Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(11): 30214–30229.
DOI:10.3934/math.20241459
Received: 27 August 2024
Revised: 16 October 2024
Accepted: 21 October 2024
Published: 24 October 2024

Research article

Novel Pareto Z-eigenvalue inclusion intervals for tensor eigenvalue complementarity problems and its applications

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Abstract: In this paper, we establish Pareto Z-eigenvalue inclusion intervals of tensor eigenvalue complementarity problems based on the spectral radius of symmetric matrices deduced from the provided tensor. Numerical examples are suggested to demonstrate the effectiveness of the results. As an application we offer adequate criteria for the strict copositivity of symmetric tensors.

Keywords: tensor eigenvalue complementarity problems; Pareto Z-eigenvalue intervals; strict copositivity; lower and upper bounds
Mathematics Subject Classification: 15A18, 15A42, 15A69

1. Introduction

Consider the tensor eigenvalue complementarity problems of finding $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n_+ \backslash \{0\}$ such that

$$0 \le x \perp (\lambda x - \mathcal{A}x^{m-1}) \ge 0 \qquad \text{and} \qquad x^\top x = 1, \tag{1.1}$$

where $a \perp b$ means that vectors a, b are perpendicular to each other, and $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$ is an m-th order n-dimensional real tensor, and $\mathcal{A}x^{m-1}$ is the vector in $\mathbb{R}^n$ with entries

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\dots,i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \ N = \{1, \dots, n\}.$$

If (1.1) holds, $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n_+ \backslash \{0\}$ is called a Pareto Z-eigenpair of tensor $\mathcal{A}$.

Pareto Z-eigenvalue problems of tensors were introduced by Song [17], which can be seen generalizations of classical tensor (matrix) eigenvalue problems [1, 5, 6, 8, 13, 14, 19–22], have broad applications in higher-order Markov chains [11] and magnetic resonance imaging [15, 26, 27]. Therefore, Pareto Z-eigenvalue problems of tensors garnered a lot of interest in the literature [4, 10, 30, 32]. To achieve Pareto Z-eigenvalues of tensor eigenvalue complementarity problems, for instance, Zeng [32] suggested a semidefinite relaxation approach. Nonetheless, there exist a huge, potentially endless number of Pareto Z-eigenvalues of tensors [2, 32]. Therefore, calculating all Pareto Z-eigenvalues is difficult. A few scholars have turned to investigating Pareto Z-eigenvalue intervals to describe the distribution of Pareto Z-eigenvalues [17, 29]. Particularly, Yang et al. [29] proposed Pareto Z-eigenvalue intervals via key tensor elements, which have a significant impact on the Pareto Z-eigenvalue estimation. It is crucial to create new Pareto Z-eigenvalue intervals that are independent of certain tensor constituents. Note that matrices can be viewed as the large elements of tensors, and the spectral radius has relative stability. Can we use the spectral radius of the related symmetric matrices instead of tensor elements to accurately characterize the Pareto Z-eigenvalue? Different from the existing Z-eigenvalue inclusion sets [16, 24, 25, 31], we investigate the relations between the tensor and its induced matrix and establish Pareto Z-eigenvalue intervals from the spectral radius of the linked symmetric matrices.

As we know, tensor $\mathcal{A}$ is strictly copositive if $\mathcal{A}x^m > 0, \forall x \in \mathbb{R}^n_+\backslash\{0\}$, which has important applications in vacuum stability of a general scalar potential [9] and polynomial optimization [3, 12]. Song et al. [17] pointed out that symmetric tensor $\mathcal{A}$ is strictly copositive if and only if its Pareto Z-eigenvalues are positive. Therefore, we can identify whether a tensor is copositive by the lower bounds of Pareto Z-eigenvalues. Inspired by the articles [17, 29], we propose some criteria for judging strict copositivity via the spectral radius of the symmetric matrices extracted from the given tensor.

The remainder of this paper is organized as follows: In Section 2, crucial definitions and preliminary results are recalled. In Section 3, we establish two tight Pareto Z-eigenvalue intervals via the spectral radius of the symmetric matrices. In Section 4, sufficient conditions are proposed for identifying strict copositivity of symmetric tensors.

2. Preliminaries

In this section, we first introduce important definitions and notations of tensors [2, 13, 29].

The set of all real numbers is denoted by $\mathbb{R}$, and the n-dimensional real Euclidean space is denoted by $\mathbb{R}^n$. For any $a \in \mathbb{R}$, we denote $[a]_+ := \max\{0, a\}$ and $[a]_- := \max\{0, -a\}$. For any $\mathcal{A} \in \mathbb{R}^{[m,n]}$, we define

$$[\mathcal{A}]_+ := ([a_{i_1 i_2 \cdots i_m}]_+) \in \mathbb{R}^{[m,n]}, \quad [\mathcal{A}]_- := ([a_{i_1 i_2 \cdots i_m}]_-) \in \mathbb{R}^{[m,n]}.$$

Definition 2.1. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, and $\sigma_Z(\mathcal{A})$ be the set of all Pareto Z-eigenvalues of $\mathcal{A}$.*

(i) The maximum Pareto Z-eigenvalue and the minimum Pareto Z-eigenvalue of $\mathcal{A}$ are denoted by

$$\rho_Z(\mathcal{A}) = \max\{\lambda : \lambda \in \sigma_Z(\mathcal{A})\} \text{ and } \tau_Z(\mathcal{A}) = \min\{\lambda : \lambda \in \sigma_Z(\mathcal{A})\}.$$

(ii) $\mathcal{A}$ is called symmetry if

$$a_{i_1 \ldots i_m} = a_{i_{\pi(1)} \ldots i_{\pi(m)}}, \ \forall \ \pi \in \Gamma_m,$$

where Γ_m is the permutation group of m indices.

(iii) $\delta_{i_1 i_2 \ldots i_m}$ is called the generalized Kronecker symbol:

$$\delta_{i_1 i_2 \ldots i_m} = \begin{cases} 1, & if\ i_1 = i_2 = \ldots i_m \\ 0, & otherwise. \end{cases}$$

We conclude this section with significant results of the symmetric matrices [7] and the bound of Pareto Z-eigenvalue.

Lemma 2.1. *Let $P \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $x \in \mathbb{R}^n$ be a unit vector, i.e., $x^\top x = 1$. $\mu_{\min}(P)$ (or $\mu_{\max}(P)$) denotes the minimum (maximum) eigenvalue of a square matrix P, and $\rho(P)$ is the spectral radius of P. Then,*

$$\mu_{\min}(P) \le x^\top Px \le \mu_{\max}(P) \text{ and } |x^\top Px| \le \rho(P).$$

Lemma 2.2. *(Theorem 2 of [29]) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\sigma(\mathcal{A}) \ne \emptyset$. Then,*

$$\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) = \bigcup_{i\in N} \Omega_i(\mathcal{A}) := \{\lambda \in \mathbb{R} : |\lambda| \le \max\{R_i(\mathcal{A})_+, R_i(\mathcal{A})_-\}\},$$

where $R_i(\mathcal{A})_+ := \sum\limits_{i_2,\dots,i_m=1}^{n} [a_{ii_2\cdots i_m}]_+, \quad R_i(\mathcal{A})_- := \sum\limits_{i_2,\dots,i_m=1}^{n} [a_{ii_2\dots i_m}]_-.$

3. Pareto Z-eigenvalues inclusion intervals

We begin with the bounds of Pareto Z-eigenvalues for a third-order tensor based on the spectral radius of the symmetric matrix $V_i := \frac{\mathcal{A}_{i::}+\mathcal{A}_{i::}^\top}{2}$.

Theorem 3.1. *Let $\mathcal{A} \in \mathbb{R}^{[3,n]}$ with $\sigma_Z(\mathcal{A}) \ne \emptyset$. Then,*

$$\sigma_Z(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}) := \left\{\lambda \in \mathbb{R} : -\sqrt{\sum_{i\in N}(\rho[V_i]_-)^2} \le \lambda \le \sqrt{\sum_{i\in N}(\rho[V_i]_+)^2}\right\}, \tag{3.1}$$

where $[V_i]_+ = \frac{[\mathcal{A}_{i::}]_+ + [\mathcal{A}_{i::}]_+^\top}{2}$, $[V_i]_- = \frac{[\mathcal{A}_{i::}]_- + [\mathcal{A}_{i::}]_-^\top}{2}$ and $\mathcal{A}_{i::}$ is the matrix by fixing i indices of $\mathcal{A}$.

Proof. Suppose that (λ, x) is a Pareto Z-eigenpair of $\mathcal{A}$. On the one hand, since $x^\top x = 1$ and $x_i \ge 0$ hold for all $i \in N$, we obtain

$$\begin{aligned}
\lambda \sum_{i\in N} x_i^2 = \mathcal{A}x^3 \le [\mathcal{A}]_+x^3 &= \sum_{i,i_2,i_3\in N} [a_{ii_2i_3}]_+ x_i x_{i_2} x_{i_3} = \sum_{i\in N}\Big(\sum_{i_2,i_3\in N} [a_{ii_2i_3}]_+ x_{i_2}x_{i_3}\Big)x_i \\
&\le \sqrt{\Big(\sum_{i_2,i_3\in N}[a_{1i_2i_3}]_+x_{i_2}x_{i_3}\Big)^2 + \cdots + \Big(\sum_{i_2,i_3\in N}[a_{ni_2i_3}]_+x_{i_2}x_{i_3}\Big)^2} \cdot \sqrt{x_1^2 + \cdots + x_n^2} \\
&= \sqrt{(x^\top[\mathcal{A}_{1::}]_+x)^2 + \cdots + (x^\top[\mathcal{A}_{n::}]_+x)^2},
\end{aligned} \tag{3.2}$$

where the second inequality holds from Cauchy–Schwarz inequality. It follows from the definition of $[V_i]_+$ and $x^\top[\mathcal{A}_{i::}]_+x = x^\top[\mathcal{A}_{i::}]_+^\top x$ that

$$x^\top[V_i]_+x = x^\top \frac{[\mathcal{A}_{i::}]_+ + [\mathcal{A}_{i::}]_+^\top}{2} x = x^\top[\mathcal{A}_{i::}]_+x. \tag{3.3}$$

Since $[V_i]_+$ is a real symmetric matrix, by (3.2), (3.3), and Lemma 2.1, we obtain

$$\lambda \le \sqrt{\sum_{i\in N}(\rho[V_i]_+)^2}. \tag{3.4}$$

On the other hand, from $x^\top x = 1$ and $x_i \geq 0$ for all $i \in N$, one has

$$\begin{aligned}
-\lambda \sum_{i\in N} x_i^2 &= -\mathcal{A}x^3 \leq [\mathcal{A}]_- x^3 = \sum_{i\in N}(\sum_{i,i_2,i_3\in N} [a_{ii_2i_3}]_- x_i x_{i_2} x_{i_3} = \sum_{i,i_2,i_3\in N} [a_{ii_2i_3}]_- x_{i_2} x_{i_3}) x_i \\
&\leq \sqrt{(\sum_{i_2,i_3\in N} [a_{1i_2i_3}]_- x_{i_2} x_{i_3})^2 + \cdots + (\sum_{i_2,i_3\in N} [a_{ni_2i_3}]_- x_{i_2} x_{i_3})^2} \cdot \sqrt{x_1^2 + \cdots + x_n^2} \\
&= \sqrt{(x^\top [\mathcal{A}_{1::}]_- x)^2 + \cdots + (x^\top [\mathcal{A}_{n::}]_- x)^2}. \qquad (3.5)
\end{aligned}$$

It follows from the definition of $[V_i]_-$ and $x^\top [\mathcal{A}_{i::}]_- x = x^\top [\mathcal{A}_{i::}]_-^\top x$ that

$$x^\top [V_i]_- x = x^\top \frac{[\mathcal{A}_{i::}]_- + [\mathcal{A}_{i::}]_-^\top}{2} x = x^\top [\mathcal{A}_{i::}]_- x. \qquad (3.6)$$

Taking into account that $[V_i]_-$ is a real symmetric matrix, by (3.5), (3.6), and Lemma 2.1, we deduce

$$\lambda \geq -\sqrt{\sum_{i\in N} (\rho[V_i]_-)^2}. \qquad (3.7)$$

Combining (3.4) with (3.7) yields

$$-\sqrt{\sum_{i\in N} (\rho[V_i]_-)^2} \leq \lambda \leq \sqrt{\sum_{i\in N} (\rho[V_i]_+)^2},$$

which implies $\lambda \in \Upsilon(\mathcal{A})$ and $\sigma_Z(\mathcal{A}) \subseteq \Upsilon(\mathcal{A})$. □

The following example is proposed to test the efficiency of the obtained results.

Example 3.1. *Consider a tensor* $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,3]}$ *defined by*

$$a_{ijk} = \begin{cases} a_{111} = 1; a_{112} = -1; a_{131} = 1; a_{133} = 1; \\ a_{211} = -1; a_{222} = 2; a_{232} = 1; \\ a_{311} = 1; a_{322} = 3; a_{323} = 1; \\ a_{ijk} = 0, \quad \textit{otherwise}. \end{cases}$$

By calculating, we have

$$[\mathcal{A}_{1::}]_+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad [\mathcal{A}_{1::}]_- = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[V_1]_+ = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}, \quad [V_1]_- = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \rho([V_1]_+) = 1.5000, \quad \rho([V_1]_-) = 0.5000.$$

Following the similar calculations to the above, one has

$$\rho([V_2]_+) = 2.1180, \quad \rho([V_2]_-) = 0.5000, \quad \rho([V_3]_+) = 3.0811, \quad \rho([V_3]_-) = 0.0000.$$

According to Theorem 3.1, we obtain

$$\Upsilon(\mathcal{A}) = \{\lambda \in \mathbb{R} : -\frac{\sqrt{2}}{2} \leq \lambda \leq 4.0285\}.$$

Recalling Theorem 1 of [29], we deduce

$$\Omega(\mathcal{A}) = \{\lambda \in \mathbb{R} : -1.0000 \leq \lambda \leq 5.0000\},$$

which implies that the bound of Theorem 3.1 is sharp.

The following example estimates the Pareto Z-eigenvalues to guarantee nonconstant trajectories of equilibrium systems.

Example 3.2. *Consider the following differential equilibrium system:*

$$\sum : \ \dot{x}_1(t) = x_1^2 + x_1x_2; \ \dot{x}_2(t) = 2x_2^2 + x_1x_3; \ \dot{x}_3(t) = x_3^2 \ \text{with} \ x_1^2 + x_2^2 + x_3^3 = 1.$$

Thus, $\sum$ *can be written as* $\dot{x}(t) = \mathcal{A}x^2$, *where* $x = (x_1, x_2, x_3)^\top$ *with* $x_1^2+x_2^2+x_3^3 = 1$ *and* $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,3]}$ *with*

$$a_{ijk} = \begin{cases} a_{111} = a_{121} = 1; a_{222} = 2; a_{231} = a_{333} = 1; \\ a_{ijkl} = 0, otherwise. \end{cases}$$

In order to ensure nonconstant trajectories of the equilibrium system, we need to find $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n_+\backslash\{0\}$ such that

$$0 \leq x \perp (\lambda x - \mathcal{A}x^2) \geq 0 \quad \text{and} \quad x^\top x = 1.$$

Using Algorithm 3.1 of [32], we obtain four Pareto Z-eigenvalues and the associated Pareto Z-eigenvectors about 3.25 seconds:

$$\lambda_1 = 0.8944, u_1 = (0.0000, 0.4472, 0.8944); \lambda_2 = 1.0000, u_2 = (1.0000, 0.0000, 0.0000);$$
$$\lambda_3 = 1.4142, u_3 = (0.7071, 0.7071, 0.0000); \lambda_4 = 2.0000, u_4 = (0.0000, 1.0000, 0.0000).$$

It follows from Theorem 3.1 that we estimate $0 \leq \lambda \leq \sqrt{6}$. We apply this estimation to Algorithm 3.1 of [32] and can calculate the above Pareto Z-eigenvalues in 2.65 seconds. Therefore, Algorithm 3.1 of [32] could be accelerated by establishing the bound of Pareto Z-eigenvalues.

Using the spectral radius of symmetric matrices extracted from the given tensor, we establish Pareto Z-eigenvalue intervals of an m-order tensor with $m \geq 4$.

Theorem 3.2. *Let* $\mathcal{A} \in \mathbb{R}^{[m,n]}$ *with* $m \geq 4$, *and* $\sigma_Z(\mathcal{A}) \neq \emptyset$. *Then,*

$$\sigma_Z(\mathcal{A}) \subseteq \Theta(\mathcal{A}) = \bigcup_{i \in N} \Theta_i(\mathcal{A}) := \Big\{\lambda \in \mathbb{R} : |\lambda| \leq \max\{\rho([B_i]_+), \rho([B_i]_-)\}\Big\},$$

where $[B_i]_+ = \frac{[A_i]_+ + [A_i]_+^\top}{2}$, $\quad [B_i]_- = \frac{[A_i]_- + [A_i]_-^\top}{2}$ *and*

$$[A_i]_+ = \begin{bmatrix} \sum\limits_{i_2,\ldots,i_{m-2} \in N} [a_{ii_2\ldots i_{m-2}11}]_+ & \cdots & \sum\limits_{i_2,\ldots,i_{m-2} \in N} [a_{ii_2\ldots i_{m-2}1n}]_+ \\ \vdots & \vdots & \vdots \\ \sum\limits_{i_2,\ldots,i_{m-2} \in N} [a_{ii_2\ldots i_{m-2}n1}]_+ & \cdots & \sum\limits_{i_2,\ldots,i_{m-2} \in N} [a_{ii_2\ldots i_{m-2}nn}]_+ \end{bmatrix},$$

$$[A_i]_- = \begin{bmatrix} \sum\limits_{i_2,\ldots,i_{m-2}\in N}[a_{ii_2\ldots i_{m-2}11}]_- & \cdots & \sum\limits_{i_2,\ldots,i_{m-2}\in N}[a_{ii_2\ldots i_{m-2}1n}]_- \\ \vdots & \vdots & \vdots \\ \sum\limits_{i_2,\ldots,i_{m-2}\in N}[a_{ii_2\ldots i_{m-2}n1}]_- & \cdots & \sum\limits_{i_2,\ldots,i_{m-2}\in N}[a_{ii_2\ldots i_{m-2}nn}]_- \end{bmatrix}.$$

Proof. Suppose that (λ, x) is a Pareto Z-eigenpair of $\mathcal{A}$. Then,

$$\lambda x_i^2 = \sum_{i_2,\ldots,i_m\in N} a_{ii_2\ldots i_m} x_i x_{i_2}\ldots x_{i_m}. \tag{3.8}$$

Denote $x_p = \max\limits_{i\in N}\{x_i\}$. Then, $0 < x_p \le 1$ as $x^\top x = 1$. Recalling the p-th equation of (3.8), we obtain

$$\lambda x_p^2 = \sum_{i_2,\ldots,i_m\in N} a_{pi_2\ldots i_m} x_p x_{i_2}\ldots x_{i_m}.$$

Taking modulus in the equation above, one has

$$\begin{aligned}
|\lambda| x_p^2 &= |\sum_{i_2,\ldots,i_m\in N}[a_{pi_2\ldots i_m}]_+ x_p x_{i_2}\ldots x_{i_m} - \sum_{i_2,\ldots,i_m\in N}[a_{pi_2\ldots i_m}]_- x_p x_{i_2}\ldots x_{i_m}| \\
&\le \max\{\sum_{i_2,\ldots,i_m\in N}[a_{pi_2\ldots i_m}]_+ x_p x_{i_2}\ldots x_{i_m}, \sum_{i_2,\ldots,i_m\in N}[a_{pi_2\ldots i_m}]_- x_p x_{i_2}\ldots x_{i_m}\} \\
&\le \max\{\sum_{i_{m-1},i_m\in N}\sum_{i_2,\ldots,i_{m-2}\in N}[a_{pi_2\ldots i_m}]_+ x_p^2 x_{i_{m-1}} x_{i_m}, \sum_{i_{m-1},i_m\in N}\sum_{i_2,\ldots,i_{m-2}\in N}[a_{pi_2\ldots i_m}]_- x_p^2 x_{i_{m-1}} x_{i_m}\} \\
&= x_p^2 \max\{x^\top [A_p]_+ x, x^\top [A_p]_- x\},
\end{aligned} \tag{3.9}$$

where $[A_p]_+$ and $[A_p]_-$ are defined in Theorem 3.2. Certainly, $x^\top[A_i]_+x = x^\top[A_i]_+^\top x$ and $x^\top[A_i]_-x = x^\top[A_i]_-^\top x$. It follows from the definitions of $[B_i]_+$ and $[B_i]_-$ that

$$x^\top[B_p]_+x = x^\top\frac{[A_p]_+ + [A_p]_+^\top}{2}x = x^\top[A_p]_+x, \quad x^\top[B_p]_-x = x^\top\frac{[A_p]_- + [A_p]_-^\top}{2}x = x^\top[A_p]_-x. \tag{3.10}$$

Since $[B_p]_+$ and $[B_p]_-$ are real symmetric matrices, by (3.9), (3.10) and Lemma 2.1, we have

$$|\lambda| \le \max\{\rho([B_p]_+), \rho([B_p]_-)\},$$

which implies $\lambda \in \Theta(\mathcal{A})$, and hence $\sigma_Z(\mathcal{A}) \subseteq \Theta(\mathcal{A})$. □

Now, we are in a position to establish tight Pareto Z-eigenvalues inclusion intervals by accurate classification of index sets.

Theorem 3.3. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ with $m \ge 4$, and $\sigma_Z(\mathcal{A}) \ne \emptyset$. Then,*

$$\sigma_Z(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) = \bigcup_{i\in N}\bigcap_{j\in N, i\ne j}\mathcal{M}_{i,j}(\mathcal{A}),$$

where

$$\mathcal{M}_{i,j}(\mathcal{A}) := \left\{\lambda \in \mathbb{R} : (|\lambda| - \rho([B_i^j]_+))|\lambda| \le \rho([D_{ij}]_-)\max\{\rho([B_j]_+), \rho([B_j]_-)\}\right\}$$

$$\bigcup\Big\{\lambda\in\mathbb{R}:(|\lambda|-\rho([B_i^j]_-))|\lambda|\le\rho([D_{ij}]_-)\max\{\rho([B_j]_+),\rho([B_j]_-)\}\Big\},$$

$[B_i^j]_+=\frac{[A_i^j]_++[A_i^j]_+^\top}{2}$, $\quad [B_i^j]_-=\frac{[A_i^j]_-+[A_i^j]_-^\top}{2}$ *and*

$$[A_i^j]_+=\begin{bmatrix}\sum\limits_{\delta_{ji_2\ldots i_{m-2}}=0}[a_{ii_2\ldots i_{m-2}11}]_+ & \cdots & \sum\limits_{\delta_{ji_2\ldots i_{m-2}}=0}[a_{ii_2\ldots i_{m-2}1n}]_+\\ \vdots & \vdots & \vdots\\ \sum\limits_{\delta_{ji_2\ldots i_{m-2}}=0}[a_{ii_2\ldots i_{m-2}n1}]_+ & \cdots & \sum\limits_{\delta_{ji_2\ldots i_{m-2}}=0}[a_{ii_2\ldots i_{m-2}nn}]_+\end{bmatrix},$$

$$[A_i^j]_-=\begin{bmatrix}\sum\limits_{\delta_{ji_2\ldots i_{m-2}}=0}[a_{ii_2\ldots i_{m-2}11}]_- & \cdots & \sum\limits_{\delta_{ji_2\ldots i_{m-2}}=0}[a_{ii_2\ldots i_{m-2}1n}]_-\\ \vdots & \vdots & \vdots\\ \sum\limits_{\delta_{ji_2\ldots i_{m-2}}=0}[a_{ii_2\ldots i_{m-2}n1}]_- & \cdots & \sum\limits_{\delta_{ji_2\ldots i_{m-2}}=0}[a_{ii_2\ldots i_{m-2}nn}]_-\end{bmatrix},$$

$$[D_{ij}]_+=\frac{[C_{ij}]_++[C_{ij}]_+^\top}{2},\quad [D_{ij}]_-=\frac{[C_{ij}]_-+[C_{ij}]_-^\top}{2},$$

$$[C_{ij}]_+=\begin{bmatrix}[a_{ij\ldots j11}]_+ & \cdots & [a_{ij\ldots j1n}]_+\\ \vdots & \vdots & \vdots\\ [a_{ij\ldots jn1}]_+ & \cdots & [a_{ij\ldots jnn}]_+\end{bmatrix},[C_{ij}]_-=\begin{bmatrix}[a_{ij\ldots j11}]_- & \cdots & [a_{ij\ldots j1n}]_-\\ \vdots & \vdots & \vdots\\ [a_{ij\ldots jn1}]_- & \cdots & [a_{ij\ldots jnn}]_-\end{bmatrix}.$$

Proof. Let (λ,x) be a Pareto Z-eigenpair of $\mathcal{A}$. Setting $0<x_p=\max\limits_{i\in N}\{x_i\}$ and referring to the p-th equation of (3.8), for any $q\in N, q\neq p$, we obtain

$$\begin{aligned}|\lambda|x_p^2&=|\sum_{i_2,\ldots,i_m\in N}a_{pi_2\ldots i_m}x_px_{i_2}\ldots x_{i_m}|\\
&=|\sum_{i_2,\ldots,i_m\in N}[a_{pi_2\ldots i_m}]_+x_px_{i_2}\ldots x_{i_m}-\sum_{i_2,\ldots,i_m\in N}[a_{pi_2\ldots i_m}]_-x_px_{i_2}\ldots x_{i_m}|\\
&\le\max\{\sum_{i_2,\ldots,i_m\in N}[a_{pi_2\ldots i_m}]_+x_px_{i_2}\ldots x_{i_m},\sum_{i_2,\ldots,i_m\in N}[a_{pi_2\ldots i_m}]_-x_px_{i_2}\ldots x_{i_m}\}\\
&=\max\{x^\top[C_{pq}]_+xx_px_q+x_p^2x^\top[A_p^q]_+x,x^\top[C_{pq}]_-xx_px_q+x_p^2x^\top[A_p^q]_-x\}.\end{aligned}\tag{3.11}$$

Clearly,

$$x^\top[C_{pq}]_+x=x^\top[C_{pq}]_+^\top x,\quad x^\top[C_{pq}]_-x=x^\top[C_{pq}]_-^\top x,$$
$$x^\top[A_p^q]_+x=x^\top[A_p^q]_+^\top x,\quad x^\top[A_p^q]_-x=x^\top[A_p^q]_-^\top x.$$

With the definitions of $[D_{ij}]_+$, $[D_{ij}]_-$, $[B_i^j]_+$ and $[B_i^j]_-$, it is easy to verify that

$$\begin{aligned}x^\top[D_{pq}]_+x=x^\top[C_{pq}]_+x,\quad x^\top[D_{pq}]_-x=x^\top[C_{pq}]_-x,\\
x^\top[B_p^q]_+x=x^\top[A_p^q]_+x,\quad x^\top[B_p^q]_-x=x^\top[A_p^q]_-x.\end{aligned}\tag{3.12}$$

Since $[D_{pq}]_+$,$[D_{pq}]_-$, $[B_p^q]_+$ and $[B_p^q]_-$ are real symmetric matrices, by (3.11), (3.12), and Lemma 2.1, we deduce

$$|\lambda|x_p^2\le\max\{\rho([D_{pq}]_+)x_px_q+x_p^2\rho([B_p^q]_+),\rho([D_{pq}]_-)x_px_q+x_p^2\rho([B_p^q]_-)\}.\tag{3.13}$$

Recalling the q-th equation of (3.8), one has

$$\begin{aligned}|\lambda|x_q^2 &= |\sum_{i_2,\ldots,i_m\in N} a_{qi_2\ldots i_m}x_qx_{i_2}\ldots x_{i_m}| \\ &\leq \max\{\sum_{i_2,\ldots,i_m\in N}[a_{qi_2\ldots i_m}]_+x_qx_{i_2}\ldots x_{i_m}, \sum_{i_2,\ldots,i_m\in N}[a_{qi_2\ldots i_m}]_-x_qx_{i_2}\ldots x_{i_m}\} \\ &\leq \max\{\sum_{i_2,\ldots,i_m\in N}[a_{qi_2\ldots i_m}]_+x_qx_px_{i_{m-1}}x_{i_m}, \sum_{i_2,\ldots,i_m\in N}[a_{qi_2\ldots i_m}]_-x_qx_px_{i_{m-1}}x_{i_m}\} \\ &= x_px_q\max\{x^\top[A_q]_+x, x^\top[A_q]_-x\} = x_px_q\max\{x^\top[B_q]_+x, x^\top[B_q]_-x\},\end{aligned} \tag{3.14}$$

where $[B_q]_+$ and $[B_q]_-$ are defined in Theorem 3.2. It follows from (3.14) and Lemma 2.1 that

$$|\lambda|x_q^2 \leq x_px_q\max\{\rho([B_q]_+), \rho([B_q]_-)\}. \tag{3.15}$$

We now break up the argument into two cases.

Case I. $|\lambda|x_p^2 \leq \rho([D_{pq}]_+)x_px_q + x_p^2\rho([B_p^q]_+)$. In this case, if $x_q > 0$, multiplying (3.13) with (3.15) and dividing $x_p^2x_q^2$ yield

$$(|\lambda| - \rho([B_p^q]_+))|\lambda| \leq \rho([D_{pq}]_+)\max\{\rho([B_q]_+), \rho([B_q]_-)\},$$

which implies $\lambda \in \mathcal{M}_{p,q}(\mathcal{A})$.
Otherwise, $x_q = 0$. From (3.13), it holds that

$$(|\lambda| - \rho([B_p^q]_+))|\lambda| \leq 0 \leq \rho([D_{pq}]_+)\max\{\rho([B_q]_+), \rho([B_q]_-)\},$$

which shows that $\lambda \in \mathcal{M}_{p,q}(\mathcal{A})$.

Case II. $|\lambda|x_p^2 \leq \rho([D_{pq}]_-)x_px_q + x_p^2\rho([B_p^q]_-)$. Following the similar arguments to the proof of Case I, we obtain $\lambda \in \mathcal{M}_{p,q}(\mathcal{A})$. Combining Cases I and II, we obtain the desired results. □

In order to illustrate the validity of Theorems 3.2 and 3.3, we employ a running example.

Example 3.3. *Consider a tensor* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ *defined by*

$$a_{ijkl} = \begin{cases} a_{1111} = 0.1; a_{1112} = -0.2; a_{1122} = -0.2; a_{1213} = -0.2; a_{1222} = 0.1; a_{1233} = 0.1; a_{1333} = -0.1; \\ a_{2111} = -1; a_{2131} = 3; a_{2211} = 1; a_{2212} = -2; a_{2222} = 1; a_{2311} = 2; a_{2333} = 1; \\ a_{3111} = 3; a_{3112} = -2; a_{3121} = 2; a_{3212} = -1; a_{3222} = 5; a_{3233} = 2; a_{3333} = -2; \\ a_{ijkl} = 0, \quad \textit{otherwise}. \end{cases}$$

From Theorem 3.2, we compute

$$\Theta(\mathcal{A}) = \bigcup_{i\in N}\Theta_i(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 5.4142\}.$$

Recalling Theorem 3.3, one has

$$\mathcal{M}(\mathcal{A}) = \bigcup_{i\in N}\bigcap_{j\in N, i\neq j}\mathcal{M}_{i,j}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 5.1620\}.$$

From Theorem 2 of [29], we obtain

$$\Omega(\mathcal{A}) = \bigcup_{i\in N} \Omega_i(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \le 12\}.$$

By virtue of Theorem 3 of [29], one has

$$\Phi(\mathcal{A}) = \bigcup_{i\in N} \bigcap_{j\in N, i\neq j} \Phi_{i,j}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \le 9.2276\}.$$

It follows from Theorem 4 of [29] that

$$\mathcal{N}(\mathcal{A}) = \bigcup_{i\in N} \bigcap_{j\in N, i\neq j} \mathcal{N}_{i,j}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \le 7.4686\}.$$

Therefore, the bounds in Theorems 3.2 and 3.3 are sharper than those Theorems 2–4 in [29].

4. Checking the strict copositivity of tensors

In this section, we focus on sufficient conditions for judging strict copositivity via the spectral radius of the symmetric matrices extracted from the given tensor. For this, we give a necessary condition for a strictly copositive tensor.

Lemma 4.1. *(Proposition 2.1 of [18]) Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$. If $\mathcal{A}$ is strictly copositive, then $a_{i\dots i} > 0, \forall i \in N$.*

Theorem 4.1. *Let $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{R}^{[3,n]}$ be symmetric with $a_{iii} > 0$ for all $i \in N$. If*

$$a_{iii}\frac{1}{\sqrt{n}} - \sqrt{\sum_{\delta_{ii_2 i_3}=0} ([a_{ii_2 i_3}]_-)^2} > 0, \forall i \in N, \tag{4.1}$$

then $\mathcal{A}$ is strictly copositive.

Proof. Suppose that (λ, x) is a Pareto Z-eigenpair of $\mathcal{A}$. Setting $0 < x_p = \max_{i\in N}\{x_i\}$ and referring to the p-th equation of (3.8), we obtain

$$\begin{aligned} \lambda x_p^2 &= \sum_{i_2, i_3 \in N} a_{pi_2 i_3} x_p x_{i_2} x_{i_3} \\ &= a_{ppp} x_p^3 + \sum_{\delta_{pi_2 i_3}=0} [a_{pi_2 i_3}]_+ x_p x_{i_2} x_{i_m} - \sum_{\delta_{pi_2 i_3}=0} [a_{pi_2 i_3}]_- x_p x_{i_2} x_{i_3}. \end{aligned}$$

Further,

$$\begin{aligned} \lambda x_p^2 &\ge a_{ppp} x_p^3 - \sum_{\delta_{pi_2 i_3}=0} [a_{pi_2 i_3}]_- x_p x_{i_2} x_{i_3} \\ &\ge a_{ppp} x_p^3 - \sum_{\delta_{pi_2 i_3}=0} [a_{pi_2 i_3}]_- x_p^2 x_{i_3}. \end{aligned} \tag{4.2}$$

Dividing both sides by x_p^2 on (4.2), we have

$$\begin{aligned}\lambda &\geq a_{ppp}x_p - \sum_{\delta_{pi_2i_3}=0}[a_{pi_2i_3}]_- x_{i_3}\\ &\geq a_{ppp}x_p - \sqrt{\sum_{\delta_{pi_21=0}}([a_{pi_21}]_-)^2 + \ldots + \sum_{\delta_{pi_2n=0}}([a_{pi_2n}]_-)^2}\cdot\sqrt{x_1^2+\ldots+x_n^2}\\ &\geq a_{ppp}x_p - \sqrt{\sum_{\delta_{pi_2i_3=0}}([a_{pi_2i_3}]_-)^2}. \end{aligned} \tag{4.3}$$

Since $x_p = \max_{i\in N}\{x_i\}$ and $x^\top x = 1$, we deduce $x_p \geq \frac{1}{\sqrt{n}}$. It follows from $a_{iii} > 0$ and (4.3) that

$$\lambda \geq a_{ppp}\frac{1}{\sqrt{n}} - \sqrt{\sum_{\delta_{pi_2i_3=0}}([a_{pi_2i_3}]_-)^2}. \tag{4.4}$$

Combining (4.1) with (4.4), we have $\lambda > 0$. Further, $\mathcal{A}$ is strictly copositive from Lemma 2.1. □

Theorem 4.2. *Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ be symmetric with $m \geq 4$ and $a_{i\ldots i} > 0$ for all $i \in N$. If*

$$a_{i\ldots i}(\frac{1}{\sqrt{n}})^{m-2} - \rho([A_i]_-) > 0, \forall i \in N, \tag{4.5}$$

then $\mathcal{A}$ is strictly copositive.

Proof. Let (λ, x) be a Pareto Z-eigenpair of $\mathcal{A}$. Setting $0 < x_p = \max_{i\in N}\{x_i\}$ and referring to the p-th equation of (3.8), we obtain

$$\begin{aligned}\lambda x_p^2 &= \sum_{i_2,\ldots,i_m=1}^{n} a_{pi_2\ldots i_m}x_px_{i_2}\ldots x_{i_m}\\ &= a_{p\ldots p}x_p^m + \sum_{\delta_{pi_2\ldots i_m}=0}[a_{pi_2\ldots i_m}]_+x_px_{i_2}\ldots x_{i_m} - \sum_{\delta_{pi_2\ldots i_m}=0}[a_{pi_2\ldots i_m}]_-x_px_{i_2}\ldots x_{i_m}.\end{aligned}$$

Further,

$$\begin{aligned}\lambda x_p^2 &\geq a_{p\ldots p}x_p^m - \sum_{\delta_{pi_2\ldots i_m}=0}[a_{pi_2\ldots i_m}]_-x_px_{i_2}\ldots x_{i_m}\\ &\geq a_{p\ldots p}x_p^m - \sum_{\delta_{pi_2\ldots i_m}=0}[a_{pi_2\ldots i_m}]_-x_p^2x_{i_{m-1}}x_{i_m}\\ &\geq a_{p\ldots p}x_p^m - x_p^2x^\top[A_p]_-x \ \geq a_{p\ldots p}x_p^m - x_p^2\rho([A_p]_-),\end{aligned} \tag{4.6}$$

where $[A_p]_-$ is defined in Theorem 3.2. Dividing both sides by x_p^2 on (4.6), we deduce

$$\lambda \geq a_{p\ldots p}x_p^{m-2} - \rho([A_p]_-). \tag{4.7}$$

Since $x_p = \max_{i\in N}\{x_i\}$ and $x^\top x = 1$, we deduce $x_p \geq \frac{1}{\sqrt{n}}$. It follows from $a_{i\ldots i} > 0$ and (4.7) that

$$\lambda \geq a_{p\ldots p}(\frac{1}{\sqrt{n}})^{m-2} - \rho([A_p]_-). \tag{4.8}$$

Combining (4.5) with (4.8), we obtain $\lambda > 0$. Further, $\mathcal{A}$ is strictly copositive from Lemma 2.1. □

A nice consequence of our results is that Theorems 4.1 and 4.2 are better than that of Theorem 5 of [29].

Lemma 4.2. *(Theorem 5 of [29]) Let* $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$ *be symmetric with* $a_{i\dots i} > 0$ *for* $i \in N$. *Then,* $\mathcal{A}$ *is strictly copositive, provided that*

$$a_{i\dots i}(\frac{1}{\sqrt{n}})^{m-2} - R_i(\mathcal{A})_- > 0, \tag{4.9}$$

where $R_i(\mathcal{A})_- = \sum\limits_{i_2,\dots,i_m \in N} [a_{ii_2\dots i_m}]_-$.

Corollary 4.1. *Let* $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$ *be symmetric with* $a_{i\dots i} > 0$ *for* $i \in N$. *Then,*

$$\begin{cases} a_{iii}\frac{1}{\sqrt{n}} - \sqrt{\sum\limits_{\delta_{ii_2i_3}=0} ([a_{ii_2i_3}]_-)^2} \geq a_{iii}\frac{1}{\sqrt{n}} - R_i(\mathcal{A})_-, & \text{if} \quad m = 3, \\ a_{i\dots i}(\frac{1}{\sqrt{n}})^{m-2} - \rho([A_i]_-) \geq a_{i\dots i}(\frac{1}{\sqrt{n}})^{m-2} - R_i(\mathcal{A})_-, & \text{otherwise} \quad m \geq 4. \end{cases}$$

Proof. It follows from $a_{i\dots i} > 0$ that $R_i(\mathcal{A})_- = \sum\limits_{\delta_{ii_2\dots i_m}=0} [a_{ii_2\dots i_m}]_-$. We now break up the argument into two cases.

Case 1. $m = 3$. It is clear that

$$\sqrt{\sum_{\delta_{ii_2i_3}=0} ([a_{ii_2i_3}]_-)^2} \leq \sum_{\delta_{ii_2i_3}=0} [a_{ii_2i_3}]_- = R_i(\mathcal{A})_-.$$

Further,

$$a_{iii}\frac{1}{\sqrt{n}} - \sqrt{\sum_{\delta_{ii_2i_3}=0} ([a_{ii_2i_3}]_-)^2} \geq a_{iii}\frac{1}{\sqrt{n}} - R_i(\mathcal{A})_-.$$

Case 2. $m \geq 4$. We obtain

$$\rho([A_i]_-) \leq \max_{1 \leq i_{m-1} \leq n} \sum_{i_2,\dots,i_{m-2},i_m \in N} [a_{ii_2\dots i_m}]_- \leq \sum_{i_2,\dots,i_m \in N} [a_{ii_2\dots i_m}]_- = R_i(\mathcal{A})_-.$$

Consequently,

$$a_{i\dots i}(\frac{1}{\sqrt{n}})^{m-2} - \rho([A_i]_-) \geq a_{i\dots i}(\frac{1}{\sqrt{n}})^{m-2} - R_i(\mathcal{A})_-.$$

Therefore, the desired results hold. □

Identifying the strict copositivity actually necessitates $\mathcal{A}$ being symmetric. Therefore, symmetry may be relatively strict for general tensors. We can solve this issue by symmetrizing the tensors $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$ as follows:

$$\widetilde{a}_{i_1 i_2 \dots i_m} = \begin{cases} a_{i_1 i_2 \dots i_m} & \text{if } i_1 = i_2 = \dots = i_m, \\ \frac{1}{m!} \sum\limits_{i_2 \dots i_m \in \Gamma_m} a_{i_1 i_2 \dots i_m} & \text{otherwise}, \end{cases}$$

where $\widetilde{\mathcal{A}} = (\widetilde{a}_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$ is the symmetrization tensor under permutation group Γ_m.

The following example shows that Theorem 4.1 can verify the strict copositivity more accurately than that of Theorem 5 of [29] for $m = 3$ tensors.

Example 4.1. *Consider a tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,2]}$ defined by*

$$a_{ijk} = \begin{cases} a_{111} = \frac{1}{2}; a_{112} = -\frac{1}{5}; a_{121} = -\frac{1}{5}; a_{122} = 0; \\ a_{222} = 1; a_{211} = -\frac{1}{5}; a_{212} = 0; a_{221} = 0. \end{cases}$$

It is easy to see that $\mathcal{A}$ is symmetric with

$$a_{111}\frac{1}{\sqrt{2}} - \sqrt{\sum_{\delta_{1i_2i_3}=0} ([a_{1i_2i_3}]_-)^2} = \frac{\sqrt{2}}{20} > 0,$$

$$a_{222}\frac{1}{\sqrt{2}} - \sqrt{\sum_{\delta_{2i_2i_3}=0} ([a_{2i_2i_3}]_-)^2} = \frac{5\sqrt{2}-2}{10} > 0,$$

which means that $\mathcal{A}$ is strictly copositive.

Referring to Theorem 5 of [29], we deduce

$$a_{111}\frac{1}{\sqrt{2}} - R_1(\mathcal{A})_- = \frac{5\sqrt{2}-8}{20} < 0.$$

Therefore, it is impossible to judge the strict copositivity of $\mathcal{A}$ with Theorem 5 of [29].

When $\mathcal{A}$ is asymmetric, we still identify the strict copositivity of tensors by Theorem 4.1.

Example 4.2. *Consider a tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,2]}$ defined by*

$$a_{ijk} = \begin{cases} a_{111} = \frac{1}{2}; a_{112} = -\frac{1}{10}; a_{121} = -\frac{2}{5}; a_{122} = 0; \\ a_{222} = 1; a_{211} = -\frac{1}{10}; a_{212} = 0; a_{221} = 0. \end{cases}$$

Observe that $\mathcal{A}$ is asymmetric from $a_{112} = -\frac{1}{10}, a_{121} = -\frac{2}{5}$ and $a_{211} = -\frac{1}{10}$. Therefore, we cannot directly use Theorem 4.1 to judge whether $\mathcal{A}$ is strictly copositive. Symmetrizing $\mathcal{A}$, we obtain $\widetilde{\mathcal{A}}$ with

$$\widetilde{a}_{ijk} = \begin{cases} \widetilde{a}_{111} = \frac{1}{2}; \widetilde{a}_{112} = -\frac{1}{5}; \widetilde{a}_{121} = -\frac{1}{5}; \widetilde{a}_{122} = 0; \\ \widetilde{a}_{222} = 1; \widetilde{a}_{211} = -\frac{1}{5}; \widetilde{a}_{212} = 0; \widetilde{a}_{221} = 0. \end{cases}$$

It is easy to see that $\widetilde{\mathcal{A}}$ is symmetric with

$$a_{111}\frac{1}{\sqrt{2}} - \sqrt{\sum_{\delta_{1i_2i_3}=0} ([a_{1i_2i_3}]_-)^2} = \frac{\sqrt{2}}{20} > 0,$$

$$a_{222}\frac{1}{\sqrt{2}} - \sqrt{\sum_{\delta_{2i_2i_3}=0} ([a_{2i_2i_3}]_-)^2} = \frac{5\sqrt{2}-2}{10} > 0,$$

which implies that $\widetilde{\mathcal{A}}$ is strictly copositive. Taking into account that $\mathcal{A}x^3 = \widetilde{\mathcal{A}}x^3 > 0$, we deduce that $\mathcal{A}$ is strictly copositive.

In what follows, we reveal that the results of Theorem 4.2 are sharper than those of Theorem 5 of [29] for $m \geq 4$ tensors.

Example 4.3. *Consider a tensor* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ *defined by*

$$a_{ijkl} = \begin{cases} a_{1111} = 36; a_{1112} = a_{1121} = a_{1211} = a_{2111} = 50; \\ a_{2222} = 66; a_{1122} = a_{1221} = a_{1212} = -10; \\ a_{2221} = a_{2212} = a_{2122} = a_{1222} = 70; \\ a_{2211} = a_{2121} = a_{2112} = -20. \end{cases}$$

First, we rewrite

$$[A_1]_- = \begin{bmatrix} 0 & 10 \\ 10 & 10 \end{bmatrix}, \quad [A_2]_- = \begin{bmatrix} 20 & 20 \\ 20 & 0 \end{bmatrix}$$

and compute

$$a_{1111}(\frac{1}{\sqrt{2}})^2 - \rho([B_1]_-) = 1.8197 > 0, \quad a_{2222}(\frac{1}{\sqrt{2}})^2 - \rho([B_2]_-) = 0.6393 > 0,$$

which means that $\mathcal{A}$ is strictly copositive.

Recalling to Theorem 5 of [29], we obtain

$$a_{1111}(\frac{1}{\sqrt{2}})^2 - R_1(\mathcal{A})_- = -12 < 0.$$

Consequently, we cannot judge the strict copositivity of $\mathcal{A}$ from Theorem 5 of [29].

5. Conclusions

In this paper, we proposed sharp Pareto Z-eigenvalue inclusion intervals for tensor eigenvalue complementarity problems via the spectral radius of symmetric matrices. Further, we proposed some criteria to confirm the strict copositivity of real tensors. It may be possible to conduct additional research to create some algorithms for tensor eigenvalue complementarity problems using Pareto Z-eigenvalue intervals, such as parametric algorithms and ADMM algorithms [5, 23, 28].

Author contributions

Xueyong Wang: Conceptualization, Methodology, Writing-original draft preparation; Gang Wang: Conceptualization, Writing-review, Project administration; Ping Yang: Software, Formal analysis. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

This work was supported by the Natural Science Foundation of China (12071250) and the Natural Science Foundation of Shandong Province (ZR2021MA088).

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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