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Research article

Jointly A-hyponormal m-tuple of commuting operators and related results

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Abstract: In this paper, we aim to investigate the class of jointly hyponormal operators related to a positive operator A on a complex Hilbert space $\mathcal{X}$, which is called jointly A-hyponormal. This notion was first introduced by Guesba et al. in [Linear and Multilinear Algebra, 69(15), 2888–2907] for m-tuples of operators that admit adjoint operators with respect to A. Mainly, we prove that if $\mathbf{B} = (B_1, \cdots, B_m)$ is a jointly A-hyponormal m-tuple of commuting operators, then $\mathbf{B}$ is jointly A-normaloid. This result allows us to establish, for a particular case when A is the identity operator, a sharp bound for the distance between two jointly hyponormal m-tuples of operators, expressed in terms of the difference between their Taylor spectra. We also aim to introduce and investigate the class of spherically A-p-hyponormal operators with $0 < p < 1$. Additionally, we study the tensor product of specific classes of multivariable operators in semi-Hilbert spaces.

Keywords: semi-Hilbert spaces; jointly A-hyponormal operators; joinlty A-normaloid operators; Taylor spectrum; spherically A-p-hyponormal; tensor product
Mathematics Subject Classification: 47A10, 47A12, 47A13, 47B20, 47B37

1. Introduction and preliminary concepts

In recent years, there has been significant progress in studying multivariable operators on Hilbert spaces, building on ideas from single-variable operator theory. These developments are discussed in [1–7] and other related works.

One interesting direction in this field is studying multivariable operators on Hilbert spaces with a semi-inner product defined by a positive semidefinite operator A. These spaces, called semi-Hilbert spaces, have attracted a lot of attention (see, for instance, [8–10] and the cited references). This research was started by the first author and others in [11] in 2018. Initially, the focus was on the joint A-numerical range and joint A-maximal numerical range. Later studies introduced and explored several new classes of operators in this setting, such as jointly A-normal, jointly A-hyponormal, jointly $(\alpha;\beta)$-A-normal operators, and (A, m)-isometric tuples, among others. For more information on these classes, see [1, 11–15] and the references therein.

In this paper, we contribute to this field by studying the class of jointly A-hyponormal commuting operator tuples. We also introduce a new class of multivariable operators called spherically A-p-hyponormal operators with $0 < p < 1$. These operators are defined and studied within the algebra of bounded linear operators that have A-adjoints.

Before delving into our specific contributions, we will explore semi-Hilbert spaces and establish essential notation and definitions pertinent to multivariable operators. The subsequent subsection will provide a comprehensive overview of semi-Hilbert spaces, covering their definitions, notation, and fundamental concepts. Throughout this paper, we work within the framework where $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ denotes a Hilbert space over the field of complex numbers $\mathbb{C}$, with $\langle\cdot,\cdot\rangle$ representing the inner product that defines the norm $\|\cdot\|$. The set of all bounded linear operators on $\mathcal{X}$, including the identity operator I, is denoted by $\mathscr{L}(\mathcal{X})$. We use $\mathbb{N}$ and $\mathbb{N}^*$ to denote the sets of non-negative and positive integers, respectively. An "operator" in this context refers to an element of $\mathscr{L}(\mathcal{X})$. For any operator B, $\mathcal{R}(B)$ denotes its range, $\mathcal{N}(B)$ its null space, and B^* its adjoint.

An operator $B \in \mathscr{L}(\mathcal{X})$ is termed positive ($B \geq 0$) if $\langle By, y\rangle \geq 0$ for all $y \in \mathcal{X}$. We denote the set of all positive operators on $\mathcal{X}$ by $\mathscr{L}(\mathcal{X})^+$. For $B, C \in \mathscr{L}(\mathcal{X})$, the notation $B \geq C$ means that $B - C \in \mathscr{L}(\mathcal{X})^+$. The square root of a positive operator $B \in \mathscr{L}(\mathcal{X})^+$ is denoted by $\sqrt{B}$.

Hereafter, let A be a non-zero positive operator defining the semi-inner product $\langle\cdot,\cdot\rangle_A : \mathcal{X}\times\mathcal{X} \longrightarrow \mathbb{C}$ by $\langle y_1, y_2\rangle_A := \langle Ay_1, y_2\rangle$ for all $(y_1, y_2) \in \mathcal{X} \times \mathcal{X}$. This construction characterizes $(\mathcal{X}, \|\cdot\|_A)$ as a semi-Hilbert space, where $\|\cdot\|_A$ denotes the semi-norm defined as $\|y\|_A = \sqrt{\langle y, y\rangle_A}$ for all $y \in \mathcal{X}$. The unit A-sphere in $\mathcal{X}$, denoted $\mathbb{S}^A_{\mathcal{X}}$, consists of elements with $\|y\|_A = 1$. When $A = I$, $\mathbb{S}_{\mathcal{X}}$ represents the unit sphere of $\mathcal{X}$. It is important to note that $(\mathcal{X}, \|\cdot\|_A)$ is generally neither a normed space nor a complete space. However, one can show that $(\mathcal{X}, \|\cdot\|_A)$ becomes a Hilbert space if and only if A is injective and $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$, i.e., A is invertible. Here, $\overline{\mathcal{R}(A)}$ denotes the closure of $\mathcal{R}(A)$ in the norm topology of $\mathcal{X}$. For further details, refer to [16].

Consider an operator $B \in \mathscr{L}(\mathcal{X})$. An operator $C \in \mathscr{L}(\mathcal{X})$ is defined as an A-adjoint of B if $\langle By_1, y_2\rangle_A = \langle y_1, Cy_2\rangle_A$ holds for all $y_1, y_2 \in \mathcal{X}$, which is equivalent to $AC = B^*A$ (see [17]). It should be emphasized that not all operators $B \in \mathscr{L}(\mathcal{X})$ possess an A-adjoint operator, and even in cases where such an adjoint exists, uniqueness cannot be guaranteed (see [17]). These nuances are particularly relevant in the study of operators on semi-Hilbert spaces.

In this context, the Douglas range inclusion theorem [18] is particularly significant. Briefly, the theorem states that for $B, C \in \mathscr{L}(\mathcal{X})$, the equation $BZ = C$ has a solution in $\mathscr{L}(\mathcal{X})$ if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(B)$, or equivalently, if there exists some $\alpha > 0$ such that $\|C^*x\| \leq \alpha\|B^*x\|$ for every $x \in \mathcal{X}$. Moreover, if either of these conditions holds, there exists a unique solution $D \in \mathscr{L}(\mathcal{X})$ to the equation $BZ = C$ such that $\mathcal{R}(D) \subseteq \overline{\mathcal{R}(B^*)}$. This unique solution D is referred to as the "reduced solution" of the equation $BZ = C$.

Consider $\mathscr{L}_A(\mathcal{X})$ as the set of operators possessing A-adjoints. According to the Douglas theorem, an operator $B \in \mathscr{L}(\mathcal{X})$ belongs to $\mathscr{L}_A(\mathcal{X})$ if and only if $\mathcal{R}(B^*A) \subseteq \mathcal{R}(A)$. Moreover, the "reduced solution" of the equation $AX = B^*A$ will be denoted by $B^{\star_A}$. If $A^\dagger$ represents the Moore-Penrose pseudo-inverse of A, then $B^{\star_A} = A^{\dagger}B^{*}A$ (see [19]). It is crucial to distinguish between $B^{\star_A}$ and B^{*}; although they share similarities, $(B^{\star_A})^{\star_A} = \mathcal{P}_A B \mathcal{P}_A$ instead of $(B^{\star_A})^{\star_A} = B$, where $\mathcal{P}_A$ denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$. However, equivalence holds when $\mathcal{R}(B) \subseteq \overline{\mathcal{R}(A)}$: $(B^{\star_A})^{\star_A} = B$ (see [17]). Notice that for any $B, C \in \mathscr{L}_A(\mathcal{X})$ and $\mu, \nu \in \mathbb{C}$, the following properties hold: $\mu B + \nu C \in \mathscr{L}_A(\mathcal{X})$ and $(\mu B+\nu C)^{\star_A} = \overline{\mu} B^{\star_A} + \overline{\nu} C^{\star_A}$. Additionally, $BC \in \mathscr{L}_A(\mathcal{X})$ and $(BC)^{\star_A} = C^{\star_A} B^{\star_A}$. Furthermore, $B^n \in \mathscr{L}_A(\mathcal{X})$ and $(B^n)^{\star_A} = (B^{\star_A})^n$ for all $n \in \mathbb{N}^*$. Let $B \in \mathscr{L}(\mathcal{X})$. The operator B is termed A-selfadjoint if $AB = B^*A$, indicating that AB is selfadjoint. It is referred to as A-positive if $AB \geq 0$, denoted as $B \geq_A 0$. The notation $B \geq_A C$ signifies that $B - C \geq_A 0$. Note that if $B \geq_A 0$, then $B \in \mathscr{L}_A(\mathcal{X})$. For proofs and further details, see [17, 19].

Let $[B, C] := BC - CB$ for $B, C \in \mathscr{L}(\mathcal{X})$. An operator $B \in \mathscr{L}_A(\mathcal{X})$ is termed A-normal if $[B^{\star_A}, B] = 0$, and A-hyponormal if $[B^{\star_A}, B] \geq 0$. Let $\mathscr{L}(\mathcal{X})^m$ denote the direct product of m copies of $\mathscr{L}(\mathcal{X})$, where $m \in \mathbb{N}^*$. Consider $\mathbf{B} = (B_1, \ldots, B_m) \in \mathscr{L}(\mathcal{X})^m$ as an m-tuple of operators. When the operators B_k commute pairwise ($[B_k, B_l] = 0$ for all $k, l \in \{1, \ldots, m\}$), we say $\mathbf{B}$ is a commuting tuple. Furthermore, $\mathbf{B}$ is termed an A-doubly commuting operator tuple if it is commuting and $[B_i^{\star_A}, B_j] = 0$ for all $1 \leq i \neq j \leq m$.

Following [13], an operator tuple $\mathbf{B} = (B_1, \ldots, B_m) \in \mathscr{L}_A(\mathcal{X})^m$ is jointly A-hyponormal if the operator matrix

$$C_A(\mathbf{B}) = \begin{pmatrix} [B_1^{\star_A}, B_1] & [B_2^{\star_A}, B_1] & \cdots & [B_m^{\star_A}, B_1] \\ [B_1^{\star_A}, B_2] & [B_2^{\star_A}, B_2] & \cdots & [B_m^{\star_A}, B_2] \\ \vdots & \vdots & \vdots & \vdots \\ [B_1^{\star_A}, B_m] & [B_2^{\star_A}, B_m] & \cdots & [B_m^{\star_A}, B_m] \end{pmatrix}$$

is $\mathbb{A}$-positive, where $\mathbb{A} = \operatorname{diag}(A, \ldots, A)$ and denotes a diagonal matrix. Consequently, $\mathbf{B} = (B_1, \ldots, B_m) \in \mathscr{L}_A(\mathcal{X})^m$ is a jointly A-hyponormal m-tuple of operators if and only if

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \langle [B_j^{\star_A}, B_i] x_j, x_i \rangle_A \geq 0, \tag{1.1}$$

holds for all $x_1, x_2, \ldots, x_m \in \mathcal{X}$.

It is important to note that the definition of jointly A-hyponormality does not require the coordinates to commute. By taking $A = I$, we recover the definition originally introduced by A. Athavale in [2]. We take this opportunity to provide a more precise clarification regarding the definition presented in [13]: The authors originally defined joint A-hyponormality stating that $C_A(\mathbf{B})$ is A-positive, whereas it should correctly be $C_A(\mathbf{B})$ being $\mathbb{A}$-positive. Furthermore, it is important to note that in [13], the authors proved in Theorem 2.2 that if $\mathbf{B} = (B_1, \ldots, B_m) \in \mathscr{L}_A(\mathcal{X})^m$ is jointly A-hyponormal, then so is $\mathbf{B}^n = (B_1^{n_1}, \ldots, B_m^{n_m})$ for all $n = (n_1, \ldots, n_m) \in \mathbb{N}^m$. Here, $\mathbb{N}^m$ denotes the Cartesian product of $\mathbb{N}$ taken m times. However, this result may not hold true even for $m = 1$ and $A = I$. Indeed, it is well-known that although B being hyponormal implies B^2 is hyponormal in some cases, in general, B^2 is not hyponormal (see Problem 209 [20] for further details).

Now, let us focus on recalling some useful concepts, particularly in the context where operators have A-adjoints. For $B \in \mathscr{L}_A(\mathcal{X})$, the A-seminorm and A-numerical radius of B are defined as follows (cf. [21]):

$$\|B\|_A := \sup_{x \in \mathbb{S}_{\mathcal{X}}^A} \|Bx\|_A \quad \text{and} \quad \omega_A(B) := \sup_{x \in \mathbb{S}_{\mathcal{X}}^A} |\langle Bx, x \rangle_A|. \tag{1.2}$$

These quantities have attracted considerable attention in recent literature, with numerous studies exploring various results and inequalities related to them (refer to, for example, the recent book [10] and its references).

The notions given in (1.2) have been extended to the multivariable setting. For $\mathbf{B} = (B_1, \ldots, B_m) \in \mathscr{L}_A(\mathcal{X})^m$, which may not necessarily commute, the following two quantities are defined in [11]:

$$\omega_A(\mathbf{B}) := \sup_{x \in \mathbb{S}_{\mathcal{X}}^A} \sqrt{\sum_{j=1}^m |\langle B_j x, x\rangle_A|^2} \quad \text{and} \quad \|\mathbf{B}\|_A := \sup_{x \in \mathbb{S}_{\mathcal{X}}^A} \sqrt{\sum_{j=1}^m \|B_j x\|_A^2}.$$

It is noteworthy that for $\mathbf{B} = (B_1, \ldots, B_m) \in \mathscr{L}_A(\mathcal{X})^m$, it was demonstrated in [12] that

$$\|\mathbf{B}\|_A = \sqrt{\left\|\sum_{j=1}^m B_j^{\star_A} B_j\right\|_A}. \tag{1.3}$$

It is clear that $\|\cdot\|_A$ and $\omega_A(\cdot)$ define two seminorms on $\mathscr{L}_A(\mathcal{X})^m$, referred to as the joint operator A-seminorm and the joint A-numerical radius of operators, respectively. These seminorms are equivalent, as demonstrated in [11], where it was shown that for every $\mathbf{B} \in \mathscr{L}_A(\mathcal{X})^m$, the following inequalities hold:

$$\frac{1}{2\sqrt{m}}\|\mathbf{B}\|_A \leq \omega_A(\mathbf{B}) \leq \|\mathbf{B}\|_A.$$

Let us now consider the concept of the joint spectral radius of semi-Hilbert space operators. Specifically, if $\mathbf{B} = (B_1, \ldots, B_m) \in \mathscr{L}_A(\mathcal{X})^m$ is an m-tuple of commuting operators, the joint A-spectral radius associated with $\mathbf{B}$ was first introduced in [12] as:

$$r_A(\mathbf{B}) := \inf_{j \in \mathbb{N}^*} \left\| \sum_{\substack{|\gamma|=j,\\ \gamma \in \mathbb{N}^m}} \frac{j!}{\gamma!} (\mathbf{B}^{\star_A})^\gamma \mathbf{B}^\gamma \right\|_A^{\frac{1}{2j}} = \lim_{j \to \infty} \left\| \sum_{\substack{|\gamma|=j,\\ \gamma \in \mathbb{N}^m}} \frac{j!}{\gamma!} (\mathbf{B}^{\star_A})^\gamma \mathbf{B}^\gamma \right\|_A^{\frac{1}{2j}}. \tag{1.4}$$

Here, $\mathbf{B}^{\star_A} = (B_1^{\star_A}, \ldots, B_m^{\star_A})$. For the multi-index $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m$, we used $\mathbf{B}^\gamma := \prod_{k=1}^m B_k^{\gamma_k}$, $|\gamma| := \sum_{j=1}^m \gamma_j$, and $\gamma! := \prod_{k=1}^m \gamma_k!$. Note that the second equality in (1.4) was established in [12]. Following [16], an m-tuple $\mathbf{B} = (B_1, \ldots, B_m) \in \mathscr{L}_A(\mathcal{X})^m$ is said to be jointly A-normaloid if

$$r_A(\mathbf{B}) = \|\mathbf{B}\|_A.$$

Several characterizations and properties of this class of operators have been stated in [16].

Consider $\mathbf{B} = (B_1, \ldots, B_m) \in \mathscr{L}_A(\mathcal{X})^m$. We define the operator $\Theta_{\mathbf{B}} : \mathscr{L}_A(\mathcal{X}) \to \mathscr{L}_A(\mathcal{X})$ by

$$\Theta_{\mathbf{B}}(X) := \sum_{j=1}^m B_j^{\star_A} X B_j,$$

with $\Theta_{\mathbf{B}}^0(X) = X$ and $\Theta_{\mathbf{B}}^n(X) = \Theta_{\mathbf{B}}[\Theta_{\mathbf{B}}^{n-1}(X)]$ inductively for all $n \geq 1$. According to [12], for a commuting m-tuple $\mathbf{B} \in \mathscr{L}_A(\mathcal{X})^m$, we have

$$r_A(\mathbf{B}) = \lim_{j \to \infty} \|\Theta_{\mathbf{B}}^j(I)\|_A^{\frac{1}{2j}}. \tag{1.5}$$

When $A = I$, the notation $r_I(\mathbf{B})$ simplifies to $r(\mathbf{B})$, which can be described in terms of the Taylor spectrum. Specifically,

$$r(\mathbf{B}) = \max\{\|\gamma\|_2, \lambda = (\gamma_1, \ldots, \gamma_m) \in \sigma_T(\mathbf{B})\},$$

where $\|\cdot\|_2$ represents the Euclidean norm on $\mathbb{C}^m$. For further details, refer to [22–25].

We close this section by summarizing the main objectives of this paper. One primary goal is to prove that if $\mathbf{B} = (B_1, \ldots, B_m) \in \mathscr{L}_A(\mathcal{X})^m$ is a jointly A-hyponormal m-tuple of commuting operators, then $\mathbf{B}$ is jointly A-normaloid. This result generalizes and extends a theorem by Chavan et al. in [26], though our techniques differ from theirs. Furthermore, this result enables us to establish, in the particular case where $A = I$, a sharp bound for the distance between two jointly hyponormal m-tuples of operators, expressed in terms of the difference between their Taylor spectra. Additionally, we aim to extend the celebrated Löwner-Heinz inequality, which states that "$B \geq C \geq 0$ ensures $B^\alpha \geq C^\alpha \geq 0$ for all $\alpha \in [0, 1]$" (cf. [27]), to the setting of semi-Hilbert space operators. As a consequence of this, we explore a new class of multivariable operators called spherically A-p-hyponormal operators, where $0 < p < 1$. Finally, we will investigate the tensor product of specific classes of multivariable operators in semi-Hilbert spaces.

2. Main results

In this section, we will present our results. To demonstrate our initial finding, we need to introduce some lemmas. Let us start with the following one.

Lemma 2.1. *Let $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$ be a jointly A-hyponormal m-tuple of commuting operators. Then,*

$$\Theta_{\mathbf{B}}^2(I) \geq_A [\Theta_{\mathbf{B}}(I)]^2.$$

Proof. Let $x \in \mathcal{X}$. Then, using the commutativity of $\mathbf{B}$, we have

$$\begin{aligned}\langle(\Theta_{\mathbf{B}}^2(I) - [\Theta_{\mathbf{B}}(I)]^2)x, x\rangle_A &= \sum_{i=1}^m \sum_{j=1}^m \langle(B_i^{\star_A} B_j^{\star_A} B_j B_i - B_i^{\star_A} B_i B_j^{\star_A} B_j)x, x\rangle_A \\ &= \sum_{i=1}^m \sum_{j=1}^m \langle B_i^{\star_A}[B_j^{\star_A}, B_i]B_j x, x\rangle_A \\ &= \sum_{i=1}^m \sum_{j=1}^m \langle [B_j^{\star_A}, B_i]B_j x, B_i x\rangle_A.\end{aligned}$$

Set $x_i := B_i x$, $i \in \{1, \ldots, m\}$. Then, (1.1) implies that

$$\langle(\Theta_{\mathbf{B}}^2(I) - [\Theta_{\mathbf{B}}(I)]^2)x, x\rangle_A = \sum_{i=1}^m \sum_{j=1}^m \langle [B_j^{\star_A}, B_i]x_j, x_i\rangle_A \geq 0.$$

Since $x \in \mathcal{X}$ was arbitrary, we conclude that $\Theta_{\mathbf{B}}^2(I) \geq_A [\Theta_{\mathbf{B}}(I)]^2$. □

Recall from [28] the following definition.

Definition 2.1. *A sequence $\{a_k\}_{k\in\mathbb{N}}$ of nonnegative numbers is said to be* log-convex *if $a_k^2 \leq a_{k-1}a_{k+1}$, for all $k \in \mathbb{N}^*$.*

It is important to note that the log-convex sequences appear naturally in many areas in mathematics, especially in moment problems, as the following example demonstrates.

Example 2.1. *A sequence* $\{a_k\}_{k\in\mathbb{N}}$ *of real numbers is said to be a* Stieltjes moment sequence *if there exists a positive Borel measure* μ *on the closed half-line* $[0,+\infty)$ *such that*

$$a_k = \int_0^{+\infty} t^k \, d\mu(t), \quad k \in \mathbb{N}.$$

The measure μ *is called a* representing measure *of* $\{a_k\}_{k\in\mathbb{N}}$.

By applying the Cauchy–Schwarz inequality, for all $k \in \mathbb{N}^*$, *we have that*

$$\begin{aligned} a_k^2 &= \left(\int_0^{+\infty} t^k \, d\mu(t)\right)^2 \\ &= \left(\int_0^{+\infty} t^{\frac{k-1}{2}} t^{\frac{k+1}{2}} \, d\mu(t)\right)^2 \\ &\leq \int_0^{+\infty} t^{k-1} \, d\mu(t) \cdot \int_0^{+\infty} t^{k+1} \, d\mu(t) \\ &= a_{k-1}a_{k+1}, \end{aligned}$$

and thus, $\{a_k\}_{k\in\mathbb{N}}$ *is* log*-convex.*

The subsequent lemma is also essential.

Lemma 2.2. *Let* $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$ *be a jointly A-hyponormal m-tuple of commuting operators and let* $x \in \mathcal{X}$. *Then, the sequence* $\{\theta_k\}_{k\in\mathbb{N}}$ *given by*

$$\theta_k := \langle \Theta_{\mathbf{B}}^k(I)x, x\rangle_A, \quad k \in \mathbb{N},$$

is log*-convex.*

Proof. Let $k \in \mathbb{N}^*$ be arbitrary. By the definition of log-convexity, we need to show that $\theta_k^2 \leq \theta_{k-1}\theta_{k+1}$. Using the Cauchy-Schwarz inequality for semi-inner products, and Lemma 2.1,

$$\begin{aligned} \theta_k^2 &= \langle \Theta_{\mathbf{B}}^k(I)x, x\rangle_A^2 \\ &= \langle \Theta_{\mathbf{B}}^{k-1}(\Theta_{\mathbf{B}}(I))x, x\rangle_A^2 \\ &= \left(\sum_{\substack{|\gamma|=k-1,\\ \gamma\in\mathbb{N}^m}} \frac{(k-1)!}{\gamma!} \langle \Theta_{\mathbf{B}}(I)\mathbf{B}^\gamma x, \mathbf{B}^\gamma x\rangle_A\right)^2 \\ &\leq \sum_{\substack{|\gamma|=k-1,\\ \gamma\in\mathbb{N}^m}} \frac{(k-1)!}{\gamma!} \|\Theta_{\mathbf{B}}(I)\mathbf{B}^\gamma x\|_A^2 \cdot \sum_{\substack{|\gamma|=k-1,\\ \gamma\in\mathbb{N}^m}} \frac{(k-1)!}{\gamma!} \|\mathbf{B}^\gamma x\|_A^2 \\ &= \sum_{\substack{|\gamma|=k-1,\\ \gamma\in\mathbb{N}^m}} \frac{(k-1)!}{|\gamma|!} \langle [\Theta_{\mathbf{B}}(I)]^2\mathbf{B}^\gamma x, \mathbf{B}^\gamma x\rangle_A \cdot \sum_{\substack{|\gamma|=k-1,\\ \gamma\in\mathbb{N}^m}} \frac{(k-1)!}{\gamma!} \langle (\mathbf{B}^{\star_A})^\gamma \mathbf{B}^\gamma x, x\rangle_A^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\substack{|\gamma|=k-1,\\ \gamma\in\mathbb{N}^m}} \frac{(k-1)!}{\gamma!}\langle \Theta_{\mathbf{B}}^2(I)\mathbf{B}^\gamma x, \mathbf{B}^\gamma x\rangle_A \cdot \langle \Theta_{\mathbf{B}}^{k-1}(I)x, x\rangle_A \\
&= \langle \Theta_{\mathbf{B}}^{k-1}(I)x, x\rangle_A \cdot \sum_{\substack{|\gamma|=k-1,\\ \gamma\in\mathbb{N}^m}} \frac{(k-1)!}{\gamma!}\langle (\mathbf{B}^{\star_A})^\gamma \Theta_{\mathbf{B}}^2(I)\mathbf{B}^\gamma x, x\rangle_A \\
&= \langle \Theta_{\mathbf{B}}^{k-1}(I)x, x\rangle_A \cdot \langle \Theta_{\mathbf{B}}^{k-1}(\Theta_{\mathbf{B}}^2(I))x, x\rangle_A \\
&= \langle \Theta_{\mathbf{B}}^{k-1}(I)x, x\rangle_A \cdot \langle \Theta_{\mathbf{B}}^{k+1}(I)x, x\rangle_A \\
&= \theta_{k-1}\theta_{k+1}.
\end{aligned}
$$

This completes the proof. □

Before we establish the primary result of this section, we also require the following general lemma.

Lemma 2.3. *Let* $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$. *Then*

$$
\left\|\Theta_{\mathbf{B}}^n(I)\right\|_A \leq \|\Theta_{\mathbf{B}}(I)\|_A^n \tag{2.1}
$$

for all $n \in \mathbb{N}^*$.

Proof. We use mathematical induction. Inequality (2.1) clearly holds for $n = 1$. Assume that it is true for some $n \in \mathbb{N}^*$. Then,

$$
\begin{aligned}
\left\|\Theta_{\mathbf{B}}^{n+1}(I)\right\|_A &= \left\|\Theta_{\mathbf{B}}(\Theta_{\mathbf{B}}^n(I))\right\|_A \\
&= \left\|\sum_{k=1}^m B_j^{\star_A}[\Theta_{\mathbf{B}}^n(I)]B_j\right\|_A \\
&= \left\|\begin{pmatrix} \sum_{j=1}^m B_j^{\star_A}[\Theta_{\mathbf{B}}^n(I)]B_j & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right\|_A \\
&= \left\|\begin{pmatrix} B_1^{\star_A} & \cdots & B_m^{\star_A} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} \Theta_{\mathbf{B}}^n(I) & & \\ & \ddots & \\ & & \Theta_{\mathbf{B}}^n(I) \end{pmatrix}\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_m & 0 & \cdots & 0 \end{pmatrix}\right\|_A \\
&\leq \left\|\begin{pmatrix} \Theta_{\mathbf{B}}^n(I) & & \\ & \ddots & \\ & & \Theta_{\mathbf{B}}^n(I) \end{pmatrix}\right\|_A \left\|\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_m & 0 & \cdots & 0 \end{pmatrix}\right\|_A^2 \\
&= \left\|\Theta_{\mathbf{B}}^n(I)\right\|_A \left\|\begin{pmatrix} B_1^{\star_A} & \cdots & B_m^{\star_A} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_m & 0 & \cdots & 0 \end{pmatrix}\right\|_A
\end{aligned}
$$

$$
\begin{aligned}
&= \left\|\Theta_{\mathbf{B}}^{n}(I)\right\|_A \left\|\sum_{j=1}^{m} B_j^{\star_A} B_j\right\|_A \\
&\leq \|\Theta_{\mathbf{B}}(I)\|_A^n \, \|\Theta_{\mathbf{B}}(I)\|_A \\
&= \|\Theta_{\mathbf{B}}(I)\|_A^{n+1},
\end{aligned}
$$

as desired. □

We are now prepared to demonstrate our primary result in this paper.

Theorem 2.1. *Let* $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$ *be a jointly A-hyponormal m-tuple of commuting operators. Then,* $\mathbf{B}$ *is jointly A-normaloid.*

Proof. It is well-known that $r_A(\mathbf{B}) \leq \|\mathbf{B}\|_A$ (see [12]). So, it suffices to prove that $r_A(\mathbf{B}) \geq \|\mathbf{B}\|_A$.

Let $k \in \mathbb{N}^*$ and $x \in \mathcal{X}$ with $\|x\|_A = 1$ be arbitrary. Using Lemma 2.2, we have that

$$
\begin{aligned}
\langle \Theta_{\mathbf{B}}^{k}(I)x, x\rangle_A^2 &\leq \langle \Theta_{\mathbf{B}}^{k-1}(I)x, x\rangle_A \cdot \langle \Theta_{\mathbf{B}}^{k+1}(I)x, x\rangle_A \\
&\leq \left\|\Theta_{\mathbf{B}}^{k-1}(I)x\right\|_A \cdot \left\|\Theta_{\mathbf{B}}^{k+1}(I)x\right\|_A.
\end{aligned}
$$

Since $\Theta_{\mathbf{B}}^{k}(I) \geq_A 0$, it immediately follows that

$$
\begin{aligned}
\|\Theta_{\mathbf{B}}^{k}(I)\|_A^2 &= \sup_{x\in\mathbb{S}_{\mathcal{X}}^A} \langle \Theta_{\mathbf{B}}^{k}(I)x, x\rangle_A^2 \\
&\leq \sup_{x\in\mathbb{S}_{\mathcal{X}}^A} \left\|\Theta_{\mathbf{B}}^{k-1}(I)x\right\|_A \cdot \sup_{x\in\mathbb{S}_{\mathcal{X}}^A} \left\|\Theta_{\mathbf{B}}^{k+1}(I)x\right\|_A \\
&= \|\Theta_{\mathbf{B}}^{k-1}(I)\|_A \cdot \|\Theta_{\mathbf{B}}^{k+1}(I)\|_A,
\end{aligned}
$$

i.e.,

$$
\|\Theta_{\mathbf{B}}^{k}(I)\|_A^2 \leq \|\Theta_{\mathbf{B}}^{k-1}(I)\|_A \cdot \|\Theta_{\mathbf{B}}^{k+1}(I)\|_A. \tag{2.2}
$$

Let us now show that

$$
\|\Theta_{\mathbf{B}}(I)\|_A^n \leq \|\Theta_{\mathbf{B}}^{n}(I)\|_A, \tag{2.3}
$$

for all $n \in \mathbb{N}^*$. The previous inequality clearly holds for $n = 1$. Assume that (2.3) holds for some $n \in \mathbb{N}^*$ such that $n > 1$. Then, using (2.2) and (2.1), we have

$$
\begin{aligned}
\|\Theta_{\mathbf{B}}(I)\|_A^{2n} &= \left(\|\Theta_{\mathbf{B}}(I)\|_A^n\right)^2 \\
&\leq \|\Theta_{\mathbf{B}}^{n}(I)\|_A^2 \\
&\leq \|\Theta_{\mathbf{B}}^{n-1}(I)\|_A \cdot \|\Theta_{\mathbf{B}}^{n+1}(I)\|_A \\
&\leq \|\Theta_{\mathbf{B}}(I)\|_A^{n-1} \cdot \|\Theta_{\mathbf{B}}^{n+1}(I)\|_A.
\end{aligned}
$$

From here, it immediately follows that

$$
\|\Theta_{\mathbf{B}}(I)\|_A^{n+1} \leq \|\Theta_{\mathbf{B}}^{n+1}(I)\|_A.
$$

By the induction principle, we have that (2.3) holds for all $n \in \mathbb{N}^*$. Finally, (1.5), (2.3), and (1.3) imply

$$
r_A(\mathbf{B}) = \lim_{n\to\infty} \|\Theta_{\mathbf{B}}^{n}(I)\|_A^{\frac{1}{2n}}
$$

$$
\begin{aligned}
&\geq \lim_{n\to\infty} \left(\|\Theta_{\mathbf{B}}(I)\|_A^n\right)^{\frac{1}{2n}} \\
&= \lim_{n\to\infty} \|\Theta_{\mathbf{B}}(I)\|_A^{\frac{1}{2}} \\
&= \left\|\sum_{k=1}^{m} B_k^{\star_A} B_k\right\|_A^{\frac{1}{2}} \\
&= \|\mathbf{B}\|_A\,.
\end{aligned}
$$

This proves that $r_A(\mathbf{B}) \geq \|\mathbf{B}\|_A$. □

Remark 2.1. *It should be mentioned that by setting $m = 1$ in Theorem 2.1, we recover a well-known result established by the second author in [29].*

By substituting $A = I$ into Theorem 2.1, we can derive the following corollary, which was originally proven in [26].

Corollary 2.1. *Let $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}(\mathcal{X})^m$ be a jointly hyponormal m-tuple of commuting operators. Then,*

$$
r(\mathbf{B}) = \|\mathbf{B}\|. \tag{2.4}
$$

As an important application of Eq (2.4), we establish a sharp bound in our next result for the distance between two jointly hyponormal d-tuples of operators, expressed in terms of the difference between their Taylor spectra. Our result generalizes [30, Theorem 1.1] (see also [31, Theorem 2.3] and [32, Corollary 2]). Before presenting our result, let us recall that for two given d-tuples of operators, $\mathbf{B} = (B_1, \ldots, B_m)$ and $\mathbf{C} = (C_1, \ldots, C_m)$, the distance between $\mathbf{B}$ and $\mathbf{C}$ is defined as $\mathbf{B} - \mathbf{C} := (B_1 - C_1, \ldots, B_m - C_m)$.

Proposition 2.1. *Let $\mathbf{B} = (B_1, \ldots, B_m) \in \mathcal{L}(\mathcal{X})^m$ and $\mathbf{C} = (C_1, \ldots, C_m) \in \mathcal{L}(\mathcal{X})^m$ be jointly hyponormal m-tuples of commuting operators. Then*

$$
\|\mathbf{B} - \mathbf{C}\| \leq \sqrt{2}\, \max\left\{\|\eta - \nu\|_2\,;\ \eta \in \sigma_T(\mathbf{B}),\ \nu \in \sigma_T(\mathbf{C})\right\}. \tag{2.5}
$$

Proof. We imitate the argument of [21, Theorem 2.3]. Let $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{C}^m$. Since $\mathbf{B} = (B_1, \ldots, B_m)$ and $\mathbf{C} = (C_1, \ldots, C_m)$ are jointly hyponormal, then by [2, Remark 2(e)], so are $\mathbf{B} - \xi I$ and $\mathbf{C} - \xi I$. So, we see from the spectral mapping property of the Taylor spectrum [22] that

$$
\begin{aligned}
\|\mathbf{B} - \mathbf{C}\| &\leq \|\mathbf{B} - \xi I\| + \|\mathbf{C} - \xi I\| \\
&= r(\mathbf{B} - \xi I) + r(\mathbf{C} - \xi I) \quad \text{(by (2.4))} \\
&= \max\{\|\eta\|_2\,;\ \eta \in \sigma_T(\mathbf{B} - \xi I)\} + \max\{\|\nu\|_2\,;\ \nu \in \sigma_T(\mathbf{C} - \xi I)\} \\
&\leq \max_{\eta\in\sigma_T(\mathbf{B})} \|\eta - \xi\|_2 + \max_{\nu\in\sigma_T(\mathbf{C})} \|\nu - \xi\|_2 \\
&\leq \sqrt{2}\, \max\left\{\|\eta - \nu\|_2\,;\ \eta \in \sigma_T(\mathbf{B}),\ \nu \in \sigma_T(\mathbf{C})\right\}.
\end{aligned}
$$

In the last inequality, we utilize [30, Theorem 2.3] applied to the compact subsets $\sigma_T(\mathbf{B})$ and $\sigma_T(\mathbf{C})$ of $\mathbb{C}^m$. □

Remark 2.2. *It should be noted that the inequality* (2.5) *is sharp. In fact, the optimality of the constant* $\sqrt{2}$ *in* (2.5) *is well-known in the simplest case when* $d = 1$ *and* $dim(\mathcal{X}) < \infty$*. In finite-dimensional Hilbert spaces, every hyponormal operator is necessarily normal. Therefore, in this scenario, when* $d = 1$*, the bound* (2.5) *can be stated as follows: If* B *and* C *are* $m \times m$ *normal commuting matrices with eigenvalues* $\eta_1, \ldots, \eta_m$ *and* $\nu_1, \ldots, \nu_m$*, respectively, then*

$$\|B - C\| \leq \sqrt{2} \max_{i,j \in \{1,\ldots,m\}} |\eta_i - \nu_j|. \tag{2.6}$$

Now, let us consider the following normal commuting matrices:

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It is not difficult to see that $\|B - C\| = 2$*,* $\sigma(B) = \{i, -i\}$*, and* $\sigma(C) = \{1, -1\}$*. Therefore, it is evident that the bound* (2.6) *is sharp.*

Now, for a given $\mathbf{B} = (B_1, \cdots, B_m) \in \mathscr{L}_A(\mathcal{X})^m$, we define the following operator:

$$\Omega_{\mathbf{B}} : \mathscr{L}_A(\mathcal{X}) \longrightarrow \mathscr{L}_A(\mathcal{X})$$
$$X \longmapsto \Omega_{\mathbf{B}}(X) := \sum_{j=1}^{m} B_j X B_j^{\star_A}.$$

Definition 2.2. *Let* $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$*. We say that* $\mathbf{B}$ *is spherically A-hyponormal if*

$$\Theta_{\mathbf{B}}(I) \geq_A \Omega_{\mathbf{B}}(I).$$

Definition 2.3. *Let* $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$*. We say that* $\mathbf{B}$ *is spherically A-p-hyponormal (*$0 < p < 1$*) if* $\Theta_{\mathbf{B}}(I), \Omega_{\mathbf{B}}(I) \in \mathcal{L}(\mathcal{X})^+$ *and*

$$[\Theta_{\mathbf{B}}(I)]^p \geq_A [\Omega_{\mathbf{B}}(I)]^p.$$

In order to support our forthcoming contributions on these novel classes of operators, we first need to establish the following key result, which extends the celebrated Löwner-Heinz inequality (cf. [27]) to the context of semi-Hilbert space operators.

Theorem 2.2. *Let* $T, S \in \mathcal{L}(\mathcal{X})^+$*. Then,*

$$T \geq_A S \geq_A 0 \quad \Longrightarrow \quad T^\gamma \geq_A S^\gamma \geq_A 0,$$

for each $\gamma \in (0, 1]$.

Proof. Assume that

$$T \geq_A S \geq_A 0.$$

Then, $AT \geq AS \geq 0$, and thus

$$AS = (AS)^* = S^* A^* = SA,$$

since both A and S are positive. Similarly, $AT = TA$ and $A(T-S) = (T-S)A$. Using the continuous functional calculus, we have that

$$A^{\frac{1}{2\gamma}}S = SA^{\frac{1}{2\gamma}}, \quad A^{\frac{1}{2\gamma}}T = TA^{\frac{1}{2\gamma}} \quad \text{and} \quad A^{\frac{1-\gamma}{2\gamma}}(T-S) = (T-S)A^{\frac{1-\gamma}{2\gamma}},$$

where $\frac{1-\gamma}{2\gamma} \geq 0$ since $\gamma \in (0,1]$. Now,

$$A^{\frac{1}{\gamma}}S = A^{\frac{1}{2\gamma}}SA^{\frac{1}{2\gamma}} \geq 0,$$

and, similarly, $A^{\frac{1}{\gamma}}T \geq 0$. Also, using the fact that $A(T-S) \geq 0$, we have

$$A^{\frac{1}{\gamma}}(T-S) = A^{\frac{1-\gamma}{2\gamma}}A(T-S)A^{\frac{1-\gamma}{2\gamma}} \geq 0.$$

Thus,

$$A^{\frac{1}{\gamma}}T \geq A^{\frac{1}{\gamma}}S \geq 0. \tag{2.7}$$

Since $A^{\frac{1}{\gamma}}$ commutes with S and T, we have that

$$(A^{\frac{1}{\gamma}}S)^{\gamma} = (A^{\frac{1}{\gamma}})^{\gamma}S^{\gamma} = AS^{\gamma},$$

and, similarly, $(A^{\frac{1}{\gamma}}T)^{\gamma} = AT^{\gamma}$. Combining this with (2.7) and the Löwner-Heinz inequality for Hilbert spaces, we have that

$$AT^{\gamma} \geq AS^{\gamma} \geq 0,$$

i.e.,

$$T^{\gamma} \geq_A S^{\gamma} \geq_A 0.$$

This completes the proof. □

Theorem 2.2 allows us to derive the following result.

Theorem 2.3. *Let $p, q \in (0,1)$ such that $q \leq p$. If $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$ is spherically A-p-hyponormal, then $\mathbf{B}$ is spherically A-q-hyponormal.*

Proof. Since $\mathbf{B}$ is spherically A-p-hyponormal, we have that $\Theta_{\mathbf{B}}(I), \Omega_{\mathbf{B}}(I) \in \mathcal{L}(\mathcal{X})^+$ and

$$[\Theta_{\mathbf{B}}(I)]^p \geq_A [\Omega_{\mathbf{B}}(I)]^p.$$

From $q \leq p$, we have that $\frac{q}{p} \in (0,1]$, and by Theorem 2.2, we have

$$[\Theta_{\mathbf{B}}(I)]^q = ([\Theta_{\mathbf{B}}(I)]^p)^{\frac{q}{p}} \geq_A ([\Omega_{\mathbf{B}}(I)]^p)^{\frac{q}{p}} = [\Omega_{\mathbf{B}}(I)]^q.$$

Hence, $\mathbf{B}$ is spherically A-q-hyponormal. □

Theorem 2.4. *Let $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$ be a jointly A-hyponormal m-tuple such that $\Theta_{\mathbf{B}}(I), \Omega_{\mathbf{B}}(I) \in \mathcal{L}(\mathcal{X})^+$. Then, $\mathbf{B}$ is spherically A-p-hyponormal for all $0 < p < 1$.*

Proof. Let $k \in \{1, \ldots, m\}$, $p \in (0, 1]$, and $x \in \mathcal{X}$ be arbitrary. Set $x_k = x$, and $x_i = 0$ for $i \in \{1, \ldots, m\} \setminus \{k\}$. Since **B** jointly A-hyponormal, (1.1) implies that

$$\langle [B_k^{\star_A}, B_k]x, x\rangle_A = \sum_{i=1}^{m}\sum_{j=1}^{m}\langle [B_j^{\star_A}, B_i]x_j, x_i\rangle_A \geq 0.$$

Thus, $B_k^{\star_A}B_k \geq_A B_kB_k^{\star_A}$ for each $k \in \{1, \ldots, m\}$. This further implies that

$$\Theta_{\mathbf{B}}(I) = \sum_{k=1}^{m} B_k^{\star_A}B_k \geq_A \sum_{k=1}^{m} B_kB_k^{\star_A} = \Omega_{\mathbf{B}}(I).$$

Thus, **B** is spherically A-hyponormal. Theorem 2.3 now implies that **B** is spherically A-p-hyponormal for all $0 < p < 1$. □

From the proof of the previous theorem, it is easy to see that the following holds.

Theorem 2.5. *Let* $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$ *be a jointly A-hyponormal m-tuple. Then,* **B** *is spherically A-hyponormal.*

From now until the end of this paper, we aim to study the tensor product of specific classes of multivariable operators in semi-Hilbert spaces. Before proceeding, we need to recall some useful facts and notions. The study of operators on tensor products of Hilbert spaces arises in various problems in both pure and applied mathematics (see [33, 34] and references therein).

Let $(\mathcal{X}, \langle\cdot,\cdot\rangle_{\mathcal{X}})$ and $(\mathcal{Y}, \langle\cdot,\cdot\rangle_{\mathcal{Y}})$ be two complex Hilbert spaces. We denote their algebraic tensor product by $\mathcal{X} \otimes \mathcal{Y}$, which is linearly spanned by elements of the form $b \otimes c$ with $b \in \mathcal{X}$ and $c \in \mathcal{Y}$. Here, $b \otimes c$ is defined algebraically to be bilinear in its arguments b and c. The completion of $\mathcal{X} \otimes \mathcal{Y}$ under the inner product $\langle b \otimes c, d \otimes e\rangle = \langle b, d\rangle_{\mathcal{X}}\langle c, e\rangle_{\mathcal{Y}}$ is denoted by $\mathcal{X}\widehat{\otimes}\mathcal{Y}$, which forms a Hilbert space.

The tensor product $B \otimes C$ of operators B on $\mathcal{X}$ and C on $\mathcal{Y}$ is defined on $\mathcal{X} \otimes \mathcal{Y}$ by $(B \otimes C)(y \otimes z) = By \otimes Cz$. Moreover, if $B \in \mathcal{L}(\mathcal{X})$ and $C \in \mathcal{L}(\mathcal{Y})$, then $B \otimes C \in \mathcal{L}(\mathcal{X} \otimes \mathcal{Y})$ and has an extension in $\mathcal{L}(\mathcal{X}\widehat{\otimes}\mathcal{Y})$ also denoted by $B \otimes C$. For $B_1, B_2 \in \mathcal{L}(\mathcal{X})$ and $C_1, C_2 \in \mathcal{L}(\mathcal{Y})$, we have $(B_1\widehat{\otimes}C_1)(B_2\widehat{\otimes}C_2) = B_1B_2\widehat{\otimes}C_1C_2$. A comprehensive overview of tensor products of operators on Hilbert spaces can be found in [35] and the references cited therein.

If $\mathbf{B} = (B_1, \cdots, B_m)$ and $\mathbf{C} = (C_1, \cdots, C_m)$ are two m-tuples of operators, we denote their tensor product by $\mathbf{B} \otimes \mathbf{C} := (B_1 \otimes C_1, \cdots, B_m \otimes C_m)$. To establish our next result, we need to recall the following lemma from [12].

Lemma 2.4. *Let* $A \in \mathcal{L}(\mathcal{X})^+$ *and* $D \in \mathcal{L}(\mathcal{Y})^+$. *If* $B \in \mathcal{L}_A(\mathcal{X})$ *and* $C \in \mathcal{L}_D(\mathcal{Y})$, *then* $B\widehat{\otimes}C \in \mathcal{L}_{A \otimes D}(\mathcal{X}\widehat{\otimes}\mathcal{Y})$ *and*

$$(B\widehat{\otimes}C)^{\star_{A\widehat{\otimes}D}} = B^{\star_A}\widehat{\otimes}C^{\star_D}.$$

Theorem 2.6. *Let* $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$ *and* $\mathbf{C} = (C_1, \cdots, C_m) \in \mathcal{L}_D(\mathcal{Y})^m$ *be two m-tuples of doubly commuting A-hyponormal and D-hyponormal operators, respectively. Then,* $\mathbf{B} \otimes \mathbf{C}$ *is an m-tuple of doubly commuting* $A \otimes D$*-hyponormal operators.*

Proof. Using the properties of the tensor product, we have

$$\left[B_i \otimes C_i, B_j \otimes C_j\right] = (B_i \otimes C_i)(B_j \otimes C_j) - (B_j \otimes C_j)(B_i \otimes C_i)$$

$$
\begin{aligned}
&= B_iB_j \otimes C_iC_j - B_jB_i \otimes C_jC_i \\
&= B_iB_j \otimes C_iC_j - B_iB_j \otimes C_iC_j \\
&= 0,
\end{aligned}
$$

for each $i, j \in \{1, \dots, m\}$. Also, by utilizing Lemma 2.4, we have

$$
\begin{aligned}
\left[(B_i \otimes C_i)^{\star_{A\otimes D}}, B_j \otimes C_j\right] &= (B_i \otimes C_i)^{\star_{A\otimes D}}(B_j \otimes C_j) - (B_j \otimes C_j)(B_i \otimes C_i)^{\star_{A\otimes D}} \\
&= (B_i^{\star_A} \otimes C_i^{\star_D})(B_j \otimes C_j) - (B_j \otimes C_j)(B_i^{\star_A} \otimes C_i^{\star_D}) \\
&= B_i^{\star_A} B_j \otimes C_i^{\star_D} C_j - B_j B_i^{\star_A} \otimes C_j C_i^{\star_D} \\
&= B_i^{\star_A} B_j \otimes C_i^{\star_D} C_j - B_i^{\star_A} B_j \otimes C_i^{\star_D} C_j \\
&= 0,
\end{aligned}
$$

for all $i, j \in \{1, \dots, m\}, i \neq j$.

Now, for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and all $k \in \{1, \dots, m\}$,

$$
\begin{aligned}
&\langle (B_k \otimes C_k)^{\star_{A\otimes D}}(B_k \otimes C_k)(x \otimes y), x \otimes y\rangle_{A\otimes D} \\
&= \langle (B_k^{\star_A} \otimes C_k^{\star_D})(B_k \otimes C_k)(x \otimes y), x \otimes y\rangle_{A\otimes D} \\
&= \langle B_k^{\star_A} B_k x \otimes C_k^{\star_D} C_k y, x \otimes y\rangle_{A\otimes D} \\
&= \langle B_k^{\star_A} B_k x, x\rangle_A \langle C_k^{\star_D} C_k y, y\rangle_D \\
&\geq \langle B_k B_k^{\star_A} x, x\rangle_A \langle C_k C_k^{\star_D} y, y\rangle_D \\
&= \langle B_k B_k^{\star_A} x \otimes C_k C_k^{\star_D} y, x \otimes y\rangle_{A\otimes D} \\
&= \langle (B_k \otimes C_k)(B_k^{\star_A} \otimes C_k^{\star_D})(x \otimes y), x \otimes y\rangle_{A\otimes D} \\
&= \langle (B_k \otimes C_k)(B_k \otimes C_k)^{\star_{A\otimes D}}(x \otimes y), x \otimes y\rangle_{A\otimes D}.
\end{aligned}
$$

In other words,

$$
(B_k \otimes C_k)^{\star_{A\otimes D}}(B_k \otimes C_k) \geq_{A\otimes D} (B_k \otimes C_k)(B_k \otimes C_k)^{\star_{A\otimes D}}
$$

for all $k \in \{1, \dots, m\}$. This implies that $\mathbf{B} \otimes \mathbf{C}$ is an m-tuple of doubly commuting $A \otimes D$-hyponormal operators. ◻

Theorem 2.7. *Let $\mathbf{B} = (B_1, \cdots, B_m) \in \mathcal{L}_A(\mathcal{X})^m$ be a spherically A-hyponormal m-tuple and $\mathbf{C} = (C, \cdots, C) \in \mathcal{L}_D(\mathcal{Y})^m$, where $C \in \mathcal{L}_D(\mathcal{Y})$ is D-hyponormal. Then, $\mathbf{B} \otimes \mathbf{C}$ is spherically $A \otimes D$-hyponormal.*

Proof. Let $x \in \mathcal{X}$, $y \in \mathcal{Y}$ be arbitrary. Then, as in the proof of the previous theorem,

$$
\begin{aligned}
\langle \sum_{k=1}^{m} (B_k \otimes C)^{\star_{A\otimes D}}(B_k \otimes C)(x \otimes y), x \otimes y\rangle_{A\otimes D} &= \sum_{k=1}^{m} \langle B_k^{\star_A} B_k x, x\rangle_A \langle C^{\star_D} C y, y\rangle_D \\
&= \langle \sum_{k=1}^{m} B_k^{\star_A} B_k x, x\rangle_A \langle C^{\star_D} S y, y\rangle_D.
\end{aligned}
$$

Using the fact that $\mathbf{B}$ is spherically A-hyponormal, and that C is D-hyponormal, we have

$$
\langle \sum_{k=1}^{m} B_k^{\star_A} B_k x, x\rangle_A \langle S^{\star_D} S y, y\rangle_D \geq \langle \sum_{k=1}^{m} B_k B_k^{\star_A} x, x\rangle_A \langle C C^{\star_D} y, y\rangle_D
$$

$$= \langle \sum_{k=1}^{m} (B_k \otimes S)(B_k \otimes S)^{\star_{A\otimes D}}(x \otimes y), x \otimes y \rangle_{A\otimes D}.$$

This implies that

$$\sum_{k=1}^{m} (B_k \otimes C)^{\star_{A\otimes D}}(B_k \otimes C) \geq \sum_{k=1}^{m} (B_k \otimes S)(B_k \otimes C)^{\star_{A\otimes D}}.$$

In other words, $\mathbf{B} \otimes \mathbf{C}$ is spherically $A \otimes D$-hyponormal. □

3. Conclusions

In this paper, we have extended the concept of jointly A-hyponormal operators, as introduced by Guesba et al. in [13], by establishing several important properties for this class of operators on complex Hilbert spaces. Specifically, we proved that any jointly A-hyponormal m-tuple of commuting operators is also jointly A-normaloid. This result enabled us to derive a sharp bound for the distance between two jointly hyponormal m-tuples of operators in terms of the difference between their Taylor spectra, in the special case where A is the identity operator.

Furthermore, we introduced and examined the class of spherically A-p-hyponormal operators, extending the analysis to the case where $0 < p < 1$. We also explored the tensor product of specific classes of multivariable operators in semi-Hilbert spaces, contributing new insights into the structure of these operators.

Our results provide a foundation for further investigations into operator theory, particularly in the study of A-hyponormal operators, their spectral properties, and their applications in semi-Hilbert spaces. We believe that the techniques and findings presented here may stimulate future research in these areas, potentially leading to new developments in the theory of operator inequalities and multivariable operator systems.

Author contributions

The work was a collaborative effort of all authors, who contributed equally to writing the article. All authors have read and approved the final manuscript.

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Conflict of interest

The authors declare that they have no competing interests.

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