AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(11): 30320–30347.
DOI: 10.3934/math.20241463
Received: 04 July 2024
Revised: 27 August 2024
Accepted: 04 October 2024
Published: 24 October 2024

Research article

On the solutions of some systems of rational difference equations

M. T. Alharthi*

Department of Mathematics and Statistics, Faculty of Science, Jeddah University, Jeddah, Saudi Arabia

* **Correspondence:** Email: Mtalharthi@uj.edu.sa.

Abstract: In this paper, we considered some systems of rational difference equations of higher order as follows

$$\begin{aligned} u_{n+1} &= \frac{v_{n-6}}{1 \pm v_n u_{n-1} v_{n-2} u_{n-3} v_{n-4} u_{n-5} v_{n-6}}, \\ v_{n+1} &= \frac{u_{n-6}}{1 \pm u_n v_{n-1} u_{n-2} v_{n-3} u_{n-4} v_{n-5} u_{n-6}}, \end{aligned}$$

where the initial conditions $u_{0,}$ u_{-1}, u_{-2}, u_{-3}, u_{-4}, u_{-5}, u_{-6}, $v_{0,}$ v_{-1}, v_{-2}, v_{-3}, v_{-4}, v_{-5} and v_{-6} were arbitrary real numbers. We obtained a closed form of the solutions for each considered system and also some periodic solutions of some systems were found. We presented some numerical examples to explain the obtained theoretical results.

Keywords: difference equations; periodic solutions; stability; recursive sequences; system of difference equations
Mathematics Subject Classification: 39A10

1. Introduction

Difference equations are a very important topic in our real life as they describe many natural phenomena as mathematical models. They also arise naturally as discrete analogues and as numerical solutions of differential as well as delay differential equations having applications in many fields like physics, biology, ecology, population biology, economics, probability theory, genetics, psychology and many other fields. See, for example [1,8–12,19,20].

There are many populations in ecosystems that do not reproduce continuously but rather seasonally. Discrete-time models of difference equations are therefore important tools to study populations with nonoverlapping generations. For example, Elaydi and Yakubu [9] investigated cycles and dispersal in discrete population models, while Franke and Yakubu [15], Friedman and

Yakubu [16], and Yakubu [29] used discrete-time models to understand infectious diseases. The Beverton-Holt model is a classic discrete-time population model which gives the expected number x_{n+1} (or density) of individuals in generation $n+1$ as a function of the number of individuals in the previous generation, $x_{n+1} = \frac{Ax_n}{B+x_n}$; this model was introduced in the context of fisheries by Beverton & Holt [7].

Kulenovic and Ladas [22] considered the difference equation $u_{n+1} = \frac{\alpha+\beta u_n+\gamma u_{n-1}}{A+Bu_n+Cu_{n-1}}$, $n = 0, 1, ...$ and many of its special cases for study and investigation, where they believe that the results about the rational difference equations are of paramount importance in their own right and, furthermore, also believe that these results offer prototypes toward the development of the basic theory of the global behavior of solutions of nonlinear difference equations of order greater than one.

The difference equation $x_{n+1} = \frac{a+bx_n}{A+x_n}, n = 0, 1, ...$ is in optics and mathematical biology and is known in the literatures as the Riccati difference equation; see Saaty [27]. When $a = 0$ and $b > A$, this equation has been proposed by Pielou in her books [25, 26] as a discrete analogue of the delay logistic equation $\frac{dN(t)}{dt} = rN(t)[1 - \frac{N(t-\tau)}{P}], t \geq 0$. The equation $x_{n+1} = \frac{1+x_n}{x_{n-1}}, n = 0, 1, ...$ was discovered by Lyness [24] while he was working on a problem in number theory.

Asiri et al. [4] provided a description and interpretation of the solutions of the system

$$u_{n+1} = \frac{v_{n-2}}{1 - v_{n-2}u_{n-1}v_n}, \quad v_{n+1} = \frac{u_{n-2}}{\pm 1 \pm u_{n-2}v_{n-1}u_n},$$

where they obtained the closed form of the solutions for some different cases of the considered system and shown that some of these solutions are periodic of period six and others of period twelve.

Kurbanli et al. [23] presented a closed form of the following system of difference equations:

$$u_{n+1} = \frac{u_{n-1}}{v_n u_{n-1} - 1}, \quad v_{n+1} = \frac{v_{n-1}}{u_n v_{n-1} - 1}, \quad w_{n+1} = \frac{u_n}{v_n w_{n-1}},$$

and then studied the boundedness and the global stability of the solutions for the considered system.

Elsayed and Alharbi [14] obtained a closed form to the solutions for the systems

$$u_{n+1} = \frac{u_n v_{n-1}}{u_n + v_n}, \quad v_{n+1} = \frac{v_n u_{n-1}}{u_n + v_n},$$

and proved that the same system has periodic solutions of period twelve. Also they presented some qualitative results for the solutions of the considered system.

Touafek and Elsayed [28] highlighted the solutions and the periodic solutions of the following systems:

$$u_{n+1} = \frac{u_{n-3}}{\pm 1 \pm u_{n-3}v_{n-1}}, \quad v_{n+1} = \frac{v_{n-3}}{\pm 1 \pm v_{n-3}u_{n-1}}.$$

El-Metwally and Elsayed [13] considered the systems

$$u_{n+1} = \frac{v_{n-2}}{\pm 1 \pm v_{n-2}u_{n-1}v_n}, \quad v_{n+1} = \frac{u_{n-2}}{\pm 1 \pm u_{n-2}v_{n-1}u_n},$$

and studied four different cases of this system where they obtained a closed form of the solutions in every case. They also discussed the periodic nature and analyzed the solutions of the systems.

Akrour et al. [2] found the closed form of the following system of difference equations:

$$u_{n+1}=\frac{av_{n-2}u_{n-1}v_n+bu_{n-1}v_{n-2}+cv_{n-2}+d}{v_{n-2}u_{n-1}v_n},\ v_{n+1}=\frac{au_{n-2}v_{n-1}u_n+bv_{n-1}u_{n-2}+cu_{n-2}+d}{u_{n-2}v_{n-1}u_n}.$$

For more studies for nonlinear difference equations and systems of rational difference equations, see [3,5,6,17–21].

In this paper, we are concerned with solving and studying the qualitative behavior of the solutions of the systems:

$$\left.\begin{aligned} u_{n+1} &= \frac{v_{n-6}}{1\pm v_nu_{n-1}v_{n-2}u_{n-3}v_{n-4}u_{n-5}v_{n-6}},\\ v_{n+1} &= \frac{u_{n-6}}{1\pm u_nv_{n-1}u_{n-2}v_{n-3}u_{n-4}v_{n-5}u_{n-6}},\end{aligned}\right\} \tag{1.1}$$

where the initial conditions u_0, u_{-1}, u_{-2}, u_{-3}, u_{-4}, u_{-5}, u_{-6}, v_0, v_{-1}, v_{-2}, v_{-3}, v_{-4}, v_{-5}, and v_{-6} are arbitrary real numbers.

In the following, we consider the following four different cases of system (1.1):

Case I.

$$\left.\begin{aligned} u_{n+1} &= \frac{v_{n-6}}{1+v_nu_{n-1}v_{n-2}u_{n-3}v_{n-4}u_{n-5}v_{n-6}},\\ v_{n+1} &= \frac{u_{n-6}}{1+u_nv_{n-1}u_{n-2}v_{n-3}u_{n-4}v_{n-5}u_{n-6}},\end{aligned}\right\}. \tag{1.2}$$

Case II.

$$\left.\begin{aligned} u_{n+1} &= \frac{v_{n-6}}{1+v_nu_{n-1}v_{n-2}u_{n-3}v_{n-4}u_{n-5}v_{n-6}},\\ v_{n+1} &= \frac{u_{n-6}}{1-u_nv_{n-1}u_{n-2}v_{n-3}u_{n-4}v_{n-5}u_{n-6}},\end{aligned}\right\}. \tag{1.3}$$

Case III.

$$\left.\begin{aligned} u_{n+1} &= \frac{v_{n-6}}{1-v_nu_{n-1}v_{n-2}u_{n-3}v_{n-4}u_{n-5}v_{n-6}},\\ v_{n+1} &= \frac{u_{n-6}}{1+u_nv_{n-1}u_{n-2}v_{n-3}u_{n-4}v_{n-5}u_{n-6}},\end{aligned}\right\}. \tag{1.4}$$

Case IV.

$$\left.\begin{aligned} u_{n+1} &= \frac{v_{n-6}}{1-v_nu_{n-1}v_{n-2}u_{n-3}v_{n-4}u_{n-5}v_{n-6}},\\ v_{n+1} &= \frac{u_{n-6}}{1-u_nv_{n-1}u_{n-2}v_{n-3}u_{n-4}v_{n-5}u_{n-6}},\end{aligned}\right\}. \tag{1.5}$$

In the following, we set $u_0=a$, $u_{-1}=b$, $u_{-2}=c$, $u_{-3}=d$, $u_{-4}=e$, $u_{-5}=f$, $u_{-6}=g$, $v_0=h$, $v_{-1}=k$, $v_{-2}=l$, $v_{-3}=p$, $v_{-4}=q$, $v_{-5}=r$, and $v_{-6}=t$.

2. Dynamics of Case I

In this section, we obtain a closed form for the solution of the following system of difference equations

$$\left.\begin{aligned} u_{n+1} &= \frac{v_{n-6}}{1+v_nu_{n-1}v_{n-2}u_{n-3}v_{n-4}u_{n-5}v_{n-6}},\\ v_{n+1} &= \frac{u_{n-6}}{1+u_nv_{n-1}u_{n-2}v_{n-3}u_{n-4}v_{n-5}u_{n-6}},\end{aligned}\right\} \tag{2.1}$$

where the initial conditions u_0, u_{-1}, u_{-2}, u_{-3}, u_{-4}, u_{-5}, u_{-6}, v_0, v_{-1}, v_{-2}, v_{-3}, v_{-4}, v_{-5}, and v_{-6} are arbitrary real numbers.

Theorem 2.1. *Assume that $\{u_n, v_n\}_{n=-6}^{\infty}$ is a solution of system (2.1). Then for $n = 0, 1, 2, \ldots$, the following formulas provide the solution for system (2.1):*

$$u_{14n-6} = \frac{g\prod_{j=1}^{2n}[7(j-1)acegkpr+1]}{\prod_{j=0}^{2n-1}[(7j+1)acegkpr+1]},$$

$$u_{14n-5} = f\prod_{j=1}^{2n}\frac{[(7j-6)bdfhlqt+1]}{[(7j-5)bdfhlqt+1]},$$

$$u_{14n-4} = e\prod_{j=1}^{2n}\frac{[(7j-5)acegkpr+1]}{[(7j-4)acegkpr+1]},$$

$$u_{14n-3} = d\prod_{j=1}^{2n}\frac{[(7j-4)bdfhlqt+1]}{[(7j-3)bdfhlqt+1]},$$

$$u_{14n-2} = c\prod_{j=1}^{2n}\frac{[(7j-3)acegkpr+1]}{[(7j-2)acegkpr+1]},$$

$$u_{14n-1} = b\prod_{j=1}^{2n}\frac{[(7j-2)bdfhlqt+1]}{[(7j-1)bdfhlqt+1]},$$

$$u_{14n} = a\prod_{j=1}^{2n}\frac{[(7j-1)acegkpr+1]}{[(7j)acegkpr+1]},$$

$$u_{14n+1} = \frac{t\prod_{j=1}^{2n}[7j)bdfhlqt+1]}{\prod_{j=0}^{2n}[(7j+1)bdfhlqt+1]},$$

$$u_{14n+2} = r\prod_{j=0}^{2n}\frac{[(7j+1)acegkpr+1]}{[(7j+2)acegkpr+1]},$$

$$u_{14n+3} = q\prod_{j=0}^{2n}\frac{[(7j+2)bdfhlqt+1]}{[(7j+3)bdfhlqt+1]},$$

$$u_{14n+4} = p\prod_{j=0}^{2n}\frac{[(7j+3)acegkpr+1]}{[(7j+4)acegkpr+1]},$$

$$u_{14n+5} = l\prod_{j=0}^{2n}\frac{[(7j+4)bdfhlqt+1]}{[(7j+5)bdfhlqt+1]},$$

$$u_{14n+6} = k\prod_{j=0}^{2n}\frac{[(7j+5)acegkpr+1]}{[7j+6)acegkpr+1]},$$

$$u_{14n+7} = h\prod_{j=0}^{2n}\frac{[(7j+6)bdfhlqt+1]}{[7(j+1)bdfhlqt+1]},$$

$$v_{14m-6} = t\prod_{j=0}^{2m-1}\frac{[(7j)bdfhlqt+1]}{[(7j+1)bdfhlqt+1]},$$

$$v_{14m-5} = r\prod_{j=0}^{2m-1}\frac{[(7j+1)acegkpr+1]}{[(7j+2)acegkpr+1]},$$

$$v_{14m-4} = q\prod_{j=0}^{2m-1}\frac{[(7j+2)bdfhlqt+1]}{[(7j+3)bdfhlqt+1]},$$

$$v_{14m-3} = p\prod_{j=0}^{2m-1}\frac{[(7j+3)acegkpr+1]}{[(7j+4)acegkpr+1]},$$

$$v_{14m-2} = l\prod_{j=0}^{2m-1}\frac{[(7j+4)bdfhlqt+1]}{[(7j+5)bdfhlqt+1]},$$

$$v_{14m-1} = k\prod_{j=0}^{2m-1}\frac{[(7j+5)acegkpr+1]}{[(7j+6)acegkpr+1]},$$

$$v_{14m} = h\prod_{j=0}^{2m-1}\frac{[(7j+6)bdfhlqt+1]}{[(7(j+1)bdfhlqt+1]},$$

$$v_{14m+1} = \frac{g\prod_{j=1}^{2m}[(7j)acegkpr+1]}{\prod_{j=0}^{2m}[(7j+1)acegkpr+1]},$$

$$v_{14n+2} = f\prod_{j=0}^{2n}\frac{[(7j+1)bdfhlqt+1]}{[(7j+2)bdfhlqt+1]},$$

$$v_{14n+3} = e\prod_{j=0}^{2n}\frac{[(7j+2)acegkpr+1]}{[(7j+3)acegkpr+1]},$$

$$v_{14n+4} = d\prod_{j=0}^{2n}\frac{[(7j+3)bdfhlqt+1]}{[(7j+4)bdfhlqt+1]},$$

$$v_{14n+5} = c\prod_{j=0}^{2n}\frac{[(7j+4)acegkpr+1]}{[(7j+5)acegkpr+1]},$$

$$v_{14n+6} = b\prod_{j=0}^{2n}\frac{[(7j+5)bdfhlqt+1]}{[(7j+6)bdfhlqt+1]},$$

and

$$v_{14n+7} = a\prod_{j=0}^{2n}\frac{[(7j+6)acegkpr+1]}{[7(j+1)acegkpr+1]}.$$

where $u_0 = a$, $u_{-1} = b$, $u_{-2} = c$, $u_{-3} = d$, $u_{-4} = e$, $u_{-5} = f$, $u_{-6} = g$, $v_0 = h$, $v_{-1} = k$, $v_{-2} = l$, $v_{-3} = p$, $v_{-4} = q$, $v_{-5} = r$ and, $v_{-6} = t$.

Proof. For n = 0, the result holds. Now suppose that n > 0 and that our assumption holds for n−1, that is,

$$u_{14n-20} = \frac{g\prod_{j=1}^{2n-2}[7(j-1)acegkpr+1]}{\prod_{j=0}^{2n-3}[(7j+1)acegkpr+1]},$$

$$u_{14n-19} = f\prod_{j=1}^{2n-2}\frac{[(7j-6)bdfhlqt+1]}{[(7j-5)bdfhlqt+1]},$$

$$u_{14n-18} = e\prod_{j=1}^{2n-2}\frac{[(7j-5)acegkpr+1]}{[(7j-4)acegkpr+1]},$$

$$u_{14n-17} = d\prod_{j=1}^{2n-2}\frac{[(7j-4)bdfhlqt+1]}{[(7j-3)bdfhlqt+1]},$$

$$u_{14n-16} = c\prod_{j=1}^{2n-2}\frac{[(7j-3)acegkpr+1]}{[(7j-2)acegkpr+1]},$$

$$u_{14n-15} = b\prod_{j=1}^{2n-2}\frac{[(7j-2)bdfhlqt+1]}{[(7j-1)bdfhlqt+1]},$$

$$u_{14n-14} = a\prod_{j=1}^{2n-2}\frac{[(7j-1)acegkpr+1]}{[(7j)acegkpr+1]},$$

$$u_{14n-13} = \frac{t\prod_{j=1}^{2n-2}[7j)bdfhlqt+1]}{\prod_{j=0}^{2n-2}[(7j+1)bdfhlqt+1]},$$

$$u_{14n-12} = r\prod_{j=0}^{2n-2}\frac{[(7j+1)acegkpr+1]}{[(7j+2)acegkpr+1]},$$

$$u_{14n-11} = q\prod_{j=0}^{2n-2}\frac{[(7j+2)bdfhlqt+1]}{[(7j+3)bdfhlqt+1]},$$

$$u_{14n-10} = p\prod_{j=0}^{2n-2}\frac{[(7j+3)acegkpr+1]}{[(7j+4)acegkpr+1]},$$

$$u_{14n-9} = l\prod_{j=0}^{2n-2}\frac{[(7j+4)bdfhlqt+1]}{[(7j+5)bdfhlqt+1]},$$

$$u_{14n-8} = k\prod_{j=0}^{2n-2}\frac{[(7j+5)acegkpr+1]}{[7j+6)acegkpr+1]},$$

$$u_{14n-7} = h\prod_{j=0}^{2n-2}\frac{[(7j+6)bdfhlqt+1]}{[7(j+1)bdfhlqt+1]},$$

and

$$v_{14n-20} = t\prod_{j=0}^{2n-3}\frac{[(7j)bdfhlqt+1]}{[(7j+1)bdfhlqt+1]},$$

$$v_{14n-19} = r\prod_{j=0}^{2n-3}\frac{[(7j+1)acegkpr+1]}{[(7j+2)acegkpr+1]},$$

$$v_{14n-18} = q\prod_{j=0}^{2n-3}\frac{[(7j+2)bdfhlqt+1]}{[(7j+3)bdfhlqt+1]},$$

$$v_{14n-17} = p\prod_{j=0}^{2n-3}\frac{[(7j+3)acegkpr+1]}{[(7j+4)acegkpr+1]},$$

$$v_{14n-16} = l\prod_{j=0}^{2n-3}\frac{[(7j+4)bdfhlqt+1]}{[(7j+5)bdfhlqt+1]},$$

$$v_{14n-15} = k\prod_{j=0}^{2n-3}\frac{[(7j+5)acegkpr+1]}{[(7j+6)acegkpr+1]},$$

$$v_{14n-14} = h\prod_{j=0}^{2n-3}\frac{[(7j+6)bdfhlqt+1]}{[(7(j+1)bdfhlqt+1]},$$

$$v_{14n-13} = \frac{g\prod_{j=1}^{2n-2}[(7j)acegkpr+1]}{\prod_{j=0}^{2n-2}[(7j+1)acegkpr+1]},$$

$$v_{14n-12} = f\prod_{j=0}^{2n-2}\frac{[(7j+1)bdfhlqt+1]}{[(7j+2)bdfhlqt+1]},$$

$$v_{14n-11} = e\prod_{j=0}^{2n-2}\frac{[(7j+2)acegkpr+1]}{[(7j+3)acegkpr+1]},$$

$$\begin{aligned}
v_{14n-10} &= d\prod_{j=0}^{2n-2}\frac{[(7j+3)bdfhlqt+1]}{[(7j+4)bdfhlqt+1]},\\
v_{14n-9} &= c\prod_{j=0}^{2n-2}\frac{[(7j+4)acegkpr+1]}{[(7j+5)acegkpr+1]},\\
v_{14n-8} &= b\prod_{j=0}^{2n-2}\frac{[(7j+5)bdfhlqt+1]}{[(7j+6)bdfhlqt+1]},\\
v_{14n-7} &= a\prod_{j=0}^{2n-2}\frac{[(7j+6)acegkpr+1]}{[7(j+1)acegkpr+1]}.
\end{aligned}$$

Now, we want to prove u_{14n-6}, so from system (2.1) we have

$$\begin{aligned}
u_{14n-6} &= \frac{v_{14n-13}}{1+v_{14n-7}u_{14n-8}v_{14n-9}u_{14n-10}v_{14n-11}u_{14n-12}v_{14n-13}}\\
&= \frac{1}{(1/v_{14n-13})+v_{14n-7}u_{14n-8}v_{14n-9}u_{14n-10}v_{14n-11}u_{14n-12}}\\
&= \frac{g\prod_{j=1}^{2n-2}[(7j)acegkpr+1]}{\prod_{j=0}^{2n-2}[(7j+1)acegkpr+1]+g\prod_{j=1}^{2n-2}[(7j)acegkpr+1]v_{14n-7}...u_{14n-12}}.
\end{aligned}$$

Now, we see that

$$\begin{aligned}
v_{14n-7}u_{14n-8}v_{14n-9}u_{14n-10}v_{14n-11}u_{14n-12} = a\prod_{j=0}^{2n-2}\frac{[(7j+6)acegkpr+1]}{[7(j+1)acegkpr+1]}\\
\times k\prod_{j=0}^{2n-2}\frac{[(7j+5)acegkpr+1]}{[7j+6)acegkpr+1]}\times c\prod_{j=0}^{2n-2}\frac{[(7j+4)acegkpr+1]}{[(7j+5)acegkpr+1]}\\
\times p\prod_{j=0}^{2n-2}\frac{[(7j+3)acegkpr+1]}{[(7j+4)acegkpr+1]}\times e\prod_{j=0}^{2n-2}\frac{[(7j+2)acegkpr+1]}{[(7j+3)acegkpr+1]}\\
\times r\prod_{j=0}^{2n-2}\frac{[(7j+1)acegkpr+1]}{[(7j+2)acegkpr+1]}\\
= acekpr\prod_{j=0}^{2n-2}\frac{[(7j+1)acegkpr+1]}{[7(j+1)acegkpr+1]}.
\end{aligned}$$

Again,

$$\prod_{j=0}^{2n-2}[(7j+1)acegkpr+1]+g\prod_{j=1}^{2n-2}[(7j)acegkpr+1]v_{14n-7}...u_{14n-12}$$

$$
\begin{aligned}
&= \prod_{j=0}^{2n-2}[(7j+1)acegkpr+1]\{1+\frac{acegkpr}{[7(2n-1)acegkpr+1]}\} \\
&= \prod_{j=0}^{2n-2}[(7j+1)acegkpr+1].\frac{[7(2n-1)acegkpr+1]+acegkpr}{[7(2n-1)acegkpr+1]} \\
&= \frac{\prod_{j=0}^{2n-2}[(7j+1)acegkpr+1].[(14n-6)acegkpr+1]}{[7(2n-1)acegkpr+1]} \\
&= \frac{\prod_{j=0}^{2n-1}[(7j+1)acegkpr+1]}{[7(2n-1)acegkpr+1]}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
& g\prod_{j=1}^{2n-2}[(7j)acegkpr+1]v_{14n-7}...u_{14n-12} \\
&= acegkpr\prod_{j=1}^{2n-2}[(7j)acegkpr+1]\prod_{j=0}^{2n-2}\frac{[(7j+1)acegkpr+1]}{[7(j+1)acegkpr+1]} \\
&= \frac{acegkpr\prod_{j=0}^{2n-2}[(7j+1)acegkpr+1]}{[7(2n-1)acegkpr+1]}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
u_{14n-6} &= g\prod_{j=1}^{2n-2}[(7j)acegkpr+1] \div \frac{\prod_{j=0}^{2n-1}[(7j+1)acegkpr+1]}{[7(2n-1)acegkpr+1]} \\
&= \frac{g\prod_{j=1}^{2n-2}[(7j)acegkpr+1].[7(2n-1)acegkpr+1]}{\prod_{j=0}^{2n-1}[(7j+1)acegkpr+1]} \\
&= \frac{g\prod_{j=1}^{2n-1}[(7j)acegkpr+1]}{\prod_{j=0}^{2n-1}[(7j+1)acegkpr+1]} = \frac{g\prod_{j=1}^{2n}[7(j-1)acegkpr+1]}{\prod_{j=0}^{2n-1}[(7j+1)acegkpr+1]}.
\end{aligned}
$$

Again, from system (2.1), we can obtain that

$$
\begin{aligned}
v_{14n-6} &= \frac{u_{14n-13}}{1+u_{14n-7}v_{14n-8}u_{14n-9}v_{14n-10}u_{14n-11}v_{14n-12}u_{14n-13}} \\
&= \frac{1}{(1/u_{14n-13})+u_{14n-7}v_{14n-8}u_{14n-9}v_{14n-10}u_{14n-11}v_{14n-12}}
\end{aligned}
$$

$$= \; = \frac{t\prod_{j=1}^{2n-2}[(7j)bdfhlqt+1]}{\prod_{j=0}^{2n-2}[(7j+1)bdfhlqt+1]+g\prod_{j=1}^{2n-2}[(7j)bdfhlqt+1]u_{14n-7}...v_{14n-12}}.$$

Now, we see that

$$u_{14n-7}v_{14n-8}u_{14n-9}v_{14n-10}u_{14n-11}v_{14n-12} = h\prod_{j=0}^{2n-2}\frac{[(7j+6)bdfhlqt+1]}{[7(j+1)bdfhlqt+1]}$$

$$\begin{aligned}
&\times b\prod_{j=0}^{2n-2}\frac{[(7j+5)bdfhlqt+1]}{[7j+6)bdfhlqt+1]}\times l\prod_{j=0}^{2n-2}\frac{[(7j+4)bdfhlqt+1]}{[(7j+5)bdfhlqt+1]}\\
&\times d\prod_{j=0}^{2n-2}\frac{[(7j+3)bdfhlqt+1]}{[(7j+4)bdfhlqt+1]}\times q\prod_{j=0}^{2n-2}\frac{[(7j+2)bdfhlqt+1]}{[(7j+3)bdfhlqt+1]}\\
&\times f\prod_{j=0}^{2n-2}\frac{[(7j+1)bdfhlqt+1]}{[(7j+2)bdfhlqt+1]}\\
= \; & bdfhlqt\prod_{j=0}^{2n-2}\frac{[(7j+1)bdfhlqt+1]}{[7(j+1)bdfhlqt+1]}.
\end{aligned}$$

Again,

$$\begin{aligned}
&\prod_{j=0}^{2n-2}[(7j+1)bdfhlqt+1]+t\prod_{j=1}^{2n-2}[(7j)bdfhlqt+1]u_{14n-7}...v_{14n-12}\\
= \; & \prod_{j=0}^{2n-2}[(7j+1)bdfhlqt+1]\{1+\frac{bdfhlqt}{[7(2n-1)bdfhlqt+1]}\}\\
= \; & \prod_{j=0}^{2n-2}[(7j+1)bdfhlqt+1].\frac{[7(2n-1)bdfhlqt+1]+bdfhlqt}{[7(2n-1)bdfhlqt+1]}\\
= \; & \frac{\prod_{j=0}^{2n-2}[(7j+1)bdfhlqt+1].[(14n-6)bdfhlqt+1]}{[7(2n-1)bdfhlqt+1]}\\
= \; & \frac{\prod_{j=0}^{2n-1}[(7j+1)bdfhlqt+1]}{[7(2n-1)bdfhlqt+1]}.
\end{aligned}$$

Thus,

$$v_{14n-6} \; = \; t\prod_{j=1}^{2n-2}[(7j)bdfhlqt+1]\div\frac{\prod_{j=0}^{2n-1}[(7j+1)bdfhlqt+1]}{[7(2n-1)bdfhlqt+1]}$$

$$= \frac{t\prod_{j=1}^{2n-2}[(7j)bdfhlqt+1].[7(2n-1)bdfhlqt+1]}{\prod_{j=0}^{2n-1}[(7j+1)bdfhlqt+1]}$$

$$= \frac{t\prod_{j=0}^{2n-1}[(7j)bdfhlqt+1]}{\prod_{j=0}^{2n-1}[(7j+1)bdfhlqt+1]}.$$

Therefore, similarly the other relations can be proven. The proof is completed.

3. Dynamics of Case II

We examine in this section the solutions of the system of two difference equations of the following form

$$\left.\begin{array}{l} u_{n+1} = \frac{v_{n-6}}{1+v_n u_{n-1} v_{n-2} u_{n-3} v_{n-4} u_{n-5} v_{n-6}}, \\ v_{n+1} = \frac{u_{n-6}}{1-u_n v_{n-1} u_{n-2} v_{n-3} u_{n-4} v_{n-5} u_{n-6}}, \end{array}\right\} \tag{3.1}$$

where the initial conditions $u_{0,}$ u_{-1}, u_{-2}, u_{-3}, u_{-4}, u_{-5}, u_{-6}, $v_{0,}$ v_{-1}, v_{-2}, v_{-3}, v_{-4}, v_{-5} and v_{-6} are arbitrary real numbers.

Theorem 3.1. *Assume that $\{u_n, v_n\}_{n=-6}^{\infty}$ is a solution of system (3.1). We observe that the solutions to the system (3.1) are periodic solutions of period fourteen and take the following forms:*

$$\begin{aligned} u_{14n-6} &= g, \\ u_{14n-5} &= f, \\ u_{14n-4} &= e, \\ u_{14n-3} &= d, \\ u_{14n-2} &= c, \\ u_{14n-1} &= b, \\ u_{14n} &= a, \end{aligned}$$

$$\begin{aligned} u_{14n+1} &= \frac{t}{bdfhlqt+1}, \\ u_{14n+2} &= -r(acegkpr-1), \\ u_{14n+3} &= \frac{q}{bdfhlqt+1}, \\ u_{14n+4} &= -p(acegkpr-1), \\ u_{14n+5} &= \frac{l}{bdfhlqt+1}, \\ u_{14n+6} &= -k(acegkpr-1), \end{aligned}$$

$$u_{14n+7} = \frac{h}{bdfhlqt+1},$$

$$\begin{aligned}
v_{14n+1} &= -\frac{g}{acegkpr-1},\\
v_{14n+2} &= f(bdfhlqt+1),\\
v_{14n+3} &= -\frac{e}{acegkpr-1},\\
v_{14n+4} &= d(bdfhlqt+1),\\
v_{14n+5} &= -\frac{c}{acegkpr-1},\\
v_{14n+6} &= b(bdfhlqt+1),
\end{aligned}$$

and

$$v_{14n+7} = -\frac{a}{acegkpr-1},$$

Proof. By using mathematical induction, the result holds for n = 0. Suppose that the result holds for $n-1$

$$\begin{aligned}
u_{14n-20} &= g,\\
u_{14n-19} &= f,\\
u_{14n-18} &= e,\\
u_{14n-17} &= d,\\
u_{14n-16} &= c,\\
u_{14n-15} &= b,\\
u_{14n-14} &= a,
\end{aligned}$$

$$\begin{aligned}
u_{14n-13} &= \frac{t}{bdfhlqt+1},\\
u_{14n-12} &= -r(acegkpr-1),\\
u_{14n-11} &= \frac{q}{bdfhlqt+1},\\
u_{14n-10} &= -p(acegkpr-1),\\
u_{14n-9} &= \frac{l}{bdfhlqt+1},\\
u_{14n-8} &= -k(acegkpr-1),\\
u_{14n-7} &= \frac{h}{bdfhlqt+1},
\end{aligned}$$

and

$$\begin{aligned}
v_{14n-20} &= t,\\
v_{14n-19} &= r,
\end{aligned}$$

$$
\begin{aligned}
v_{14n-18} &= q, \\
v_{14n-17} &= p, \\
v_{14n-16} &= l, \\
v_{14n-15} &= k, \\
v_{14n-14} &= h,
\end{aligned}
$$

$$
\begin{aligned}
v_{14n-13} &= -\frac{g}{acegkpr-1}, \\
v_{14n-12} &= f(bdfhlqt+1), \\
v_{14n-11} &= -\frac{e}{acegkpr-1}, \\
v_{14n-10} &= d(bdfhlqt+1), \\
v_{14n-9} &= -\frac{c}{acegkpr-1}, \\
v_{14n-8} &= b(bdfhlqt+1), \\
v_{14n-7} &= -\frac{a}{acegkpr-1},
\end{aligned}
$$

From system (3.1), we have

$$
\begin{aligned}
u_{14n-6} &= \frac{v_{14n-13}}{1+v_{14n-7}u_{14n-8}v_{14n-9}u_{14n-10}v_{14n-11}u_{14n-12}v_{14n-13}} \\
&= \frac{-\frac{g}{acegkpr-1}}{1+\frac{-a}{acegkpr-1}\cdot -k(acegkpr-1)\cdot\frac{-c}{acegkpr-1}\cdot -p(acegkpr-1)\cdot\frac{-e}{acegkpr-1}\cdot -r(acegkpr-1)\cdot\frac{-g}{acegkpr-1}} \\
&= \frac{-\frac{g}{acegkpr-1}}{1-\frac{acegkpr}{acegkpr-1}} = g.
\end{aligned}
$$

Also,

$$
\begin{aligned}
v_{14n-6} &= \frac{u_{14n-13}}{1-u_{14n-7}v_{14n-8}u_{14n-9}v_{14n-10}u_{14n-11}v_{14n-12}u_{14n-13}} \\
&= \frac{\frac{t}{bdfhlqt+1}}{1-\frac{h}{bdfhlqt+1}\cdot b(bdfhlqt+1)\cdot\frac{l}{bdfhlqt+1}\cdot d(bdfhlqt+1)\cdot\frac{q}{bdfhlqt+1}\cdot f(bdfhlqt+1)\cdot\frac{t}{bdfhlqt+1}} \\
&= \frac{\frac{t}{bdfhlqt+1}}{1-\frac{bdfhlqt}{bdfhlqt+1}} = t.
\end{aligned}
$$

Similarly, we can prove other relations and the proof is completed.

4. Dynamics of Case III

In this part, we deal with the solutions of the following system of difference equations:

$$\left.\begin{aligned} u_{n+1} &= \frac{v_{n-6}}{1 - v_n u_{n-1} v_{n-2} u_{n-3} v_{n-4} u_{n-5} v_{n-6}}, \\ v_{n+1} &= \frac{u_{n-6}}{1 + u_n v_{n-1} u_{n-2} v_{n-3} u_{n-4} v_{n-5} u_{n-6}}. \end{aligned}\right\} \tag{4.1}$$

Theorem 4.1. *Suppose that* $\{u_n, v_n\}_{n=-6}^{\infty}$ *is a solution of system (4.1). We observe that the solutions for system (4.1) are periodic solutions of period fourteen and take the following forms:*

$$\begin{aligned} u_{14n-6} &= g, \\ u_{14n-5} &= f, \\ u_{14n-4} &= e, \\ u_{14n-3} &= d, \\ u_{14n-2} &= c, \\ u_{14n-1} &= b, \\ u_{14n} &= a, \\ u_{14n+1} &= -\frac{t}{bdfhlqt - 1}, \\ u_{14n+2} &= r(acegkpr + 1), \\ u_{14n+3} &= -\frac{q}{bdfhlqt - 1}, \\ u_{14n+4} &= p(acegkpr + 1), \\ u_{14n+5} &= -\frac{l}{bdfhlqt - 1}, \\ u_{14n+6} &= k(acegkpr + 1), \\ u_{14n+7} &= -\frac{h}{bdfhlqt - 1}, \\ v_{14n-6} &= t, \\ v_{14n-5} &= r, \\ v_{14n-4} &= q, \\ v_{14n-3} &= p, \\ v_{14n-2} &= l, \end{aligned}$$

$$\begin{aligned} v_{14n-1} &= k, \\ v_{14n} &= h, \\ v_{14n+1} &= \frac{g}{acegkpr + 1}, \\ v_{14n+2} &= -f(bdfhlqt - 1), \\ v_{14n+3} &= \frac{e}{acegkpr + 1}, \\ v_{14n+4} &= -d(bdfhlqt - 1), \\ v_{14n+5} &= \frac{c}{acegkpr + 1}, \end{aligned}$$

$$v_{14n+6} = -b(bdfhlqt-1),$$

and

$$v_{14n+7} = \frac{a}{acegkpr+1},$$

Proof. For n = 0, the result holds. Now, suppose that n > 0 and that our assumption holds for n−1, that is

$$\begin{aligned}
u_{14n-20} &= g,\\
u_{14n-19} &= f,\\
u_{14n-18} &= e,\\
u_{14n-17} &= d,\\
u_{14n-16} &= c,\\
u_{14n-15} &= b,\\
u_{14n-14} &= a,
\end{aligned}$$

$$\begin{aligned}
u_{14n-13} &= -\frac{t}{bdfhlqt-1},\\
u_{14n-12} &= r(acegkpr+1),\\
u_{14n-11} &= -\frac{q}{bdfhlqt-1},\\
u_{14n-10} &= p(acegkpr+1),\\
u_{14n-9} &= -\frac{l}{bdfhlqt-1},\\
u_{14n-8} &= k(acegkpr+1),\\
u_{14n-7} &= -\frac{h}{bdfhlqt-1},
\end{aligned}$$

$$\begin{aligned}
v_{14n-20} &= t,\\
v_{14n-19} &= r,\\
v_{14n-18} &= q,\\
v_{14n-17} &= p,\\
v_{14n-16} &= l,\\
v_{14n-15} &= k,\\
v_{14n-14} &= h,
\end{aligned}$$

$$\begin{aligned}
v_{14n-13} &= \frac{g}{acegkpr+1},\\
v_{14n-12} &= -f(bdfhlqt-1),\\
v_{14n-11} &= \frac{e}{acegkpr+1},\\
v_{14n-10} &= -d(bdfhlqt-1),
\end{aligned}$$

$$v_{14n-9} = \frac{c}{acegkpr+1},$$
$$v_{14n-8} = -b(bdfhlqt-1),$$
$$v_{14n-7} = \frac{a}{acegkpr+1}.$$

Next, one can obtain from system (4.1) that

$$\begin{aligned} u_{14n-6} &= \frac{v_{14n-13}}{1 - v_{14n-7}u_{14n-8}v_{14n-9}u_{14n-10}v_{14n-11}u_{14n-12}v_{14n-13}} \\ &= \frac{\frac{g}{acegkpr+1}}{1 - \frac{a}{acegkpr+1} \cdot k(acegkpr+1) \cdot \frac{c}{acegkpr+1} \cdot p(acegkpr+1) \cdot \frac{e}{acegkpr+1} \cdot r(acegkpr+1) \cdot \frac{g}{acegkpr+1}} \\ &= \frac{\frac{g}{acegkpr+1}}{1 - \frac{acegkpr}{acegkpr+1}} = g. \end{aligned}$$

and

$$\begin{aligned} v_{14n-6} &= \frac{u_{14n-13}}{1 + u_{14n-7}v_{14n-8}u_{14n-9}v_{14n-10}u_{14n-11}v_{14n-12}u_{14n-13}} \\ &= \frac{\frac{-t}{bdfhlqt-1}}{1 + \frac{-h}{bdfhlqt+1} \cdot -b(bdfhlqt+1) \cdot \frac{-l}{bdfhlqt+1} \cdot -d(bdfhlqt-1) \cdot \frac{-q}{bdfhlqt-1} \cdot -f(bdfhlqt-1) \cdot \frac{t}{bdfhlqt-1}} \\ &= \frac{\frac{-t}{bdfhlqt-1}}{1 - \frac{bdfhlqt}{bdfhlqt-1}} = t. \end{aligned}$$

Accordingly, the remaining relations can be proven in a similar way. Hence, the proof has been achieved.

5. Dynamics of Case IV

This section is allocated to determine formulas for the solutions of the following nonlinear system of difference equations:

$$\left.\begin{aligned} u_{n+1} &= \frac{v_{n-6}}{1 - v_n u_{n-1} v_{n-2} u_{n-3} v_{n-4} u_{n-5} v_{n-6}}, \\ v_{n+1} &= \frac{u_{n-6}}{1 - u_n v_{n-1} u_{n-2} v_{n-3} u_{n-4} v_{n-5} u_{n-6}}. \end{aligned}\right\} \tag{5.1}$$

Theorem 5.1. *Assume that $\{u_n, v_n\}$ is a solution for system (5.1). Then, for n = 0, 1,..., we have*

$$u_{14n-6} = \frac{-g \prod_{j=1}^{2n} [7(j-1)acegkpr - 1]}{\prod_{j=0}^{2n-1} [(7j+1)acegkpr - 1]},$$

$$u_{14n-5} = f\prod_{j=1}^{2n}\frac{[(7j-6)bdfhlqt-1]}{[(7j-5)bdfhlqt-1]},$$

$$u_{14n-4} = e\prod_{j=1}^{2n}\frac{[(7j-5)acegkpr-1]}{[(7j-4)acegkpr-1]},$$

$$u_{14n-3} = d\prod_{j=1}^{2n}\frac{[(7j-4)bdfhlqt-1]}{[(7j-3)bdfhlqt-1]},$$

$$u_{14n-2} = c\prod_{j=1}^{2n}\frac{[(7j-3)acegkpr-1]}{[(7j-2)acegkpr-1]},$$

$$u_{14n-1} = b\prod_{j=1}^{2n}\frac{[(7j-2)bdfhlqt-1]}{[(7j-1)bdfhlqt-1]},$$

$$u_{14n} = a\prod_{j=1}^{2n}\frac{[(7j-1)acegkpr-1]}{[(7j)acegkpr-1]},$$

$$u_{14n+1} = \frac{-t\prod_{j=1}^{2n}[7j)bdfhlqt-1]}{\prod_{j=0}^{2n}[(7j+1)bdfhlqt-1]},$$

$$u_{14n+2} = r\prod_{j=0}^{2n}\frac{[(7j+1)acegkpr-1]}{[(7j+2)acegkpr-1]},$$

$$u_{14n+3} = q\prod_{j=0}^{2n}\frac{[(7j+2)bdfhlqt-1]}{[(7j+3)bdfhlqt-1]},$$

$$u_{14n+4} = p\prod_{j=0}^{2n}\frac{[(7j+3)acegkpr+1]}{[(7j+4)acegkpr+1]},$$

$$u_{14n+5} = l\prod_{j=0}^{2n}\frac{[(7j+4)bdfhlqt-1]}{[(7j+5)bdfhlqt-1]},$$

$$u_{14n+6} = k\prod_{j=0}^{2n}\frac{[(7j+5)acegkpr-1]}{[7j+6)acegkpr-1]},$$

$$u_{14n+7} = h\prod_{j=0}^{2n}\frac{[(7j+6)bdfhlqt-1]}{[7(j+1)bdfhlqt-1]},$$

and

$$v_{14n-6} = -t\prod_{j=0}^{2n-1}\frac{[(7j)bdfhlqt-1]}{[(7j+1)bdfhlqt-1]},$$

$$
\begin{aligned}
v_{14n-5} &= r\prod_{j=0}^{2n-1}\frac{[(7j+1)acegkpr-1]}{[(7j+2)acegkpr-1]},\\
v_{14n-4} &= q\prod_{j=0}^{2n-1}\frac{[(7j+2)bdfhlqt-1]}{[(7j+3)bdfhlqt-1]},\\
v_{14n-3} &= p\prod_{j=0}^{2n-1}\frac{[(7j+3)acegkpr-1]}{[(7j+4)acegkpr-1]},
\end{aligned}
$$

$$
\begin{aligned}
v_{14n-2} &= l\prod_{j=0}^{2n-1}\frac{[(7j+4)bdfhlqt-1]}{[(7j+5)bdfhlqt-1]},\\
v_{14n-1} &= k\prod_{j=0}^{2n-1}\frac{[(7j+5)acegkpr-1]}{[(7j+6)acegkpr-1]},\\
v_{14n} &= h\prod_{j=0}^{2n-1}\frac{[(7j+6)bdfhlqt-1]}{[(7(j+1)bdfhlqt-1]},\\
v_{14n+1} &= \frac{-g\prod_{j=1}^{2n}[(7j)acegkpr-1]}{\prod_{j=0}^{2n}[(7j+1)acegkpr-1]},
\end{aligned}
$$

$$
\begin{aligned}
v_{14n+2} &= f\prod_{j=0}^{2n}\frac{[(7j+1)bdfhlqt-1]}{[(7j+2)bdfhlqt-1]},\\
v_{14n+3} &= e\prod_{j=0}^{2n}\frac{[(7j+2)acegkpr-1]}{[(7j+3)acegkpr-1]},\\
v_{14n+4} &= d\prod_{j=0}^{2n}\frac{[(7j+3)bdfhlqt-1]}{[(7j+4)bdfhlqt-1]},\\
v_{14n+5} &= c\prod_{j=0}^{2n}\frac{[(7j+4)acegkpr-1]}{[(7j+5)acegkpr-1]},\\
v_{14n+6} &= b\prod_{j=0}^{2n}\frac{[(7j+5)bdfhlqt-1]}{[(7j+6)bdfhlqt-1]},\\
v_{14n+7} &= a\prod_{j=0}^{2n}\frac{[(7j+6)acegkpr-1]}{[7(j+1)acegkpr-1]}.
\end{aligned}
$$

Proof. By using mathematical induction, the result holds for n = 0. Suppose that the result holds for $n-1$

$$u_{14n-20} = \frac{-g\prod_{j=1}^{2n-2}[7(j-1)acegkpr-1]}{\prod_{j=0}^{2n-3}[(7j+1)acegkpr-1]},$$

$$u_{14n-19} = f\prod_{j=1}^{2n-2}\frac{[(7j-6)bdfhlqt-1]}{[(7j-5)bdfhlqt-1]},$$

$$u_{14n-18} = e\prod_{j=1}^{2n-2}\frac{[(7j-5)acegkpr-1]}{[(7j-4)acegkpr-1]},$$

$$u_{14n-17} = d\prod_{j=1}^{2n-2}\frac{[(7j-4)bdfhlqt-1]}{[(7j-3)bdfhlqt-1]},$$

$$u_{14n-16} = c\prod_{j=1}^{2n-2}\frac{[(7j-3)acegkpr-1]}{[(7j-2)acegkpr-1]},$$

$$u_{14n-15} = b\prod_{j=1}^{2n-2}\frac{[(7j-2)bdfhlqt-1]}{[(7j-1)bdfhlqt-1]},$$

$$u_{14n-14} = a\prod_{j=1}^{2n-2}\frac{[(7j-1)acegkpr-1]}{[(7j)acegkpr-1]},$$

$$u_{14n-13} = \frac{-t\prod_{j=1}^{2n-2}[7j)bdfhlqt-1]}{\prod_{j=0}^{2n-2}[(7j+1)bdfhlqt-1]},$$

$$u_{14n-12} = r\prod_{j=0}^{2n-2}\frac{[(7j+1)acegkpr-1]}{[(7j+2)acegkpr-1]},$$

$$u_{14n-11} = q\prod_{j=0}^{2n-2}\frac{[(7j+2)bdfhlqt-1]}{[(7j+3)bdfhlqt-1]},$$

$$u_{14n-10} = p\prod_{j=0}^{2n-2}\frac{[(7j+3)acegkpr+1]}{[(7j+4)acegkpr+1]},$$

$$u_{14n-9} = l\prod_{j=0}^{2n-2}\frac{[(7j+4)bdfhlqt-1]}{[(7j+5)bdfhlqt-1]},$$

$$
\begin{aligned}
u_{14n-8} &= k\prod_{j=0}^{2n-2}\frac{[(7j+5)acegkpr-1]}{[7j+6)acegkpr-1]},\\
u_{14n-7} &= h\prod_{j=0}^{2n-2}\frac{[(7j+6)bdfhlqt-1]}{[7(j+1)bdfhlqt-1]},
\end{aligned}
$$

and

$$
\begin{aligned}
v_{14n-20} &= -t\prod_{j=0}^{2n-3}\frac{[(7j)bdfhlqt-1]}{[(7j+1)bdfhlqt-1]},\\
v_{14n-19} &= r\prod_{j=0}^{2n-3}\frac{[(7j+1)acegkpr-1]}{[(7j+2)acegkpr-1]},\\
v_{14n-18} &= q\prod_{j=0}^{2n-3}\frac{[(7j+2)bdfhlqt-1]}{[(7j+3)bdfhlqt-1]},\\
v_{14n-17} &= p\prod_{j=0}^{2n-3}\frac{[(7j+3)acegkpr-1]}{[(7j+4)acegkpr-1]},
\end{aligned}
$$

$$
\begin{aligned}
v_{14n-16} &= l\prod_{j=0}^{2n-3}\frac{[(7j+4)bdfhlqt-1]}{[(7j+5)bdfhlqt-1]},\\
v_{14n-15} &= k\prod_{j=0}^{2n-3}\frac{[(7j+5)acegkpr-1]}{[(7j+6)acegkpr-1]},\\
v_{14n-14} &= h\prod_{j=0}^{2n-3}\frac{[(7j+6)bdfhlqt-1]}{[(7(j+1)bdfhlqt-1]},\\
v_{14n-13} &= \frac{-g\prod_{j=1}^{2n-2}[(7j)acegkpr-1]}{\prod_{j=0}^{2n-2}[(7j+1)acegkpr-1]},
\end{aligned}
$$

$$
\begin{aligned}
v_{14n-12} &= f\prod_{j=0}^{2n-2}\frac{[(7j+1)bdfhlqt-1]}{[(7j+2)bdfhlqt-1]},\\
v_{14n-11} &= e\prod_{j=0}^{2n-2}\frac{[(7j+2)acegkpr-1]}{[(7j+3)acegkpr-1]},\\
v_{14n-10} &= d\prod_{j=0}^{2n-2}\frac{[(7j+3)bdfhlqt-1]}{[(7j+4)bdfhlqt-1]},\\
v_{14n-9} &= c\prod_{j=0}^{2n-2}\frac{[(7j+4)acegkpr-1]}{[(7j+5)acegkpr-1]},
\end{aligned}
$$

$$v_{14n-8} = b\prod_{j=0}^{2n-2}\frac{[(7j+5)bdfhlqt-1]}{[(7j+6)bdfhlqt-1]},$$
$$v_{14n-7} = a\prod_{j=0}^{2n-2}\frac{[(7j+6)acegkpr-1]}{[7(j+1)acegkpr-1]}.$$

Now, from system (5.1) we get

$$\begin{aligned}
u_{14n-6} &= \frac{v_{14n-13}}{1-v_{14n-7}u_{14n-8}v_{14n-9}u_{14n-10}v_{14n-11}u_{14n-12}v_{14n-13}} \\
&= \frac{1}{(1/v_{14n-13})-v_{14n-7}u_{14n-8}v_{14n-9}u_{14n-10}v_{14n-11}u_{14n-12}} \\
&= \frac{-g\prod_{j=1}^{2n-2}[(7j)acegkpr-1]}{\prod_{j=0}^{2n-2}[(7j+1)acegkpr-1]+g\prod_{j=1}^{2n-2}[(7j)acegkpr-1]v_{14n-7}...u_{14n-12}}.
\end{aligned}$$

Now, we see that

$$v_{14n-7}u_{14n-8}v_{14n-9}u_{14n-10}v_{14n-11}u_{14n-12} = a\prod_{j=0}^{2n-2}\frac{[(7j+6)acegkpr-1]}{[7(j+1)acegkpr-1]}$$

$$\begin{aligned}
&\times k\prod_{j=0}^{2n-2}\frac{[(7j+5)acegkpr-1]}{[7j+6)acegkpr-1]}\times c\prod_{j=0}^{2n-2}\frac{[(7j+4)acegkpr-1]}{[(7j+5)acegkpr-1]} \\
&\times p\prod_{j=0}^{2n-2}\frac{[(7j+3)acegkpr-1]}{[(7j+4)acegkpr-1]}\times e\prod_{j=0}^{2n-2}\frac{[(7j+2)acegkpr-1]}{[(7j+3)acegkpr-1]} \\
&\times r\prod_{j=0}^{2n-2}\frac{[(7j+1)acegkpr-1]}{[(7j+2)acegkpr-1]} \\
= \ & acekpr\prod_{j=0}^{2n-2}\frac{[(7j+1)acegkpr-1]}{[7(j+1)acegkpr-1]}.
\end{aligned}$$

Again,

$$\begin{aligned}
&\prod_{j=0}^{2n-2}[(7j+1)acegkpr-1]+g\prod_{j=1}^{2n-2}[(7j)acegkpr-1]v_{14n-7}...u_{14n-12} \\
= &\prod_{j=0}^{2n-2}[(7j+1)acegkpr-1]\{1+\frac{acegkpr}{[7(2n-1)acegkpr-1]}\} \\
= &\prod_{j=0}^{2n-2}[(7j+1)acegkpr-1].\frac{[7(2n-1)acegkpr-1]+acegkpr}{[7(2n-1)acegkpr-1]}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\prod_{j=0}^{2n-2}[(7j+1)acegkpr-1].[(14n-6)acegkpr-1]}{[7(2n-1)acegkpr-1]} \\
&= \frac{\prod_{j=0}^{2n-1}[(7j+1)acegkpr-1]}{[7(2n-1)acegkpr-1]}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
& g\prod_{j=1}^{2n-2}[(7j)acegkpr-1]v_{14n-7}...u_{14n-12} \\
&= acegkpr\prod_{j=1}^{2n-2}[(7j)acegkpr-1]\prod_{j=0}^{2n-2}\frac{[(7j+1)acegkpr-1]}{[7(j+1)acegkpr-1]} \\
&= \frac{acegkpr\prod_{j=0}^{2n-2}[(7j+1)acegkpr-1]}{[7(2n-1)acegkpr-1]}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
u_{14n-6} &= -g\prod_{j=1}^{2n-2}[(7j)acegkpr-1] \div \frac{\prod_{j=0}^{2n-1}[(7j+1)acegkpr-1]}{[7(2n-1)acegkpr-1]} \\
&= \frac{-g\prod_{j=1}^{2n-2}[(7j)acegkpr-1].[7(2n-1)acegkpr-1]}{\prod_{j=0}^{2n-1}[(7j+1)acegkpr-1]} \\
&= \frac{-g\prod_{j=1}^{2n-1}[(7j)acegkpr-1]}{\prod_{j=0}^{2n-1}[(7j+1)acegkpr-1]} = \frac{-g\prod_{j=1}^{2n}[7(j-1)acegkpr-1]}{\prod_{j=0}^{2n-1}[(7j+1)acegkpr-1]}.
\end{aligned}
$$

Again, from system (5.1) we can obtain that

$$
\begin{aligned}
v_{14n-6} &= \frac{u_{14n-13}}{1-u_{14n-7}v_{14n-8}u_{14n-9}v_{14n-10}u_{14n-11}v_{14n-12}u_{14n-13}} \\
&= \frac{1}{(1/u_{14n-13})-u_{14n-7}v_{14n-8}u_{14n-9}v_{14n-10}u_{14n-11}v_{14n-12}} \\
&= = \frac{-t\prod_{j=1}^{2n-2}[(7j)bdfhlqt-1]}{\prod_{j=0}^{2n-2}[(7j+1)bdfhlqt-1]+g\prod_{j=1}^{2n-2}[(7j)bdfhlqt-1]u_{14n-7}...v_{14n-12}}.
\end{aligned}
$$

Now, we see that

$$u_{14n-7}v_{14n-8}u_{14n-9}v_{14n-10}u_{14n-11}v_{14n-12} = h\prod_{j=0}^{2n-2}\frac{[(7j+6)bdfhlqt-1]}{[7(j+1)bdfhlqt-1]}$$

$$\begin{aligned}
&\times b\prod_{j=0}^{2n-2}\frac{[(7j+5)bdfhlqt-1]}{[7j+6)bdfhlqt-1]}\times l\prod_{j=0}^{2n-2}\frac{[(7j+4)bdfhlqt-1]}{[(7j+5)bdfhlqt-1]}\\
&\times d\prod_{j=0}^{2n-2}\frac{[(7j+3)bdfhlqt-1]}{[(7j+4)bdfhlqt-1]}\times q\prod_{j=0}^{2n-2}\frac{[(7j+2)bdfhlqt-1]}{[(7j+3)bdfhlqt-1]}\\
&\times f\prod_{j=0}^{2n-2}\frac{[(7j+1)bdfhlqt-1]}{[(7j+2)bdfhlqt-1]}\\
= \; & bdfhlqt\prod_{j=0}^{2n-2}\frac{[(7j+1)bdfhlqt-1]}{[7(j+1)bdfhlqt-1]}.
\end{aligned}$$

Again,

$$\begin{aligned}
&\prod_{j=0}^{2n-2}[(7j+1)bdfhlqt-1]+t\prod_{j=1}^{2n-2}[(7j)bdfhlqt-1]u_{14n-7}...v_{14n-12}\\
= \; &\prod_{j=0}^{2n-2}[(7j+1)bdfhlqt-1]\{1+\frac{bdfhlqt}{[7(2n-1)bdfhlqt-1]}\}\\
= \; &\prod_{j=0}^{2n-2}[(7j+1)bdfhlqt-1].\frac{[7(2n-1)bdfhlqt-1]+bdfhlqt}{[7(2n-1)bdfhlqt-1]}\\
= \; &\frac{\prod_{j=0}^{2n-2}[(7j+1)bdfhlqt-1].[(14n-6)bdfhlqt-1]}{[7(2n-1)bdfhlqt-1]}\\
= \; &\frac{\prod_{j=0}^{2n-1}[(7j+1)bdfhlqt-1]}{[7(2n-1)bdfhlqt-1]}.
\end{aligned}$$

Thus,

$$\begin{aligned}
v_{14n-6} &= -t\prod_{j=1}^{2n-2}[(7j)bdfhlqt-1]\div\frac{\prod_{j=0}^{2n-1}[(7j+1)bdfhlqt-1]}{[7(2n-1)bdfhlqt-1]}\\
&= \frac{-t\prod_{j=1}^{2n-2}[(7j)bdfhlqt-1].[7(2n-1)bdfhlqt-1]}{\prod_{j=0}^{2n-1}[(7j+1)bdfhlqt-1]}
\end{aligned}$$

$$= \frac{-t\prod_{j=0}^{2n-1}[(7j)bdfhlqt-1]}{\prod_{j=0}^{2n-1}[(7j+1)bdfhlqt-1]}.$$

Then, we can prove other relations similarly. Hence, the proof is complete.

6. Numerical Examples

We will now introduce a strong verification and confirmation of our theoretical discussion. This confirmation is embodied in presenting some numerical examples.

Example 1. The graph of system (2.1) is illustrated with the initial values: $u_{-6} = 3$, $u_{-5} = 4.2$, $u_{-4} = 3.5$, $u_{-3} = 2$, $u_{-2} = 9$, $u_{-1} = 5$, $u_0 = 4$, $v_{-6} = 0.6$, $v_{-5} = 8$, $v_{-4} = 4$, $v_{-3} = 2.75$, $v_{-2} = 7.5$, $v_{-1} = 5.75$, and $v_0 = 12$. See Figure 1.

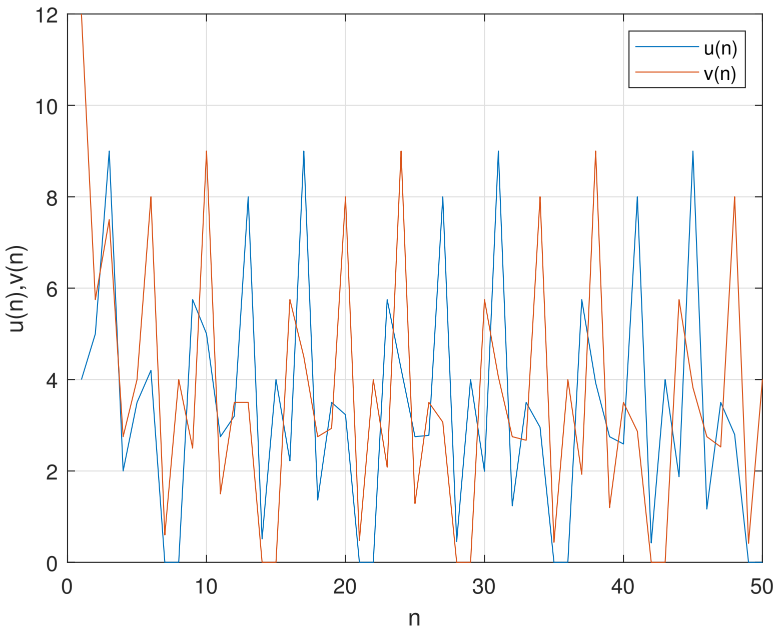

Figure 1. A periodic solution of period fourteen for system (2.1) with random initial values.

Example 2. This example presents the behavior of the solutions of system (3.1) which are periodic solutions of period fourteen . Here, we take $u_{-6} = 1.4$, $u_{-5} = 0.2$, $u_{-4} = 0.5$, $u_{-3} = 1.2$, $u_{-2} = 0.9$, $u_{-1} = .35$, $u_0 = 0.4$, $v_{-6} = 0.8$, $v_{-5} = 1.8$, $v_{-4} = 0.14$, $v_{-3} = 0.12$, $v_{-2} = 1.5$, $v_{-1} = 2.2$, and $v_0 = 1.22$. See Figure 2.

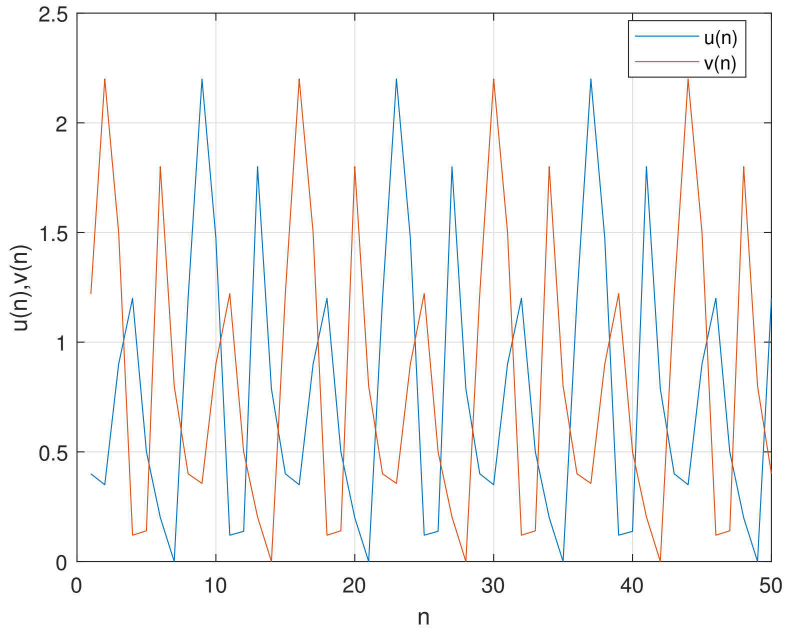

Figure 2. A periodic solution of period fourteen of system (3.1) with random initial values.

Example 3. Figure 3 shows the behavior of solutions of system (4.1) when we let $u_{-6} = 1.4$, $u_{-5} = 0.2$, $u_{-4} = 0.65$, $u_{-3} = 1.2$, $u_{-2} = 0.9$, $u_{-1} = 0.5$, $u_0 = 2.4$, $v_{-6} = 0.8$, $v_{-5} = 1.8$, $v_{-4} = 0.14$, $v_{-3} = 0.2$, $v_{-2} = 1.5$, $v_{-1} = 2.2$, and $v_0 = 1.22$. Note that the behavior of solutions are periodic solutions of period fourteen.

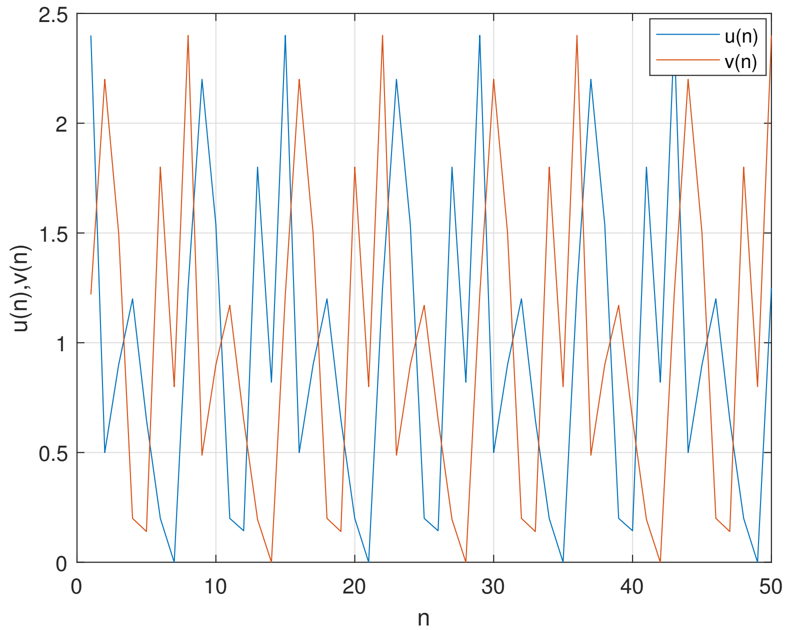

Figure 3. A periodic solution of period fourteen of system (4.1) with random initial values.

Example 4. The solutions of system (5.1) are presented in Figure 4. The considered initial conditions here are $u_{-6} = 3.4$, $u_{-5} = 4.2$, $u_{-4} = 5$, $u_{-3} = 1.2$, $u_{-2} = 0.9$, $u_{-1} = 3.5$, $u_0 = 2.4$, $v_{-6} = 0.8$, $v_{-5} = 1.8$, $v_{-4} = 4$, $v_{-3} = 2$, $v_{-2} = 1.5$, $v_{-1} = 2.2$, and $v_0 = 1.22$.

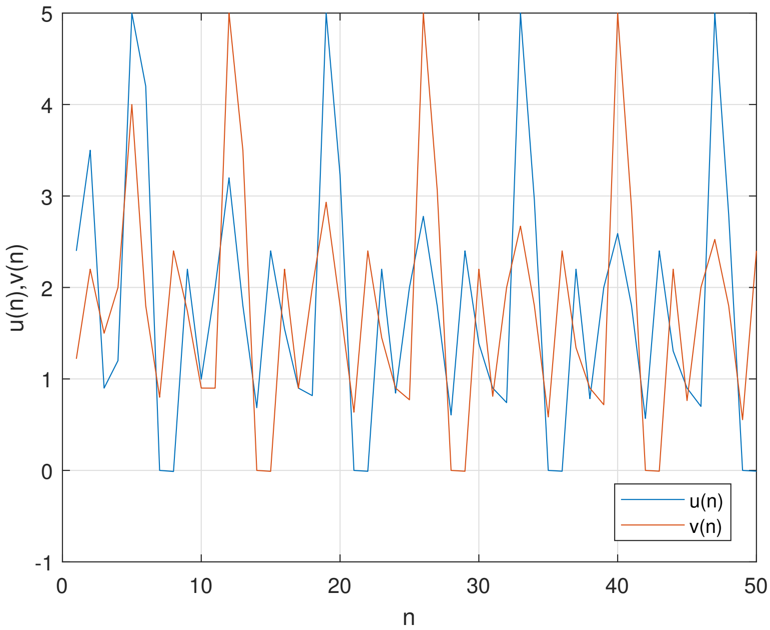

Figure 4. A periodic solution of period fourteen of system 5.1 with random initial values.

7. Conclusions

The closed form of the solutions of some rational systems of the difference equation of higher order have been presented. In Section 2, we obtained the form of the solutions of system $u_{n+1} = \frac{v_{n-6}}{1 + v_n u_{n-1} v_{n-2} u_{n-3} v_{n-4} u_{n-5} v_{n-6}}$, $v_{n+1} = \frac{u_{n-6}}{1 + u_n v_{n-1} u_{n-2} v_{n-3} u_{n-4} v_{n-5} u_{n-6}}$. Also in Section 3, we found the solution's form of the system $u_{n+1} = \frac{v_{n-6}}{1+v_n u_{n-1} v_{n-2} u_{n-3} v_{n-4} u_{n-5} v_{n-6}}$, $v_{n+1} = \frac{u_{n-6}}{1-u_n v_{n-1} u_{n-2} v_{n-3} u_{n-4} v_{n-5} u_{n-6}}$, . In Section 4 and 5 , we got the solution of the following systems, respectively, $u_{n+1} = \frac{v_{n-6}}{1 - v_n u_{n-1} v_{n-2} u_{n-3} v_{n-4} u_{n-5} v_{n-6}}$, $v_{n+1} = \frac{u_{n-6}}{1 + u_n v_{n-1} u_{n-2} v_{n-3} u_{n-4} v_{n-5} u_{n-6}}$ and $u_{n+1} = \frac{v_{n-6}}{1 - v_n u_{n-1} v_{n-2} u_{n-3} v_{n-4} u_{n-5} v_{n-6}}$, $v_{n+1} = \frac{u_{n-6}}{1 - u_n v_{n-1} u_{n-2} v_{n-3} u_{n-4} v_{n-5} u_{n-6}}$. Finally, we gave some numerical examples to illustrate the obtained results.

Conflict of interest

The author declares there is no conflict of interest.

References

1. H. N. Agiza, E. M. Elabbasy, H. El-Metwally, A. A. Elsadany, Chaotic dynamics of a discrete prey-predator model with Holling type II, *Nonlinear Anal. Real*, **10** (2009), 116–129. https://doi.org/10.1016/j.nonrwa.2007.08.029
2. Y. Akrour, N. Touafek, Y. Halim, On a system of difference equations of third order solved in closed form, *J. Innov. Appl. Math. Comput. Sci.*, **1** (2021), 1–15. https://doi.org/10.58205/jiamcs.v1i1.8
3. E. M. Elsayed, M. T. Alharthi, The form of the solutions of fourth order rational system of difference equations, *Ann. Commun. Math.*, **5** (2022), 161–180. https://doi.org/10.62072/acm.2022.050304
4. A. Asiri, M. M. El-Dessoky, E. M. Elsayed, Solution of a third order fractional system of difference equations, *J. Comput. Anal. Appl.*, **24** (2018), 444–453.
5. N. Battaloglu, C. Cinar, I. Yalcınkaya, The dynamics of the difference equation, *Ars Comb.*, **97** (2010), 281–288.
6. F. Belhannache, Asymptotic stability of a higher order rational difference equation, *Electron. J. Math. Anal. Appl.*, **7** (2019), 1–8.
7. R. J. Beverton, S. J. Holt, *On the dynamics of exploited fish populations*, Dordrecht : Springer Science & Business Media, 1993. https://doi.org/10.1007/978-94-011-2106-4
8. E. M. Elabbasy, H. N. Agiza, A. A. Elsadany, H. El-Metwally, The dynamics of triopoly game with heterogeneous players, *Int. J. Nonlinear Sci.*, **3** (2007), 83–90.
9. S. Elaydi, A. A. Yakubu, Open problems and conjectures, *J. Differ. Equ. Appl.*, **8** (2002), 755–760. https://doi.org/10.1080/1023619021000000762
10. I. M. Elbaz, H. El-Metwally, M. A. Sohaly, Dynamics of delayed Nicholson′s blowflies models, *J. Biol. Syst.*, **30** (2022), 741–759. https://doi.org/10.1142/S0218339022500279
11. H. El-Metwally, Global behavior of an economic model, *Chaos Soliton. Fract.*, **33** (2007), 994–1005. https://doi.org/10.1016/j.chaos.2006.01.060
12. H. El-Metwally, M. A. Sohaly, I. M. Elbaz, Stochastic global exponential stability of disease-free equilibrium of HIV/AIDS model, *Eur. Phys. J. Plus*, **135** (2020), 840. https://doi.org/10.1140/epjp/s13360-020-00856-0
13. H. El-Metwally, E. M. Elsayed, Form of solutions and periodicity for systems of difference equations, *J. Comput. Anal. Appl.*, **15** (2013), 1.
14. K. N Alharbi, E. M Elsayed, The expressions and behavior of solutions for nonlinear systems of rational difference equations, *J. Innov. Appl. Math. Comput. Sci.*, **2** (2022), 78–91. https://doi.org/10.58205/jiamcs.v2i1.24
15. J. E. Franke, A. A. Yakubu, Sis epidemic attractors in periodic environments, *J. Biol. Dyn.*, **1** (2007), 394–412. https://doi.org/10.1080/17513750701605630
16. A. Friedman, A. A. Yakubu, Host demographic Allee effect, fatal disease and migration, Persistence or extinction, *SIAM. J. Appl. Math.*, **72** (2012), 1644–1666. https://doi.org/10.1137/120861382

17. M. Gumus, O. Ocalan, The qualitative analysis of a rational system of difference equations, *J. Fract. Calc. Appl.*, **9** (2018), 113–126.

18. N. Haddad, N. Touafek, J. F. T. Rabago, Well-defined solutions of a system of difference equations, *J. Appl. Math. Comput.*, **56** (2018), 439–458. https://doi.org/10.1007/s12190-017-1081-8

19. M. Y. Hamada, T. El-Azab, H. El-Metwally, Bifurcations analysis of a two-dimensional discrete time predator-prey model, *Math. Method. Appl. Sci.*, **46** (2023), 4815–4833. https://doi.org/10.1002/mma.8807

20. M. Y. Hamada, Tamer El- Azab, H. El-Metwally, Bifurcations and dynamics of a discrete predator-prey model of ricker type, *J. Appl. Math. Comput.*, **69** (2023), 113–135. https://doi.org/10.1007/s12190-022-01737-8

21. A. Khelifa, Y. Halim, M. Berkal, Solutions of a system of two higher-order difference equations in terms of Lucas sequence, *Univ. J. Math. Appl.*, **2** (2019), 202–211. https://doi.org/10.32323/ujma.610399

22. M. R. S. Kulenovic, G. Ladas, *Dynamics of second order rational difference equations*, New York: Chapman and Hall, 2001. https://doi.org/10.1201/9781420035384

23. A. S. Kurbanli, C. Cinar, M. E. Erdogan, On the behavior of solutions of the system of rational difference equations $u_{n+1} = \frac{u_{n-1}}{v_n u_{n-1}-1}$, $v_{n+1} = \frac{v_{n-1}}{u_n v_{n-1}-1}$, $w_{n+1} = \frac{u_n}{v_n w_{n-1}}$, *Appl. Math.*, **2** (2011), 1031–1038. https://doi.org/10.4236/am.2011.28143

24. R. C. Lyness, 1581. Cycles, *Math. Gaz.*, **26** (1942), 268. https://doi.org/10.2307/3606036

25. E. C. Pielou, *An introduction to mathematical ecology*, 1969.

26. E. C. Pielou, *Population and community ecology: Principles and methods*, CRC Press, 1974.

27. T. L. Saaty, *Modern nonlinear equations*, 1967.

28. N. Touafek, E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, *B. Math. Soc. Sci. Math.*, **2** (2012), 217–224.

29. A. A. Yakubu, Allee effects in a discrete-time sis epidemicmodel with infected newborns, *J. Differ. Equ. Appl.*, **13** (2007), 341–356. https://doi.org/10.1080/10236190601079076

AIMS Press