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Existence and regularity results for critical $(p, 2)$-Laplacian equation

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Abstract: In this paper, we study a class of $(p, 2)$-Laplacian equation with Hartree-type nonlinearity and critical exponents. Under some general assumptions and based on variational tools, we establish the existence, regularity, and symmetry of nontrivial solutions for such a problem.

Keywords: $(p, 2)$-Laplacian equation; Hartree-type nonlinearity; variational methods; regularity and symmetry; critical exponent
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1. Introduction and main results

Let us consider the following critical $(p, 2)$-Laplacian equation

$$-\Delta_p u - \Delta u + u + |u|^{p-2}u = (I_\alpha * F(u))f(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geqslant 3$, $1 < p < N$, $0 < \alpha < N$, and Δ_p is the p-Laplacian with $\Delta_p = \nabla(|\nabla u|^{p-2}\nabla u)$. I_α is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \pi^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right) |x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \backslash \{0\}.$$

Equation (1.1) is closely related to the following nonlocal quasilinear equation:

$$-\Delta_p u - \mu(x)\Delta_q u + |u|^{p-2}u + |u|^{q-2}u = (I_\alpha * F(u))f(u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $1 < p, q < N$ and $\mu : \mathbb{R}^N \mapsto [0, \infty)$ is supposed to be Lipschitz continuous. The operator involved in (1.2) is the so-called double phase operator whose behavior switches between two different elliptic situations. The pioneering work to treat such operators comes from Zhikov [36, 37], who introduced such classes to provide models of strongly anisotropic materials. For more details and recent works about double phase problems, we refer to [15, 22].

When $N = 3$, $p = q = 2$, $\mu = 1$, and $F(u) = u^2$, then Eq (1.1) is the well-known Choquard equation:

$$-\Delta u + u = (I_\alpha * |u|^2)u, \quad x \in \mathbb{R}^3. \tag{1.3}$$

Equation (1.3) appears in several physical models like the quantum theory of polarons [29], Hartree-Fock theory [18], and self-gravitating matter [30]. After the pioneer work of Lieb [18] and Lions [20], the existence of weak solutions for Choquard equations have been a fascinating topic in past decades. For more related work, we refer to [2, 31] for the subcritical case, [8, 13] for the upper critical case, [9, 26] for the lower critical case, and [21, 32] for the double critical case.

When $q = 2 \neq p$ and $\mu = 1$, Eq (1.2) reduces to (1.1). It appears in many different disciplines of physics and has a wide range of applications, such as chemical reaction design [5], quantum field theory [4], biophysics [10], and plasma physics [33]. From a mathematical point of view, the main difficulty in (1.1) is the non-homogeneity of the operator $-\Delta_p - \Delta$. For this reason, equations involving such operator or its variant have been received increasing attention from various authors. In particular, Gasiński-Papageorgiou [14] considered Eq (1.1) when the nonlinearity takes the following form:

$$\begin{cases} -\Delta_p u - \Delta u = f(x, u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.4}$$

where $p > 2$ and $\Omega \in \mathbb{R}^N$ is a bounded C^2 domain. Under the assumption that $f(x, u)$ exhibits asymmetric behaviour as $u \to \pm\infty$, more precisely $f(x, u)$ is superlinear in the positive direction (without satisfying the Ambrosetti-Rabinowitz condition) and sublinear resonant in the negative direction, the authors obtained the existence and multiplicity results of (1.4) via variational tools and Morse theory methods. Later, Papageorgiou-Rădulescu-Repovš [28] imposed certain assumptions on $f(x, u)$ to make it double resonant at both $\pm\infty$ and 0. By virtue of variational tools and critical groups, the authors obtained the existence and multiplicity results of (1.4).

In [27], the authors considered the following Dirichlet problem:

$$\begin{cases} -\Delta_p u - \Delta u = \lambda |u|^{p-2}u + f(x, u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.5}$$

where $p > 2$, $\lambda > 0$, $\Omega \subset \mathbb{R}^N$ with a C^2 boundary, and $f(x, u)$ is a Carathéodory function. Based on critical point theory, together with suitable truncation and comparison techniques, Papageorgiou-Rădulescu-Repovš [27] obtained the existence and multiplicity results of (1.5) when λ is near the principal eigenvalue $\lambda_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$. Subsequently, their work was extended by Bhattacharya-Emamizadeh-Farjudian [6] to the case of $1 < p < 2$. By applying the fibering method and spectrum analysis, a priori bounds and regularity results of (1.5) were investigated. Moameni-Wong [24] studied the case of $f(x, u)$ in (1.5) satisfying supercritical growth. By using a variational principle on convex subsets of a Banach space, the authors proved the existence of at least one nontrivial solution of (1.5). Equation (1.5) with Neumann boundary condition ($\frac{\partial u}{\partial \nu} = 0$) has been considered recently in Mihăilescu [23]. The authors showed that the eigenvalue set of this problem consists of 0 and an unbounded open interval from the first eigenvalue of $-\Delta_p - \Delta$ ($p > 2$) to infinity. After that, Fărcăşeanu et al. [12] extended the results in [23] to $0 < p < 2$ by means of the determination of a critical point on the Nehari manifold [3]. For more results related to the $(p, 2)$-Laplacian equation, one can refer to [1, 16]

Recently, Moroz-Van Schaftingen [25] established the $W^{2,q}_{loc}(\mathbb{R}^N)$ regularity ($q > 1$) and Pohožaev identity of weak solutions for the following generalized Choquard equation:

$$-\Delta u + u = (I_\alpha * F(u))f(u), \quad x \in \mathbb{R}^N, \tag{1.6}$$

where $N \geqslant 3$, $\alpha \in (0, N)$, and F satisfies the subcritical Berestycki-Lions type condition, namely:

(H_1) There exists $t_0 \in \mathbb{R}\backslash\{0\}$ such that $F(t_0) \neq 0$, where $F : t \in \mathbb{R} \to \int_0^t f(\zeta)\mathrm{d}\zeta$.

(H_2) There exists $C > 0$ such that for every $t \in \mathbb{R}$, $|tf(t)| \leqslant C(|t|^{\frac{N+\alpha}{N}} + |t|^{\frac{N+\alpha}{N-2}})$.

(H_3)

$$\lim_{t\to 0} \frac{F(t)}{|t|^{\frac{N+\alpha}{N}}} = 0 \text{ and } \lim_{t\to\infty} \frac{F(t)}{|t|^{\frac{N+\alpha}{N-2}}} = 0.$$

Li-Ma [17] studied Eq (1.6) with a perturbation. By virtue of the subcritical approximation and the Pohožaev constraint method, they obtained the regularity and Pohožaev identity of weak solutions. Cassani-Du-Liu [7] studied Eq (1.6) with $N = 2$ and $I_\alpha = \ln \frac{1}{|x|}$. By using an asymptotic approximation approach, the existence of positive solutions of (1.6) is obtained.

Up to our knowledge, no results have been reported regarding the existence and regularity of weak solutions for the $(p, 2)$-Laplacian equation with critical Hartree-type nonlinearity. Inspired by the above cited results, the main objective of this paper is to fill this gap. The novelty of this paper lies in two aspects. On one hand, due to the existence of the $(p, 2)$-Laplacian operator, problem (1.1) becomes non-homogeneous. Therefore, the method used in [25] is invalid. To overcome this difficulty, we introduce some new ideas and establish new estimates to improve the integrability of weak solutions of Eq (1.1). On the other hand, we are the first to consider a class of $(p, 2)$-Laplacian equation with critical Hartree-type nonlinearity.

Before we present our results, we suppose that f satisfies the following conditions:

(F_1) There exists $C > 0$ such that for every $t \in \mathbb{R}$, $|tf(t)| \leqslant C(|t|^{2^\sharp_\alpha} + |t|^{2^*_\alpha})$, where $2^\sharp_\alpha = \frac{N+\alpha}{N}$ and $2^*_\alpha = \frac{N+\alpha}{N-2}$.

(F_2) $F(u) = \frac{1}{2^\sharp_\alpha}|u|^{2^\sharp_\alpha} + \frac{\lambda}{2^*_\alpha}|u|^{2^*_\alpha}$.

Now we can formulate our main results in this paper.

Theorem 1.1. *Let $N \geqslant 3$, $1 < p < N$, $0 < \alpha < N$, and condition (F_1) holds. If u is a nontrivial solution of Eq* (1.1)*, then*

(i) $u \in L^q(\mathbb{R}^N)$ for any $q \in [2, \infty]$;

(ii) the following Pohožaev identity holds:

$$\begin{aligned}&\frac{N-2}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{N}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{N-p}{p}\|u\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{N}{p}\|u\|^p_{L^p(\mathbb{R}^N)}\\ =&\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x.\end{aligned}$$

Theorem 1.2. *Let $N \geqslant 3$, $1 < p < N$, $0 < \alpha < N$, and condition (F_2) hold. Then, there exists $\Lambda > 0$ such that for any $\lambda \in (0, \Lambda)$, Eq (1.1) possesses a nonnegative radially symmetric ground state solution, where*

$$\Lambda = \frac{2_\alpha^*(\alpha+2)^{\frac{2_\alpha^*-1}{2}}(N-2)^{\frac{1}{2}}(N+\alpha)^{\frac{N+\alpha}{\alpha(N-2)}}\mathcal{S}_2^{\frac{2_\alpha^*}{2}}}{\alpha^{\frac{2_\alpha^*-1}{2}}\left[N\cdot(2_\alpha^\sharp)^2\right]^{\frac{2_\alpha^*-1}{2(2_\alpha^\sharp-1)}}\mathcal{S}_1^{\frac{2_\alpha^\sharp(2_\alpha^*-1)}{2(2_\alpha^\sharp-1)}}}.$$

At the end of this section, we outline our method. We introduce this into two parts.

Regularity: First, by applying the Minty-Browder theorem [11] and the decomposition of Riesz potential, we improve the integrability of weak solutions to Eq (1.1). Then, under a different range of 2_α^*, we use two different iteration approaches to establish an $L^\infty(\mathbb{R}^N)$ estimate for weak solutions of Eq (1.1). As a result, the Pohožaev identity of Eq (1.1) is established.

Existence: With delicate analysis and optimal range of λ, we give an exact estimate of the minimum on the Pohožaev manifold. Using this fact, one can show that the minimizing sequences in Pohožaev manifold are non-vanishing in $L^2(\mathbb{R}^N)$ and $L^{2^*}(\mathbb{R}^N)$. This, together with a compactness lemma (see Proposition 2.2), the existence of ground state solutions of Eq (1.1) is obtained. Finally, we prove these ground solutions are radially symmetric.

This paper is organized as follows. In Section 2, we introduce some basic notations and technical lemmas. In Section 3, we study the regularity of weak solutions and Pohožaev identity of Eq (1.1). In Section 4, we study the existence and symmetry of ground state solutions Eq (1.1).

2. Preliminaries

In this section, we give some definitions and results which will be used later. C, C_i $(i = 1, 2, \cdots)$ denote positive constants which can be changed line by line. Let X be a Banach space, and use X_{rad} to denote the radial subspace of X.

In this work, our working space can be defined by

$$E = H^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$$

equipped with the norm

$$\|u\|_E = \|u\|_{H^1(\mathbb{R}^N)} + \|u\|_{W^{1,p}(\mathbb{R}^N)}.$$

Proposition 2.1. ([19]) *Let $s, t > 1$, and $\alpha \in (0, N)$ with $\frac{1}{s} + \frac{1}{t} = 1 + \frac{\alpha}{N}$. Then, there exists $C(N, \alpha, s, t) > 0$ such that for any $u \in L^s(\mathbb{R}^N)$ and $v \in L^t(\mathbb{R}^N)$,*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u(x)v(y)}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y\right| \leqslant C(N, \alpha, s, t)\|u\|_{L^s(\mathbb{R}^N)}\|v\|_{L^t(\mathbb{R}^N)}.$$

If $s = t = \frac{2N}{N+\alpha}$, then $C(N, \alpha, s, t) = C_{N,\alpha} = \pi^{\frac{N-\alpha}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})}\left[\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right]^{-\frac{\alpha}{N}}$.

Proposition 2.2. ([34]) *Let $N \geqslant 3$, and $\{u_n\} \subset E$ be any bounded sequence satisfying*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^2\mathrm{d}x > 0 \ \ \textit{and} \ \ \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x > 0.$$

Then, the sequence $\{u_n\}$ converges weakly and a.e. to $u \not\equiv 0$ in $L^2_{loc}(\mathbb{R}^N)$.

The following inequalities can be viewed as a consequence of Proposition 2.1, which is useful in the following estimation:

$$\mathcal{S}_1\left[\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^\sharp_\alpha})|u|^{2^\sharp_\alpha}\mathrm{d}x\right]^{\frac{1}{2^\sharp_\alpha}}\leqslant\|u\|^2_{L^2(\mathbb{R}^N)},\ \ u\in L^2(\mathbb{R}^N)\tag{2.1}$$

and

$$\mathcal{S}_2\left[\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_\alpha})|u|^{2^*_\alpha}\mathrm{d}x\right]^{\frac{1}{2^*_\alpha}}\leqslant\|u\|^2_{D^{1,2}(\mathbb{R}^N)},\ \ u\in D^{1,2}(\mathbb{R}^N),\tag{2.2}$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are the embedding constants.

Lemma 2.1. *For any $x,y\in\mathbb{R}^N$, the following assertions are valid:*
(i) If $1<p<2$, then

$$\frac{|x-y|^2}{(|x|+|y|)^{2-p}}\leqslant C(|x|^{p-2}x-|y|^{p-2}y)(x-y);$$

$$\left||x|^{p-2}x-|y|^{p-2}y\right|\leqslant C|x-y|^{p-1}.$$

(ii) If $2\leqslant p<\infty$, then

$$|x-y|^p\leqslant C(|x|^{p-2}x-|y|^{p-2}y)(x-y);$$

$$\left||x|^{p-2}x-|y|^{p-2}y\right|\leqslant C(|x|+|y|)^{p-2}|x-y|.$$

3. Regularity of weak solutions and the Pohožaev identity

In this section, we study the regularity of weak solutions of Eq (1.1).

Lemma 3.1. ([25]) *Let $q,r,w,t\in[1,\infty)$ and $\zeta\in[0,2]$ such that*

$$1+\frac{\alpha}{N}-\frac{1}{w}-\frac{1}{t}=\frac{\zeta}{q}+\frac{2-\zeta}{r}.$$

If $\mu\in(0,2)$ satisfies

$$\min(q,r)\left(\frac{\alpha}{N}-\frac{1}{w}\right)<\mu<\max(q,r)\left(1-\frac{1}{w}\right),$$

$$\min(q,r)\left(\frac{\alpha}{N}-\frac{1}{t}\right)<2-\mu<\max(q,r)\left(1-\frac{1}{t}\right),$$

then for every $H\in L^w(\mathbb{R}^N)$, $K\in L^t(\mathbb{R}^N)$, and $u\in L^q(\mathbb{R}^N)\cap L^r(\mathbb{R}^N)$,

$$\begin{aligned}&\int_{\mathbb{R}^N}(I_\alpha*(H|u|^\mu))G|u|^{2-\mu}\mathrm{d}x\\ \leqslant& C\left(\int_{\mathbb{R}^N}|H|^w\mathrm{d}x\right)^{\frac{1}{w}}\left(\int_{\mathbb{R}^N}|G|^t\mathrm{d}x\right)^{\frac{1}{t}}\left(\int_{\mathbb{R}^N}|u|^q\mathrm{d}x\right)^{\frac{\zeta}{q}}\left(\int_{\mathbb{R}^N}|u|^r\mathrm{d}x\right)^{\frac{2-\zeta}{r}}.\end{aligned}$$

Similar to the proof of [25, Lemma 3.2], we get the following lemma without proof.

Lemma 3.2. *Let $N\geqslant 3$, $0<\alpha<N$, and $0<\theta<2$. If $H,G\in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)+L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)$, and $\frac{\alpha}{N}<\theta<2$, then for every $\epsilon>0$ there exists $C_{\epsilon,\theta}\in\mathbb{R}$ such that, for every $u\in E$,*

$$\int_{\mathbb{R}^N}[I_\alpha*(H|u|^\theta)]G|u|^{2-\theta}\mathrm{d}x\leqslant\epsilon^2\|u\|^2_{D^{1,2}(\mathbb{R}^N)}+C_{\epsilon,\theta}\|u\|^2_{L^2(\mathbb{R}^N)}.$$

Proposition 3.1. ([11]) *Let X be a reflexive Banach space. Let Φ be a (nonlinear) continuous mapping from X into its dual space X^{-1} such that*

(i) $\langle \Phi u - \Phi v, u - v\rangle > 0, \quad \forall u, v \in X, \ u \neq v;$

(ii) $\lim\limits_{\|u\|_X \to \infty} \frac{\langle \Phi u, u\rangle}{\|u\|_X} = +\infty.$

Then, for every $g \in X^{-1}$, there exists a unique $u \in X$ such that $\Phi u = g$.

Lemma 3.3. *Let*

$$\begin{aligned} \langle \Phi u, v\rangle = &\int_{\mathbb{R}^N} \nabla u \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla v \mathrm{d}x + \tau \int_{\mathbb{R}^N} uv \mathrm{d}x \\ &+ \int_{\mathbb{R}^N} |u|^{p-2} uv \mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * Hu) Gv \mathrm{d}x, \quad \forall u, v \in E. \end{aligned} \tag{3.1}$$

Then, Φ satisfies the following conditions:

(i) $\langle \Phi u - \Phi v, u - v\rangle > 0, \quad \forall u, v \in E, \ u \neq v;$

(ii) $\lim\limits_{\|u\|_E \to \infty} \frac{\langle \Phi u, u\rangle}{\|u\|_E} = +\infty.$

Proof. Under direct calculation, we can compute

$$\begin{aligned} \langle \Phi(u) - \Phi(v), u - v\rangle = &\|u - v\|^2_{D^{1,2}(\mathbb{R}^N)} + \tau \|u - v\|^2_{L^2(\mathbb{R}^N)} \\ &+ \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v)(\nabla u - \nabla v) \mathrm{d}x \\ &+ \int_{\mathbb{R}^N} (|u|^{p-2} u - |v|^{p-2} v)(u - v) \mathrm{d}x - \int_{\mathbb{R}^N} [I_\alpha * H(u - v)] G(u - v) \mathrm{d}x. \end{aligned}$$

Now, we give the verifications of (i)–(ii). By Lemma 2.1, we have that for $1 < p < 2$,

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v)(\nabla u - \nabla v) \mathrm{d}x + \int_{\mathbb{R}^N} (|u|^{p-2} u - |v|^{p-2} v)(u - v) \mathrm{d}x \\ \geqslant & C \Big\{ \Big[\int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v)(\nabla u - \nabla v) \mathrm{d}x \Big] \Big[\int_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p) \mathrm{d}x \Big]^{\frac{2-p}{p}} \\ &+ \Big[\int_{\mathbb{R}^N} (|u|^{p-2} u - |v|^{p-2} v)(u - v) \mathrm{d}x \Big] \Big[\int_{\mathbb{R}^N} (|u|^p + |v|^p) \mathrm{d}x \Big]^{\frac{2-p}{p}} \Big\} \\ \geqslant & C \Big\{ \Big[\int_{\mathbb{R}^N} \Big| (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v)(\nabla u - \nabla v) \Big|^{\frac{p}{2}} (|\nabla u|^{2-p} + |\nabla v|^{2-p})^{\frac{p}{2}} \mathrm{d}x \Big]^{\frac{2}{p}} \\ &+ \Big[\int_{\mathbb{R}^N} \Big| (|u|^{p-2} u - |v|^{p-2} v)(u - v) \Big|^{\frac{p}{2}} (|u|^{2-p} + |v|^{2-p})^{\frac{p}{2}} \mathrm{d}x \Big]^{\frac{2}{p}} \Big\} \\ \geqslant & C \Big\{ \Big[\int_{\mathbb{R}^N} \Big| (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v)(\nabla u - \nabla v) \Big|^{\frac{p}{2}} (\nabla u + \nabla v)^{\frac{p(2-p)}{2}} \mathrm{d}x \Big]^{\frac{2}{p}} \\ &+ \Big[\int_{\mathbb{R}^N} \Big| (|u|^{p-2} u - |v|^{p-2} v)(u - v) \Big|^{\frac{p}{2}} (u + v)^{\frac{p(2-p)}{2}} \mathrm{d}x \Big]^{\frac{2}{p}} \Big\} \\ \geqslant & C \Big[\Big(\int_{\mathbb{R}^N} |\nabla (u - v)|^p \mathrm{d}x \Big)^{\frac{2}{p}} + \Big(\int_{\mathbb{R}^N} |u - v|^p \mathrm{d}x \Big)^{\frac{2}{p}} \Big], \end{aligned} \tag{3.2}$$

and for $2 \leqslant p < \infty$,

$$\begin{aligned}&\int_{\mathbb{R}^N}(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)(\nabla u-\nabla v)\mathrm{d}x+\int_{\mathbb{R}^N}(|u|^{p-2}u-|v|^{p-2}v)(u-v)\mathrm{d}x\\&\geqslant C\left(\int_{\mathbb{R}^N}|\nabla(u-v)|^p\mathrm{d}x+\int_{\mathbb{R}^N}|u-v|^p\mathrm{d}x\right).\end{aligned}\tag{3.3}$$

Combining (3.2) and (3.3), for $p \in (1,\infty)$ there exists $C > 0$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N}(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)(\nabla u-\nabla v)\mathrm{d}x+\int_{\mathbb{R}^N}(|u|^{p-2}u-|v|^{p-2}v)(u-v)\mathrm{d}x\\&\geqslant C\left(\int_{\mathbb{R}^N}|\nabla(u-v)|^p\mathrm{d}x+\int_{\mathbb{R}^N}|u-v|^p\mathrm{d}x\right).\end{aligned}\tag{3.4}$$

In view of Lemma 3.2 with $\theta = 1$, there exists $\tau > 0$ such that for each $v \in E$, we have

$$\int_{\mathbb{R}^N}(I_\alpha * G|v|)H|v|\mathrm{d}x \leqslant \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x+\frac{\tau}{2}\int_{\mathbb{R}^N}|v|^2\mathrm{d}x.\tag{3.5}$$

Taking this together with (3.4) and (3.5), we obtain

$$\begin{aligned}\langle\Phi u-\Phi\varphi,u-\varphi\rangle \geqslant&\frac{1}{2}\|u-\varphi\|^2_{D^{1,2}(\mathbb{R}^N)}+\frac{\tau}{2}\|u-\varphi\|^2_{L^2(\mathbb{R}^N)}\\&+C\left(\|u-\varphi\|^p_{D^{1,p}(\mathbb{R}^N)}+\|u-\varphi\|^p_{L^p(\mathbb{R}^N)}\right)\\>&0.\end{aligned}$$

So, condition (i) follows. By Lemma 3.2, it is easy to verify condition (ii). The proof is complete. □

Lemma 3.4. *Suppose that* $H, G \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)$ *and* $u \in E$ *solve*

$$-\Delta_p u-\Delta u+u+|u|^{p-2}u=(I_\alpha * Hu)G.\tag{3.6}$$

Then, $u \in L^q(\mathbb{R}^N)$ *for each* $q \in \left[2, \frac{2^*N}{\alpha}\right]$.

Proof. Using Lemma 3.2 with $\theta = 1$, there exists $\tau > 0$ such that, for every $\varphi \in E$,

$$\int_{\mathbb{R}^N}(I_\alpha * |H\varphi|)|G\varphi|\mathrm{d}x \leqslant \frac{1}{2}\|\varphi\|^2_{D^{1,2}(\mathbb{R}^N)}+\frac{\tau}{2}\|\varphi\|^2_{L^2(\mathbb{R}^N)}.\tag{3.7}$$

Let sequences $\{H_n\}, \{G_n\} \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ such that $|H_n| \leqslant |H|$ and $|G_n| \leqslant |G|$, and $H_n \to H$ and $G_n \to G$ almost everywhere in $\mathbb{R}^N$. In what follows, we claim that there exists a unique solution $u_n \in E$ satisfying

$$-\Delta_p u_n-\Delta u_n+\tau u_n+|u_n|^{p-2}u_n=[I_\alpha * (H_n u_n)]G_n+(\tau-1)u,\tag{3.8}$$

where $u \in E$ is the given solution of (3.6). The duality is given in this case by

$$\begin{aligned}\langle\Psi u,\varphi\rangle=&\int_{\mathbb{R}^N}\nabla u\nabla\varphi\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\varphi\mathrm{d}x+\tau\int_{\mathbb{R}^N}u\varphi\mathrm{d}x\\&+\int_{\mathbb{R}^N}|u|^{p-2}u\varphi\mathrm{d}x-\int_{\mathbb{R}^N}(I_\alpha * Hu)G\varphi\mathrm{d}x,\quad\forall u,\varphi\in E.\end{aligned}\tag{3.9}$$

In view of Lemma 3.3, it is easy to verify that Ψ satisfies all the conditions described in Proposition 3.1. Applying Proposition 3.1 with $g(u) = (\tau - 1)u$, we get the desired results.

Moreover, we also claim that the sequence $\{u_n\}$ converges weakly to u in E as $n \to \infty$. Multiplying both sides of (3.8) by u_n and integrating it over $\mathbb{R}^N$, then

$$\begin{aligned}&\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} + \tau\|u_n\|^2_{L^2(\mathbb{R}^N)} + \|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \|u_n\|^p_{L^p(\mathbb{R}^N)}\\ =&\int_{\mathbb{R}^N}[I_\alpha * (H_n u_n)]G_n u_n \mathrm{d}x + (\tau - 1)\int_{\mathbb{R}^N} u_n u \mathrm{d}x.\end{aligned}$$

Combining this with (3.7), the Hölder inequality, and the Young inequality, one has

$$\begin{aligned}&\frac{1}{2}\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{\tau}{2}\|u_n\|^2_{L^2(\mathbb{R}^N)} + \|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \|u_n\|^p_{L^p(\mathbb{R}^N)}\\ \leqslant&(\tau-1)\left(\int_{\mathbb{R}^N}|u_n|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|u|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ \leqslant&\frac{\tau-1}{2}\left(\|u_n\|^2_{L^2(\mathbb{R}^N)} + \|u\|^2_{L^2(\mathbb{R}^N)}\right).\end{aligned}$$

By this, we obtain

$$\|u_n\|^2_{H^1(\mathbb{R}^N)} + \|u_n\|^p_{W^{1,p}(\mathbb{R}^N)} \leqslant C\|u\|^2_{L^2(\mathbb{R}^N)}, \tag{3.10}$$

which implies that $\{u_n\}$ is bounded in E. Then, there exists $\tilde{u} \in E$ such that $u_n \rightharpoonup \tilde{u}$ in E and $u_n \to \tilde{u}$ almost everywhere in $\mathbb{R}^N$. By $H_n \in L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)$, it is easy to verify $H_n u_n$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Hence, we get $H_n u_n \rightharpoonup H\tilde{u}$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Moreover, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, by $|G_n| \leqslant |G|$ and the Lebesgue dominated convergence theorem, we can deduce $G_n\varphi \to G\varphi$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Then, we have

$$\int_{\mathbb{R}^N}[I_\alpha * (H_n u_n)]G_n\varphi \mathrm{d}x \to \int_{\mathbb{R}^N}[I_\alpha * (H\tilde{u})]G\varphi \mathrm{d}x, \ \ \forall\, \varphi \in C_0^\infty(\mathbb{R}^N).$$

Thus, $\tilde{u}$ is a weak solution of

$$-\Delta_p\tilde{u} - \Delta\tilde{u} + \tau\tilde{u} + |\tilde{u}|^{p-2}\tilde{u} = [I_\alpha * (H\tilde{u})]G + (\tau-1)u. \tag{3.11}$$

By Proposition 3.1, we know that Eq (3.11) admits a unique solution. Then, $u = \tilde{u}$.

For $\theta > 0$, we define the truncation $u_{n,\theta} : \mathbb{R}^N \to \mathbb{R}$ by

$$u_{n,\theta}(x) = \begin{cases} -\theta, & u_n \leqslant -\theta,\\ u_n, & -\theta < u_n < \theta,\\ \theta, & u_n \geqslant \theta.\end{cases}$$

For any $q > 2$, it is easy to check $|u_{n,\theta}|^{q-2}u_{n,\theta} \in E$. Taking $|u_{n,\theta}|^{q-2}u_{n,\theta} \in E$ as a test function in Eq (3.8), we can see that

$$\begin{aligned}&\int_{\mathbb{R}^N}\nabla u_n\nabla\left(|u_{n,\theta}|^{q-2}u_{n,\theta}\right)\mathrm{d}x + \tau\int_{\mathbb{R}^N}\left||u_{n,\theta}|^{\frac{q}{2}}\right|^2\mathrm{d}x\\ \leqslant&\int_{\mathbb{R}^N}\nabla u_n\nabla\left(|u_{n,\theta}|^{q-2}u_{n,\theta}\right)\mathrm{d}x + \int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\left(|u_{n,\theta}|^{q-2}u_{n,\theta}\right)\mathrm{d}x\\ &+\tau\int_{\mathbb{R}^N}|u_{n,\theta}|^{q-2}u_{n,\theta}u_n\mathrm{d}x + \int_{\mathbb{R}^N}|u_{n,\theta}|^{q-2}u_{n,\theta}|u_n|^{p-2}u_n\mathrm{d}x\\ =&\int_{\mathbb{R}^N}[I_\alpha * (H_n u_n)](G_n|u_{n,\theta}|^{q-2}u_{n,\theta})\mathrm{d}x + (\tau-1)\int_{\mathbb{R}^N}|u_{n,\theta}|^{q-2}u_{n,\theta}u_n\mathrm{d}x.\end{aligned}$$

Applying Lemma 3.2 with $\theta = \frac{2}{q}$, where $q \in [2, \frac{2N}{\alpha})$, there then exists $C > 0$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N} [I_\alpha * |H_n u_{n,\theta}|](|G_n||u_{n,\theta}|^{q-2}u_{n,\theta})\mathrm{d}x\\ \leqslant&\int_{\mathbb{R}^N} [I_\alpha * (|H||u_{n,\theta}|)](|G||u_{n,\theta}|^{q-1})\mathrm{d}x\\ \leqslant&\frac{2(q-1)}{q^2}\int_{\mathbb{R}^N}\left|\nabla\left(|u_n|^{\frac{q}{2}}\right)\right|^2\mathrm{d}x + C\int_{\mathbb{R}^N}\left||u_{n,\theta}|^{\frac{q}{2}}\right|^2\mathrm{d}x.\end{aligned}$$

Taking this together with the above two chain of inequalities and making use of the Hölder inequality and the Young inequality, we can infer

$$\begin{aligned}\int_{\mathbb{R}^N}\left|\nabla\left(|u_n|^{\frac{q}{2}}\right)\right|^2\mathrm{d}x \leqslant& C\int_{\mathbb{R}^N}(|u_n|^q + |u|^q)\,\mathrm{d}x\\ &+ C\int_{\{|u_n|>\theta\}}[I_\alpha * (|H_n u_n|)](|G_n||u_n|^{q-1})\mathrm{d}x.\end{aligned} \tag{3.12}$$

By $q \in [2, \frac{2N}{\alpha})$ and Proposition 2.1, then

$$\int_{\{|u_n|>\theta\}}[I_\alpha * (|H_n u_n|)](|G_n||u_n|^{q-1})\mathrm{d}x \leqslant C\left(\int_{\mathbb{R}^N}|H_n u_n|^s\mathrm{d}x\right)^{\frac{1}{s}}\left(\int_{\mathbb{R}^N}\left||G_n||u_n|^{q-1}\right|^t\mathrm{d}x\right)^{\frac{1}{t}},$$

with $\frac{1}{s} = \frac{N+\alpha}{2N} - \frac{1}{2} + \frac{1}{q}$ and $\frac{1}{t} = \frac{N+\alpha}{2N} + \frac{1}{2} - \frac{1}{q}$.

Using the fact that $u_n \in L^q(\mathbb{R}^N)$ and $H_n, G_n \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$, we get $|H_n u_n| \in L^s(\mathbb{R}^N)$ and $|G_n||u_n|^{q-1} \in L^t(\mathbb{R}^N)$. By applying the Lebesgue dominated convergence theorem, we have

$$\lim_{\theta\to\infty}\int_{\{|u_n|>\theta\}}[I_\alpha * (|H_n u_n|)](|G_n||u_n|^{q-1})\mathrm{d}x = 0.$$

Inserting this into (3.12) and taking $\theta \to \infty$, by the Sobolev embedding theorem we can deduce

$$\left(\int_{\mathbb{R}^N}|u_n|^{\frac{qN}{N-2}}\mathrm{d}x\right)^{\frac{N-2}{N}} \leqslant C\int_{\mathbb{R}^N}(|u_n|^q + |u|^q)\mathrm{d}x. \tag{3.13}$$

Taking into account (3.10), (3.13), and the Fatou lemma, we get that

$$\left(\int_{\mathbb{R}^N}|u|^{\frac{qN}{N-2}}\mathrm{d}x\right)^{\frac{N-2}{N}} \leqslant C\int_{\mathbb{R}^N}|u|^q\mathrm{d}x, \tag{3.14}$$

which means that $u \in L^q(\mathbb{R}^N)$ for any $q \in \left[2, \frac{2^*N}{\alpha}\right]$. The proof is complete. □

Lemma 3.5. ([19]) *Let* $1 \leqslant s \leqslant \infty$, $g \in L^{t_1}(\mathbb{R}^N)$, *and* $h \in L^{t_2}(\mathbb{R}^N)$. *Then, there exists* $C > 0$ *such that*

$$\|g * h\|_{L^s(\mathbb{R}^N)} \leqslant C\|g\|_{L^{t_1}(\mathbb{R}^N)}\|h\|_{L^{t_2}(\mathbb{R}^N)},$$

where

$$\frac{1}{t_1} + \frac{1}{t_2} = 1 + \frac{1}{s}.$$

Lemma 3.6. *Suppose that all the conditions described in Theorem 1.1 are satisfied. Let $u \in E$ be a nontrivial solution of Eq* (1.1)*. Then,*

$$\|I_\alpha * F(u)\|_{L^\infty(\mathbb{R}^N)} \leqslant C$$

Proof. In view of Lemma 3.4, we obtain $u \in L^q(\mathbb{R}^N)$ for every $q \in \left[2, \frac{2^*N}{\alpha}\right]$. By condition (F_1), one can infer $F(u) \in L^{\tilde{p}}(\mathbb{R}^N)$ for every $\tilde{p} \in \left[\frac{2N}{N+\alpha}, \frac{2N^2}{\alpha(N+\alpha)}\right]$.

Fixing $\epsilon \in \left(0, \frac{\alpha N}{2N+\alpha}\right)$, I_α can be decomposed as

$$I_\alpha = I_\alpha^1 + I_\alpha^2,$$

where $I_\alpha^1 \in L^{\frac{N-\epsilon}{N-\alpha}}(\mathbb{R}^N)$ and $I_\alpha^2 \in L^{\frac{N+\epsilon}{N-\alpha}}(\mathbb{R}^N)$. Let $s = \infty$ in Lemma 3.5. It follows from $I_\alpha^1 \in L^{\frac{N-\epsilon}{N-\alpha}}(\mathbb{R}^N)$ that

$$\|I_\alpha^1 * F(u)\|_{L^\infty(\mathbb{R}^N)} \leqslant C\|I_\alpha^1\|_{L^{\frac{N-\epsilon}{N-\alpha}}(\mathbb{R}^N)}\|F(u)\|_{L^{\frac{N-\epsilon}{\alpha-\epsilon}}(\mathbb{R}^N)}. \tag{3.15}$$

Similar to (3.15), by $I_\alpha^2 \in L^{\frac{N+\epsilon}{N-\alpha}}(\mathbb{R}^N)$ we can infer that

$$\|I_\alpha^2 * F(u)\|_{L^\infty(\mathbb{R}^N)} \leqslant C\|I_\alpha^2\|_{L^{\frac{N+\epsilon}{N-\alpha}}(\mathbb{R}^N)}\|F(u)\|_{L^{\frac{N+\epsilon}{\alpha+\epsilon}}(\mathbb{R}^N)}. \tag{3.16}$$

In view of $\epsilon \in \left(0, \frac{\alpha N}{2N+\alpha}\right)$, we derive

$$\frac{2N}{N+\alpha} < \frac{N+\epsilon}{\alpha+\epsilon} < \frac{N-\epsilon}{\alpha-\epsilon} < \frac{2N^2}{\alpha(N+\alpha)}. \tag{3.17}$$

It follows from (3.15)–(3.17) that

$$I_\alpha^1 * F(u) \in L^\infty(\mathbb{R}^N) \text{ and } I_\alpha^2 * F(u) \in L^\infty(\mathbb{R}^N).$$

The proof is completed. □

Lemma 3.7. *Suppose that all the conditions described in Theorem 1.1 are satisfied. Let $u \in E$ be a nontrivial solution of Eq* (1.1)*. For each $L > 2$, define*

$$u_L(x) = \begin{cases} -L, & u(x) < -L; \\ u(x), & |u(x)| \leqslant L; \\ L, & u(x) > L. \end{cases}$$

For $\tau > 1$, we set $\tilde{u}_L = u u_L^{2(\tau-1)}$. Then, for any $s \in [2, 2^*]$, we have

$$\left(\int_{\mathbb{R}^N} |u u_L^{\tau-1}|^s \mathrm{d}x\right)^{\frac{2}{s}} \leqslant C\tau^2 \left(\int_{\mathbb{R}^N} u^{2_\alpha^\sharp - 2}|u u_L^{\tau-1}|^2 \mathrm{d}x + \int_{\mathbb{R}^N} u^{2_\alpha^* - 2}|u u_L^{\tau-1}|^2 \mathrm{d}x\right).$$

Proof. Multiplying both sides of Eq (1.1) by $\tilde{u}_L$ and integrating, it follows that

$$\int_{\mathbb{R}^N} \nabla u \nabla \tilde{u}_L \mathrm{d}x + \int_{\mathbb{R}^N} u\tilde{u}_L \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla \tilde{u}_L \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{p-2} u \tilde{u}_L \mathrm{d}x$$
$$= \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u) \tilde{u}_L \mathrm{d}x.$$

Combining the above relation with Lemma 3.6, this leads to

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|uu_L^{\tau-1}|^s\mathrm{d}x\right)^{\frac{2}{s}}\leqslant&C\left[\int_{\mathbb{R}^N}\left|\nabla(uu_L^{\tau-1})\right|^2\mathrm{d}x+\int_{\mathbb{R}^N}u\tilde{u}_L\mathrm{d}x\right]\\ \leqslant&C\tau^2\left(\int_{\mathbb{R}^N}u^{2_\alpha^\sharp-2}|uu_L^{\tau-1}|^2\mathrm{d}x+\int_{\mathbb{R}^N}u^{2_\alpha^*-2}|uu_L^{\tau-1}|^2\mathrm{d}x\right).\end{aligned}$$

The proof is completed. □

We now are ready to establish Theorem 1.1.

Proof of Theorem 1.1. **(i) (L^∞ estimate)** We consider the following two cases separately.

Case 1. $2_\alpha^*\leqslant 2\Leftrightarrow N\geqslant 4+\alpha$.

In this case, we should keep in mind that $\tau=\frac{2^*}{2}$.

Step 1. Clearly, we have

$$2<2_\alpha^\sharp+2(\tau-1)<2_\alpha^*+2(\tau-1)<\frac{2^*N}{\alpha}.$$

In view of Lemma 3.4, we have $u\in L^q(\mathbb{R}^N)$ for any $q\in\left[2,\frac{2^*N}{\alpha}\right]$. That is,

$$\int_{\mathbb{R}^N}u^{2_\alpha^\sharp-2}|uu_L^{\tau-1}|^2\mathrm{d}x<\infty\ \text{ and }\ \int_{\mathbb{R}^N}u^{2_\alpha^*-2}|uu_L^{\tau-1}|^2\mathrm{d}x<\infty.$$

For any $0<R<\infty$, we set

$$\begin{aligned}B_\tau=&\int_{\mathbb{R}^N}u^{2_\alpha^\sharp-2}|uu_L^{\tau-1}|^2\mathrm{d}x\\ =&\int_{\{u\leqslant R\}}u^{2_\alpha^\sharp-2}|uu_L^{\tau-1}|^2\mathrm{d}x+\int_{\{u>R\}}u^{2_\alpha^\sharp-2}|uu_L^{\tau-1}|^2\mathrm{d}x\\ =&B_\tau(R)+B_\tau^c(R)\end{aligned}$$

and

$$\begin{aligned}\tilde{B}_\tau=&\int_{\mathbb{R}^N}u^{2_\alpha^*-2}|uu_L^{\tau-1}|^2\mathrm{d}x\\ =&\int_{\{u\leqslant R\}}u^{2_\alpha^*-2}|uu_L^{\tau-1}|^2\mathrm{d}x+\int_{\{u>R\}}u^{2_\alpha^*-2}|uu_L^{\tau-1}|^2\mathrm{d}x\\ =&\tilde{B}_\tau(R)+\tilde{B}_\tau^c(R).\end{aligned}$$

Obviously, we have

$$\lim_{R\to\infty}B_\tau(R)=B_\tau,\ \ \lim_{R\to0}B_\tau(R)=0$$

and

$$\lim_{R\to\infty}\tilde{B}_\tau(R)=\tilde{B}_\tau,\ \ \lim_{R\to0}\tilde{B}_\tau(R)=0.$$

Clearly, if it holds that $B_\tau=B_\tau(R)$ or $\tilde{B}_\tau=\tilde{B}_\tau(R)$, then we have $u\in L^\infty(\mathbb{R}^N)$. This completes our proof. To this end, we just need to consider the following case

$$B_\tau(R)<B_\tau\ \text{ and }\ \tilde{B}_\tau(R)<\tilde{B}_\tau.$$

Without loss of generality, we set $R = 1$. Then, there exist $0 < C_1, \tilde{C}_1 < \infty$ such that

$$B_\tau(1) = C_1 B_\tau \ \text{ and } \ \tilde{B}_\tau(1) = \tilde{C}_1 \tilde{B}_\tau. \tag{3.18}$$

From $2^\sharp_\alpha < 2$, we deduce

$$B^c_\tau(1) = \int_{\{u>1\}} u^{2^\sharp_\alpha - 2} |u u_L^{\tau-1}|^2 \mathrm{d}x \leqslant \int_{\{u>1\}} |u u_L^{\tau-1}|^2 \mathrm{d}x.$$

It follows from (3.18) that

$$B_\tau = B_\tau(1) + B^c_\tau(1) = \frac{1}{1 - C_1} B^c_\tau(1) \leqslant \frac{1}{1 - C_1} \int_{\{u>1\}} |u u_L^{\tau-1}|^2 \mathrm{d}x. \tag{3.19}$$

Similarly, one can infer

$$\tilde{B}_\tau = \tilde{B}_\tau(1) + \tilde{B}^c_\tau(1) = \frac{1}{1 - \tilde{C}_1} \tilde{B}^c_\tau(1) \leqslant \frac{1}{1 - \tilde{C}_1} \int_{\{u>1\}} |u u_L^{\tau-1}|^2 \mathrm{d}x. \tag{3.20}$$

Combining (3.19), (3.20), and Lemma 3.7, we obtain

$$\left(\int_{\mathbb{R}^N} |u u_L^{\tau-1}|^s \mathrm{d}x \right)^{\frac{2}{s}} \leqslant \left(\frac{C}{1 - C_1} + \frac{C}{1 - \tilde{C}_1} \right) \int_{\mathbb{R}^N} |u u_L^{\tau-1}|^2 \mathrm{d}x.$$

Let $L \to \infty$ in the above expression. Then,

$$\left(\int_{\mathbb{R}^N} |u|^{s\tau} \mathrm{d}x \right)^{\frac{2}{s}} \leqslant \left(\frac{C}{1 - C_1} + \frac{C}{1 - \tilde{C}_1} \right) \int_{\mathbb{R}^N} |u|^{2\tau} \mathrm{d}x. \tag{3.21}$$

By $\tau = \frac{2^*}{2}$, we have

$$\|u\|_{L^{\frac{2^* s}{2}}(\mathbb{R}^N)} \leqslant \left(\frac{C}{1 - C_1} + \frac{C}{1 - \tilde{C}_1} \right)^{\frac{1}{2^*}} \|u\|_{L^{2^*}(\mathbb{R}^N)} < \infty.$$

Since $s \in [2, 2^*]$, we get $u \in L^{p_1}(\mathbb{R}^N)$, where $p_1 \in [2, \frac{(2^*)^2}{2}]$.

Step 2. Obviously, we have

$$2 < 2^\sharp_\alpha + 2(\tau^2 - 1) < 2^*_\alpha + 2(\tau^2 - 1) < \frac{(2^*)^2}{2}$$

and for B_τ and $\tilde{B}_\tau$, we have

$$B_{\tau^2} < \infty \ \text{ and } \ \tilde{B}_{\tau^2} < \infty.$$

Similar to Step 1, we just need to show the case

$$B_{\tau^2}(1) < B_{\tau^2} \ \text{ and } \ \tilde{B}_{\tau^2}(1) < \tilde{B}_{\tau^2}.$$

Moreover, we have

$$B_\tau(1) = \int_{\{u\leqslant1\}} u^{2^\sharp_\alpha - 2} |u u_L^{\tau-1}|^2 \mathrm{d}x \geqslant \int_{\{u\leqslant1\}} u^{2^\sharp_\alpha - 2} \left| u u_L^{\tau-1} (u_L^{\tau^2 - \tau}) \right|^2 \mathrm{d}x = B_{\tau^2}(1) \tag{3.22}$$

and

$$B_\tau^c(1) = \int_{\{u>1\}} u^{2_\alpha^\sharp - 2}|uu_L^{\tau-1}|^2 \mathrm{d}x \leqslant \int_{\{u>1\}} u^{2_\alpha^\sharp - 2}\left|uu_L^{\tau-1}(u_L^{\tau^2-\tau})\right|^2 \mathrm{d}x = B_{\tau^2}^c(1). \tag{3.23}$$

In view of (3.19), (3.20), (3.22), and (3.23), it follows that

$$B_{\tau^2}(1) \leqslant B_\tau(1) = \frac{C_1}{1-C_1} B_\tau^c(1) \leqslant \frac{C_1}{1-C_1} B_{\tau^2}^c(1),$$

which implies

$$B_{\tau^2} \leqslant \frac{1}{1-C_1}\int_{\{u>1\}} \left|uu_L^{\tau^2-1}\right|^2 \mathrm{d}x.$$

Similarly, we have

$$\tilde{B}_{\tau^2} \leqslant \frac{1}{1-\tilde{C}_1}\int_{\{u>1\}} \left|uu_L^{\tau^2-1}\right|^2 \mathrm{d}x.$$

Taking $L \to \infty$ and making use of Lemma 3.7 again, we get

$$\|u\|_{L^{s\cdot\left(\frac{2^*}{2}\right)^2}(\mathbb{R}^N)} \leqslant \left(\frac{C}{1-C_1} + \frac{C}{1-\tilde{C}_1}\right)^{\frac{1}{2\cdot\left(\frac{2^*}{2}\right)^2}} \|u\|_{L^{\frac{(2^*)^2}{2}}(\mathbb{R}^N)} < \infty,$$

which implies $u \in L^{p_2}(\mathbb{R}^N)$, where $p_2 \in \left[2, 2_s^* \cdot \left(\frac{2_s^*}{2}\right)^2\right]$.

Step 3. Iterating the above procedure, for any $n \in \mathbb{N}^*$ we conclude

$$\|u\|_{L^{s\cdot\left(\frac{2^*}{2}\right)^n}(\mathbb{R}^N)} \leqslant \left(\frac{C}{1-C_1} + \frac{C}{1-\tilde{C}_1}\right)^{\frac{1}{2\cdot\left(\frac{2^*}{2}\right)^n}} \|u\|_{L^{\frac{(2^*)^n}{2^{n-1}}}(\mathbb{R}^N)}.$$

Let $s = 2^*$. Then,

$$\|u\|_{L^{\frac{(2^*)^{n+1}}{2^n}}(\mathbb{R}^N)} \leqslant \left(\frac{C}{1-C_1} + \frac{C}{1-\tilde{C}_1}\right)^{\sum_{i=1}^{n}\frac{1}{2\cdot\left(\frac{2^*}{2}\right)^i}} \|u\|_{L^{2^*}(\mathbb{R}^N)}. \tag{3.24}$$

Obviously, we have

$$\lim_{i\to\infty} \frac{2\cdot\left(\frac{2^*}{2}\right)^i}{2\cdot\left(\frac{2^*}{2}\right)^{i+1}} = \frac{2}{2^*} < 1.$$

This means that the series $\sum_{i=1}^{n} \frac{1}{2\cdot\left(\frac{2^*}{2}\right)^i}$ converges absolutely.

Let $n \to \infty$ in (3.24). Then, it holds that

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant C\|u\|_{L^{2^*}(\mathbb{R}^N)} < \infty.$$

Case 2. $2_\alpha^* > 2 \Leftrightarrow N < 4+\alpha$**.**

Step 1. Let $\tau_1 \in \left[1 + \frac{2-2_\alpha^\sharp}{2}, 1 + \frac{\frac{2^* N}{\alpha} - 2_\alpha^\sharp}{2}\right]$. Then, we claim

$$\left(1 + \int_{\mathbb{R}^N} |u|^{2_\alpha^* \tau_1} \mathrm{d}x\right)^{\frac{2}{2_\alpha^*(\tau_1 - 1)}} < \infty. \tag{3.25}$$

By the definition of u_L, we obtain

$$\int_{\mathbb{R}^N} |u|^{2^\sharp_\alpha-2}|uu_L^{\tau_1-1}|^2\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |u|^{2^\sharp_\alpha+2(\tau_1-1)}\mathrm{d}x.$$

Let $l > 0$ be chosen later. By the Hölder inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} u^{2^*_\alpha-2}|uu_L^{\tau_1-1}|^2\mathrm{d}x\\ \leqslant& l^{2^*_\alpha-2^\sharp_\alpha}\int_{\{u\leqslant l\}} u^{2^\sharp_\alpha-2}|uu_L^{\tau_1-1}|^2\mathrm{d}x + \int_{\{u>l\}} u^{2^*_\alpha-2}|uu_L^{\tau_1-1}|^2\mathrm{d}x\\ \leqslant& l^{2^*_\alpha-2^\sharp_\alpha}\int_{\mathbb{R}^N} |u|^{2^\sharp_\alpha+2(\tau_1-1)}\mathrm{d}x + \left(\int_{\{u>l\}} |u|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{2^*_\alpha-2}{2^*_\alpha}}\left(\int_{\mathbb{R}^N} |uu_L^{\tau_1-1}|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}}.\end{aligned}$$

By $2^*_\alpha \in \left[2, \frac{2^*N}{\alpha}\right]$, we can choose suitable $l > 0$ such that

$$\left(\int_{\{u>l\}} |u|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{2^*_\alpha-2}{2^*_\alpha}} \leqslant \frac{1}{2C\tau_1^2}.$$

It follows from the inequalities and Lemma 3.7 that

$$\left(\int_{\mathbb{R}^N} |uu_L^{\tau_1-1}|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}} \leqslant 2C\tau_1^2\left(\int_{\mathbb{R}^N} u^{2^\sharp_\alpha-2}|uu_L^{\tau_1-1}|^2\mathrm{d}x + l^{2^*_\alpha-2^\sharp_\alpha}\int_{\mathbb{R}^N} |u|^{2^\sharp_\alpha+2(\tau_1-1)}\mathrm{d}x\right).$$

Let $L \to \infty$. The above inequality becomes

$$\left(\int_{\mathbb{R}^N} |u|^{2^*_\alpha\tau_1}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}} \leqslant 2C\tau_1^2\left(1 + l^{2^*_\alpha-2^\sharp_\alpha}\right)\int_{\mathbb{R}^N} |u|^{2^\sharp_\alpha+2(\tau_1-1)}\mathrm{d}x. \tag{3.26}$$

In view of $2^\sharp_\alpha + 2(\tau_1 - 1) \in \left[2, \frac{2^*N}{\alpha}\right]$ and (3.26), we conclude (3.25).

Step 2. Let $\tau_2 = 1 + \frac{2^*_\alpha}{2}(\tau_1 - 1)$. We claim

$$\left(1 + \int_{\mathbb{R}^N} |u|^{2^*_\alpha\tau_2}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha(\tau_2-1)}} \leqslant (C\tau_2)^{\frac{2}{\tau_2-1}}\left(1 + \int_{\mathbb{R}^N} |u|^{2^*_\alpha\tau_1}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha(\tau_1-1)}}.$$

We choose $\tau \in [\tau_1, \tau_2]$. Then,

$$2 \leqslant 2^\sharp_\alpha + 2(\tau - 1) < 2^*_\alpha + 2(\tau - 1) \leqslant 2^*_\alpha\tau_1.$$

Combining (3.25) and Lemma 3.7, we obtain

$$\left(\int_{\mathbb{R}^N} |u|^{2^*_\alpha\tau}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}} \leqslant C\tau^2\left(\int_{\mathbb{R}^N} |u|^{2^\sharp_\alpha+2(\tau-1)}\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{2^*_\alpha+2(\tau-1)}\mathrm{d}x\right) < \infty. \tag{3.27}$$

Let $\tau = \tau_2$ in (3.27). Then,

$$\left(\int_{\mathbb{R}^N} |u|^{2^*_\alpha\tau_2}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}} \leqslant C\tau_2^2\left(\int_{\mathbb{R}^N} |u|^{2^\sharp_\alpha+2(\tau_2-1)}\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{2^*_\alpha+2(\tau_2-1)}\mathrm{d}x\right) < \infty.$$

Making use of the Young inequality, it holds that

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^{2^\sharp_\alpha+2(\tau_2-1)}\mathrm{d}x=&\int_{\mathbb{R}^N}|u|^a|u|^b\mathrm{d}x\\ \leqslant&\frac{a}{2^*_\alpha}\int_{\mathbb{R}^N}|u|^{2^*_\alpha}\mathrm{d}x+\frac{2^*_\alpha-a}{2^*_\alpha}\int_{\mathbb{R}^N}|u|^{2^*_\alpha+2(\tau_2-1)}\mathrm{d}x\\ \leqslant&C\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha+2(\tau_2-1)}\mathrm{d}x\right),\end{aligned}$$

where $a=\frac{2^*_\alpha\left(2^*_\alpha-2^\sharp_\alpha\right)}{2(\tau_2-1)}$ and $b=2^\sharp_\alpha+2(\tau_2-1)-\frac{2^*_\alpha\left(2^*_\alpha-2^\sharp_\alpha\right)}{2(\tau_2-1)}$. Thus, we get

$$\left(\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_2}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}}\leqslant C\tau_2^2\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha+2(\tau_2-1)}\mathrm{d}x\right).$$

Moreover, by $\frac{2}{2^*_\alpha}<1$, it is easy to observe

$$(x_1+x_2)^{\frac{2}{2^*_\alpha}}\leqslant x_1^{\frac{2}{2^*_\alpha}}+x_2^{\frac{2}{2^*_\alpha}},\quad \forall x_1,x_2>0.$$

Then,

$$\begin{aligned}\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_2}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}}\leqslant&1+\left(\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_2}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}}\\ \leqslant&C\tau_2^2\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha+2(\tau_2-1)}\mathrm{d}x\right),\end{aligned}$$

which implies

$$\begin{aligned}\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_2}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha(\tau_2-1)}}\leqslant&(C\tau_2)^{\frac{2}{\tau_2-1}}\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_1}\mathrm{d}x\right)^{\frac{1}{\tau_2-1}}\\ =&(C\tau_2)^{\frac{2}{\tau_2-1}}\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_1}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha(\tau_1-1)}}.\end{aligned}$$

Step 3. We iterate the above procedure and set

$$\tau_{i+1}-1=\frac{2^*_\alpha}{2}(\tau_i-1),\quad \forall i\geqslant 1\ \text{ and }\ i\in\mathbb{N}^*. \tag{3.28}$$

Then,

$$\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_{i+1}}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha(\tau_{i+1}-1)}}\leqslant(C\tau_{i+1})^{\frac{2}{\tau_{i+1}-1}}\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_i}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha(\tau_i-1)}},$$

which further gives

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_{n+1}}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha(\tau_{n+1}-1)}}\leqslant&\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_{n+1}}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha(\tau_{n+1}-1)}}\\ \leqslant&\prod_{i=1}^{n}(C\tau_{i+1})^{\frac{2}{\tau_{i+1}-1}}\left(1+\int_{\mathbb{R}^N}|u|^{2^*_\alpha\tau_1}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha(\tau_1-1)}}.\end{aligned}$$

This yields that

$$\|u\|_{L^{2^*_\alpha \tau_{n+1}}(\mathbb{R}^N)} \leqslant \left[\prod_{i=1}^{n} (C\tau_{i+1})^{\frac{2}{\tau_{i+1}-1}} \left(1 + \int_{\mathbb{R}^N} |u|^{2^*_\alpha \tau_1} \mathrm{d}x \right)^{\frac{2}{2^*_\alpha(\tau_1-1)}} \right]^{\frac{\tau_{n+1}-1}{2\tau_{n+1}}}. \tag{3.29}$$

According to (3.28), we deduce

$$\tau_{n+1} = 1 + \left(\frac{2^*_\alpha}{2} \right)^n (\tau_1 - 1). \tag{3.30}$$

Taking into account (3.29) and (3.30), it follows that

$$\|u\|_{L^{2^*_\alpha \tau_{n+1}}(\mathbb{R}^N)} \leqslant \left[\prod_{i=1}^{n} (C\tau_{i+1})^{\frac{2}{\tau_{i+1}-1}} \left(1 + \int_{\mathbb{R}^N} |u|^{2^*_\alpha \tau_1} \mathrm{d}x \right)^{\frac{2}{2^*_\alpha(\tau_1-1)}} \right]^{\frac{(2^*_\alpha)^n(\tau_1-1)}{2[2^n+(2^*_\alpha)^n(\tau_1-1)]}}. \tag{3.31}$$

By a straightforward calculation, we can infer

$$\lim_{n\to\infty} \prod_{i=1}^{n} (C\tau_{i+1})^{\frac{2}{\tau_{i+1}-1}} = \lim_{n\to\infty} e^{2\sum_{i=1}^{n}\left(\frac{\ln C}{\tau_{i+1}-1} + \frac{\ln \tau_{i+1}}{\tau_{i+1}-1}\right)}. \tag{3.32}$$

For the series $\sum_{i=1}^{\infty} \frac{\ln C}{\tau_{i+1}-1}$, we have

$$\lim_{i\to\infty} \sqrt[i]{\frac{\ln C}{\tau_{i+1}-1}} = \lim_{i\to\infty} \sqrt[i]{\frac{2^i \ln C}{(2^*_\alpha)^i(\tau_1-1)}} = \frac{2}{2^*_\alpha} < 1. \tag{3.33}$$

This means $\sum_{i=1}^{\infty} \frac{\ln C}{\tau_{i+1}-1}$ converges absolutely.

For the series $\sum_{i=1}^{\infty} \frac{\ln \tau_{i+1}}{\tau_{i+1}-1}$, it follows that

$$\begin{aligned} \lim_{i\to\infty} \frac{\ln \tau_{i+2}}{\tau_{i+2}-1} \cdot \frac{\tau_{i+1}-1}{\ln \tau_{i+1}} =& \frac{2}{2^*_\alpha} \lim_{i\to\infty} \frac{\ln\left[1 + \frac{2^*_\alpha}{2}(\tau_{i+1}-1)\right]}{\ln \tau_{i+1}} \\ \leqslant& \frac{2}{2^*_\alpha} \lim_{i\to\infty} \frac{\ln\left[\frac{2^*_\alpha}{2} + \frac{2^*_\alpha}{2}(\tau_{i+1}-1)\right]}{\ln \tau_{i+1}} \\ =& \frac{2}{2^*_\alpha} \lim_{i\to\infty} \left(\frac{\ln \frac{2^*_\alpha}{2}}{\ln \tau_{i+1}} + \frac{\ln \tau_{i+1}}{\ln \tau_{i+1}} \right) \\ <& 1, \end{aligned} \tag{3.34}$$

which implies $\sum_{i=1}^{\infty} \frac{\ln \tau_{i+1}}{\tau_{i+1}-1}$ converges absolutely.

Together with (3.32)–(3.34), we conclude $\prod_{i=1}^{\infty} (C\tau_{i+1})^{\frac{2}{\tau_{i+1}-1}} < \infty$. Letting $n \to \infty$ in (3.31), we obtain

$$\|u\|_{L^\infty(\mathbb{R}^N)} < \infty.$$

(ii) (Pohožaev indentity) Observe that, by $u \in L^\infty(\mathbb{R}^N)$, Lemma 3.6, and condition (F_1), there exists $C > 0$ such that

$$-\Delta_p u - \Delta u = -u - |u|^{p-2}u + (I_\alpha * F(u))f(u) \leqslant C(|u|^{2^\sharp_\alpha - 2}u + |u|^{2^*_\alpha - 2}u).$$

Set $l(u) = C(|u|^{2^\sharp_\alpha-2}u + |u|^{2^*_\alpha-2}u)$. By a classical bootstrapping argument for subcritical local problems in [33], we infer that $u \in W^{2,q}_{loc}(\mathbb{R}^N)$ for every $q \geqslant 1$, and hence we have $u \in C^{1,\beta}_{loc}(\mathbb{R}^N)$ for any $0 < \beta < 1$ by the Sobolev embedding theorem. Under the classical strategy used in [25, Theorem 3], one can show that

$$\begin{aligned}
&\frac{N-2}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{N}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{N-p}{p}\|u\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{N}{p}\|u\|^p_{L^p(\mathbb{R}^N)} \\
=&\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x.
\end{aligned}$$

This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by virtue of the Pohožaev manifold method and a generalized version of a Lions-type theorem.

Under condition (F_2), Eq (1.1) turns into a $(p, 2)$-Laplacian equation as follows:

$$-\Delta_p u - \Delta u + u + |u|^{p-2}u = \left[I_\alpha * \left(\frac{1}{2^\sharp_\alpha}|u|^{2^\sharp_\alpha} + \frac{\lambda}{2^*_\alpha}|u|^{2^*_\alpha}\right)\right](|u|^{2^\sharp_\alpha-2}u + \lambda|u|^{2^*_\alpha-2}u), \quad x \in \mathbb{R}^N. \qquad (\mathcal{D})$$

Then, the corresponding energy functional of Eq $(\mathcal{D})$ can be defined as

$$\begin{aligned}
J(u) =&\frac{1}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{p}\|u\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{1}{p}\|u\|^p_{L^p(\mathbb{R}^N)} \\
&- \frac{1}{2\cdot(2^\sharp_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^\sharp_\alpha})|u|^{2^\sharp_\alpha}\mathrm{d}x - \frac{\lambda^2}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha}\mathrm{d}x \\
&- \frac{\lambda}{2^\sharp_\alpha\cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^\sharp_\alpha})|u|^{2^*_\alpha}\mathrm{d}x.
\end{aligned}$$

It is easy to check $J \in C^1(E, \mathbb{R})$. Obviously, the critical points of J are weak solutions of Eq $(\mathcal{D})$ and satisfy the following Pohožaev identity:

$$\begin{aligned}
P(u) =&\frac{N-2}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{N}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{N-p}{p}\|u\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{N}{p}\|u\|^p_{L^p(\mathbb{R}^N)} \\
&- \frac{N+\alpha}{2\cdot(2^\sharp_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^\sharp_\alpha})|u|^{2^\sharp_\alpha}\mathrm{d}x - \frac{\lambda^2(N+\alpha)}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha}\mathrm{d}x \\
&- \frac{\lambda(N+\alpha)}{2^\sharp_\alpha\cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^\sharp_\alpha})|u|^{2^*_\alpha}\mathrm{d}x.
\end{aligned}$$

We define the Pohožaev manifold and its minimum as follows:

$$\mathcal{P} = \left\{u \in E\backslash\{0\}\middle|P(u) = 0\right\} \quad \text{and} \quad m = \inf_{u\in\mathcal{P}} J(u).$$

Lemma 4.1. *Assume that all conditions described in Theorem 1.2 are satisfied. Let $C_1, C_2, C_3 > 0$. Define a function $k : \mathbb{R}^+ \to \mathbb{R}$ as*

$$k(t) = C_1 t^{N-2} + C_2 t^N + C_3 t^{N-p} - C_4 t^{N+\alpha}.$$

Then, $k(t)$ has a unique critical point which corresponds to its maximum.

Proof. By the definition of $k(\cdot)$, we have

$$k'(t) = C_1(N-2)t^{N-3} + C_2 N t^{N-1} + C_3(N-p)t^{N-p-1} - C_4(N+\alpha)t^{N+\alpha-1}.$$

From the above expression, it is easy to see that $k'(t) > 0$ for $t > 0$ small, and $k'(t) < 0$ for $t > 0$ large. This yields that $k(t)$ possesses at least one maximum point. Next, we claim that the maximun point corresponding to $k(t)$ is unique. Otherwise, we suppose that there exists $t_1 \neq t_2 > 0$ such that

$$k'(t_1) = C_1(N-2)t_1^{N-3} + C_2 N t_1^{N-1} + C_3(N-p)t_1^{N-p-1} - C_4(N+\alpha)t_1^{N+\alpha-1} = 0$$

and

$$k'(t_2) = C_1(N-2)t_2^{N-3} + C_2 N t_2^{N-1} + C_3(N-p)t_2^{N-p-1} - C_4(N+\alpha)t_2^{N+\alpha-1} = 0.$$

Combining the above two equalities, it holds that

$$C_1(N-2)(t_1^{-2} - t_2^{-2}) + C_3(N-p)(t_1^{-p} - t_2^{-p}) = C_4(N+\alpha)(t_1^{\alpha} - t_2^{\alpha}),$$

which further gives $t_1 = t_2$. The proof is complete. □

Lemma 4.2. *Assume that all conditions described in Theorem 1.2 are satisfied. Then, for every $u \in E$, there exists a unique $t_u > 0$ such that $P(u_{t_u}) = 0$, where $u_t = u\left(\frac{x}{t}\right)$. Moreover, $J(u_{t_u}) = \max_{t \geqslant 0} J(u_t)$.*

Proof. For every $u \in E \backslash \{0\}$, one has

$$\begin{aligned} J(u_t) = & \frac{t^{N-2}}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{t^N}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{t^{N-p}}{p}\|u\|^2_{D^{1,p}(\mathbb{R}^N)} + \frac{t^N}{p}\|u\|^p_{L^p(\mathbb{R}^N)} \\ & - \frac{t^{N+\alpha}}{2\cdot(2^\sharp_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^\sharp_\alpha})|u|^{2^\sharp_\alpha}\mathrm{d}x - \frac{\lambda^2 t^{N+\alpha}}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha}\mathrm{d}x \\ & - \frac{\lambda t^{N+\alpha}}{2^\sharp_\alpha \cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^\sharp_\alpha})|u|^{2^*_\alpha}\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} P(u_t) = & \frac{(N-2)t^{N-2}}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{Nt^N}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{(N-p)t^{N-p}}{p}\|u\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{Nt^N}{p}\|u\|^p_{L^p(\mathbb{R}^N)} \\ & - \frac{(N+\alpha)t^{N+\alpha}}{2\cdot(2^\sharp_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^\sharp_\alpha})|u|^{2^\sharp_\alpha}\mathrm{d}x - \frac{\lambda^2(N+\alpha)t^{N+\alpha}}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha}\mathrm{d}x \\ & - \frac{\lambda(N+\alpha)t^{N+\alpha}}{2^\sharp_\alpha \cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^\sharp_\alpha})|u|^{2^*_\alpha}\mathrm{d}x. \end{aligned}$$

Combining the above two formulas, it is easy to see that $P(u_t) = tJ'(u_t) = 0$. By Lemma 4.1, we complete the proof. □

Lemma 4.3. *Suppose that all conditions described in Theorem 1.2 are satisfied. Then, $m > 0$.*

Proof. For every $u \in \mathcal{P}$, it follows from Proposition 2.1 that

$$\begin{aligned}
&\frac{N-2}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{N}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{N-p}{p}\|u\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{N}{p}\|u\|^p_{L^p(\mathbb{R}^N)} \\
=&\frac{N+\alpha}{2\cdot(2^\sharp_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^\sharp_\alpha})|u|^{2^\sharp_\alpha}\mathrm{d}x + \frac{\lambda^2(N+\alpha)}{2^\sharp_\alpha\cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^\sharp_\alpha})|u|^{2^*_\alpha}\mathrm{d}x \\
&+\frac{\lambda(N+\alpha)}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha}\mathrm{d}x \\
\leqslant& C\|u\|_E^{2\cdot 2^\sharp_\alpha} + C\|u\|_E^{2^\sharp_\alpha+2^*_\alpha} + C\|u\|_E^{2\cdot 2^*_\alpha},
\end{aligned}$$

which implies that $\|u\|_E \geqslant C$. Then, it holds that

$$\begin{aligned}
&J(u) - \frac{1}{N+\alpha}P(u) \\
=&\frac{\alpha+2}{2(N+\alpha)}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{\alpha}{2(N+\alpha)}\|u\|^2_{L^2(\mathbb{R}^N)} \\
&+\frac{\alpha+p}{p(N+\alpha)}\|u\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{\alpha}{p(N+\alpha)}\|u\|^p_{L^p(\mathbb{R}^N)} \\
\geqslant& C\|u\|_E^2 > 0.
\end{aligned} \tag{4.1}$$

The proof is complete. □

Lemma 4.4. *Assume that all conditions described in Theorem 1.2 are satisfied. Then, we have*

$$0 < m < m^* = \min\left\{\frac{\alpha}{2(N+\alpha)}\left[\frac{N\cdot(2^\sharp_\alpha)^2}{N+\alpha}\right]^{\frac{1}{2^\sharp_\alpha-1}}\mathcal{S}_1^{\frac{2^\sharp_\alpha}{2^\sharp_\alpha-1}}, \frac{\alpha+2}{2(N+\alpha)}\left[\frac{(2^*_\alpha)^2(N-2)}{\lambda^2(N+\alpha)}\right]^{\frac{1}{2^*_\alpha-1}}\mathcal{S}_2^{\frac{2^*_\alpha}{2^*_\alpha-1}}\right\}.$$

Proof. From $\lambda < \Lambda$, we can easily get

$$\frac{\alpha}{2(N+\alpha)}\left[\frac{N\cdot(2^\sharp_\alpha)^2}{N+\alpha}\right]^{\frac{1}{2^\sharp_\alpha-1}}\mathcal{S}_1^{\frac{2^\sharp_\alpha}{2^\sharp_\alpha-1}} < \frac{\alpha+2}{2(N+\alpha)}\left[\frac{(2^*_\alpha)^2(N-2)}{\lambda^2(N+\alpha)}\right]^{\frac{1}{2^*_\alpha-1}}\mathcal{S}_2^{\frac{2^*_\alpha}{2^*_\alpha-1}},$$

where Λ is defined in Threorem 1.2.

The extremal function of inequalities (2.1) can be defined as

$$\mu_\sigma = \frac{C\sigma^{\frac{N}{2}}}{(\sigma^2+|x|^2)^{\frac{N}{2}}}.$$

Let $t_\sigma > 0$ satisfy

$$J((\mu_\sigma)_{t_\sigma}) = \max_{t>0} J((\mu_\sigma)_t).$$

By the definition of m, it is easy to see that

$$0 < m < J((\mu_\sigma)_{t_\sigma}).$$

A straightforward calculation shows that

$$\|\mu_\sigma\|^2_{L^2(\mathbb{R}^N)} = \|\mu_1\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (I_\alpha * |\mu_1|^{2^\sharp_\alpha})|\mu_1|^{2^\sharp_\alpha} \mathrm{d}x = \int_{\mathbb{R}^N} (I_\alpha * |\mu_\sigma|^{2^\sharp_\alpha})|\mu_\sigma|^{2^\sharp_\alpha} \mathrm{d}x = \mathcal{S}_1^{\frac{2^\sharp_\alpha}{2^\sharp_\alpha - 1}}.$$

Moreover, we can compute

$$\int_{\mathbb{R}^N} |\nabla \mu_\sigma|^2 \mathrm{d}x = \sigma^{-2} \int_{\mathbb{R}^N} |\nabla \mu_1|^2 \mathrm{d}x, \quad \int_{\mathbb{R}^N} |\nabla \mu_\sigma|^p \mathrm{d}x = \sigma^{\frac{(2-p)N}{2} - p} \int_{\mathbb{R}^N} |\nabla \mu_1|^p \mathrm{d}x$$

and

$$\int_{\mathbb{R}^N} |\mu_\sigma|^p \mathrm{d}x = \sigma^{\frac{(2-p)N}{2}} \int_{\mathbb{R}^N} |\mu_1|^p \mathrm{d}x$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |\mu_\sigma|^{2^*_\alpha})|\mu_\sigma|^{2^*_\alpha} \mathrm{d}x = \sigma^{-2\cdot 2^*_\alpha} \int_{\mathbb{R}^N} (I_\alpha * |\mu_1|^{2^*_\alpha})|\mu_1|^{2^*_\alpha} \mathrm{d}x$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |\mu_\sigma|^{2^\sharp_\alpha})|\mu_\sigma|^{2^*_\alpha} \mathrm{d}x = \sigma^{-2^*_\alpha} \int_{\mathbb{R}^N} (I_\alpha * |\mu_1|^{2^\sharp_\alpha})|\mu_1|^{2^*_\alpha} \mathrm{d}x.$$

It follows that

$$\begin{aligned}
0 =& P((\mu_\sigma)_{t_\sigma}) \\
=& \frac{(N-2)\sigma^{-2} t_\sigma^{N-2}}{2} \|\mu_1\|^2_{D^{1,2}(\mathbb{R}^N)} + \left[\frac{N t_\sigma^N}{2} - \frac{(N+\alpha) t_\sigma^{N+\alpha}}{2 \cdot (2^\sharp_\alpha)^2} \right] \|\mu_1\|^2_{L^2(\mathbb{R}^N)} \\
&+ \frac{(N-p)\sigma^{\frac{(2-p)N}{2} - p} t_\sigma^{N-p}}{p} \|\mu_1\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{N \sigma^{\frac{(2-p)N}{2}} t_\sigma^N}{p} \|\mu_1\|^p_{L^p(\mathbb{R}^N)} \\
&- \frac{\lambda^2 (N+\alpha) \sigma^{-2\cdot 2^*_\alpha} t_\sigma^{N+\alpha}}{2 \cdot (2^*_\alpha)^2} \int_{\mathbb{R}^N} (I_\alpha * |\mu_1|^{2^*_\alpha})|\mu_1|^{2^*_\alpha} \mathrm{d}x \\
&- \frac{\lambda (N+\alpha) \sigma^{-2^*_\alpha} t_\sigma^{N+\alpha}}{2^\sharp_\alpha \cdot 2^*_\alpha} \int_{\mathbb{R}^N} (I_\alpha * |\mu_1|^{2^\sharp_\alpha})|\mu_1|^{2^*_\alpha} \mathrm{d}x.
\end{aligned} \tag{4.2}$$

Taking the limit superior as $\sigma \to \infty$ in (4.1), we further obtain

$$\begin{aligned}
&\limsup_{\sigma\to\infty} \left[\frac{(N-2)\sigma^{-2} t_\sigma^{N-2}}{2} \|\mu_1\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{N t_\sigma^N}{2} \|\mu_1\|^2_{L^2(\mathbb{R}^N)} \right. \\
&\left. + \frac{(N-p)\sigma^{\frac{(2-p)N}{2} - p} t_\sigma^{N-p}}{p} \|\mu_1\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{N \sigma^{\frac{(2-p)N}{2}} t_\sigma^N}{p} \|\mu_1\|^p_{L^p(\mathbb{R}^N)} \right] \\
=& \limsup_{\sigma\to\infty} \left[\frac{(N+\alpha) t_\sigma^{N+\alpha}}{2 \cdot (2^\sharp_\alpha)^2} \|\mu_1\|^2_{L^2(\mathbb{R}^N)} + \frac{\lambda^2 (N+\alpha) \sigma^{-2\cdot 2^*_\alpha} t_\sigma^{N+\alpha}}{2 \cdot (2^*_\alpha)^2} \int_{\mathbb{R}^N} (I_\alpha * |\mu_1|^{2^*_\alpha})|\mu_1|^{2^*_\alpha} \mathrm{d}x \right. \\
&\left. + \frac{\lambda (N+\alpha) \sigma^{-2^*_\alpha} t_\sigma^{N+\alpha}}{2^\sharp_\alpha \cdot 2^*_\alpha} \int_{\mathbb{R}^N} (I_\alpha * |\mu_1|^{2^\sharp_\alpha})|\mu_1|^{2^*_\alpha} \mathrm{d}x \right].
\end{aligned} \tag{4.3}$$

Let $s_\infty = \limsup_{\sigma\to\infty} t_\sigma$. We can prove $0 < t_\sigma < \infty$. Otherwise, we suppose $t_\sigma = \infty$. Then,

$$\begin{aligned}
&\limsup_{\sigma\to\infty}\Bigg[\frac{(N-2)\sigma^{-2}t_\sigma^{N-2}}{2}\|\mu_1\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{Nt_\sigma^N}{2}\|\mu_1\|^2_{L^2(\mathbb{R}^N)} \\
&\quad + \frac{(N-p)\sigma^{\frac{(2-p)N}{2}-p}t_\sigma^{N-p}}{p}\|\mu_1\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{N\sigma^{\frac{(2-p)N}{2}}t_\sigma^N}{p}\|\mu_1\|^p_{L^p(\mathbb{R}^N)}\Bigg] \\
&< \limsup_{\sigma\to\infty} t_\sigma{}^N\left[\frac{(N+\alpha)t_\sigma^\alpha}{2\cdot(2^\sharp_\alpha)^2}\|\mu_1\|^2_{L^2(\mathbb{R}^N)}\right] \\
&\leqslant \limsup_{\sigma\to\infty}\Bigg[\frac{(N+\alpha)t_\sigma^{N+\alpha}}{2\cdot(2^\sharp_\alpha)^2}\|\mu_1\|^2_{L^2(\mathbb{R}^N)} + \frac{\lambda^2(N+\alpha)\sigma^{-2\cdot 2^*_\alpha}t_\sigma^{N+\alpha}}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |\mu_1|^{2^*_\alpha})|\mu_1|^{2^*_\alpha}\mathrm{d}x \\
&\quad + \frac{\lambda(N+\alpha)\sigma^{-2^*_\alpha}t_\sigma^{N+\alpha}}{2^\sharp_\alpha\cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\mu_1|^{2^\sharp_\alpha})|\mu_1|^{2^*_\alpha}\mathrm{d}x\Bigg].
\end{aligned}$$

This yields a contradiction with (4.3).

Now, we show $s_\infty > 0$. Arguing by contradiction, we assume $s_\infty = 0$. Therefore, there exists $\hat{\sigma} > 0$ large such that $s_{\hat{\sigma}} > 0$ small enough. Then,

$$\begin{aligned}
&\frac{(N-2)\hat{\sigma}^{-2}s_{\hat{\sigma}}^{N-2}}{2}\|\mu_1\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{Ns_{\hat{\sigma}}^N}{2}\|\mu_1\|^2_{L^2(\mathbb{R}^N)} + \frac{(N-p)\hat{\sigma}^{\frac{(2-p)N}{2}-p}s_{\hat{\sigma}}^{N-p}}{p}\|\mu_1\|^p_{D^{1,p}(\mathbb{R}^N)} \\
&\quad + \frac{N\hat{\sigma}^{\frac{(2-p)N}{2}}s_{\hat{\sigma}}^N}{p}\|\mu_1\|^p_{L^p(\mathbb{R}^N)} \\
&> \frac{Ns_{\hat{\sigma}}^N}{2}\|\mu_1\|^2_{L^2(\mathbb{R}^N)} \\
&\geqslant s_{\hat{\sigma}}^N\Bigg[\frac{(N+\alpha)s_{\hat{\sigma}}^\alpha}{2\cdot(2^\sharp_\alpha)^2}\|\mu_1\|^2_{L^2(\mathbb{R}^N)} + \frac{\lambda^2(N+\alpha)\hat{\sigma}^{-2\cdot 2^*_\alpha}s_{\hat{\sigma}}^\alpha}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |\mu_1|^{2^*_\alpha})|\mu_1|^{2^*_\alpha}\mathrm{d}x \\
&\quad + \frac{\lambda(N+\alpha)\hat{\sigma}^{-2^*_\alpha}s_{\hat{\sigma}}^\alpha}{2^\sharp_\alpha\cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\mu_1|^{2^\sharp_\alpha})|\mu_1|^{2^*_\alpha}\mathrm{d}x\Bigg] \\
&= \frac{(N+\alpha)s_{\hat{\sigma}}^{N+\alpha}}{2\cdot(2^\sharp_\alpha)^2}\|\mu_1\|^2_{L^2(\mathbb{R}^N)} + \frac{\lambda^2(N+\alpha)\hat{\sigma}^{-2\cdot 2^*_\alpha}s_{\hat{\sigma}}^{N+\alpha}}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |\mu_1|^{2^*_\alpha})|\mu_1|^{2^*_\alpha}\mathrm{d}x \\
&\quad + \frac{\lambda(N+\alpha)\hat{\sigma}^{-2^*_\alpha}s_{\hat{\sigma}}^{N+\alpha}}{2^\sharp_\alpha\cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\mu_1|^{2^\sharp_\alpha})|\mu_1|^{2^*_\alpha}\mathrm{d}x,
\end{aligned}$$

which contradicts with (4.2). Hence, we get $0 < s_\infty < \infty$.

In view of $0 < s_\infty < \infty$ and taking the limit superior as $\sigma \to \infty$ in (4.2) again, it holds that

$$\limsup_{\sigma\to\infty}\left[\frac{Nt_\sigma^N}{2} - \frac{(N+\alpha)t_\sigma^{N+\alpha}}{2\cdot(2^\sharp_\alpha)^2}\right]\|\mu_1\|^2_{L^2(\mathbb{R}^N)} = 0.$$

Then, we have

$$s_\infty = \left[\frac{N\cdot(2^\sharp_\alpha)^2}{N+\alpha}\right]^{\frac{1}{\alpha}}.$$

Applying this for any $\bar{\sigma} > 0$ large, it follows that

$$\begin{aligned} J((\mu_{\bar{\sigma}})_{s_{\bar{\sigma}}}) =& \frac{\bar{\sigma}^{-2} s_{\bar{\sigma}}^{N-2}}{2} \|\mu_1\|^2_{D^{1,2}(\mathbb{R}^N)} + \left(\frac{s_{\bar{\sigma}}^N}{2} - \frac{s_{\bar{\sigma}}^{N+\alpha}}{2 \cdot (2^\sharp_\alpha)^2} \right) \|\mu_1\|^2_{L^2(\mathbb{R}^N)} \\ &+ \frac{\bar{\sigma}^{\frac{(2-p)N}{2} - p} s_{\bar{\sigma}}^{N-p}}{p} \|\mu_1\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{\bar{\sigma}^{\frac{(2-p)N}{2}} s_{\bar{\sigma}}^{N}}{p} \|\mu_1\|^p_{L^p(\mathbb{R}^N)} \\ &- \frac{\lambda^2 \bar{\sigma}^{-2 \cdot 2^*_\alpha} s_{\bar{\sigma}}^{N+\alpha}}{2 \cdot (2^*_\alpha)^2} \int_{\mathbb{R}^N} (I_\alpha * |\mu_1|^{2^*_\alpha}) |\mu_1|^{2^*_\alpha} \mathrm{d}x \\ &- \frac{\lambda \bar{\sigma}^{-2^*_\alpha} s_{\bar{\sigma}}^{N+\alpha}}{2^\sharp_\alpha \cdot 2^*_\alpha} \int_{\mathbb{R}^N} (I_\alpha * |\mu_1|^{2^\sharp_\alpha}) |\mu_1|^{2^*_\alpha} \mathrm{d}x \\ &< \max_{s>0} \left[\frac{s^N}{2} - \frac{s^{N+\alpha}}{2 \cdot (2^\sharp_\alpha)^2} \right] \|\mu_1\|^2_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{4.4}$$

Set

$$h(t) = \frac{t^N}{2} - \frac{t^{N+\alpha}}{2 \cdot (2^\sharp_\alpha)^2}.$$

Then, we know $h'(s_\infty) = 0$ and s_∞ is the unique maxium point of $h(\cdot)$. By (4.4), one has

$$J((\mu_{\bar{\sigma}})_{s_{\bar{\sigma}}}) < \left[\frac{s_\infty^N}{2} - \frac{s_\infty^{N+\alpha}}{2 \cdot (2^\sharp_\alpha)^2} \right] \|\mu_1\|^2_{L^2(\mathbb{R}^N)} = \frac{\alpha}{2(N+\alpha)} \left[\frac{N \cdot (2^\sharp_\alpha)^2}{N+\alpha} \right]^{\frac{1}{2^\sharp_\alpha - 1}} \mathcal{S}_1^{\frac{2^\sharp_\alpha}{2^\sharp_\alpha - 1}}.$$

The proof is completed. □

Lemma 4.5. *Suppose that all conditions described in Theorem 1.2 hold. Let $\{u_n\}$ be a bounded minimizing sequence of J satisfying*

$$J(u_n) \to m \ \text{ and } \ P(u_n) \to 0, \ \text{ as } n \to \infty.$$

Then, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x > 0 \ \text{ and } \ \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \mathrm{d}x > 0.$$

Proof. First, we show $\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x > 0$. Otherwise, we suppose

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x = 0. \tag{4.5}$$

Combining Proposition 2.1 and (4.5), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^\sharp_\alpha}) |u_n|^{2^\sharp_\alpha} \mathrm{d}x \leqslant \lim_{n\to\infty} C_{N,\alpha} \left(\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x \right)^{\frac{N+\alpha}{N}} = 0$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^\sharp_\alpha}) |u_n|^{2^*_\alpha} \mathrm{d}x \leqslant \lim_{n\to\infty} C_{N,\alpha} \left(\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x \right)^{\frac{N+\alpha}{2N}} \left(\int_{\mathbb{R}^N} |u_n|^{2^*} \mathrm{d}x \right)^{\frac{N+\alpha}{2N}} = 0.$$

Combining the above two inequalities, one has

$$m + o_n(1) = \frac{1}{2}\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{1}{2}\|u_n\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{p}\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{1}{p}\|u_n\|^p_{L^p(\mathbb{R}^N)} - \frac{\lambda^2}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha}\mathrm{d}x \tag{4.6}$$

and

$$o_n(1) = \frac{N-2}{2}\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{N}{2}\|u_n\|^2_{L^2(\mathbb{R}^N)} + \frac{N-p}{p}\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{N}{p}\|u_n\|^p_{L^p(\mathbb{R}^N)} - \frac{\lambda^2(N+\alpha)}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha}\mathrm{d}x. \tag{4.7}$$

Combining (4.6) and (4.7), it follows that

$$\begin{aligned} m + o_n(1) \geqslant& \frac{\alpha+2}{2(N+\alpha)}\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{\alpha}{2(N+\alpha)}\|u_n\|^2_{L^2(\mathbb{R}^N)} + \frac{\alpha+p}{p(N+\alpha)}\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} \\ &+ \frac{\alpha}{p(N+\alpha)}\|u_n\|^p_{L^p(\mathbb{R}^N)} \\ \geqslant& \frac{\alpha+2}{2(N+\alpha)}\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)}. \end{aligned} \tag{4.8}$$

It follows from (2.2) and (4.7) that

$$\begin{aligned} (N-2)\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} \leqslant& \frac{\lambda^2(N+\alpha)}{(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha}\mathrm{d}x \\ \leqslant& \frac{\lambda^2(N+\alpha)}{(2^*_\alpha)^2}\left(\frac{1}{\mathcal{S}_2}\right)^{2^*_\alpha}\|u_n\|^{2\cdot 2^*_\alpha}_{D^{1,2}(\mathbb{R}^N)}. \end{aligned}$$

This implies

$$\left[\frac{(2^*_\alpha)^2(N-2)}{\lambda^2(N+\alpha)}\right]^{\frac{1}{2^*_\alpha-1}}\mathcal{S}_2^{\frac{2^*_\alpha}{2^*_\alpha-1}} \leqslant \|u_n\|^2_{D^{1,2}(\mathbb{R}^N)}. \tag{4.9}$$

In view of (4.8) and (4.9), we can derive

$$m + o_n(1) \geqslant \frac{\alpha+2}{2(N+\alpha)}\left[\frac{(2^*_\alpha)^2(N-2)}{\lambda^2(N+\alpha)}\right]^{\frac{1}{2^*_\alpha-1}}\mathcal{S}_2^{\frac{2^*_\alpha}{2^*_\alpha-1}}.$$

This yields a contradiction with Lemma 4.4.

Next, we show $\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x > 0$. On the contrary, it suffices to show

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x = 0. \tag{4.10}$$

From Proposition 2.1 and (4.10), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha}\mathrm{d}x \leqslant \lim_{n\to\infty}C_{N,\alpha}\left(\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x\right)^{\frac{N+\alpha}{N}} = 0$$

and

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{2^\sharp_\alpha})|u_n|^{2^*_\alpha}\mathrm{d}x\\ \leqslant&\lim_{n\to\infty}C_{N,\alpha}\left(\int_{\mathbb{R}^N}|u_n|^2\mathrm{d}x\right)^{\frac{N+\alpha}{2N}}\left(\int_{\mathbb{R}^N}|u_n|^{2^*}\mathrm{d}x\right)^{\frac{N+\alpha}{2N}}=0.\end{aligned}$$

Together with the above two expressions, we get

$$\begin{aligned}m+o_n(1)=&\frac{1}{2}\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)}+\frac{1}{2}\|u_n\|^2_{L^2(\mathbb{R}^N)}+\frac{1}{p}\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)}+\frac{1}{p}\|u_n\|^p_{L^p(\mathbb{R}^N)}\\&-\frac{\lambda^2}{2\cdot(2^\sharp_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{2^\sharp_\alpha})|u_n|^{2^\sharp_\alpha}\mathrm{d}x\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}o_n(1)=&\frac{N-2}{2}\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)}+\frac{N}{2}\|u_n\|^2_{L^2(\mathbb{R}^N)}+\frac{N-p}{p}\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)}+\frac{N}{p}\|u_n\|^p_{L^p(\mathbb{R}^N)}\\&-\frac{\lambda^2(N+\alpha)}{2\cdot(2^\sharp_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{2^\sharp_\alpha})|u_n|^{2^\sharp_\alpha}\mathrm{d}x.\end{aligned}\tag{4.12}$$

From (4.11) and (4.12), we get that

$$m+o_n(1)\geqslant\frac{\alpha}{2(N+\alpha)}\|u_n\|^2_{L^2(\mathbb{R}^N)}.\tag{4.13}$$

Observe that, by (2.1) and (4.12), we have

$$\begin{aligned}\|u_n\|^2_{L^2(\mathbb{R}^N)}\leqslant&\frac{\lambda^2(N+\alpha)}{N\cdot(2^\sharp_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{2^\sharp_\alpha})|u_n|^{2^\sharp_\alpha}\mathrm{d}x\\ \leqslant&\frac{\lambda^2(N+\alpha)}{N\cdot(2^\sharp_\alpha)^2}\left(\frac{1}{\mathcal{S}_1}\right)^{2^\sharp_\alpha}\|u_n\|^{2\cdot2^\sharp_\alpha}_{L^2(\mathbb{R}^N)}.\end{aligned}$$

This shows

$$\left[\frac{N\cdot(2^\sharp_\alpha)^2}{\lambda^2(N+\alpha)}\right]^{\frac{1}{2^\sharp_\alpha-1}}\mathcal{S}_1^{\frac{2^\sharp_\alpha}{2^\sharp_\alpha-1}}\leqslant\|u\|^2_{L^2(\mathbb{R}^N)}.\tag{4.14}$$

Combining (4.13) and (4.14), we infer that

$$m\geqslant\frac{\alpha}{2(N+\alpha)}\left[\frac{N\cdot(2^\sharp_\alpha)^2}{\lambda^2(N+\alpha)}\right]^{\frac{1}{2^\sharp_\alpha-1}}\mathcal{S}_1^{\frac{2^\sharp_\alpha}{2^\sharp_\alpha-1}}.$$

This contradicts with Lemma 4.5, the proof is complete. □

We now can conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\{u_n\}$ be a minimizing sequence of J satisfying

$$J(u_n)\to m\ \ and\ \ P(u_n)\to0,\ \ as\ n\to\infty.$$

It is easy to verify $\{u_n\}$ is bounded in E. Taking into account Lemmas 4.4 and 4.5 and Proposition 2.2, we deduce that $\{u_n\}$ converges weakly and a.e. to $u \not\equiv 0$ in $L^2_{loc}(\mathbb{R}^N)$. Using the Brézis-Lieb lemma [35], we obtain

$$m \leqslant J(u) = J(u) - \frac{1}{N+\alpha}P(u) \leqslant \lim_{n\to\infty}\left(J(u_n) - \frac{1}{N+\alpha}P(u_n)\right) = \lim_{n\to\infty} J(u_n) = m,$$

which implies $J(u) = m$. Moreover, we can choose $u \geqslant 0$. Therefore, u is a nonnegative ground state solution of Eq ($\mathcal{D}$).

Let u be a nonnegative ground state solution of Eq ($\mathcal{D}$). In order to show u is radial, it suffices to prove $m = m_{rad}$, where

$$\mathcal{P}_{rad} = \left\{u \in E_{rad}\backslash\{0\}\middle| P(u) = 0\right\}$$

and

$$m_{rad} = \inf_{u\in\mathcal{P}_{rad}} J(u).$$

On one hand, by $E_{rad} \subset E$, it follows that $m \leqslant m_{rad}$. On the other hand, for any $v \in \mathcal{P}$, by J being even, we know $|v| \in \mathcal{P}$. Let $|v|^*$ be the decreasing rearrangement of $|v|$. In view of [19, Theorem 3.7], one has

$$\||v|^*\|^2_{D^{1,2}(\mathbb{R}^N)} \leqslant \||v|\|^2_{D^{1,2}(\mathbb{R}^N)} \text{ and } \||v|^*\|^2_{L^2(\mathbb{R}^N)} \leqslant \|v\|^2_{L^2(\mathbb{R}^N)} \tag{4.15}$$

and

$$\||v|^*\|^p_{D^{1,p}(\mathbb{R}^N)} \leqslant \||v|\|^p_{D^{1,p}(\mathbb{R}^N)} \text{ and } \||v|^*\|^p_{L^p(\mathbb{R}^N)} \leqslant \|v\|^p_{L^p(\mathbb{R}^N)} \tag{4.16}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)|^{2^\sharp_\alpha}|v(y)|^{2^\sharp_\alpha}}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{||v(x)|^*|^{2^\sharp_\alpha}||v(y)|^*|^{2^\sharp_\alpha}}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \tag{4.17}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)|^{2^\sharp_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{||v(x)|^*|^{2^\sharp_\alpha}||v(y)|^*|^{2^*_\alpha}}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \tag{4.18}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{||v(x)|^*|^{2^*_\alpha}||v(y)|^*|^{2^*_\alpha}}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \tag{4.19}$$

From (4.15)–(4.19), for any $t > 0$, it follows that

$$J((|v|^*)_t) \leqslant J((|v|_t)). \tag{4.20}$$

By Lemma 4.2, there exists $t_v > 0$ such that $(|v|^*)_{t_v} \in \mathcal{P}$. We have

$$\begin{aligned}
&\frac{(N-2)t_v^{N-2}}{2}\||v|^*\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{Nt_v^N}{2}\||v|^*\|^2_{L^2(\mathbb{R}^N)} + \frac{(N-p)t_v^{N-p}}{p}\||v|^*\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{Nt_v^N}{p}\||v|^*\|^p_{L^p(\mathbb{R}^N)}\\
=&\frac{(N+\alpha)t_v^{N+\alpha}}{2\cdot(2^\sharp_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * ||v|^*|^{2^\sharp_\alpha})||v|^*|^{2^\sharp_\alpha}\mathrm{d}x + \frac{\lambda(N+\alpha)t_v^{N+\alpha}}{2^\sharp_\alpha\cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * ||v|^*|^{2^\sharp_\alpha})||v|^*|^{2^*_\alpha}\mathrm{d}x\\
&+\frac{\lambda^2(N+\alpha)t_v^{N+\alpha}}{2\cdot(2^*_\alpha)^2}\int_{\mathbb{R}^N}(I_\alpha * ||v|^*|^{2^*_\alpha})||v|^*|^{2^*_\alpha}\mathrm{d}x.
\end{aligned}$$

From the above and $|v| \in \mathcal{P}$, it is easy to observe $t_v \in (0, 1]$. In view of (4.20), one has

$$m \leqslant J((|v^*|)_{t_v}) \leqslant J((|v|)_{t_v}) \leqslant \max_{t \geqslant 0} J((|v|)_t) = J(|v|).$$

Therefore, for any $|v| \in \mathcal{P}$, there exists $t_v > 0$ such that $t_v|v|^* \in \mathcal{P}$ and

$$J((|v|^*)_{t_v}) \leqslant J(|v|).$$

This implies $m_{rad} \leqslant m$. Hence, we have $m_{rad} = m$. This means u is a radially symmetric ground state solution of Eq ($\mathcal{D}$). The proof is complete. □

Author contributions

Lixiong Wang: Conceptualization, Methodology, Formal analysis, Writing-original draft preparation, Writing-reviewing and editing, Validation, Funding acquisition; Ting Liu: Writing-reviewing and editing, Visualization, Investigation, Supervision. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare there are no conflicts of interest.

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