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*Research article*

# On the nonlinear Schrödinger equation with critical source term: global well-posedness, scattering and finite time blowup

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**Abstract:** This study explored the time asymptotic behavior of the Schrödinger equation with an inhomogeneous energy-critical nonlinearity. The approach follows the concentration-compactness method due to Kenig and Merle. To address the primary challenge posed by the singular inhomogeneous term, we utilized Caffarelli-Kohn-Nirenberg weighted inequalities. This work notably expanded the existing literature by applying these techniques to higher spatial dimensions without requiring any spherically symmetric assumption.

**Keywords:** Schrödinger problem; energy-critical; nonlinear equations; fixed point method; global existence; scattering; blowup
**Mathematics Subject Classification:** 35Q55

## 1. Introduction

We consider the Cauchy problem for the Schrödinger equation with an inhomogeneous nonlinearity

$$\mathrm{i}\frac{\partial}{\partial t}u + \Delta u + |x|^{-\tau}|u|^{p-1}u = 0; \tag{INLS}$$

$$u_{|t=0} = u_0. \tag{1.1}$$

In this context, the wave function $u$ is a complex-valued function defined on the variable $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. Additionally, the singular inhomogeneous term is given by $|\cdot|^{-\tau}$, where $\tau > 0$.

The inhomogeneous nonlinear equation of Schrödinger type describes beam propagation in nonlinear optics and plasma physics. In fact, stable high-power propagation can be realized in a plasma by introducing a preliminary laser beam that forms a channel with reduced electron density, thereby decreasing the nonlinearity within that channel [1–3]. In the context of the optical nonlinear Schrödinger equation, light energy can be confined, enabling the transmission of complex structured beams and solitons [4, 5]. Additional references include [6–8].

The well-posedness of the inhomogeneous nonlinear Schrödinger equation (INLS) has been explored by numerous authors. The existence of energy subcritical solutions was first established in [9]. This result was later revisited in [10], where solutions in Strichartz spaces were examined under additional restrictions for $N = 2, 3$. The dichotomy between global existence and scattering versus finite time blowup below the ground state threshold was addressed in [11–13] using the concentration-compactness argument by Kenig and Merle [14]. This work was further developed in [15] employing the Dodson-Murphy method [16], and the spherically symmetric assumption was relaxed in [17]. Additional discussions on more general inhomogeneous terms can be found in [18, 19]. The finite time blowup of solutions without radial or finite time variance assumptions was investigated in [20, 21]. Recently, the Sobolev critical regime has also been considered, with local well-posedness studied in [22–25]. Scattering for spherically symmetric initial data was demonstrated in [26, 27] in the three-dimensional case, while the radial assumption for this case was removed in [28]. A result indicating non-scattering was presented in [29]. For a numerical perspective, a quantitative analysis of solutions to the three-dimensional cubic nonlinear Schrödinger equation above the mass-energy threshold is provided in [33], which introduces a new blowup criterion and predicts the asymptotic behavior of solutions across various initial data classes, including modulated ground states, Gaussian, super-Gaussian, off-centered Gaussian, and oscillatory Gaussian, along with several conjectures regarding the scattering threshold.

The motivation of this note is to extend the findings of [26–28] to higher spatial dimensions and to eliminate the radial assumption. Specifically, the scattering threshold was demonstrated in [26, 27] for three spatial dimensions. The novel contribution here is to establish the scattering threshold for $N \geq 4$ without assuming spherical symmetry. This indicates that every energy-critical solution to (INLS) asymptotically approaches a solution of the linear Schrödinger equation as $t \to \infty$. The methodology follows the roadmap laid out by Kenig and Merle in [14].

The remainder of the paper is organized as follows: Section 2 presents the main result along with some useful estimates. Section 3 provides auxiliary results. Section 4 is dedicated to proving global existence and scattering. Finally, Section 5 addresses the finite time blowup.

Here and henceforth, the Lebesgue and Sobolev spaces equipped with the standard norms are denoted by

$$L^r := L^r(\mathbb{R}^N), \quad \dot H^1 := \{f \in S'(\mathbb{R}^N), \nabla f \in L^2\}, \quad \dot H^1_{rd} := \{f \in \dot H^1, \quad f(\cdot) = f(|\cdot|)\};$$
$$\|\cdot\|_r := \|\cdot\|_{L^r}, \quad \|\cdot\| := \|\cdot\|_2, \quad \|\cdot\|_{\dot H^1} := \|\nabla\cdot\|.$$

Finally, one denotes by $(T^-, T^+)$ the maximal existence interval of an eventual energy solution to (INLS).

## 2. Background and main result

This section contains the main contribution of this note and some useful standard estimates.

*2.1. Preliminary*

Let us denote the free Schrödinger kernel:

$$e^{\mathrm{i}t\Delta}u := \mathcal{F}^{-1}\Big(e^{-\mathrm{i}t|\cdot|^2}\mathcal{F}u\Big), \tag{2.1}$$

where $\mathcal{F}$ is the Fourrier transform. Thanks to the Duhamel formula, solutions to (INLS) are fixed points of the integral function

$$f(u(t)) := e^{\mathrm{i}t\Delta}u_0 - \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Big(|x|^{-\tau}|u|^{p-1}u\Big)\,ds. \tag{2.2}$$

Solutions of the problem (INLS), formally satisfy the conservation of the energy

$$E(u(t)) := \int_{\mathbb{R}^N}|\nabla u(t,x)|^2\,dx - \frac{2}{1+p}\int_{\mathbb{R}^N}|x|^{-\tau}|u(t,x)|^{1+p}\,dx = E(u_0). \tag{2.3}$$

If $u$ resolves the equation (INLS), then so does the family $u_\kappa := \kappa^{\frac{2-\tau}{p-1}}u(\kappa^2\cdot,\kappa\cdot)$, $\kappa > 0$. Moreover, there is only one invariant Sobolev norm under the above dilatation, precisely

$$\|u_\kappa(t)\|_{\dot H^{s_c}} = \|u(\kappa^2 t)\|_{\dot H^{s_c}}, \quad s_c := \frac{N}{2} - \frac{2-\tau}{p-1}.$$

In the sequel, we will focus on the energy-critical regime

$$s_c = 1 \Leftrightarrow p = p^c := 1 + \frac{2(2-\tau)}{N-2}, \quad N \geq 3. \tag{2.4}$$

We will consider the next assumption on the inhomogeneous term

$$0 < \tau < \min\Big\{\frac{6-N}{2}, \frac{4}{N}\Big\} \quad \text{or} \quad \frac{2+N}{N} < \tau < 2. \tag{2.5}$$

Let us define the potential energy

$$P[u] := \int_{\mathbb{R}^N}|x|^{-\tau}|u|^{1+p}\,dx. \tag{2.6}$$

Take the associated ground state

$$\varphi(x) := \Big(1 + \frac{|x|^{2-\tau}}{(N-\tau)(N-2)}\Big)^{-\frac{N-2}{2-\tau}}. \tag{2.7}$$

Thanks to [35, Theorem 4.3] and [34, Remark 2.1], one has

$$\Delta\varphi + |x|^{-\tau}\varphi^p = 0; \tag{2.8}$$

$$\frac{1}{C_*} := \inf_{0\neq u\in\dot H^1}\frac{\|\nabla u\|}{(P[u])^{\frac{1}{1+p}}} = \frac{\|\nabla\varphi\|}{(P[\varphi])^{\frac{1}{1+p}}}. \tag{2.9}$$

Finally, we denote for short the Sobolev embedding exponent

$$2^* := \frac{2N}{N-2}, N \geq 3. \tag{2.10}$$

From now on, we hide the time variable for simplicity, spreading it out only when necessary.

### 2.2. Main result

The main contribution of this note is the next dichotomy of global existence and scattering versus finite time blowup of energy critical solutions under the ground state threshold.

**Theorem 2.1.** *Let $N > 3$, $\tau$ satisfy* (2.5)*, and $p = p^c$. Let $u_0 \in \dot H^1$, satisfying:*

$$E(u_0) < E(\varphi). \tag{2.11}$$

Then,

1) The solution of (INLS) is global and scatters if

$$\|\nabla u_0\| < \|\nabla \varphi\|. \tag{2.12}$$

2) The solution of (INLS) blows up in finite or infinite time if $u_0 \in L^2$ and

$$\|\nabla u_0\| > \|\nabla \varphi\|. \tag{2.13}$$

In view of the results stated in the above theorem, some comments are in order.

- $\varphi$ denotes a ground state solution to (2.7);
- This work complements [26–28] to higher space dimensions $N > 3$;
- The local existence obtained here complements [22–25], where the data was supposed to be in $H^1$;
- Due to the use of fixed point argument in the small data theory, for $\tau < 1$, one needs the condition $p \geq 2$. This gives the restriction $0 < 2\tau \leq 6 - N$. So one assumes that $N \leq 5$. Moreover, the use of the Sobolev embedding $\dot H^1 \hookrightarrow L^{\frac{2N}{N-2}}$ restricts the space dimension to $N \in \{3, 4, 5\}$;
- For $\tau < 1$, the condition $\tau < \frac{4}{N}$ is because one needs the inequality $p - 1 - \tau > 0$ in the local theory;
- $$\min\Big\{\frac{6-N}{2}, \frac{4}{N}\Big\} = \begin{cases} \frac{4}{3}, & \text{if} \quad N = 3, \\ 1, & \text{if} \quad N = 4, \\ \frac{1}{2}, & \text{if} \quad N = 5; \end{cases}$$
- The condition $\tau < \frac{6-N}{2}$ doesn't appear in [26] because only $N = 3$ is treated and so $0 < \tau < \min\{\frac{6-N}{2}, \frac{4}{N}\}$ reads $0 < \tau < \frac{4}{3}$;
- In the local theory, for $\tau > 1$, we use some weighted Strichartz spaces in the spirit of [27]. The choice of $\gamma = (\frac{-2+N}{2})^-$ done in [27] is not possible for $N \geq 4$ because of the necessary condition $\gamma < 1$. So, the proof is different and we get the extra restriction $\tau > \frac{2+N}{N}$;
- The blowup in finite or infinite time means that $\sup_{[0,T^+)} \|\nabla u(t)\| = \infty$;
- The radial assumption is not needed for the blowup;
- If one assumes that $xu_0 \in L^2$ or $u_0$ radial, the finite time blowup holds;
- This work complements [14] to the inhomogeneous case.

### 2.3. Sketch of the proof of the first part of Theorem 2.1

By contradiction, if the first part of Theorem 2.1 fails, then there is a minimal non-scattering solution under the ground state threshold which possesses certain compactness properties as follows.

**Proposition 2.1.** *If the first part of Theorem 2.1 fails, there exists a maximal solution to* (INLS), *denoted by* $u \in C([0,T^+),\dot{H}^1)$ *and a frequency scale function* $\lambda : [0,T^+) \mapsto \mathbb{R}_+$, *such that* $\inf_{t\in[0,T^+)} \lambda(t) \geq 1$, *and*

$$\sup_{t\in[0,T^+)} \|\nabla u(t)\| < \|\nabla\varphi\|; \tag{2.14}$$

$$\|u\|_{S(0,T^+)} = \infty; \tag{2.15}$$

$$\Big\{\frac{1}{\lambda(t)^{\frac{N-2}{2}}} u(t, \frac{\cdot}{\lambda(t)}), \quad t \in [0,T^+)\Big\} \quad \textit{is pre-compact in} \quad \dot{H}^1. \tag{2.16}$$

To complete the proof of the first part of Theorem 2.1, one proves that the type of solution appearing in the statement of Proposition 2.1 cannot exist. This is achieved in Subsection 4.2.

**Remark 2.1.** *In Proposition 2.1, there is no moving spatial center* $x(t)$ *in the parametrization of the minimal non-scattering solution. Indeed, thanks to Proposition 4.1, the profiles with* $\frac{|x_n|}{\lambda_n} \to \infty$ *correspond to scattering solutions. By arguments in [36], we can arrange the frequency scale function to be bounded below.*

**Remark 2.2.** *Proposition 2.1 is an adaptation of [28, Theorem 1.2], and the idea of the proof is somehow similar. Indeed, we aim to generalize [28, Theorem 1.2] for higher space dimensions and for more general inhomogeneous term, namely* $N \geq 3$ *and* $b$ *satisfying* (2.5) *rather than* $N = 3$ *and* $\tau = 1$.

### 2.4. Useful tools

For the reader's convenience, we recall some known and useful tools which play an important role in the proof of the main result. To start, we recall the homogeneous Sobolev embedding [37, Theorem 1.38], for $N \geq 3$,

$$\|u\|_{2^*} \leq C_N \|\nabla u\|, \quad \text{for all} \quad u \in \dot{H}^1. \tag{2.17}$$

The following Caffarelli-Kohn-Nirenberg weighted interpolation inequalities [38, 39], will be useful.

**Lemma 2.1.** *Let* $N \geq 1$, $1 < p \leq q < \infty$, *and* $-\frac{N}{q} < b \leq a < \frac{N}{q'}$. *Assume that* $a - b - 1 = N(\frac{1}{q} - \frac{1}{p})$. *Then,*

$$\||\cdot|^b f\|_q \leq C\||\cdot|^a \nabla f\|_p.$$

Recall the associated Bernstein estimates to the standard Littlewood–Paley projections $P_M$, see [40, Subsection 11.2],

$$\||\nabla|^s P_M f\|_r \simeq M^s \|P_M f\|_r, \quad \text{for all} \quad 1 \leq r \leq \infty; \tag{2.18}$$

$$\|P_M f\|_{r_1} \simeq M^{N(\frac{1}{r_2}-\frac{1}{r_1})} \|P_M f\|_{r_2}, \quad \text{for all} \quad 1 \leq r_2 \leq r_1 \leq \infty. \tag{2.19}$$

The next refined Fatou argument [41, Lemma 11.3] will be useful.

**Lemma 2.2.** *Let a functional sequence satisfy:*

$$\limsup_{n\to\infty} \|f_n\|_r < \infty \quad \text{and} \quad f_n \to f \quad \text{almost everywhere on} \quad \mathbb{R}^N.$$

Then,

$$\lim_n \int_{\mathbb{R}^N} \Big(|f_n|^r - |f_n - f|^r - |f|^r\Big)\,dx = 0. \tag{2.20}$$

The next linear profile decomposition for bounded radial sequences in $\dot{H}^1$ is a key tool for the scattering proof [41, 42].

**Proposition 2.2.** *Take $(u_n)$ as a bounded sequence in $\dot{H}^1$. Then, for any $M \in \mathbb{N}$, there exist a subsequence denoted also by $(u_n)$ and*

1) For any $1 \le j \le M$, a profile $\psi^j \in \dot{H}^1$;
2) For any $1 \le j \le M$, a sequence $(t_n^j, \lambda_n^j, x_n^j) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^N$ satisfying

$$t_n^j \equiv 0 \quad \text{or} \quad t_n^j \to \infty \quad \text{and} \quad x_n^j \equiv 0 \quad \text{or} \quad |x_n^j| \to \infty, \tag{2.21}$$

and for $1 \le i \ne j \le M$ and $n \to \infty$,

$$\log\Big(\frac{\lambda_n^j}{\lambda_n^k} + \frac{t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2}{\lambda_n^j\lambda_n^k} + \frac{|x_n^j - x_n^k|}{\lambda_n^j\lambda_n^k}\Big) \to \infty; \tag{2.22}$$

3) A sequence of remainders $W_n^M \in \dot{H}^1$, such that

$$\begin{aligned} u_n &= \sum_{j=1}^{M} f_n^j(e^{it_n^j\Delta}\psi^j) + W_n^M \\ &:= \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}} \sum_{j=1}^{M} [e^{it_n^j\Delta}\psi^j](\frac{\cdot - x_n^j}{\lambda_n^j}) + W_n^M. \end{aligned} \tag{2.23}$$

Moreover,

$$\lim_{M\to\infty}\Big[\limsup_{n\to\infty} \|\nabla e^{i\cdot\Delta} W_n^M\|_{S(\mathbb{R})}\Big] = 0. \tag{2.24}$$

For fixed $M$, one has the next Pythagorean expansions

$$\|\nabla u_n\|^2 = \sum_{j=1}^{M} \|\nabla\psi^j\|^2 + \|\nabla W_n^M\|^2 + o_n(1);$$

$$E(u_n) = \sum_{j=1}^{M} E\Big(e^{it_n^j\Delta}\psi^j\Big) + E(W_n^M) + o_n(1).$$

Now, let us collect some standard estimates related to the Schrödinger problem.

**Definition 2.1.**
*A pair* $(q,r)$ *is said admissible if* $q,r\ge 2$, $(q,r,N)\neq(2,\infty,2)$ *and* $N(\frac{1}{2}-\frac{1}{r})=\frac{2}{q}$. *One says for short* $(q,r)\in\Lambda$. *Let* $I\subset\mathbb{R}$ *be an interval, and one denotes the Strichartz space by*

$$\Omega(I):=\bigcap_{(q,r)\in\Lambda}L^q(I,L^r).$$

Let us now state some Strichartz estimates [43].

**Proposition 2.3.** *Let* $N\ge 2$ *and* $u_0\in L^2$. *Then,*

1) $\|e^{i\cdot\Delta}u_0\|_{\Omega(I)}\lesssim\|u_0\|$;
2) $\|u-e^{it\Delta}u_0\|_{\Omega(I)}\lesssim\inf_{(\tilde{q},\tilde{r})\in\Lambda}\|i\frac{\partial}{\partial t}u+\Delta u\|_{L^{\tilde{q}'}(I,L^{\tilde{r}'})}$.

Let us give some Strichartz estimates adapted to the weighted Lebesgue spaces [24].

**Definition 2.2.** *Take* $N\ge 3$ *and* $0\le\gamma<1$. *A pair of real numbers* $(q,r)$ *is* $\gamma$*-admissible if*

$$\begin{cases} N(\frac{1}{2}-\frac{1}{r})+\gamma=\frac{2}{q};\\ \frac{\gamma}{2}<\frac{1}{q}\le\frac{1}{2};\\ \frac{\gamma}{2}\le\frac{1}{r}<\frac{1}{2}.\end{cases}$$

Take the set $\Lambda_\gamma:=\{(q,r),\ \gamma\text{-admissible}\ \}$ and the weighted Strichartz norm

$$\|\cdot\|_{S^\gamma(I)}:=\sup_{(q,r)\in\Lambda_\gamma}\|\cdot\|_{L^q\left(I,L^r(|x|^{-r\gamma})\right)}.$$

The next Strichartz estimate was proved in [24, Proposition 1.5].

**Proposition 2.4.** *Let* $N\ge 3$ *and* $0\le\gamma,\tilde{\gamma}<1$, *and a time slab* $I\subset\mathbb{R}$. *Take* $(q,r)\in\Lambda_\gamma$ *and* $(\tilde{q},\tilde{r})\in\Lambda_{\tilde{\gamma}}$. *Then,*

$$\|e^{i\cdot\Delta}f\|_{L^q\left(I,L^r(|x|^{-r\gamma})\right)}\lesssim\|f\|;\tag{2.25}$$

$$\Big\|\int_0^{\cdot}e^{i(\cdot-\tau)\Delta}h(\tau,\cdot)\,d\tau\Big\|_{L^q\left(I,L^r(|x|^{-r\gamma})\right)}\lesssim\|h\|_{L^{\tilde{q}'}\left(I,L^{\tilde{r}'}(|x|^{\tilde{\gamma}\tilde{r}'})\right)},\quad\text{for}\quad q>\tilde{q}'.\tag{2.26}$$

Define the variance potential

$$V_\psi:=\int_{\mathbb{R}^N}\psi(x)|u(\cdot,x)|^2\,dx,\tag{2.27}$$

where $\psi:\mathbb{R}^N\to\mathbb{R}$ is a smooth function. Let also the Morawetz action be

$$M_\psi = 2\Im \int_{\mathbb{R}^N} \bar{u}(\nabla\psi \cdot \nabla u)\, dx := 2\Im \int_{\mathbb{R}^N} \bar{u}(\psi_j u_j)\, dx, \tag{2.28}$$

where here and in the sequel, the repeated index are summed. Let us give a Morawetz type estimate [45, Lemma 4.5].

**Proposition 2.5.** *Take $u \in C([0,T], \dot{H}^1)$ as the local solution to* (INLS). *Let $\psi : \mathbb{R}^N \to \mathbb{R}$ be a smooth function. Then, the following equality holds on* $[0,T]$,

$$\begin{aligned} V''_\psi[u] = M'_\psi[u] &= 4\int_{\mathbb{R}^N} \partial_l\partial_k\psi \Re(\partial_k u \partial_l \bar{u})\, dx - \int_{\mathbb{R}^N} \Delta^2\psi |u|^2\, dx \\ &- 2\frac{p^c-2}{p^c}\int_{\mathbb{R}^N} \Delta\psi |x|^{-\tau}|u|^{p^c}\, dx + \frac{4}{p^c}\int_{\mathbb{R}^N} \nabla\psi\cdot\nabla(|x|^{-\tau})|u|^{p^c}\, dx. \end{aligned}$$

## 3. Auxiliary results

This section proves the profile decomposition, a local theory and a variational analysis.

### *3.1. Profile decomposition*

In this subsection, one proves Proposition 2.2. Taking account of [41, 42], it is sufficient to prove that

$$P[u_n] = \sum_{j=1}^{M} P\Big[e^{it_n^j\Delta}\psi^j\Big] + P[W_n^M] + o_n(1).$$

For this, denoting the sequence $(\tilde{\psi}_l^1)_n := e^{it_n^1\Delta}\psi^1$, one needs to establish that

$$\lim_n \int_{\mathbb{R}^N} |x|^{-\tau}\Big(|u_n|^{1+p} - |u_n - (\tilde{\psi}_l^1)_n|^{1+p} - |(\tilde{\psi}_l^1)_n|^{1+p}\Big)\, dx := \lim_n I_n = 0.$$

One of the two next scenarios happens. The first one is $t_n^1 \to \infty$. The second one is $t_n^1 \equiv 0$. Take the first case. Recall two useful inequalities. The first one [35] reads, for any $m \geq 2$,

$$||x|^m - |x-y|^m - |y|^m| \leq m2^{m-1}\Big(|x-y|^{m-1}|y| + |x-y||y|^{m-1}\Big). \tag{3.1}$$

The second one follows from Lemma 2.1,

$$\||x|^{-1}f\|_r \lesssim \|\nabla f\|_r, \quad \text{for all} \quad 1 < r < N. \tag{3.2}$$

Now, by (3.2) and (3.1), via Sobolev embeddings and Hölder estimate, one writes

$$\begin{aligned} I_n &\leq (1+p)2^p \int_{\mathbb{R}^N} |x|^{-\tau}\Big(|u_n - (\tilde{\psi}_l^1)_n|^p|(\tilde{\psi}_l^1)_n| + |u_n - (\tilde{\psi}_l^1)_n||(\tilde{\psi}_l^1)_n|^p\Big)\, dx \\ &\lesssim \int_{\mathbb{R}^N} \Big((|x|^{-1}|u_n - (\tilde{\psi}_l^1)_n|)^\tau |u_n - (\tilde{\psi}_l^1)_n|^{p-\tau}|(\tilde{\psi}_l^1)_n| + (|x|^{-1}|(\tilde{\psi}_l^1)_n|)^\tau|u_n - (\tilde{\psi}_l^1)_n||(\tilde{\psi}_l^1)_n|^{p-\tau}\Big)\, dx \\ &\lesssim \||x|^{-1}(u_n - (\tilde{\psi}_l^1)_n)\|^\tau \|u_n - (\tilde{\psi}_l^1)_n\|_{2^*}^{p-\tau}\|(\tilde{\psi}_l^1)_n\|_{2^*} + \||x|^{-1}(\tilde{\psi}_l^1)_n\|^\tau\|u_n - (\tilde{\psi}_l^1)_n\|_{2^*}\|(\tilde{\psi}_l^1)_n\|_{2^*}^{p-\tau} \end{aligned}$$

$$\lesssim \|u_n - (\tilde{\psi}_l^1)_n\|_{\dot H^1}^{\tau}\|u_n - (\tilde{\psi}_l^1)_n\|_{2^*}^{p-\tau}\|(\tilde{\psi}_l^1)_n\|_{2^*} + \|(\tilde{\psi}_l^1)_n\|_{\dot H^1}^{\tau}\|u_n - (\tilde{\psi}_l^1)_n\|_{2^*}\|(\tilde{\psi}_l^1)_n\|_{2^*}^{p-\tau}. \tag{3.3}$$

Using the free Schrödinger operator dispersive estimate [44], $\|e^{it\Delta}\cdot\|_r \leq \frac{C}{t^{N(\frac{1}{2}-\frac{1}{r})}}\|\cdot\|_{r'}$, for all $r \geq 2$, one gets $\lim_n I_n = 0$. In the second case, the claim follows by (3.3) via (2.24) and (2.20). This finishes the proof.

### *3.2. Local Theory*

One discusses two cases depending on the inhomogeneous index.

#### 3.2.1. First case:

$0 < \tau < \min\left\{\frac{6-N}{2}, \frac{4}{N}\right\}$. Here and hereafter, one takes, for an interval $I \subset \mathbb{R}$, the spaces

$$S(I) := L^{\frac{2(2+N)}{N-2}}\left(I \times \mathbb{R}^N\right); \tag{3.4}$$

$$W(I) := L^{\frac{2(2+N)}{N-2}}\left(I, L^{\frac{2N(2+N)}{4+N^2}}\right); \tag{3.5}$$

$$\nabla W(I) := \left\{u : \quad \nabla_x u \in W(I)\right\}. \tag{3.6}$$

This choice implies in particular that

$$\nabla W(I) \hookrightarrow S(I); \tag{3.7}$$

$$\Big(\frac{2(2+N)}{N-2}, \frac{2N(2+N)}{4+N^2}\Big) \in \Lambda. \tag{3.8}$$

Take also the spaces

$$W_\tau(I) := L^{\frac{2(2+N)(1+\tau)}{N(1+\tau)-2}}\left(I, L^{\frac{2N(2+N)(1+\tau)}{4+N^2(1+\tau)}}\right); \tag{3.9}$$

$$\nabla W_\tau(I) := \left\{u : \quad \nabla_x u \in W_\tau(I)\right\}. \tag{3.10}$$

This choice implies in particular that

$$W_0(I) = W(I); \tag{3.11}$$

$$\Big(\frac{2(2+N)(1+\tau)}{N(1+\tau)-2}, \frac{2N(2+N)(1+\tau)}{4+N^2(1+\tau)}\Big) \in \Lambda. \tag{3.12}$$

This subsection contains two parts.

• Global solution for small data. The problem (INLS) has a local solution in the energy space which is global for small data.

**Proposition 3.1.** *Let* $0 \in I$ *a real interval and* $u_0 \in \dot H^1$*. Then, there exists* $\delta > 0$ *such that if* $\|e^{i\cdot\Delta}u_0\|_{S(I)} \leq \delta$*, then, there is a unique solution to* (INLS) *in* $C(I, \dot H^1)$*. Moreover,* $\|u\|_{S(I)} \leq 2\delta$ *and* $\|\nabla u\|_{W_\tau(I)\cap W(I)} < \infty$.

*Proof.* We proceed with a fixed point argument. Take the Duhamel integral function (2.2). Let also, for $a, b > 0$, the space

$$X_{a,b} := \left\{u \in C(I, \dot H^1), \quad \|u\|_{L^\infty(I,\dot H^1)} \leq 2A, \quad \|u\|_{S(I)} \leq a, \quad \|\nabla u\|_{W_\tau(I)\cap W(I)} \leq b\right\},$$

endowed with the complete distance $d(u,v) := \|u-v\|_{S(I)}$. By the Strichartz estimate, one writes for $u, v \in X_{a,b}$ and $w := u - v$,

$$\begin{aligned} d(f(u), f(v)) &\lesssim \left\| \nabla \Big[ |x|^{-\tau} (|u|^{p-1}u - |v|^{p-1}v) \Big] \right\|_{L^2(I, L^{\frac{2N}{2+N}})} \\ &\lesssim \||x|^{-\tau-1}(|u|^{p-1} + |v|^{p-1})w\|_{L^2(I, L^{\frac{2N}{2+N}})} + \||x|^{-\tau}|u|^{p-1}\nabla w\|_{L^2(I, L^{\frac{2N}{2+N}})} \\ &+ \||x|^{-\tau}(|u|^{p-2} + |v|^{p-2})\nabla v w\|_{L^2(I, L^{\frac{2N}{2+N}})} \\ &:= (I) + (II) + (III). \end{aligned}$$

Thus, by (3.2), one gets

$$\begin{aligned} (II) &\lesssim \|(|x|^{-1}u)^\tau |u|^{p-1-\tau}\nabla w\|_{L^2(I, L^{\frac{2N}{2+N}})} \\ &\lesssim \||x|^{-1}u\|^\tau_{W_\tau(I)} \|u\|^{p-1-\tau}_{S(I)} \|\nabla w\|_{W_\tau(I)} \\ &\lesssim \|\nabla u\|^\tau_{W_\tau(I)} \|u\|^{p-1-\tau}_{S(I)} \|\nabla w\|_{W_\tau(I)} \\ &\lesssim b^\tau a^{p-1-\tau} d(u,v). \end{aligned} \tag{3.13}$$

Here, one used the Hölder estimate via the identities

$$\frac{1}{2} = \frac{p-1-\tau}{\frac{2(2+N)}{N-2}} + \frac{1+\tau}{\frac{2(2+N)(1+\tau)}{N(1+\tau)-2}}; \tag{3.14}$$

$$\frac{2+N}{2N} = \frac{p-1-\tau}{\frac{2(2+N)}{N-2}} + \frac{1+\tau}{\frac{2N(2+N)(1+\tau)}{4+N^2(1+\tau)}}. \tag{3.15}$$

Moreover, in order to estimate $(I)$, it is sufficient to consider the following term, by use of (3.2) and the Hölder estimate via (3.14)–(3.15),

$$\begin{aligned} (I)_1 &:= \||x|^{-\tau-1}|u|^{p-1}w\|_{L^2(I, L^{\frac{2N}{2+N}})} \\ &= \|(|x|^{-1}u)^\tau |u|^{p-1-\tau}(|x|^{-1}w)\|_{L^2(I, L^{\frac{2N}{2+N}})} \\ &\lesssim \||x|^{-1}u\|^\tau_{W_\tau(I)} \|u\|^{p-1-\tau}_{S(I)} \||x|^{-1}w\|_{W_\tau(I)} \\ &\lesssim \|\nabla u\|^\tau_{W_\tau(I)} \|u\|^{p-1-\tau}_{S(I)} \|\nabla w\|_{W_\tau(I)} \\ &\lesssim b^\tau a^{p-1-\tau} d(u,v). \end{aligned} \tag{3.16}$$

Furthermore, in order to estimate $(III)$, it is sufficient to consider the following term, by use of (3.2), Sobolev embeddings, and the Hölder estimate via (3.14)–(3.15),

$$\begin{aligned} (III)_1 &:= \||x|^{-\tau}|u|^{p-2}\nabla v w\|_{L^2(I, L^{\frac{2N}{2+N}})} \\ &= \|(|x|^{-1}w)^\tau |u|^{p-2}\nabla v |w|^{1-\tau}\|_{L^2(I, L^{\frac{2N}{2+N}})} \\ &\lesssim \||x|^{-1}w\|^\tau_{W_\tau(I)} \|u\|^{p-2}_{S(I)} \|\nabla v\|_{W_\tau(I)} \|w\|^{1-\tau}_{S(I)} \\ &\lesssim \|u\|^{p-2}_{S(I)} \|\nabla v\|_{W_\tau(I)} \|w\|_{W_\tau(I)} \\ &\lesssim b a^{p-2} d(u,v). \end{aligned} \tag{3.17}$$

Here, one takes the case $\tau \leq 1$ and used the assumption $p^c \geq 2$, which reads

$$\tau \leq \frac{6-N}{2}. \tag{3.18}$$

Now, if $\tau > 1$, one has $p \geq 2$ and $p > 1+\tau$, so, one writes

$$\begin{aligned}
(III)_1 &:= \||x|^{-\tau}|u|^{p-2}\nabla v w\|_{L^2(I,L^{\frac{2N}{2+N}})} \\
&= \||u|^{p-1-\tau}(|x|^{-1}u)^{\tau-1}\nabla v(|x|^{-1}w)\|_{L^2(I,L^{\frac{2N}{2+N}})} \\
&\lesssim \||x|^{-1}u\|_{W_\tau(I)}^{\tau-1}\|u\|_{S(I)}^{p-1-\tau}\|\nabla v\|_{W_\tau(I)}\||x|^{-1}w\|_{W_\tau(I)} \\
&\lesssim \|\nabla u\|_{W_\tau(I)}^{\tau-1}\|u\|_{S(I)}^{p-1-\tau}\|\nabla v\|_{W_\tau(I)}\|\nabla w\|_{W_\tau(I)} \\
&\lesssim b^\tau a^{p-1-\tau} d(u,v). \qquad (3.19)
\end{aligned}$$

Now, regrouping the identities (3.13) to (3.19), it follows that

$$d(f(u),f(v)) \leq c\Big(b^\tau a^{p-1-\tau} + b a^{p-2}\Big) d(u,v). \tag{3.20}$$

So, $f$ is a contraction for small $0 < a, b \ll 1$. Next, one proves the stability $f(X_{a,b}) \subset X_{a,b}$. Taking $v = 0$ in (3.20) via Strichartz estimates, one gets for the choice $b := 2cA$ and $0 < a \ll 1$,

$$\begin{aligned}
\|\nabla f(u)\|_{W_\tau(I)\cap W(I)} &\leq c\|\nabla u_0\| + d(f(u), f(0)) \\
&\leq cA + c\Big(b^\tau a^{p-1-\tau} + b a^{p-2}\Big) b \qquad (3.21) \\
&\leq \Big[\frac{1}{2} + c\Big((2cA)^\tau a^{p-1-\tau} + 2cA a^{p-2}\Big)\Big] b \qquad (3.22) \\
&< b. \qquad (3.23)
\end{aligned}$$

Now, by (3.2) via the Hölder estimate and Sobolev embedding via an absorption argument, one gets for $\delta := \frac{a}{2} \ll 1$,

$$\begin{aligned}
\|f(u)\|_{S(I)} &\leq \|e^{i\cdot\Delta}u_0\|_{S(I)} + \|(|x|^{-1}u)^\tau |u|^{p-1-\tau} u\|_{L^2(I,L^{\frac{2N}{2+N}})} \\
&\leq \delta + c\||x|^{-1}u\|_{W_\tau(I)}^\tau \|u\|_{S(I)}^{p-1-\tau}\|u\|_{W_\tau(I)} \\
&\leq \delta + c\|\nabla u\|_{W_\tau(I)}^\tau \|u\|_{S(I)}^{p-1-\tau}\|u\|_{W_\tau(I)} \\
&\leq \delta + c b^{1+\tau} a^{p-1-\tau} \\
&\leq 2\delta = a. \qquad (3.24)
\end{aligned}$$

Finally, with Sobolev embeddings and arguing as in (3.23), for $0 < a \ll 1$, one gets

$$\begin{aligned}
\|f(u)\|_{L^\infty(I,\dot H^1)} &\leq A + c\Big(b^\tau a^{p-1-\tau} + b a^{p-2}\Big) b \\
&\leq 2A. \qquad (3.25)
\end{aligned}$$

The stability $f(X_{a,b}) \subset X_{a,b}$ follows by (3.23)–(3.25). The proof is closed via a classical Picard argument. □

• Long-time perturbation. The second part of this section deals with the next result.

**Proposition 3.2.** *Let $T > 0$ and $I := [0,T]$. Take $u \in C(I,\dot H^1)$ as a solution to* (INLS) *and $\tilde u \in L^\infty(I,\dot H^1)$, satisfying for some $\epsilon, A > 0$,*

$$\|\tilde u\|_{L^\infty_T(\dot H^1)\cap S(I)} \le A; \tag{3.26}$$
$$i\tilde u_t + \Delta\tilde u + |x|^{-\tau}|\tilde u|^{p-1}\tilde u = e;$$
$$\max\Big\{\|\nabla e\|_{\Omega'(I)}, \|e^{i\cdot\Delta}[u_0-\tilde u_0]\|_{W_\tau(I)}\Big\} \le \epsilon. \tag{3.27}$$

Then, there exists $\epsilon_0 := \epsilon_0(A)$, satisfying for any $0<\epsilon<\epsilon_0$,

$$\|u\|_{S(I)} \le C(A).$$

*Proof.* Taking $w := u - \tilde u$ and $\mathcal{N}[u] := |x|^{-\tau}|u|^{p-1}u$, one gets

$$\begin{aligned} iw_t + \Delta w &= i\frac{\partial}{\partial t}u + \Delta u - (i\tilde u_t + \Delta\tilde u) \\ &= \mathcal{N}[\tilde u] - \mathcal{N}[w+\tilde u] - e. \end{aligned}$$

Taking account of the Duhamel integral formula (2.2), one writes

$$\begin{aligned} w(t) &= e^{i(t-t_k)\Delta}w(t_k) + i\int_{t_k}^t e^{i(t-\tau)\Delta}\Big(\mathcal{N}[\tilde u] - \mathcal{N}[w+\tilde u]\Big)\,d\tau \\ &+ i\int_{t_k}^t e^{i(t-\tau)\Delta}e(\tau)\,d\tau. \end{aligned} \tag{3.28}$$

Here, one picks a partition

$$t_0 = 0, \quad I := \bigcup_{0\le k\le K}[t_k, t_{1+k}) := \bigcup_k I_k; \tag{3.29}$$
$$\|\nabla\tilde u\|_{W_\tau(t_k,t_{1+k})} < \eta \ll 1, \quad \text{for all} \quad 0\le k\le K. \tag{3.30}$$

Indeed, arguing as in the local theory with a Bootstrap argument and Sobolev embeddings via (3.13), (3.26) and (3.27), one has

$$\begin{aligned} \|\nabla\tilde u\|_{W_\tau(t_k,t_{1+k})} &\lesssim \|\tilde u(t_k)\|_{\dot H^1} + \|\nabla\mathcal{N}[\tilde u]\|_{L^2(I_k,L^{\frac{2N}{2+N}})} + \|\nabla e\|_{\Omega'(I_k)} \\ &\lesssim A + \|\nabla\tilde u\|^{1+\tau}_{W_\tau(I_k)}\|\tilde u\|^{p-1-\tau}_{S(I_k)} + \epsilon \lesssim A. \end{aligned}$$

With a Picard fixed point argument and arguing as in the the local theory, one solves the previous integral equation in $I_0$. So, with Proposition 3.1,

$$\|w\|_{S(I_0)} \le 2\epsilon \quad \text{and} \quad \|\nabla w\|_{W_\tau(I_0)} \le C(\epsilon, A).$$

Letting $t = t_1$ in the previous integral equality (3.28) and applying $e^{i(t-t_1)\Delta}$, one gets

$$\begin{aligned} e^{i(t-t_1)\Delta}w(t_1) &= e^{i(t-t_0)\Delta}w(t_0) + i\int_{t_0}^{t_1} e^{i(t-\tau)\Delta}\Big(\mathcal{N}[\tilde{u}] - \mathcal{N}[w+\tilde{u}]\Big)\,d\tau \\ &+ i\int_{t_0}^{t_1} e^{i(t-\tau)\Delta}e(\tau)\,d\tau. \end{aligned}$$

Taking account of the proof of Proposition 3.1 via (3.27), one gets

$$\|e^{i(t-t_1)\Delta}w(t_1)\|_{S(I_1)} \le \|e^{i(t-t_0)\Delta}w(t_0)\|_{S(I)} + 2c\epsilon \le c\epsilon + 2c\epsilon.$$

Iterating this process, it follows that

$$\|e^{i(t-t_k)\Delta}w(t_k)\|_{S(I_k)} \le \|e^{i(t-t_0)\Delta}w(t_0)\|_{S(I)} + c2^k\epsilon \le c2^{1+k}\epsilon.$$

Now, applying Proposition 3.1 to the above Duhamel integral formula (3.28), we get

$$\begin{aligned} &\|w\|_{S(I_k)} \le c\epsilon 2^{2+k}; \\ &\|w\|_{S(I)} \le c\epsilon\sum_{k=0}^{K} 2^{2+k} \le 4c(-1+2^{1+K})\epsilon. \end{aligned} \tag{3.31}$$

Finally, one ends the proof by the triangle inequality in (3.31) via (3.26),

$$\|u\|_{S(I)} \le \|w\|_{S(I)} + \|\tilde{u}\|_{S(I)} \lesssim C(A).$$

□

#### 3.2.2. Second case

$\frac{2+N}{N} < \tau < 2$.

• Global solution for small data. The problem (INLS) has a local solution in the energy space which is global for small data. We start with some notations.

Let $0 < \varepsilon \ll 1$, $\gamma := 1 - \varepsilon$, and the real numbers

$$r_0 := \frac{2N}{N-2-\varepsilon} = \Big(\frac{2N}{N-2}\Big)^+; \tag{3.32}$$

$$\frac{1}{r_1} = \frac{1}{r_0} + \frac{1}{N} \Longleftrightarrow r_1 = \frac{2N}{N-\varepsilon}; \tag{3.33}$$

$$(q_0, r_1) \in \Lambda_\gamma \Longleftrightarrow q_0 = \frac{4}{2-\varepsilon}; \tag{3.34}$$

$$(q_2, r_2) \in \Lambda_\gamma \quad \text{to be picked later.} \tag{3.35}$$

Here and hereafter, if $I$ is a real interval, one takes the weighted Lebesgue spaces

$$S(I) := L^{q_0}\Big(I, L^{r_0}(|x|^{-r_0\gamma})\Big); \tag{3.36}$$

$$W(I) := L^{q_0}\Big(I, L^{r_1}(|x|^{-r_0\gamma})\Big); \tag{3.37}$$

$$M(I) := L^{q_2}\Big(I, L^{r_2}(|x|^{-r_2\gamma})\Big). \tag{3.38}$$

Since $N \geq 4$ gives $\frac{N}{r_0} > \gamma$, by Lemma 2.1, one gets

$$\|\cdot\|_{S(I)} \lesssim \|\nabla \cdot\|_{W(I)}. \tag{3.39}$$

Let us state the small data global existence result.

**Proposition 3.3.** *Let $N \geq 4$, $\frac{2+N}{N} < \tau < 2$, $0 \in I$ be a real interval and $u_0 \in \dot{H}^1$. Then, there exists $\delta > 0$ such that if $\|e^{i\cdot\Delta}u_0\|_{S(I)} \leq \delta$, there is a unique solution to* (INLS) *in $C(I, \dot{H}^1)$. Moreover, $\|u\|_{S(I)} \leq 2\delta$ and $\|\nabla u\|_{M(I)\cap W(I)} < \infty$.*

*Proof.* Let $f$ be the Duhamel integral function given in (2.2). Let also, for $a, b > 0$, the space

$$X_{a,b} := \Big\{u \in C(I, \dot{H}^1), \quad \|u\|_{L^\infty(I,\dot{H}^1)} \leq 2A, \quad \|u\|_{S(I)} \leq a, \quad \|\nabla u\|_{M(I)\cap W(I)} \leq b\Big\},$$

endowed with the complete distance

$$d(u, v) := \|u - v\|_{S(I)}. \tag{3.40}$$

Take $0 \leq \tilde{\gamma} < 1$ and $(\tilde{q}, \tilde{r}) \in \Lambda_{\tilde{\gamma}}$ to be picked later. Let $u, v \in X_{a,b}$ and $w := u - v$. By the Strichartz estimate in Proposition 2.4 via (3.39), one writes

$$\begin{aligned} d(f(u), f(v)) &\lesssim \Big\|\nabla\Big[|x|^{-\tau}(|u|^{p-1}u - |v|^{p-1}v)\Big]\Big\|_{L^{\tilde{q}'}\big(I,L^{\tilde{r}'}(|x|^{\tilde{\gamma}\tilde{r}'})\big)} \\ &\lesssim \||x|^{-\tau-1}(|u|^{p-1} + |v|^{p-1})w\|_{L^{\tilde{q}'}\big(I,L^{\tilde{r}'}(|x|^{\tilde{\gamma}\tilde{r}'})\big)} + \||x|^{-\tau}|u|^{p-1}\nabla w\|_{L^{\tilde{q}'}\big(I,L^{\tilde{r}'}(|x|^{\tilde{\gamma}\tilde{r}'})\big)} \\ &+ \||x|^{-\tau}(|u|^{p-2} + |v|^{p-2})\nabla v w\|_{L^{\tilde{q}'}\big(I,L^{\tilde{r}'}(|x|^{\tilde{\gamma}\tilde{r}'})\big)} \\ &:= (I) + (II) + (III). \end{aligned} \tag{3.41}$$

In order to estimate $(I)$, it is sufficient to consider the following term

$$\begin{aligned} (I)_1 &:= \||x|^{-\tau-1}|u|^{p-1}w\|_{L^{\tilde{q}'}\big(I,L^{\tilde{r}'}(|x|^{\tilde{\gamma}\tilde{r}'})\big)} \\ &= \||x|^{-\tau-1+\tilde{\gamma}}|u|^{p-1}w\|_{L^{\tilde{q}'}\big(I,L^{\tilde{r}'}\big)}. \end{aligned} \tag{3.42}$$

Using the Hölder estimate via Lemma 2.1 and (3.42), we write

$$\begin{aligned} (I)_1 &= \|(|x|^{-\gamma-1}u)^{p-1-\theta}(|x|^{-\gamma}u)^\theta|x|^{-\gamma-1}w\|_{L^{\tilde{q}'}\big(I,L^{\tilde{r}'}\big)} \\ &\lesssim \||x|^{-\gamma-1}u\|^{p-1-\theta}_{L^{q_2}(I,L^{r_2})}\||x|^{-\gamma}u\|^\theta_{L^{q_0}(I,L^{r_0})}\||x|^{-\gamma-1}w\|_{L^{q_2}(I,L^{r_2})} \\ &\lesssim \||x|^{-\gamma-1}w\|^{p-\theta}_{L^{q_2}L^{r_2}}\||x|^{-\gamma}u\|^\theta_{L^{q_0}L^{r_0}} \\ &\lesssim \|\nabla u\|^{p-1-\theta}_{M(I)}\|u\|^\theta_{S(I)}\|\nabla w\|_{M(I)} \\ &\lesssim b^{p-1-\theta}a^\theta d(u, v). \end{aligned} \tag{3.43}$$

Here, one needs the identities

$$\frac{1}{\tilde{r}'} = \frac{\theta}{r_0} + \frac{p-\theta}{r_2}; \tag{3.44}$$

$$\frac{1}{\tilde{q}'} = \frac{\theta}{q_0} + \frac{p-\theta}{q_2}; \tag{3.45}$$

$$0 < \theta = \tilde{\gamma} + p(1+\gamma) - 1 - \tau < p-1, \tag{3.46}$$

with the inequalities

$$0 < \gamma, \tilde{\gamma} < 1; \tag{3.47}$$

$$\frac{\gamma}{2} < \frac{1}{q_0} \leq \frac{1}{2}, \quad \frac{\gamma}{2} \leq \frac{1}{r_1} < \frac{1}{2}; \tag{3.48}$$

$$0 < \frac{\tilde{\gamma}}{2} < \frac{1}{\tilde{q}} \leq \frac{1}{2}, \quad \frac{\tilde{\gamma}}{2} \leq \frac{1}{\tilde{r}} < \frac{1}{2}. \tag{3.49}$$

Note that (3.44) with (3.46) gives (3.45). Now, one picks

$$\frac{2+N}{N} < \tau < 2 \quad \text{and} \quad 0 < \varepsilon < \frac{\tau N - N - 2}{N-2}. \tag{3.50}$$

The choice (3.50) implies that

$$\tilde{\gamma} := \tau - p - \varepsilon \in (0,1). \tag{3.51}$$

Moreover, one picks

$$\theta := \tilde{\gamma} + p(1+\gamma) - 1 - \tau = p(1-\varepsilon) - 1 - \varepsilon. \tag{3.52}$$

With (3.51) and (3.52), the inequalities in (3.46) are satisfied for $0 < \varepsilon \ll 1$. Now, let us choose

$$\begin{aligned} \frac{1}{\tilde{q}} &= \frac{1}{2}\Big(2 - \frac{2\theta}{q_0} - \gamma(p-\theta)\Big)^- \\ &= \frac{1}{2}\Big(2 - \theta[N(\frac{1}{2} - \frac{1}{r_0} - \frac{1}{N}) + \gamma] - \gamma(p-\theta)\Big)^-. \end{aligned} \tag{3.53}$$

By (3.32), (3.53) implies that

$$\begin{aligned} \frac{1}{\tilde{q}} &= \frac{1}{2}\Big(2 - \theta(\frac{\varepsilon}{2} + \gamma) - \gamma(p-\theta)\Big)^- \\ &= \frac{1}{2}\Big(2 - \theta\frac{\varepsilon}{2} - \gamma p\Big)^- \\ &= \frac{1}{2}\Big(2 - (1-\varepsilon)p - \frac{\varepsilon}{2}(p(1-\varepsilon) - 1 - \varepsilon)\Big)^- \\ &= \frac{1}{2}\Big(2 - p + \frac{\varepsilon}{2}(1+p)(1+\varepsilon)\Big)^- \\ &:= \frac{2 - p + \varepsilon'}{2}. \end{aligned} \tag{3.54}$$

Now, one checks the requested assumptions on the above choice. Compute

$$\begin{aligned}\frac{1}{r_1} &= \frac{1}{N} + \frac{1}{r_0}\\ &= \frac{1}{2} - \frac{\varepsilon}{2N} \in [\frac{\gamma}{2}, \frac{1}{2}) = [\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2}).\end{aligned} \tag{3.55}$$

Moreover,

$$\begin{aligned}\frac{2}{q_0} &= N(\frac{1}{2} - \frac{1}{N} - \frac{1}{r_0}) + \gamma\\ &= 1 - \frac{\varepsilon}{2} \in (\gamma, 1].\end{aligned} \tag{3.56}$$

Furthermore, by (3.51) and (3.54), because $\tau < 2$ implies that $2 - p \in (\tau - p, 1]$, it follows that for $\varepsilon_0 \to 0$,

$$(\tilde{\gamma}, 1] \ni \frac{2}{\tilde{q}} = 2 - p + \varepsilon'. \tag{3.57}$$

Also, by taking $\varepsilon, \varepsilon' \to 0$, one gets $\frac{2}{\tilde{r}} = 1 - \frac{2}{N}(2 - \tau + \varepsilon' + \varepsilon) < 1$ because $\tau < 2$. So, we need to check that

$$\begin{aligned}\tilde{\gamma} \leq \frac{2}{\tilde{r}} &\Longleftrightarrow -\tilde{\gamma} + 1 - \frac{2}{N}\Big(2 - p + \varepsilon' - \tilde{\gamma}\Big) \geq 0\\ &\Longleftrightarrow (N-2)\tau < N - 4 + Np.\end{aligned} \tag{3.58}$$

The last line is clearly satisfied because

$$\tau < 2. \tag{3.59}$$

Let us see the couple $(q_2, r_2)$. By (3.54) and (3.34), for $0 < \varepsilon \ll 1$, one writes

$$(\gamma, 1] \ni \frac{2}{q_2} = \frac{1}{p-\theta}\Big(\frac{2}{\tilde{q}'} - \frac{2\theta}{q_0}\Big) \tag{3.60}$$

$$\begin{aligned}&= \frac{1}{p-\theta}\Big(2 - (2 - p + \varepsilon') - \theta\frac{2-\varepsilon}{2}\Big)\\ &= 1 - \frac{1}{p-\theta}\Big(\varepsilon' - \theta\frac{\varepsilon}{2}\Big)\\ &\Longleftrightarrow \varepsilon\theta < 2\varepsilon' < \varepsilon(2p - \theta).\end{aligned} \tag{3.61}$$

The identity (3.61) is possible because of (3.52) and taking $\varepsilon \ll 1$. Moreover, the equality $\gamma + N(\frac{1}{2} - \frac{1}{r_2}) = \frac{2}{q_2}$ via (3.61) implies that

$$[\gamma, 1) \ni \frac{2}{r_2} = 1 - \frac{2}{N}\Big(1 - \frac{1}{p-\theta}\Big(\varepsilon' - \theta\frac{\varepsilon}{2}\Big) - \gamma\Big)$$

$$= 1 - \frac{2}{N}\Big(\varepsilon - \frac{1}{p-\theta}\Big(\varepsilon' - \theta\frac{\varepsilon}{2}\Big)\Big). \tag{3.62}$$

Now, the identity (3.62) is equivalent to

$$\varepsilon(p-\theta) > \varepsilon' - \frac{1}{2}\varepsilon\theta; \tag{3.63}$$

$$\frac{2}{N}\Big(\varepsilon - \frac{1}{p-\theta}\Big(\varepsilon' - \theta\frac{\varepsilon}{2}\Big)\Big) < \varepsilon. \tag{3.64}$$

The condition (3.63) is satisfied by (3.61) and (3.64) is equivalent to

$$\varepsilon(2p - \theta - N(p-\theta)) < 2\varepsilon'. \tag{3.65}$$

This is clearly possible via (3.61) because $\theta < p$ by (3.52),

Now, (3.55) and (3.56) imply that $(q_0, r_1) \in \Lambda^{\gamma}$. Also, (3.57) and (3.58) imply that $(\tilde{q}, \tilde{r}) \in \Lambda^{\tilde{\gamma}}$. Finally, (3.60) and (3.61) imply that $(q_2, r_2) \in \Lambda^{\gamma}$. Thus, (3.43) follows under the assumption

$$\frac{2+N}{N} < \tau < 2. \tag{3.66}$$

Now, using again the Hölder estimate and Lemma 2.1 via (3.41), one has

$$\begin{aligned}
(II) &= \|(|x|^{-\gamma-1}u)^{p-1-\theta}(|x|^{-\gamma}u)^{\theta}(|x|^{-\gamma}\nabla w)\|_{L^{\tilde{q}'}(I,L^{\tilde{r}'})} \\
&\lesssim \||x|^{-\gamma-1}u\|^{p-1-\theta}_{L^{q_2}(I,L^{r_2})}\||x|^{-\gamma}u\|^{\theta}_{L^{q_0}(I,L^{r_0})}\||x|^{-\gamma}\nabla w\|_{L^{q_2}(I,L^{r_2})} \\
&\lesssim \|\nabla u\|^{p-1-\theta}_{M(I)}\|u\|^{\theta}_{S(I)}\|\nabla w\|_{M(I)} \\
&\lesssim b^{p-1-\theta}a^{\theta}d(u,v).
\end{aligned} \tag{3.67}$$

In order to estimate $(III)$, it is sufficient to consider the following term, where using again the Hölder estimate and Lemma 2.1 via (3.41), one obtains

$$\begin{aligned}
(III)_1 &= \||x|^{-\tau+\tilde{\gamma}}|u|^{p-2}\nabla v w\|_{L^{\tilde{q}'}(I,L^{\tilde{r}'})} \\
&= \|(|x|^{-\gamma}\nabla v)(|x|^{-\gamma}u)^{p-2}(|x|^{-\gamma-1}w)^{p-1-\theta}(|x|^{-\gamma}w)^{\theta-p+2}\|_{L^{\tilde{q}'}(I,L^{\tilde{r}'})} \\
&\lesssim \||x|^{-\gamma}\nabla v\|_{L^{q_2}(I,L^{r_2})}\||x|^{-\gamma}u\|^{p-2}_{L^{q_0}(I,L^{r_0})}\||x|^{-\gamma-1}w\|^{p-1-\theta}_{L^{q_2}(I,L^{r_2})}\||x|^{-\gamma}w\|^{\theta-p+2}_{L^{q_0}(I,L^{r_0})} \\
&\lesssim \|\nabla v\|_{M(I)}\|\nabla w\|^{p-1-\theta}_{M(I)}\|u\|^{p-2}_{S(I)}\|w\|^{\theta-p+2}_{S(I)} \\
&\lesssim b^{p-1-\theta}a^{\theta}d(u,v).
\end{aligned} \tag{3.68}$$

Indeed, (3.46) implies that $p-2 < \theta < p-1$. Now, regrouping the identities (3.41), (3.43), (3.67) and (3.68), it follows that

$$d(f(u), f(v)) \lesssim b^{p-1-\theta}a^{\theta}d(u,v). \tag{3.69}$$

So, $f$ is a contraction for small $0 < a \ll 1$. Next, one proves the stability $f(X_{a,b}) \subset X_{a,b}$. Taking $v = 0$ in (3.69) via Strichartz estimates, one gets for the choice $b := 2cA$ and $0 < a \ll 1$,

$$\|\nabla f(u)\|_{M(I)\cap W(I)} \leq c\|\nabla u_0\| + d(f(u), f(0))$$

$$
\begin{aligned}
&\le cA + cb^{p-\theta}a^{\theta} \\
&\le \Big(\frac{1}{2} + (2cA)^{p-1-\theta}a^{\theta}\Big)b \\
&< b.
\end{aligned}
\tag{3.70}
$$

Arguing as previously, for $0 < a \ll 1$, one gets

$$\|f(u)\|_{L^\infty(I,\dot H^1)} \le A + cb^{p-\theta}a^{\theta} \le 2A. \tag{3.71}$$

Now, for $\delta := \frac{a}{2} \ll 1$, one uses (2.2) via (3.39) and (3.70) to write

$$
\begin{aligned}
\|f(u)\|_{S(I)} &\le \|e^{i\cdot\Delta}u_0\|_{S(I)} + \|\nabla f(u)\|_{M(I)} \\
&\le \delta + ca^{\theta}b^{p-\theta} \\
&\le 2\delta = a.
\end{aligned}
\tag{3.72}
$$

The stability $f(X_{a,b}) \subset X_{a,b}$ follows by (3.70), (3.71) and (3.72). The proof is closed via a classical Picard argument. □

• Long-time perturbation. The second part of this section deals with the next result.

**Proposition 3.4.** *Let $T > 0$ and $I := [0,T]$. Take $u \in C(I,\dot H^1)$ as a solution to* (INLS) *and $\tilde u \in L^\infty(I,\dot H^1)$, satisfying for some $\epsilon, A > 0$,*

$$\|\tilde u\|_{L^\infty_T(\dot H^1)\cap S(I)} \le A; \tag{3.73}$$

$$i\tilde u_t + \Delta\tilde u + |x|^{-\tau}|\tilde u|^{p-1}\tilde u = e;$$

$$\max\Big\{\|\nabla e\|_{L^{\tilde q'}\big(I,L^{\tilde r'}(|x|^{\tilde\gamma\tilde r'})\big)}, \|\nabla e^{i\cdot\Delta}[u_0 - \tilde u_0]\|_{M(I)\cap W(I)}\Big\} \le \epsilon. \tag{3.74}$$

Then, there exists $\epsilon_0 := \epsilon_0(A)$, satisfying for any $0 < \epsilon < \epsilon_0$,

$$\|u\|_{S(I)} \le C(A).$$

The proof is omitted because it follows like Proposition 3.2.

### *3.3. Variational analysis*

In this section, one prepares some estimates related to the stability of the assumptions (2.11)–(2.13) by the flow of(INLS). Take $\varphi \in \dot H^1$ to be the ground state of (2.8), which is a minimizer of (2.9). The Eq (2.7) gives

$$\|\nabla\varphi\|^2 = P[\varphi]; \tag{3.75}$$

$$\|\nabla\varphi\| = C_*^{-\frac{1+p}{p-1}} = C_*^{-\frac{N-\tau}{2-\tau}}; \tag{3.76}$$

$$E(\varphi) = \frac{2-\tau}{N-\tau}C_*^{-2\frac{N-\tau}{2-\tau}}. \tag{3.77}$$

Let us give the first result of this section.

**Lemma 3.1.** *For $\delta \in (0,1)$, there exists $\tilde{\delta} := \tilde{\delta}(\delta, N) \in (0,1)$ such that if $u \in \dot{H}^1$ satisfies*

$$\|\nabla u\| < \|\nabla \varphi\|; \tag{3.78}$$
$$E(u) < (1-\delta)E(\varphi), \tag{3.79}$$

then,

$$\|\nabla u\|^2 < (1-\tilde{\delta})\|\nabla \varphi\|^2; \tag{3.80}$$
$$\|\nabla u\|^2 - P[u] \geq \tilde{\delta}\|\nabla u\|^2; \tag{3.81}$$
$$E(u) \geq 0. \tag{3.82}$$

*Proof.* Take the real function $f(x) := x - \frac{2}{1+p}C_*^{1+p}x^{\frac{1+p}{2}}$. Then,

$$\begin{aligned} f(\|\nabla u\|^2) &= \|\nabla u\|^2 - \frac{2}{1+p}C_*^{1+p}\|\nabla u\|^{1+p} \\ &\leq \|\nabla u\|^2 - \frac{2}{1+p}P[u] \\ &\leq E(u) \qquad (3.83) \\ &\leq (1-\delta)E(\varphi). \qquad (3.84) \end{aligned}$$

The equation $f'(x) = 0$ is equivalent to $x = x^* = C_*^{-\frac{2(1+p)}{p-1}} = \|\nabla \varphi\|^2$. Moreover, by (3.77), one has $f(x^*) = E(\varphi)$. Now, since $f$ is positive and strictly increasing on $[0, x^*]$, one has (3.80) and (3.82) by (3.83) and (3.84). Now, let the real function $g(x) := x - C_*^{1+p}x^{\frac{1+p}{2}}$. Then,

$$\begin{aligned} \|\nabla u\|^2 - P[u] &\geq \|\nabla u\|^2 - (C_*\|\nabla u\|)^{1+p} \\ &= g(\|\nabla u\|^2). \qquad (3.85) \end{aligned}$$

Moreover, $g(x) = 0$ if, and only if, $x = 0$ or $x = x^*$. Thus, $g(x) \gtrsim x$ on $[0, (1-\tilde{\delta})x^*]$. So, (3.80) gives (3.81). □

**Corollary 3.1.** *If $u \in \dot{H}^1$ satisfies $\|\nabla u\| < \|\nabla \varphi\|$. Then, $E(u) \geq 0$.*

*Proof.* The case $E(u) \geq E(\varphi) = \frac{2-\tau}{N-\tau}C_*^{-2\frac{N-\tau}{2-\tau}} > 0$ is clear. Otherwise, Lemma 3.1 gives the result. □

With Lemma 3.1 via a continuity argument and the conservation of the energy, one has the following energy trapping.

**Proposition 3.5.** *For $\delta \in (0,1)$, there exists $\tilde{\delta} \in (0,1)$ such that if $u_0 \in \dot{H}^1$ satisfies:*

$$\|\nabla u_0\| < \|\nabla \varphi\|; \tag{3.86}$$
$$E(u_0) < (1-\delta)E(\varphi), \tag{3.87}$$

then the maximal solution to (INLS) satisfies, for any $t \in [0, T^+)$,

$$\|\nabla u(t)\|^2 < (1-\tilde{\delta})\|\nabla\varphi\|^2; \tag{3.88}$$
$$\|\nabla u(t)\|^2 - P[u(t)] \geq \tilde{\delta}\|\nabla u(t)\|^2; \tag{3.89}$$
$$E(u) \geq 0; \tag{3.90}$$
$$E(u(t)) \simeq \|\nabla u(t)\|^2 \simeq \|\nabla u_0\|^2. \tag{3.91}$$

*Proof.* For the last point, since $E(u(t)) \leq \|\nabla u(t)\|^2$, by (3.89), one has

$$\begin{aligned} E(u(t)) &\geq (1-\frac{2}{1+p})\|\nabla u(t)\|^2 + \frac{2}{1+p}(\|\nabla u(t)\|^2 - P[u(t)]) \\ &\geq (1-\frac{2}{1+p})\|\nabla u(t)\|^2. \end{aligned}$$

The rest of the proof follows by Lemma 3.1 via the conservation of the energy and a continuity argument. □

Now, one gives a result similar to Lemma 3.1, in the complementary of the assumption (3.78).

**Lemma 3.2.** *For $\delta \in (0,1)$, there exists $\tilde{\delta} := \tilde{\delta}(\delta, N) \in (0,1)$ such that if $u \in \dot{H}^1$ satisfies* (3.79) *and*

$$\|\nabla u\| > \|\nabla\varphi\|, \tag{3.92}$$

then

$$\|\nabla u\|^2 > (1+\tilde{\delta})\|\nabla\varphi\|^2; \tag{3.93}$$
$$\|\nabla u\|^2 - P[u] \leq -\tilde{\delta}\|\nabla\varphi\|^2. \tag{3.94}$$

*Proof.* The proof of (3.93) is omitted because it is similar to Lemma 3.1. For (3.94), one writes

$$\begin{aligned} 2\Big(\|\nabla u\|^2 - P[u]\Big) &= (1+p)E(u) - (p-1)\|\nabla u\|^2 \\ &< (1+p)(1-\delta)E(\varphi) - (p-1)\|\nabla\varphi\|^2 \\ &< (1-\delta)(p-1)\|\nabla\varphi\|^2 - (p-1)\|\nabla\varphi\|^2 \\ &< -\delta(p-1)\|\nabla\varphi\|^2. \end{aligned}$$

This closes the proof. □

## 4. Energy critical scattering

In this section, one proves the global existence and energy scattering, namely, the first part of Theorem 2.1. The proof follows with contradiction. To begin, one proves that if the first part of Theorem 2.1 fails, then Proposition 2.1 holds. Then, one shows that the scenarios in Proposition 2.1 don't happen.

*4.1. Sketch of the proof of Proposition 2.1*

We treat the case $0 < \tau < \min\{\frac{6-N}{2}, \frac{4}{N}\}$ because the case $\frac{2+N}{N} < \tau < 2$ follows similarly. For $0 < \lambda_n < \infty$ and $x_n \in \mathbb{R}^N$, one defines the operator

$$f_n[\psi] := \frac{1}{\lambda_n^{\frac{N-2}{2}}}\psi(\frac{\cdot - x_n}{\lambda_n}). \tag{4.1}$$

The next result is essential in proving Proposition 2.1 was established in [28, Proposition 3.3] in three space dimensions.

**Proposition 4.1.** *Let the sequences* $0 < \lambda_n < \infty$*,* $x_n \in \mathbb{R}^N$*, and* $t_n \in \mathbb{R}$*, such that*

$$|\frac{x_n}{\lambda_n}| \to \infty, \quad \text{and} \quad t_n \equiv 0 \quad \text{or} \quad t_n \to \pm\infty. \tag{4.2}$$

Take $\psi \in \dot{H}^1$ and the sequence

$$\psi_n := f_n[e^{\mathrm{i}t_n\Delta}\psi] = e^{\mathrm{i}\lambda_n^2 t_n\Delta} f_n[\psi]. \tag{4.3}$$

Then, for any $n >> 1$, there is a global solution to (INLS) denoted by $v_n \in C(\mathbb{R}, \dot{H}^1)$ satisfying

$$v_n(0) = \psi_n, \quad \|v_n\|_{\nabla W_\tau(\mathbb{R})} \lesssim 1. \tag{4.4}$$

Moreover, for all $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ and $\chi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^N)$ such that

$$\|\lambda_n^{\frac{N-2}{2}} v_n(\lambda_n^2(t - t_n), \lambda_n x + x_n) - \chi\|_{\nabla W_\tau(\mathbb{R})} < \varepsilon, \quad \forall n > n_0. \tag{4.5}$$

*Proof.* Let a smooth function be

$$\chi_n(x) := \begin{cases} 1, & |x + \frac{x_n}{\lambda_n}| \geq \frac{1}{2}|\frac{x_n}{\lambda_n}|; \\ 0, & |x + \frac{x_n}{\lambda_n}| < \frac{1}{4}|\frac{x_n}{\lambda_n}|. \end{cases}, \quad |\partial^\alpha \chi_n| \lesssim |\frac{x_n}{\lambda_n}|^{-|\alpha|}. \tag{4.6}$$

Take the sequence of slabs

$$I_{n,T} := [a_{n,T}^-, a_{n,T}^+] := [-\lambda_n^2(t_n + T), \lambda_n^2(-t_n + T)]; \tag{4.7}$$

$$I_{n,T}^+ := (a_{n,T}^+, \infty), \quad I_{n,T}^- := (-\infty, a_{n,T}^-). \tag{4.8}$$

Taking account of [40, Appendix A.2], let a Littlewood-Paley frequency cutoff be

$$P_n := P_{|\frac{x_n}{\lambda_n}|^{-\theta} \leq \cdot \leq |\frac{x_n}{\lambda_n}|^\theta}, \quad \theta \in (0, 1). \tag{4.9}$$

Let us denote the sequence of approximate solutions to (INLS),

$$\tilde{v}_{n,T}(t) := \begin{cases} f_n[\chi_n P_n e^{\mathrm{i}(\lambda_n^{-2}t + t_n)\Delta}\psi], & t \in I_{n,T}; \\ e^{\mathrm{i}(t - a_{n,T}^+)\Delta}[\tilde{v}_{n,T}(a_{n,T}^+)], & t \in I_{n,T}^+; \\ e^{\mathrm{i}(t - a_{n,T}^-)\Delta}[\tilde{v}_{n,T}(a_{n,T}^-)], & t \in I_{n,T}^-. \end{cases} \tag{4.10}$$

Using the long-time perturbation result in Proposition 3.2, one proves the existence of the solutions $v_n$.

• Proof of the condition

$$\limsup_{T\to\infty}\limsup_{n\to\infty}\|\tilde{v}_{n,T}\|_{L^\infty(\mathbb{R},\dot{H}^1)\cap\nabla W_\tau(\mathbb{R})} \lesssim 1. \tag{4.11}$$

One has directly from (4.6) via the Hölder estimate,

$$\begin{aligned}\|\chi_n\|_\infty + \|\nabla\chi_n\|_N &\lesssim 1 + |\frac{x_n}{\lambda_n}|^{-1}|B(|\frac{x_n}{\lambda_n}|)|^{\frac{1}{N}}\\ &\lesssim 1. \end{aligned} \tag{4.12}$$

Now, by the Hölder estimate, it follows that on $I_{n,T}$,

$$\begin{aligned}\|\tilde{v}_{n,T}\|_{\dot{H}^1} &= \|f_n[\chi_n P_n e^{i(\lambda_n^{-2}t+t_n)\Delta}\psi]\|_{\dot{H}^1}\\ &\lesssim \|\chi_n(\frac{\cdot - x_n}{\lambda_n})e^{i(t+\lambda_n^2 t_n)\Delta}[f_n(P_n\psi)]\|_{\dot{H}^1}\\ &\lesssim \|\chi_n\|_\infty\|e^{i(t+\lambda_n^2 t_n)\Delta}[f_n(P_n\psi)]\|_{\dot{H}^1} + \|\nabla\chi_n\|_N\|e^{i(t+\lambda_n^2 t_n)\Delta}[f_n(P_n\psi)]\|_{2^*}. \end{aligned} \tag{4.13}$$

Take $\phi_{P_n}$, a bump function associated to the projector $P_n$. By Strichartz and Bernstein estimates and Sobolev embedding via (4.13), it follows that on $I_{n,T}$,

$$\begin{aligned}\|\tilde{v}_{n,T}\|_{\dot{H}^1} &\lesssim \|f_n(P_n\psi)\|_{\dot{H}^1}\\ &\lesssim \|P_n\psi\|_{\dot{H}^1}\\ &\lesssim |\frac{x_n}{\lambda_n}|^{-\theta}\|\phi_{P_n}\|_N\|\psi\|_{2^*}\\ &\lesssim 1. \end{aligned} \tag{4.14}$$

Also, by the Hölder and Strichartz estimates via Sobolev embedding and (4.13), it follows that

$$\begin{aligned}\|\tilde{v}_{n,T}\|_{\nabla W_\tau(I_{n,T})} &= \|f_n[\chi_n P_n e^{i(\lambda_n^{-2}t+t_n)\Delta}\psi]\|_{\nabla W_\tau(I_{n,T})}\\ &\lesssim \|\chi_n(\frac{\cdot - x_n}{\lambda_n})e^{i(t+\lambda_n^2 t_n)\Delta}[f_n(P_n\psi)]\|_{\nabla W_\tau(I_{n,T})}\\ &\lesssim \|\chi_n\|_\infty\|e^{i(t+\lambda_n^2 t_n)\Delta}[f_n(P_n\psi)]\|_{\nabla W_\tau(I_{n,T})} + \|\nabla\chi_n\|_N\|e^{i(t+\lambda_n^2 t_n)\Delta}[f_n(P_n\psi)]\|_{S(I_{n,T})}\\ &\lesssim \|f_n(P_n\psi)\|_{\dot{H}^1}. \end{aligned} \tag{4.15}$$

Thus, by (4.14) and (4.15), one gets

$$\|\tilde{v}_{n,T}\|_{L^\infty(I_{n,T},\dot{H}^1)\cap\nabla W_\tau(I_{n,T})} \lesssim 1. \tag{4.16}$$

Thus, (4.11) follows by (4.10) and (4.16) via Strichartz estimates.

• Proof of the condition

$$\limsup_{T\to\infty}\limsup_{n\to\infty}\|\tilde{v}_{n,T}(0) - \psi_n\|_{\dot{H}^1} = 0. \tag{4.17}$$

Let us take two cases: the first one is $t_n = 0 \in I_{n,T}$. So,

$$\|\tilde{v}_{n,T}(0) - \psi_n\|_{\dot{H}^1} = \|(1 - \chi_n P_n)\psi\|_{\dot{H}^1} \to 0. \tag{4.18}$$

In a second case, one assumes that $t_n \to +\infty$ and $0 \in I^+_{n,T}$. So, by (4.10), one writes

$$\begin{aligned}
\|\tilde{v}_{n,T}(0) - \psi_n\|_{\dot{H}^1} &= \|e^{-ia^+_{n,T}\Delta}[\tilde{v}_{n,T}(a^+_{n,T})] - \psi_n\|_{\dot{H}^1} \\
&= \|e^{-ia^+_{n,T}\Delta}[f_n(\chi_n P_n e^{\mathrm{i}(\lambda_n^{-2}a^+_{n,T}+t_n)\Delta}\psi] - \psi_n\|_{\dot{H}^1} \\
&= \|f_n[e^{\mathrm{i}t_n\Delta}e^{-iT\Delta}\chi_n P_n e^{\mathrm{i}T\Delta}\psi] - f_n[e^{\mathrm{i}t_n\Delta}\psi]\|_{\dot{H}^1} \\
&= \|(1 - \chi_n P_n)e^{\mathrm{i}T\Delta}\psi\|_{\dot{H}^1} \to 0.
\end{aligned} \tag{4.19}$$

• Proof of the condition

$$\begin{aligned}
\limsup_{T\to\infty}\limsup_{n\to\infty}\|\tilde{e}_{n,T}\|_{L^2(\mathbb{R},\dot{W}^{1,\frac{2N}{2+N}})} &:= \limsup_{T\to\infty}\limsup_{n\to\infty}\|(i\partial_t + \Delta)\tilde{v}_{n,T} + |x|^{-\tau}|\tilde{v}_{n,T}|^{p^c-1}\tilde{v}_{n,T}\|_{L^2(\mathbb{R},\dot{W}^{1,\frac{2N}{2+N}})} \\
&= 0.
\end{aligned} \tag{4.20}$$

Let us split the error into two parts as follows $\tilde{e}_{n,T} := \tilde{e}^l_{n,T} + \tilde{e}^{nl}_{n,T}$. First one writes on $I_{n,T}$,

$$\begin{aligned}
\tilde{e}^l_{n,T} &= (i\partial_t + \Delta)\tilde{v}_{n,T} \\
&= (i\partial_t + \Delta)\Big(f_n[\chi_n P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi]\Big) \\
&= (i\partial_t + \Delta)\Big([\chi_n(\frac{x - x_n}{\lambda_n})e^{\mathrm{i}(\lambda_n^2 t_n+t)\Delta}f_n P_n\psi]\Big) \\
&= \Delta(\chi_n(\frac{x - x_n}{\lambda_n}))e^{\mathrm{i}(\lambda_n^2 t_n+t)\Delta}[f_n P_n\psi] + 2\nabla(\chi_n(\frac{x - x_n}{\lambda_n}))\cdot e^{\mathrm{i}(\lambda_n^2 t_n+t)\Delta}\nabla[f_n P_n\psi].
\end{aligned} \tag{4.21}$$

Moreover,

$$\begin{aligned}
\tilde{e}^{nl}_{n,T} &= |x|^{-\tau}|\tilde{v}_{n,T}|^{p^c-1}\tilde{v}_{n,T} \\
&= |x|^{-\tau}|f_n[\chi_n P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi]|^{p^c-1}f_n[\chi_n P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi] \\
&= \lambda_n^{-(2-\tau)}f_n\Big(|\lambda_n x + x_n|^{-\tau}\chi_n^{p^c}|P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi|^{p^c-1}P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi\Big).
\end{aligned} \tag{4.22}$$

Now, by Hölder and Bernstein estimates via (4.6) and (4.21), one writes for $0 < \theta \ll 1$,

$$\begin{aligned}
\|\nabla\tilde{e}^l_{n,T}\|_{\Omega'(I_{n,T})} &\lesssim \sum_{k=1}^{3}\|\partial^k[\chi_n(\frac{x - x_n}{\lambda_n})]e^{\mathrm{i}(\lambda_n^2 t_n+t)\Delta}\partial^{3-k}[f_n P_n\psi]\|_{L^1(I_{n,T},L^2)} \\
&\lesssim |I_{n,T}|\sum_{k=1}^{3}\lambda_n^{-k}\|\partial^k\chi_n(\frac{x - x_n}{\lambda_n})\|_\infty\|\partial^{3-k}[f_n P_n\psi]\|_{L^\infty(I_{n,T},L^2)} \\
&\lesssim \lambda_n^2 T\sum_{k=1}^{3}\lambda_n^{-k}|\frac{x_n}{\lambda_n}|^{-k}\lambda_n^{k-2}\|\partial^{3-k}[P_n\psi]\|_{L^\infty(I_{n,T},L^2)} \\
&\lesssim T\sum_{k=1}^{3}|\frac{x_n}{\lambda_n}|^{-k+(3-k)\theta} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{4.23}$$

Moreover, by (4.22),

$$\|\nabla\tilde{e}^{nl}_{n,T}\|_{\Omega'(I_{n,T})} \lesssim \lambda_n^{-(2-\tau)}\Big\|\nabla\Big[f_n\Big(|\lambda_n x + x_n|^{-\tau}\chi_n^{p^c}|P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi|^{p^c-1}P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi\Big)\Big]\Big\|_{L^2(I_{n,T},L^{\frac{2N}{2+N}})}$$
$$\lesssim \lambda_n^{\tau}\sqrt{T}\Big\|\nabla\Big[\chi_n^{p^c}|\lambda_n x + x_n|^{-\tau}|P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi|^{p^c}\Big]\Big\|_{L^\infty(I_{n,T},\dot{W}^{1,\frac{2N}{2+N}})}. \tag{4.24}$$

Now, taking account of (4.6), one has

$$\|\chi_n^{p^c}|\lambda_n x + x_n|^{-\tau}\|_\infty \lesssim |x_n|^{-\tau}; \tag{4.25}$$
$$\|\nabla[\chi_n^{p^c}|\lambda_n x + x_n|^{-\tau}]\|_\infty \lesssim |x_n|^{-\tau}|\frac{x_n}{\lambda_n}|^{-1}. \tag{4.26}$$

So, by Hölder and Bernstein estimates via (4.24) and (4.26), one writes

$$\|\nabla\tilde{e}^{nl}_{n,T}\|_{\Omega'(I_{n,T})} \lesssim \lambda_n^{\tau}|x_n|^{-\tau}|\frac{x_n}{\lambda_n}|^{-1}\|[P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi]^{p^c}\|_{L^\infty(I_{n,T},L^{\frac{2N}{2+N}})}$$
$$+ \lambda_n^{\tau}\|\chi_n^{p^c}|\lambda_n x + x_n|^{-\tau}|P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi|^{p^c-1}\nabla(P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi)\|_{L^\infty(I_{n,T},L^{\frac{2N}{2+N}})}$$
$$:= (A) + (B). \tag{4.27}$$

Using Hölder and Bernstein estimates and Sobolev embedding, one gets

$$(A) = \lambda_n^{\tau}|x_n|^{-\tau}|\frac{x_n}{\lambda_n}|^{-1}\||P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi|^{p^c}\|_{L^\infty(I_{n,T},L^{\frac{2N}{2+N}})}$$
$$\lesssim \lambda_n^{\tau}|x_n|^{-\tau}|\frac{x_n}{\lambda_n}|^{-1}\|P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi\|^{p^c}_{L^\infty(I_{n,T},\dot{H}^1)}$$
$$\lesssim |\frac{x_n}{\lambda_n}|^{-\tau-1+\theta}\|\psi\|^{p^c}_{\dot{H}^1} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.28}$$

Also, by Hölder and Bernstein estimates and Sobolev embedding, one gets, via the Strichartz estimates,

$$(B) = \lambda_n^{\tau}\|\chi_n^{p^c}|\lambda_n x + x_n|^{-\tau}|P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi|^{p^c-1}\nabla(P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi)\|_{L^\infty(I_{n,T},L^{\frac{2N}{2+N}})}$$
$$\lesssim |\frac{x_n}{\lambda_n}|^{-\tau}\|P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi\|^{p^c-1}_{L^\infty(I_{n,T},L^{N(p^c-1)})}\|\nabla(P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi)\|_{L^\infty(I_{n,T},L^2)}$$
$$\lesssim |\frac{x_n}{\lambda_n}|^{-\tau+\theta\frac{(N-2)(1-\tau)}{2(2-\tau)}}\|e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi\|^{p^c-1}_{L^\infty(I_{n,T},L^{2^*})}\|\nabla(P_n e^{\mathrm{i}(\lambda_n^{-2}t+t_n)\Delta}\psi)\|_{L^\infty(I_{n,T},L^2)}$$
$$\lesssim |\frac{x_n}{\lambda_n}|^{-\tau+\theta(p^c+\frac{(N-2)(1-\tau)}{2(2-\tau)})}\|\psi\|^{p^c}_{\dot{H}^1} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.29}$$

Taking $\theta < \min\{1+b, \frac{b}{p^c+\frac{(N-2)(1-\tau)}{2(2-\tau)}}\}$, the proof is finished by (4.27)–(4.29). Now, one turns on $I^+_{n,T}$. In such a case, one has via (4.10),

$$\tilde{e}_{n,T} = (i\partial_t + \Delta)\tilde{v}_{n,T} + |x|^{-\tau}|\tilde{v}_{n,T}|^{p^c-1}\tilde{v}_{n,T}$$
$$= |x|^{-\tau}|\tilde{v}_{n,T}|^{p^c-1}\tilde{v}_{n,T}$$
$$= |x|^{-\tau}|e^{\mathrm{i}(t-a^+_{n,T})\Delta}[\tilde{v}_{n,T}(a^+_{n,T})]|^{p^c-1}e^{\mathrm{i}(t-a^+_{n,T})\Delta}[\tilde{v}_{n,T}(a^+_{n,T})]. \tag{4.30}$$

Arguing as in (3.13), one writes, via (4.11),

$$\begin{aligned}\|\tilde{e}_{n,T}\|_{\Omega'(I_{n,T})} &\lesssim \|e^{\mathrm{i}(t-a^+_{n,T})\Delta}[\tilde{v}_{n,T}(a^+_{n,T})]\|^{p^c-1-\tau}_{S(I_{n,T})}\|e^{\mathrm{i}(t-a^+_{n,T})\Delta}[\tilde{v}_{n,T}(a^+_{n,T})]\|^{1+b}_{W_\tau(I_{n,T})}\\ &\lesssim \|e^{\mathrm{i}(t-a^+_{n,T})\Delta}[\tilde{v}_{n,T}(a^+_{n,T})]\|^{p^c-1-\tau}_{S(I_{n,T})}\\ &\lesssim \|e^{\mathrm{i}t\Delta}[f_n(\chi_n P_n e^{\mathrm{i}T\Delta}\psi)]\|^{p^c-1-\tau}_{S(0,\infty)}\\ &\lesssim \|f_n e^{\mathrm{i}\lambda_n^{-2}t\Delta}[\chi_n P_n e^{\mathrm{i}T\Delta}\psi]\|^{p^c-1-\tau}_{S(0,\infty)}. \end{aligned} \tag{4.31}$$

Moreover, with a change of variable via (4.31) and Sobolev embedding with Strichartz estimates, one gets

$$\begin{aligned}\|\tilde{e}_{n,T}\|_{\Omega'(I_{n,T})} &\lesssim \|e^{\mathrm{i}t\Delta}[\chi_n P_n e^{\mathrm{i}T\Delta}\psi]\|^{p^c-1-\tau}_{S(0,\infty)}\\ &\lesssim \|e^{\mathrm{i}t\Delta}[\chi_n P_n e^{\mathrm{i}T\Delta}\psi]\|^{p^c-1-\tau}_{\nabla W_\tau(0,\infty)}\\ &\lesssim \|\nabla[\chi_n P_n - 1]e^{\mathrm{i}T\Delta}\psi\|^{p^c-1-\tau} + \|e^{\mathrm{i}t\Delta}\psi\|^{p^c-1-\tau}_{W_\tau(T,\infty)}. \end{aligned} \tag{4.32}$$

The proof is achieved by the dominated convergence theorem and (4.32).

Now, (4.4) follows with a direct application of Proposition 3.4 which gives also

$$\limsup_{T\to\infty}\limsup_{n}\|\tilde{v}_{n,T} - v_n\|_{\nabla W_\tau(\mathbb{R})} = 0. \tag{4.33}$$

In the rest, one proves (4.5), which is reduced via (4.33) to

$$\limsup_{T\to\infty}\limsup_{n}\|\chi_n e^{\mathrm{i}t\Delta}P_n\psi - \chi\|_{\nabla W_\tau(\mathbb{R})} = 0. \tag{4.34}$$

For large $T >> 1$ and $-T < t < T$, by (4.10) and the dominated convergence theorem, one has for $n \to \infty$,

$$\begin{aligned}\|\lambda_n^{\frac{N-2}{2}}\tilde{v}_{n,T}(\lambda_n^2(t-t_n), \lambda_n x + x_n) - \chi\|_{\nabla W_\tau(\mathbb{R})} &= \|\chi_n e^{\mathrm{i}t\Delta}P_n\psi - \chi\|_{\nabla W_\tau(\mathbb{R})}\\ &\to \|e^{\mathrm{i}t\Delta}\psi - \chi\|_{\nabla W_\tau(\mathbb{R})}. \end{aligned} \tag{4.35}$$

So, (4.5) follows by (4.35) via a density argument. Moreover, for $t > T$, one has via (4.10),

$$\begin{aligned}\lambda_n^{\frac{N-2}{2}}\tilde{v}_{n,T}(\lambda_n^2(t-t_n), \lambda_n x + x_n) &= f_n^{-1}e^{\mathrm{i}\lambda_n^2(t-T)\Delta}f_n\chi_n e^{\mathrm{i}T\Delta}P_n\psi\\ &= e^{\mathrm{i}t\Delta}(e^{-iT\Delta}\chi_n e^{\mathrm{i}T\Delta})P_n\psi. \end{aligned} \tag{4.36}$$

So, (4.5) follows by (4.36) via a density argument. □

Now, one returns to the proof of Proposition 2.1. One says that the statement $(SC)(u_0)$ holds if: For $u_0 \in \dot{H}^1$ satisfying (2.11) and (2.12), the corresponding solution to (INLS) is global and satisfies:

$$u \in S(\mathbb{R}), \tag{4.37}$$

where $S(\mathbb{R})$ is the space defined in (3.4) and (3.36), respectively. Using Sobolev embeddings and the Strichartz estimate, one writes for $0 < T < T^+$,

$$\|e^{\mathrm{i}\cdot\Delta}u_0\|_{S(0,T)} \lesssim \|\nabla e^{\mathrm{i}\cdot\Delta}u_0\|_{W_\tau(0,T)} \lesssim \|\nabla u_0\|. \tag{4.38}$$

Thus, if $\|\nabla u_0\| \ll 1$, by the small data theory in Proposition 3.3, $(SC)(u_0)$ holds. Now, for each $\delta > 0$, one defines the quantities

$$S_\delta := \Big\{u_0 \in \dot{H}^1, \quad E(u_0) < \delta \quad \text{and} \quad \|\nabla u_0\| < \|\nabla\varphi\|\Big\}; \tag{4.39}$$

$$E_c := \sup\Big\{\delta > 0 \quad \text{s. t} \quad u_0 \in S_\delta \Rightarrow (SC)(u_0) \quad \text{holds}\Big\}. \tag{4.40}$$

If the first part of Theorem 2.1 fails, it follows that

$$E_c < E(\varphi). \tag{4.41}$$

Then, there is a sequence $u_n$ of solutions to (INLS) such that the data $u_{n,0} \in \dot{H}^1$ satisfies

$$\|\nabla u_{n,0}\| < \|\nabla\varphi\|; \tag{4.42}$$

$$E(u_{n,0}) \to E_c \quad \text{as} \quad n \to \infty; \tag{4.43}$$

$$\|u_n\|_{S(\mathbb{R})} = \infty \quad \text{ for any } n. \tag{4.44}$$

Now, using the profile decomposition in Proposition 2.2, one writes

$$\begin{aligned} u_{n,0} &= \sum_{j=1}^{M} f_n^j(e^{\mathrm{i}t_n^j\Delta}\psi^j) + W_n^M \\ &:= \frac{1}{(\lambda_n^j)^{\frac{N-2}{2}}}\sum_{j=1}^{M}[e^{\mathrm{i}t_n^j\Delta}\psi^j](\frac{\cdot - x_n^j}{\lambda_n^j}) + W_n^M. \end{aligned} \tag{4.45}$$

Using Proposition 3.4 and Proposition 4.1 and following lines in [28, Theorem 1.2], one has only one profile, $(t_n, x_n) \equiv (0, 0)$ and $\|W_n^1\|_{\dot{H}^1} \to 0$. Then, there exists $0 < \lambda_n < \infty$ such that $\|\lambda_n^{\frac{N-2}{2}} u_{0,n}(\lambda_n\cdot) - \psi\|_{\dot{H}^1} \to 0$. Taking account of Propositions 3.4 and 3.5, the solution to (INLS) with datum $\psi$ is the solution needed. See [28, Theorem 1.2] for more details.

### 4.2. *Preclusion of compact solutions*

Let $u \in C([0, T^+), \dot{H}^1)$ and a frequency scale function $\lambda : [0, T^+) \mapsto \mathbb{R}_+$, such that $\inf_{t\in[0,T^+)} \lambda(t) \geq 1$, given in Proposition 2.1. One discusses two cases.

#### 4.2.1. Finite-time blowup scenario

To preclude the finite-time blowup scenario, one needs the following reduced Duhamel formula [40, Proposition 8.7], which is a consequence of the compactness properties.

**Lemma 4.1.** *The following weak limit holds in $\dot{H}^1$ for $T \to T^+$,*

$$i\int_t^T e^{\mathrm{i}(t-s)\Delta}\Big[|x|^{-\tau}|u|^{p^c-1}u\Big]\,ds \rightharpoonup u(t). \tag{4.46}$$

Now, assume that $T^+ < \infty$. By (4.46) via Hölder, Hardy, and Bernstein estimates, one writes for $M > 0$,

$$\|P_M u(t)\| \lesssim \|P_M[|x|^{-\tau}|u|^{p^c-1}u]\|_{L^1((0,T^+),L^2)}$$

$$\lesssim M(T^+ - t)\|(|x|^{-1}u)^\tau |u|^{p^c-\tau}\|_{L^\infty((0,T^+),L^{\frac{2N}{2+N}})}$$
$$\lesssim M(T^+ - t)\||x|^{-1}u\|^\tau_{L^\infty((0,T^+),L^2)}\|u\|^{p^c-\tau}_{L^\infty((0,T^+),L^{2^*})}$$
$$\lesssim M(T^+ - t)\|u\|^\tau_{L^\infty((0,T^+),\dot H^1)}\|u\|^{p^c-\tau}_{L^\infty((0,T^+),L^{2^*})}. \tag{4.47}$$

So, by the Bernstein inequality for the high frequencies via (4.47), one gets

$$\|u(t)\| \lesssim \|P_M u(t)\| + \|(1 - P_M)u(t)\|$$
$$\lesssim M(T^+ - t) + M^{-1}. \tag{4.48}$$

Now, taking account of the mass conservation and letting $t$ be close to $T^+$, it follows that $u = 0$ which contradicts $T^+ < \infty$ and closes the proof.

4.2.2. Soliton-like scenario

In this subsection, one assumes that $T^+ = \infty$. Let us give some notations in the spirit of [16]. Take, for $R >> 1$, the radial function defined on $\mathbb{R}^N$ by

$$\zeta : x \mapsto \begin{cases} \frac{1}{2}|x|^2, & \text{if} \quad |x| \le R/2; \\ R|x|, & \text{if} \quad |x| > R. \end{cases}$$

Moreover, one assumes that in the centered annulus $C(R/2, R) := \{x \in \mathbb{R}^N,\ R/2 < |x| < R\}$,

$$\partial_r\zeta > 0, \quad \partial_r^2\zeta \ge 0 \quad \text{and} \quad |\partial^\alpha\zeta| \le C_\alpha R|\cdot|^{1-\alpha},\ \forall\,|\alpha| \ge 1.$$

Here, $\partial_r\zeta := \frac{\cdot}{|\cdot|}\cdot\nabla\zeta$ denotes the radial derivative. Note that on the centered ball of radius $R/2$, one has

$$\zeta_{jk} = \delta_{jk}, \quad \Delta\zeta = N \quad \text{and} \quad \Delta^2\zeta = 0.$$

Moreover, by the radial identity

$$\partial_j\partial_k = \Big(\frac{\delta_{jk}}{r} - \frac{x_jx_k}{r^3}\Big)\partial_r + \frac{x_jx_k}{r^2}\partial_r^2, \tag{4.49}$$

one gets for $|x| > R$,

$$\zeta_{jk} = \frac{R}{|x|}\Big(\delta_{jk} - \frac{x_jx_k}{|x|^2}\Big); \tag{4.50}$$
$$\Delta\zeta = \frac{(N-1)R}{|x|}; \tag{4.51}$$
$$|\Delta^2\zeta| \lesssim \frac{R}{|x|^3}. \tag{4.52}$$

Using Cauchy Schwarz and Hardy estimates via (3.89) and (3.91), one has

$$|M_\zeta| = 2\Big|\Im\int_{\mathbb{R}^N} \bar u(\nabla\zeta\cdot\nabla u)\,dx\Big| \lesssim R^2E(u). \tag{4.53}$$

Taking account of the identity (4.49), one has

$$\begin{aligned}\Re\Big(\int_{B^c(R/2)}\partial_l\partial_k\zeta\partial_k u\partial_l\bar{u}\,dx\Big) &= \Re\int_{B^c(R/2)}\Big[\Big(\frac{\delta_{lk}}{r}-\frac{x_lx_k}{r^3}\Big)\partial_r\zeta+\frac{x_lx_k}{r^2}\partial_r^2\zeta\Big]\partial_k u\partial_l\bar{u}\,dx\\ &= \int_{B^c(R/2)}\Big(|\nabla u|^2-\frac{|x\cdot\nabla u|^2}{|x|^2}\Big)\frac{\partial_r\zeta}{|x|}\,dx+\int_{B^c(R/2)}\frac{|x\cdot\nabla u|^2}{|x|^2}\partial_r^2\zeta\,dx\\ &= \int_{B^c(R/2)}|\not\nabla u|^2\frac{\partial_r\zeta}{|x|}\,dx+\int_{B^c(R/2)}\frac{|x\cdot\nabla u|^2}{|x|^2}\partial_r^2\zeta\,dx, \end{aligned} \tag{4.54}$$

where the angular gradient is

$$\not\nabla := \nabla-\frac{x\cdot\nabla}{|x|^2}x.$$

Now, by (4.54) via Proposition 2.5, one writes

$$\begin{aligned}M'_\zeta[u] &= 4\Big(\|\nabla u\|^2_{L^2(B(R/2))}-\int_{B(R/2)}|x|^{-\tau}|u|^{p^c}\,dx\Big)\\ &\quad+\int_{B^c(R/2)}|\not\nabla u|^2\frac{\partial_r\zeta}{|x|}\,dx+\int_{B^c(R/2)}\frac{|x\cdot\nabla u|^2}{|x|^2}\partial_r^2\zeta\,dx-\int_{B^c(R/2)}\Delta^2\zeta|u|^2\,dx\\ &\quad-\int_{B^c(R/2)}\Big(\frac{4b}{p^c}\frac{\nabla\zeta\cdot x}{|x|^2}+2\frac{p^c-2}{p^c}\Delta\zeta\Big)|x|^{-\tau}|u|^{p^c}\,dx. \end{aligned} \tag{4.55}$$

So, (4.55) via (3.89), (3.91) and Sobolev embeddings implies that

$$\begin{aligned}M'_\zeta[u] &\geq 4\Big(\|\nabla u\|^2_{L^2(B(R/2))}-\int_{B(R/2)}|x|^{-\tau}|u|^{p^c}\,dx\Big)-c\int_{B^c(R/2)}\Big(|x|^{-2}|u|^2+|x|^{-\tau}|u|^{p^c}\Big)dx\\ &\geq 4\Big(\|\nabla u\|^2-\int_{\mathbb{R}^N}|x|^{-\tau}|u|^{p^c}\,dx\Big)-c\int_{B^c(R/2)}\Big(|\nabla u|^2+|x|^{-2}|u|^2+|x|^{-\tau}|u|^{p^c}\Big)dx\\ &\gtrsim E(u)-c\int_{B^c(R/2)}\Big(|\nabla u|^2+|x|^{-2}|u|^2+|x|^{-\tau}|u|^{p^c}\Big)dx. \end{aligned} \tag{4.56}$$

So, (4.53) via (4.56) gives

$$E(u)\lesssim\frac{R^2}{T}+\int_{B^c(R/2)}\Big(|\nabla u|^2+|x|^{-2}|u|^2+|x|^{-\tau}|u|^{p^c}\Big)dx. \tag{4.57}$$

Finally, one picks $T := R^3\to\infty$, so (2.16) via (4.57) gives $E(u) = 0$. This contradiction finishes the proof.

## 5. Blowup

In this section, one proves the second part of Theorem 2.1. Let us denote $\phi_A := A^2\phi(\frac{\cdot}{A})$, for $A > 0$, where $\phi\in C_0^\infty(\mathbb{R}^N)$ is radial and satisfies

$$\phi(x)=\begin{cases}\frac{1}{2}|x|^2, & |x|\leq 1;\\ 0, & |x|\geq 2,\end{cases}\qquad\text{and}\qquad \phi''\leq 1.$$

A calculus gives

$$\phi_A'' \leq 1, \quad \phi_A'(r) \leq r \quad \text{and} \quad \Delta\phi_A \leq N.$$

By the localized variance identity [46, Corollary 3.2], one has

$$\begin{aligned} M_A'' &= -\int_{\mathbb{R}^N} \Delta^2\phi_A |u|^2\,dx + 4\int_{\mathbb{R}^N} \partial_l\partial_k\phi_A \Re(\partial_k u \partial_l \bar{u})\,dx \\ &+ \frac{4}{1+p}\int_{\mathbb{R}^N} \nabla\phi_A \cdot \nabla(|x|^{-\tau})|u|^{1+p}\,dx - 2\frac{p-1}{1+p}\int_{\mathbb{R}^N} \Delta\phi_A |x|^{-\tau}|u|^{1+p}\,dx \\ &= 4\Big(\|\nabla u\|^2_{L^2(|x|<A)} - \int_{|x|<A} |x|^{-\tau}|u|^{1+p}\,dx\Big) - \int_{\mathbb{R}^N} \Delta^2\phi_A|u|^2\,dx \\ &+ \frac{4}{1+p}\int_{|x|>A} \nabla\phi_A \cdot \nabla(|x|^{-\tau})|u|^{1+p}\,dx - 2\frac{p-1}{1+p}\int_{|x|>A} \Delta\phi_A|x|^{-\tau}|u|^{1+p}\,dx \\ &+ 4\int_{|x|>A} \partial_l\partial_k\phi_A \Re(\partial_k u\partial_l\bar{u})\,dx. \end{aligned} \tag{5.1}$$

Recall the next radial identities

$$\frac{\partial^2}{\partial x_j \partial x_k} := \partial^2_{jk} = \Big(\frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3}\Big)\partial_r + \frac{x_j x_k}{r^2}\partial_r^2; \tag{5.2}$$

$$\Delta = \partial_r^2 + \frac{N-1}{r}\partial_r; \tag{5.3}$$

$$\nabla = \frac{x}{r}\partial_r. \tag{5.4}$$

Since $\phi$ is radial, we have from the above identities

$$\partial^2_{jk}\phi_A \partial_k u(t)\partial_j \bar{u}(t) = |\nabla u|^2\frac{\phi_A'}{r} + \Big(\phi_A'' - \frac{\phi_A'}{r}\Big)\frac{|x\cdot\nabla u|^2}{r^2}; \tag{5.5}$$

$$\nabla\phi_A \cdot \nabla(|x|^{-\tau}) = -\tau|x|^{-\tau}\frac{\phi_A'}{r}. \tag{5.6}$$

Thus, (5.1), (5.5), and (5.6) give

$$\begin{aligned} M_A'' &= 4\Big(\|\nabla u\|^2 - P[u]\Big) - \int_{\mathbb{R}^N} \Delta^2\phi_A|u|^2\,dx - 4\Big(\int_{|x|>A} |\nabla u|^2\,dx - \int_{|x|>A} |x|^{-\tau}|u|^{1+p}\,dx\Big) \\ &+ 4\int_{|x|>A} \Big(|\nabla u|^2\frac{\phi_A'}{r} + \Big(\phi_A'' - \frac{\phi_A'}{r}\Big)\frac{|x\cdot\nabla u|^2}{r^2}\Big)\,dx \\ &- \frac{4\tau}{1+p}\int_{|x|>A} \frac{\phi_A'}{r}|x|^{-\tau}|u|^{1+p}\,dx - 2\frac{p-1}{1+p}\int_{|x|>A} \Delta\phi_A|x|^{-\tau}|u|^{1+p}\,dx \\ &\leq 4\Big(\|\nabla u\|^2 - P[u]\Big) - \int_{\mathbb{R}^N} \Delta^2\phi_A|u|^2\,dx + 4\int_{|x|>A} \Big(\frac{\phi_A'}{r} - 1\Big)\Big(|\nabla u|^2 - \frac{|x\cdot\nabla u|^2}{r^2}\Big)\,dx \\ &- \frac{4\tau}{1+p}\int_{|x|>A} \frac{\phi_A'}{r}|x|^{-\tau}|u|^{1+p}\,dx - 2\frac{p-1}{1+p}\int_{|x|>A} \Delta\phi_A|x|^{-\tau}|u|^{1+p}\,dx + 4\int_{|x|>A} |x|^{-\tau}|u|^{1+p}\,dx \\ &\lesssim \|\nabla u\|^2 - P[u] + A^{-2} + A^{-\tau}\|u\|^{1+p}_{1+p}. \end{aligned} \tag{5.7}$$

Now, if we assume that $\sup_{[0,T^+)} \|\nabla u(t)\| < \infty$, (5.7) implies that

$$M_A'' \lesssim \|\nabla u\|^2 - P[u] + A^{-2} + A^{-\tau}. \tag{5.8}$$

Thus, (3.94) and (5.8) give $M_A'' \leq -c < 0$, for large $A >> 1$. Integrating this inequality twice in time, it follows that $u$ is non-global. This ends the proof of the second part of Theorem 2.1.

## 6. Conclusions and discussions

The primary contribution of this note is Theorem 2.1, which complements the findings of [26–28] to higher spatial dimensions and removes the radial assumption. While the scattering threshold was established in [26–28] for three spatial dimensions, the novelty of this work lies in demonstrating the scattering threshold for space dimensions larger than four without the requirement of spherical symmetry. The approach follows the road-map outlined by Kenig and Merle in [14].

## Author contributions

Saleh Almuthaybiri: formal analysis, funding acquisition; Radhia Ghanmi: investigation, methodology, mriting; Tarek Saanouni: project administration, resources, supervision, validation, review. All authors have read and approved the final version of the manuscript for publication.

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## Conflict of interest

The authors declare no conflict of interest.

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