g^c(U_0^c, X_0^c)Y_0^v\\ &- g^c(U_0^v, X_0^c)Y_0^c + \ddot{R}^c(X_0^c, Y_0^c)U_0^c.\end{aligned} \tag{3.76}$$

Obtaining an inner product with V_0 in the above equation, we get

$$\begin{aligned}&D_0^c\Big(\eta^c(Y_0^c)g^c(X_0^v, V_0^c) + \eta^v(Y_0^c)g^c(X_0^c, V_0^c) - \eta^c(X_0^c)g^c(Y_0^v, V_0^c) - \eta^v(X_0^c)g^c(Y_0^c, V_0^c)\Big)\\ &= g^c(U_0^c, Y_0^c)g^c(X_0^v, V_0^c) + g^c(U_0^v, Y_0^c)g^c(X_0^c, V_0^c) - g^c(U_0^c, X_0^c)g^c(Y_0^v, V_0^c)\\ &- g^c(U_0^v, X_0^c)g^c(Y_0^c, V_0^c) + \ddot{R}^c(X_0^c, Y_0^c, U_0^c, V_0^c).\end{aligned} \tag{3.77}$$

Suppose $\{e_1, e_2, ...e_{2n+1}\}$ is a local orthonormal basis of vector fields in M. Then, by setting $X_0 = V_0 = e_i$ in Eq (3.77) and summing up with respect to i, $1 \leq i \leq 2n + 1$, we get

$$\ddot{S}^c(Y_0^c, U_0^c) = -(n-1)\Big[g^c(Y_0^c, U_0^c) - \eta^c(Y_0^c)D^v(U_0^c) - \eta^v(Y_0^c)D^c(U_0^c)\Big]. \tag{3.78}$$

We suppose that the vector field ρ is in co-direction with vector field ξ. Then their associated 1-forms will be equal, i.e. $D = \eta$, and so the above equation becomes

$$\ddot{S}^c(Y_0^c, U_0^c) = -(n-1)\Big[g^c(Y_0^c, U_0^c) - \eta^c(Y_0^c)\eta^v(U_0^c) - \eta^v(Y_0^c)\eta^c(U_0^c)\Big]. \tag{3.79}$$

Hence, we can claim the following theorem.

Theorem 3.12. *If the complete lifts of the Kenmotsu manifolds associated with the GTWC are recurrent and the associated 1-form D is equal to the associated 1-form η, then the manifold is an η-Einstein manifold in the tangent bundle.*

4. Complete lifts of the Ricci soliton on Kenmotsu manifolds endowed with the GTWC in the tangent bundle

The Ricci soliton of Kenmotsu manifolds associated with the GTWC is given by [11]

$$(\breve{L}_{V_0}g)(X_0, Y_0) + 2\breve{S}(X_0, Y_0) + 2\gamma g(X_0, Y_0) = 0. \tag{4.1}$$

Obtaining the complete lift of Eq (4.1) by using mathematical operators, we get

$$(\breve{L}^c_{V_0^c}g^c)(X_0^c, Y_0^c) + 2\breve{S}^c(X_0^c, Y_0^c) + 2\gamma g^c(X_0^c, Y_0^c) = 0. \tag{4.2}$$

If the complete lift of the potential vector field V_0^c is the structure of the vector field ξ^c in the tangent bundle, and since the complete lift of ξ^c is a parallel vector field with respect to the GTWC (from (3.4)) in the tangent bundle causing the first term of Eq (4.2) to vanish, then the manifold reduces to an Einstein manifold. If the complete lift of V_0^c is pointwise collinear with the structure vector field ξ^c in the tangent bundle such that $V_0^c = a\xi^c$, where a is a function on $M^{(2n+1)}$, then Eq (4.2) implies that

$$\begin{aligned} &ag^c(\breve{\nabla}^c_{X_0^c}\xi^c, Y_0^c) + a(X_0^c\eta^v(Y_0^c) + X_0^v\eta^c(Y_0^c)) + ag^c(X_0^c, \breve{\nabla}^c_{Y_0^c}\xi^c) \\ &+ a(Y_0^c\eta^v(X_0^c) + Y_0^v\eta^c(X_0^c)) + 2\breve{S}^c(X_0^c, Y_0^c) + 2\gamma g^c(X_0^c, Y_0^c) \\ &= 0. \end{aligned} \tag{4.3}$$

Employing Eqs (3.4) and (3.8) in the above equation, we get

$$\begin{aligned} &a(\eta^c(Y_0^c)X_0^v + \eta^v(Y_0^c)X_0^c) + a(\eta^c(X_0^c)Y_0^v + \eta^v(X_0^c)Y_0^c) \\ &+ 2\ddot{S}^c(X_0^c, Y_0^c) + 2(2n + \gamma)g^c(X_0^c, Y_0^c) \\ &= 0. \end{aligned} \tag{4.4}$$

By setting $Y_0 = \xi$ in the above equation and employing Eq (2.28), we get

$$a(X_0^c) = -(2\gamma + a\xi)\eta^c(X_0^c). \tag{4.5}$$

Interchanging X_0 with ξ in the above equation, we get

$$a(\xi^c) = -\gamma. \tag{4.6}$$

Employing the above equation in Eq (4.5), we get

$$a(X_0^c) = -\gamma\eta^c(X_0^c). \tag{4.7}$$

By differentiating the above equation, we get

$$\gamma d\eta^c = 0. \tag{4.8}$$

Since, $d\eta \neq 0$ from the above equation, we get

$$\gamma = 0. \tag{4.9}$$

Employing the above equation in Eq (4.7), we can say that a is a constant. Thus, it is verified from Eq (4.4) that

$$\ddot{S}^c(X_0^c, Y_0^c) = -(2n+\gamma)g^c(X_0^c, Y_0^c) + \gamma\Big(\eta^v(X_0^c)\eta^c(Y_0^c)\Big). \tag{4.10}$$

Thus, we state the following theorem.

Theorem 4.1. *If the complete lifts of the vector field X_0^c on the Ricci soliton of Kenmotsu manifolds associated with the GTWC are pointwise collinear with the complete lifts of the vector field ξ^c in the tangent bundle, then the manifold is an η-Einstein manifold and the data provided by the complete lifts of the Ricci soliton is found to be steady in the tangent bundle.*

5. Example

This section is devoted to a comprehensive examination of a particular example. It will provide a detailed analysis of the manifold's structural properties and the relevant mathematical framework. Let M be a five-dimensional manifold defined as

$$M = \Big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5\Big\},$$

where $\mathbb{R}$ represents the set of real numbers. Consider $\{E_1, E_2, E_3, E_4, E_5\}$ as a linearly independent global frame on M where

$$\begin{gathered} E_1 = e^{-x_5}\frac{\partial}{\partial x_1}, \qquad E_2 = e^{-x_5}\frac{\partial}{\partial x_2}, \qquad E_3 = e^{-x_5}\frac{\partial}{\partial x_3}, \\ E_4 = e^{-x_5}\frac{\partial}{\partial x_4}, \qquad E_5 = e^{-x_5}\frac{\partial}{\partial x_5}. \end{gathered} \tag{5.1}$$

Let the 1-form η be given by

$$\eta(Y_0) = g(Y_0, \xi). \tag{5.2}$$

The Riemannian metric g is defined by

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & otherwise. \end{cases} \tag{5.3}$$

Let Φ be the $(1, 1)$ tensor field defined by

$$\Phi E_i = \begin{cases} E_3, & i = 1 \\ E_4, & i = 2 \\ -E_1, & i = 3 \\ -E_2, & i = 4 \\ 0, & i = 5. \end{cases} \tag{5.4}$$

Using the linearity of Φ and g, we acquire $\eta(E_5) = 1$, $\Phi^2 X_0 = -X_0 + \eta(X_0)E_5$ and $g(\Phi X_0, \Phi Y_0) = g(X_0, Y_0) - \eta(X_0)\eta(Y_0)$. Thus, for $E_5 = \xi$, the structure (Φ, ξ, η, g) is then an almost contact metric structure on M, and M is called an almost contact metric manifold. In addition, M satisfies

$$(\ddot{\nabla}_{X_0}\Phi)Y_0 = g(\Phi X_0, Y_0)E_5 - \eta(Y_0)\Phi X_0. \tag{5.5}$$

Here, for $E_5 = \xi$, M is a Kenmotsu manifold. Let the complete and vertical lifts of E_1, E_2, E_3, E_4, E_5 be $E_1^c, E_2^c, E_3^c, E_4^c, E_5^c$ and $E_1^v, E_2^v, E_3^v, E_4^v, E_5^v$, respectively, in the tangent bundle T_0M of manifold M, and let the complete lift of the Riemannian metric g be g^c on T_0M such that

$$g^c(X_0^v, E_5^c) = \left(g^c(X_0, E_5)\right)^v = \left(\eta(X_0)\right)^v \tag{5.6}$$

$$g^c(X_0^c, E_5^c) = \left(g^c(X_0, E_5)\right)^c = \left(\eta(X_0)\right)^c \tag{5.7}$$

$$g^c(E_5^c, E_5^c) = 1, \quad g^v(X_0^v, E_5^c) = 0, \quad g^v(E_5^v, E_5^c) = 0, \tag{5.8}$$

and so on. Let the complete and vertical lifts of the $(1, 1)$ tensor field Φ_0 be Φ_0^c and Φ_0^v, respectively, and be defined by

$$\Phi^v(E_5^v) = \Phi^c(E_5^c) = 0, \tag{5.9}$$

$$\Phi^v(E_1^v) = E_3^v, \quad \Phi^c(E_1^c) = E_3^c, \tag{5.10}$$

$$\Phi^v(E_2^v) = E_4^v, \quad \Phi^c(E_2^c) = E_4^c, \tag{5.11}$$

$$\Phi^v(E_3^v) = -E_1^v, \quad \Phi^c(E_3^c) = -E_1^c, \tag{5.12}$$

$$\Phi^v(E_4^v) = -E_2^v, \quad \Phi^c(E_4^c) = -E_2^c. \tag{5.13}$$

Using the linearity of Φ and g, we infer that

$$(\Phi^2 X_0)^c = -X_0^c + \eta^c(X_0^c)E_5^v + \eta^v(X_0^c)E_5^c, \tag{5.14}$$

$$g^c\Big((\Phi E_5)^c, (\Phi E_4)^c\Big) = g^c(E_5^c, E_4^c) - \eta^c(E_5^c)\eta^v(E_4^c) - \eta^v(E_5^c)\eta^c(E_4^c). \tag{5.15}$$

Thus, for $E_5 = \xi$ in (5.6)–(5.8) and (5.14)–(5.15), the structure $(\Phi^c, \xi^c, \eta^c, g^c)$ is an almost contact metric structure on T_0M and satisfies the relation

$$\begin{aligned}(\nabla^c_{E_5^c}\Phi^c)E_4^c = {} & g^c\Big((\Phi E_5)^c, E_4^c\Big)\xi^v + g^c\Big((\Phi E_5)^v, E_4^c\Big)\xi^c \\ & - \eta^c(E_4^c)(\Phi E_5)^v - \eta^v(E_4^c)(\Phi E_5)^c.\end{aligned} \tag{5.16}$$

Thus, $(\Phi^c, \xi^c, \eta^c, g^c, T_0M)$ is a Kenmotsu manifold.

6. Conclusions

In this study, we delve into the properties of the complete lifts of Kenmotsu manifolds within the tangent bundle, particularly those endowed with the generalized Tanaka-Webster connection (GTWC). Our examination covers a broad spectrum of curvature aspects associated with these lifts, including the Ricci semi-symmetric tensor, projective curvature tensor, conharmonic curvature tensor, concircular curvature tensor, and the Weyl projective curvature tensor. Additionally, we explore the recurrent conditions pertinent to these complete lifts and present proofs for several theorems proposed in this framework.

Moreover, we extend our investigation to the complete lifts of Ricci solitons within Kenmotsu manifolds associated with the GTWC in the tangent bundle. Our results reveal that the data derived from the Ricci solitons exhibit stability within the tangent bundle, and that the manifold is an η-Einstein manifold when the complete lifts of the vector field V_0^c are pointwise collinear with ξ^c in the tangent bundle. An example of the complete lifts of a 5-dimensional Kenmotsu manifold in the tangent bundle is provided.

Author contributions

Rajesh Kumar: Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Visualization, Writing – review & editing. Sameh Shenawy: Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Visualization, Writing – review & editing. Lalnunenga Colney: Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing. Nasser Bin Turki: Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Visualization, Funding acquisition, Project administration.

Acknowledgments

This project was supported by the Researchers Supporting Project number (RSP2024R413), King Saud University, Riyadh, Saudi Arabia.

The third author is thankful to the Government of India, Ministry of Tribal Affairs (Scholarship Division), New Delhi, India for financial support in the form of NFST award no. 202122-NFST-MIZ-00686.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. D. E. Blair, *Contact manifolds in Riemannian geometry*, Berlin, Heidelberg: Springer, 1976. https://doi.org/10.1007/BFb0079307
2. A. Ghosh, R. Sharma, J. T. Cho, Contact metric manifolds with η-parallel torsion tensor, *Ann. Glob. Anal. Geom.*, **3**4 (2008), 287–299. https://doi.org/10.1007/s10455-008-9112-1

3. G. Ghosh, U. Chand De, Kenmotsu manifolds with generalized Tanaka-Webster connection, *Publ. de I'Institut Math.*, **1**02 (2017), 221–230. https://doi.org/10.2298/PIM1716221G

4. R. S. Hamilton, The Ricci flow on surface, Mathematics and general relativity, *Contemp. Math.*, **7**1 (1988), 237–262.

5. C. He, M. Zhu, *Ricci solitons on Sasakian manifolds*, Available from: https://doi.org/10.48550/arXiv.1109.4407

6. K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tohoku Math. J.*, **2**4 (1972), 93–103. https://doi.org/10.2748/tmj/1178241594

7. M. N. I. Khan, Liftings from a para-sasakian manifold to its tangent bundles, *Filomat*, **3**7 (2023), 6727–6740. https://doi.org/10.2298/FIL2320727K

8. M. N. I. Khan, L. S. K. Das, Lifts of hypersurfaces on a Sasakian manifold with a Quarter-Symmetric Semi-Metric Connection (QSSC) to its tangent bundle, *Univ. J. Math. Appl.*, **6** (2023), 170–175.

9. M. N. I. Khan, F. Mofarreh, R. A. Khan, Liftings from Lorentzian para-Sasakian manifolds to its tangent bundle, *Results Nonlinear Anal.*, **6** (2023), 74–82. https://doi.org/10.31838/rna/2023.06.04.008

10. M. N. I. Khan, U. C. De, L. S. Velimirovic, Lifts of a quarter-symmetric metric connection from a Sasakian manifold to its tangent bundle, *Mathematics*, **1**1 (2023), 53. https://doi.org/10.3390/math11010053

11. D. L. Kumar, U. Manjulamma, S. Shashidhar, Study on Kenmotsu manifolds admitting generalized Tanaka-Webster connection, *Ital. J. Pure Appl. Mat.*, **4**7 (2022), 721–723.

12. R. Kumar, L. Colney, M. N. I. Khan, Lifts of a semi-symmetric non-metric connection (SSNMC) from statistical manifolds to the tangent bundle, *Results Nonlinear Anal.*, **6** (2023), 50–65.

13. R. Kumar, L. Colney, M. N. I. Khan, Proposed theorems on the lifts of Kenmotsu manifolds admitting a non-symmetric non-metric connection (NSNMC) in the tangent bundle, *Symmetry*, **1**5 (2023), 2037. https://doi.org/10.3390/sym15112037

14. R. Kumar, L. Colney, S. Shenawy, N. Bin Turki, Tangent bundles endowed with quarter-symmetric non-metric connection (QSNMC) in a Lorentzian Para-Sasakian manifold, *Mathematics*, **1**1 (2023), 4163. https://doi.org/10.3390/math11194163

15. R. Kumar, S. Shenawy, N. B. Turki, L. Colney, U. C. De, Lifts of a semi-symmetric metric connection from Sasakian statistical manifolds to tangent bundle, *Mathematics*, **1**2 (2024), 226. https://doi.org/10.3390/math12020226

16. S. Pandey, A. Singh, R. Prasad, $\eta *$-Ricci solitons on Sasakian manifolds, *Differ, Geom.-Dyn. Syst.*, **2**4 (2022), 164–176.

17. B. E. Acet, S. Y. Perktas, E. Kilic, Kenmotsu manifolds with generalized Tanaka-Webster connection, *Adıyaman Üniversitesi Fen Bilimleri Dergisi*, **3** (2013), 79–93.

18. D. G. Prakasha, B. S. Hadimani, On the conharmonic curvature tensor of Kenmotsu manifold with generalized Tanaka-Webster connection, *Miskolc Math Notes*, **1**9 (2018), 491–503. https://doi.org/10.18514/MMN.2018.1596

19. A. Singh, Pankaj, R. Prasad, S. Patel, Ricci soliton on Sasakian manifold with quarter-symmetric non-metric connection, *Ganita*, **73** (2023), 59–74.
20. M. Tani, Prolongations of hypersurfaces to tangent bundles, *Kodai Math. Sem. Rep.*, **2**1 (1969), 85–96.
21. R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures, *J. Math. Soc. Japan*, **2**7 (1975), 45–53. https://doi.org/10.2969/jmsj/02710043
22. S. Tanno, Variational problems on contact-Riemannian manifolds, *Trans. Amer. Math. Soc.*, **3**14 (1989), 349–379. https://doi.org/10.1090/S0002-9947-1989-1000553-9
23. N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, *Jpn. J. Math.*, **2** (1976), 131–190. https://doi.org/10.4099/math1924.2.131
24. S. M. Webster, Pseudo-Hermitian structures on a real hypersurface, *J. Differ. Geom.*, **1**3 (1978), 25–41.
25. K. Yano, S. Ishihara, *Tangent and cotangent bundles: differential geometry*, New York, NY, USA: Marcel Dekker, Inc., 1973.
26. K. Yano, S. Kobayashi, Prolongations of tensor fields and connections to tangent bundles I, *J. Math. Soc. Jpn.*, **1**8 (1966), 194–210. https://doi.org/10.2969/jmsj/01820194