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Research article

A precise solution to the shortest path optimization problem in graphs using Z-numbers

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Abstract: Communication networks are exposed to internal or external risks that can affect all or part of the system. The most important components that form the infrastructure of these systems are routers, which act as nodes. In the field of graph theory, there are sophisticated techniques that can be used to optimize the path of a packet as it travels through various routers from its origin to its destination. A notable example of such an algorithm is Dijkstra's algorithm, which is designed to efficiently determine the shortest path. The algorithm works under the assumption that the system operates under ideal conditions. Real-time systems can perform better if risk factors and optimal conditions are taken into account. The relationship between the nodes can be expressed by various metrics such as distance, delay, and bandwidth. The aforementioned metrics facilitate the calculation of the optimal path, with the ultimate objective of achieving low-latency networks characterized by rapid response times. Round-trip time (RTT) can be employed as a metric for measuring enhancements in a range of latency types, including those associated with processing, transmission, queuing, and propagation. The use of Z-numbers was employed in this study to incorporate risk into the optimal path metric. RTT was the preferred metric and reliability was represented by fuzzy linguistic qualifiers. A comparison of several scenarios was shown using a numerical example of a communication network. It is expected that this study will have a significant impact on the evolution from models that consider only ideal conditions to real-time systems that include risks using Z-numbers.

Keywords: Z-number; Dijkstra algorithm; Z-graph, Z-cost; Z-ranking
Mathematics Subject Classification: 90C70

1. Introduction

Uncertainties are common in real-life information. In order to effectively incorporate this information into decision-making processes, it is essential to represent uncertainties and natural language expressions in a scientifically accurate way. In 1965, Lotfi A. Zadeh introduced the concept of a fuzzy representation of situations in the real world [1]. Fuzzy numbers, which are often used in fuzzy logic applications, represent uncertainty by a singular membership function. This means that the uncertainty measure is not taken into account. A first attempt to overcome this limitation was the introduction of the idea of a type-2 fuzzy set [2]. In 2011, Zadeh defined the concept of the Z-number by assigning a reliability measure, which is also a fuzzy number, to fuzzy numbers [3]. The Z-number, which is referred as $Z = (A, B)$, consists of two components. The first component, A, represents a number with inherent uncertainty, while the second component, B, serves as a measure of the reliability associated with A. The value of B can be viewed as a probability measure for A. For example, the statement "The estimated arrival time from city X to city Y is usually about 5 hours" is a Z-number. In this context, the statement "about 5 hours" refers to the fuzzy number A. The term "usually" refers to the fuzzy number B, which indicates an uncertain estimate of the reliability of A.

Z-numbers have numerous applications to overcome the deficiency mentioned above [4–7]. However, due to the complexity of calculations with Z-numbers, certain difficulties arise in their application. To overcome these difficulties, a number of methods have been proposed, such as computing Z-numbers by converting them into fuzzy numbers [8]. However, the use of these techniques significantly reduces the advantage of using Z-numbers.

The first studies on arithmetic operations for the use of Z-numbers in applications were carried out by R. A. Aliev et al. [9–11]. On the other hand, several methods have been proposed for ranking Z-numbers [12–15].

One of the main topics of graph theory is the optimization of the shortest path. The Dijkstra algorithm is a widely used approach to solve this problem. It has been used in communication networks as part of the OSPF (open shortest path first) protocol to optimize the path between routers [16].

Dijkstra's algorithm utilizes crisp numbers, however there are also studies in the literature using fuzzy numbers to explain the uncertainty [17–20]. Biswas suggested that Z-numbers can solve the shortest path problem using Dijkstra's algorithm [21]. Veerammal also solved the numerical example using Biswas' method given in [22]. However, in these studies, the probability distributions underlying the second components of the numbers in addition to the Z-numbers were not considered.

In this study, Z-numbers are employed to perform the requisite addition and ranking operations for Dijkstra's algorithm, in accordance with the methodologies described in 2.3 and 2.4. The incorporation of Z-numbers, which are uniquely capable of capturing both the value of uncertain information and the degree of confidence in that information, demonstrates that they offer a more effective means of representing uncertainty than crisp and fuzzy numbers. This is corroborated by numerical solutions across a range of scenarios, which show how Z-numbers provide a more comprehensive and accurate reflection of uncertainty.

The results of the study highlight the strengths of Z-numbers in handling uncertainty in computational processes, particularly in complex systems such as network optimization and pathfinding algorithms like Dijkstra's.

2. Preliminaries

The process of adding Z-numbers includes operations on random variables in accordance with the probability constraint in the second component. Therefore, this section includes explanations about random variables as well as basic concepts about Z-numbers.

2.1. Random variables

Random variables and their operations are provided in [23, 24]. This section contains some definitions of random variables and the convolution theorem for the sum of continuous random variables.

Definition 2.1. *A random variable X is a function that assigns a numerical value to every random event in the sample space S. There are two types of random variables: Continuous and discrete random variables.*

Provided that X is a random variable,

- *If X has a finite number of possible outcomes, it is a discrete random variable.*
- *If X has an infinite number of possible outcomes, it is a continuous random variable [9].*

To determine a probability that a continuous random variable X takes any value in a closed interval $[a, b]$, denoted by $P(a \leq X \leq b)$, the concept of probability distribution is used. A probability distribution or probability density function (pdf) is a function $p(x)$ such that for any two numbers a and b with $a \leq b$:

$$P(a \leq X \leq b) = \int_a^b p(x)dx,$$

where $p(x) \geq 0$ and $\int_{-\infty}^{\infty} p(x)dx = 1$.

Definition 2.2. *The continuous random variable X follows a normal (Gaussian) distribution if its pdf is defined as:*

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},$$

where μ is the mean of X and σ^2 is the variance of X. For a normal pdf, a typical notation is used: $p = N(\mu, \sigma)$ [11].

Theorem 2.1. *Let X_1 and X_2 be independent random variables. Then, the pdf of $X_{12} = X_1 + X_2$ is the convolution of the pdfs of X_1 and X_2:*

$$p_{12}(x_{12}) = \int_{-\infty}^{\infty} p_1(x_1) p_2(x_{12} - x_1) dx_1. \tag{2.1}$$

If $X_i, i = 1, 2$, is a normal random distribution with $\mu_{12} = \mu_1 + \mu_2$ and $\sigma_{12} = \sqrt{\sigma_1{}^2 + \sigma_2{}^2}$ [23] , then the pdf of convolution p_{12} is

$$p_{12}(x) = \frac{1}{\sqrt{2\pi(\sigma_1{}^2 + \sigma_2{}^2)}} e^{-\frac{1}{2}\frac{(x-(\mu_1+\mu_2))^2}{\sigma_1{}^2+\sigma_2{}^2}}. \tag{2.2}$$

2.2. Z-numbers

In this section, the definition of a Z-number and some basic concepts related to Z-numbers are given.

Definition 2.3. *A fuzzy set A whose membership function is* $\mu_A : R \to [0,1]$ *is a fuzzy number if it satisfies the following conditions:*

a. A is normal, that is $\exists u_0 \in R$ *with* $\mu_A(u_0) = 1$,

b. A is convex, that is $\mu_A((1-\lambda)u + \lambda v) \geq min\{\mu_A(u), \mu_A(v)\}$, $\forall u, v \in R$, $\lambda \in [0,1]$,

c. A is upper semicontinuous, that is $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $\mu_A(x) - \mu_A(x_0) < \varepsilon$, $|x - x_0| < \delta$,

d. $cl(A) = \{x \in R : \mu_A(x) > 0\}$ *is compact, where* $cl(A)$ *denotes the closure of the set A [25].*

Note that the notation A represents the fuzzy number and $\mu_A(x)$ represents the membership function of A.

Definition 2.4. *A trapezoidal fuzzy number is represented by four real numbers as follows:* $A = (a,b,c,d)$*, with* $a \leq b \leq c \leq d$*, and its membership function* μ_A *is given by*

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & , a \leq x \leq b, \\ 1 & , b \leq x \leq c, \\ \frac{d-x}{d-c} & , c \leq x \leq d, \\ 0 & , otherwise. \end{cases}$$

If $A = (a,b,c,d)$ *is a trapezoidal fuzzy number and* $b = c$*, then A is a triangular fuzzy number, which is represented by the form* $A = (a,b,d)$ *[26].*

Definition 2.5. *A fuzzy number A with* $\mu_A(x) = 0$ *for* $\forall x < 0$ *is called a positive fuzzy number [27].*

Definition 2.6. *Let X be a real-valued uncertain variable, A and B are two fuzzy numbers, and the Z-number is defined as an ordered pair of fuzzy numbers* $Z = (A,B)$*. The first component A is a fuzzy restriction on the values of X, a real-valued uncertain variable. The second component B is a measure of reliability for A.*

The basic form of $Z = (A,B)$ *on the random variable X, "X is* (A,B)*", can be expressed as follows:*

$$Prob(X\ is\ A)\ is\ B\ or\ (X\ is\ A)\ isp\ B,$$

where B is the restriction probability measure of X being represented by the fuzzy number A. Also, "isp" is the abbreviation for "is probability" [5].

Illustrative Example: *Let us consider that we have the information that the average life expectancy will likely be around* 80 *years in the near future. The phrase "around* 80 *years" in this linguistic expression indicates a fuzzy number. The phrase "likely" serves as a fuzzy restrictive probability measure of the possibility of this fuzzy number. If we take the expressions "around* 80 *years" and "likely" in natural language as triangular fuzzy numbers* $(70, 80, 90)$ *and* $(0.5, 0.725, 0.95)$ *respectively, we get the* $Z = (A,B) = ((70,80,90),(0.5,0.725,0.95))$ *Z-number. In this Z-number, the fuzzy number* $B = (0.5, 0.725, 0.95)$*, which corresponds to the phrase "likely", can be interpreted as a fuzzy probability value that indicates the degree of confidence in the information that the average life expectancy will be "around* 80 *years" in the near future. The graphs of the membership functions* *of the fuzzy numbers A and B, which are the components of the* $Z = (A, B)$ *Z-number, are shown in Figure 1.*

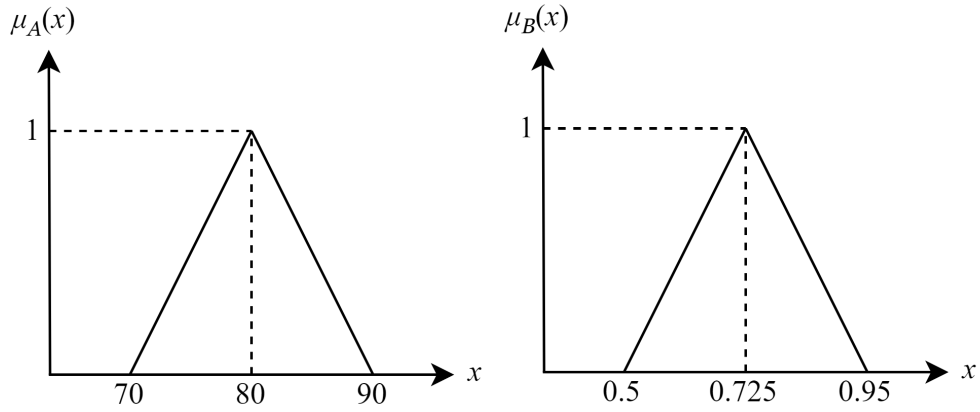


Figure 1. Graph of A and B in Z-number $Z = (A, B)$.

Definition 2.7. $Z = (A, B)$*, whose components A and B are continuous fuzzy numbers, is called a continuous Z-number [9].*

Definition 2.8. *If A and B are positive fuzzy numbers,* $Z = (A, B)$ *is called a positive Z-number [28].*

Definition 2.9. *The Z-number is expressed as the dual* $Z^+ = (A, R)$*. Here A plays the same role as in the Z-number* $Z = (A, B)$*. R is a random number defined by the probability density function* p_R *[9]. Thus:*

$$\int_R \mu_A(x) p_R(x) dx \text{ is } B$$

$$\mu_{p_R}(p_R) = \mu_B \left(\int_R \mu_A(x) p_R(x) dx \right). \tag{2.3}$$

In the continuous Z-number $Z = (A, B)$, there are many probability distributions underlying the fuzzy number B, which expresses the probability measure. In this study, normal probability distribution is used for the convenience of calculations for continuous Z-numbers.

To simplify the calculations for two Z-numbers $Z_j^+ = (A_j, R_j),\ j = 1, 2$, the membership degrees given in (2.3) can be discretized as follows:

$$\mu_{p_j}(p_{jl}) = \mu_{B_j} \left(\sum_{i=1}^{n_j} \mu_{A_j}(x_{ji}) p_{jl}(x_{ji}) \right),\ j = 1, 2. \tag{2.4}$$

To calculate the discretized membership degrees, $supp(B_j)$ must be divided by $b_{jl},\ l = 1, m$ discrete subintervals. This produces the goal programming equation

$$b_{jl} = P(A_j) = \sum_{i=1}^{n_j} \mu_{A_j}(x_{ji}) p_{jl}(x_{ji}), \tag{2.5}$$

which contains the values of the probability measure of the fuzzy number A_j. It is known that this equation satisfies the following compatibility conditions:

$$\sum_{i=1}^{n_j} p_{jl}(x_{ji}) = 1,\ p_{jl}(x_{ji}) \geq 0, \tag{2.6}$$

$$\sum_{i=1}^{n_j} p_{jl}(x_{ji})x_{ji} = \frac{\sum_{i=1}^{n_j} \mu_{A_j}(x_{ji})x_{ji}}{\sum_{i=1}^{n_j} \mu_{A_j}(x_{ji})}. \tag{2.7}$$

2.3. Addition with continuous Z-numbers (Z-addition)

In this section, the steps of addition with continuous Z-numbers described in [11] are given.

Let $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ be continuous Z-numbers. The steps of the $Z_{12} = Z_1 + Z_2 = (A_1 + A_2, B_1 + B_2) = (A_{12}, B_{12})$ addition process are as follows:

Step 1. The addition of $A_{12} = A_1 + A_2$ is calculated. Let $A_1 = (l_1, m_1, u_1)$ and $A_2 = (l_2, m_2, u_2)$ be triangular fuzzy numbers. In this case, $A_1 + A_2 = (l_1 + l_2, m_1 + m_2, u_1 + u_2)$.

Step 2. Switch to Z-numbers and use (2.1) to compute $R_{12} = R_1 + R_2$ via convolution of $p_{12} = p_1 \circ p_2$. The probability distributions p_1 and p_2, which underlie R_1 and R_2, are treated as normal pdfs in this study, and the calculation is performed using the (2.2) formula.

Step 3. The membership degrees of the discretized $\mu_{p_j},\ j = 1, 2$, are calculated using (2.4). To do this, linear programming equations written using (2.5) are solved, taking into account the compatibility conditions given in (2.6) and (2.7).

Step 4. The membership degrees of the discretized $\mu_{p_{12}}$ are determined with the use of $\mu_{p_{12}}(p_{12s}) = \left[\mu_{p_1}(p_{1l_1}) \wedge \mu_{p_2}(p_{2l_2})\right]$, where p_{12s} is the $p_{1l_1} \circ p_{2l_2}$ convolution.

Step 5. The value of $\mu_{B_{12}}$ is determined using the $\mu_{B_{12}}(b_{12s}) = \mu_{p_{12s}}(p_{12s})$ equation with $\mu_{p_{12}}$ and $\mu_{A_{12}}$. This value is designated as $b_{12s} = \sum_{i=1}^{n} \mu_{A_{12}}(x_i)p_{12s}(x_i)$. Using the obtained $\mu_{B_{12}}$, the B_{12} triangle fuzzy number is determined.

So $Z_{12} = Z_1 + Z_2 = (A_{12}, B_{12})$ is calculated.

2.4. Ranking of Z-numbers (Z-ranking)

Ranking of Z-numbers was studied in [12–15]. In this section, the Z-numbers ranking algorithm by a sigmoid function based on the combination of the convex method in [15] is briefly introduced.

The $Rank(Z)$ value of a $Z = (A, B)$ Z-number, including $G = (x_{A^*} \times |\alpha|) + STD_{A^*}$, can be calculated using the formula provided below.

$$Rank(Z) = \begin{cases} \frac{1}{1+e^{-G}} & , G > 0, \\ 0 & , G = 0, \\ \frac{-1}{1+e^{-G}} & , G < 0. \end{cases}$$

In this algorithm; α represents the conversion of B (reliability) into a crisp number, which is calculated according to the following formula:

$$\alpha = \frac{\int_X x\mu_B dx}{\int_X \mu_B dx}.$$

A^* represents the standardized generalized version of the trapezoidal fuzzy number A. The defuzzified value of A^*, denoted by x_{A^*}, is calculated according to the following formula, which combines the definition of the mean value and the convex combination:

$$x_{A^*} = \frac{\lambda(a_2^* + a_3^*) + (1-\lambda)(a_1^* + a_4^*)}{\lambda\delta_1 + (1-\lambda)\,\delta_2}.$$

STD_{A^*} represents the spread value of A^* and is calculated as follows.

$$STD_{A^*} = \sqrt{\frac{\lambda\left(\left(a_2^* - x_{A^*}\right)^2 + \left(a_3^* - x_{A^*}\right)^2\right) + (1-\lambda)\left(\left(a_1^* - x_{A^*}\right)^2 + \left(a_4^* - x_{A^*}\right)^2\right)}{\lambda\delta_1 + (1-\lambda)\,\delta_2}}.$$

Various convex combinations for trapezoidal fuzzy numbers are shown in [15]. In this study, the value of $Rank(Z) \in (-1, 1)$ was calculated using $\lambda(a_2^*+a_3^*)+(1-\lambda)(a_1^*+a_4^*)$ for $\lambda = 0.5$. Since triangular fuzzy numbers were used in this paper, $a_2 = a_3$ was taken.

As the $Rank(Z)$ value increases, so does the value of the Z-number. In other words, if u and v are Z-numbers and $Rank(u) > Rank(v)$, then u is considered greater than v. Additionally, if w is a positive Z-number, then $Rank(u) < Rank(u + w)$.

2.5. *Z-number weighted graph (Z-graph)*

This section contains the basic concepts of the graph and the definition of the Z-graph.

Definition 2.10. *A graph is comprised of two sets: The vertex set, V, and the edge set, E. An edge, denoted by xy, is an unordered pair of vertices, where x and y are elements of V. G or $G = (V, E)$ will be frequently used as shorthand [29].*

Definition 2.11. *A weighted graph $G = (V, E, w)$ is a graph in which each edge is associated with a real number. The weight of the edge xy is denoted as $w(xy)$ [29].*

Definition 2.12. *A graph is considered a Z-graph if at least one of its edges has a Z-number weight [21].*

3. Z-number based approach for the shortest path problem

The proposed Z-number based approach for shortest path optimization and some explanations required for it are presented in this section.

3.1. *Shortest path algorithm (Dijkstra's algorithm)*

This section contains an explanation of the shortest path algorithm, which is a fundamental method in graph theory and is employed to determine the most efficient route between two vertices in a given network.

Dijkstra's algorithm is known as one of the best-known algorithms for this particular objective [30]. The operation of this method is based on scanning neighboring vertices from the marked vertex at each step and updating the tags of the scanned vertices. The process begins with the selection of the source vertex, which is then assigned a distance value of zero. Subsequently, an analysis is conducted on all adjacent vertices of the initial vertex, whereby distances are assigned to them in accordance with their respective edge weights. The algorithm then selects the vertex with the shortest distance and works by assigning or updating tags to the neighbors of this vertex. The procedure described above is executed iteratively until all vertices have been visited or until the target vertex has been reached.

The objective of the algorithm is to find the path with the lowest cumulative weight from the source vertex to the destination vertex, which is achieved by backward tracking using labels [29].

3.2. Metrics between nodes

This section contains a description of round-trip time (RTT), a metric used to represent the Z-number cost values (referred to as Z-cost) associated with communication links between routers (also referred to as nodes) in a network.

RTT is used as a cost metric for the connections between nodes. Each node is considered as a router in computer networks. Round-trip time, the time it takes to send a packet to a given destination and receive a response, is often used as a metric to evaluate network performance [31].

The objective is to achieve high performance in networks, characterized by low latency and fast response time. The term "network latency" is typically used to describe a range of factors that can impede communication over a given network, ultimately affecting its overall performance. In a network, a data packet must traverse numerous links and nodes (such as cables, fibers, routers, and switches) to reach its intended destination. Each of these steps introduces a certain degree of delay. The primary sources of delay can be broadly classified into four categories: processing, transmission, queuing, and propagation delay. RTT is a complex metric that can be used to quantify the cumulative impact of these delays [32].

In this study, the RTT values, which indicate the delays between nodes, are represented by Z-numbers to incorporate the uncertainties associated with the delays in the actual operation of the communication network.

3.3. The proposed method

In this section, the Dijkstra algorithm in [29, 33, 34] has been rearranged to include operations with Z-numbers and its accuracy has been proven.

Let $G = (V, E, w)$ be a Z-graph. The set of vertices of G is denoted by V, while E represents the set of edges that connect neighboring vertices. The function w assigns a positive Z-number value $w(xy)$ to each xy edge.

The Dijkstra algorithm assigns to each vertex $v \in V$ in the graph G an ordered binary label $L(v) = (x, \delta(v))$. Where Vertex x is the last vertex visited before reaching v, and $\delta(v)$ is the Z-number weight of the shortest path used to reach vertex v from the starting point. At each stage of the algorithm, a set of previously unmarked neighboring vertices, denoted by N, is considered.

The steps to find $\rho(v)$, the shortest path from the starting vertex to each $v \in V$ in the graph G, and $\delta(v)$, the Z-number weight of this path, are as follows:

Step 1*. Assign the label $L(x)$ to each vertex x of G. When labeling for the first time, set $x = u$ as $L(x) = (-, 0)$ for the starting corner and $L(x) = (-, \infty)$ for the other corners. Mark the start vertex.

Step 2*. Define N as the neighbors of vertex u. For each $v \in N$, update $L(v) = (u, w(uv))$. Mark the

vertex v with the lowest weight $Rank(\delta(v))$. Assign $u = v$.
Step 3*. Update N by adding neighbors of vertex u to N. Then for each $v \in N$, calculate the Z-addition $\delta(u) + w(uv)$.

$$\text{If } Rank(\delta(u) + w(uv)) < Rank(\delta(v)), \text{ update } L(v) = (u, \delta(u) + w(uv)).$$

Otherwise, do not change $L(v)$.

Note that for vertices visited for the first time, the $Rank(\delta(v))$ value is taken to be ∞.
Step 4*. Mark the edge uv and the vertex $v \in N$ with the lowest weight in the Z-ranking result. Assign $u = v$.
Step 5*. Repeat Steps 3* and 4* until all vertices have been visited. During each iteration, update labels only for vertices that are adjacent to the last marked vertex.
Step 6*. The shortest path from the starting vertex u to any vertex x, denoted as $\rho(x)$, is determined by backtracking with the assistance of the first component of $L(x)$. The cumulative Z-number weight of the path is represented by the second component of $L(x)$.

If weight is interpreted as cost, then $w(xy)$ can be thought of as the Z-cost value of the xy edge. In this case, the cost from u to x is defined as the minimum weight of a path directed from u to x. If $P(u, x) = (u = v_0, v_1, ..., v_t = x)$ represents the directed path from vertex u to vertex x, then the Z-cost of the path P is equal to $\sum_{i=0}^{t-1} w(v_i v_{i+1})$, which is the sum of the cost values of all the edges in the path. In addition, $Rank(Z)$ values, which are real numbers, are used to compare Z-cost values.

The correctness theorem of Dijkstra's algorithm for the Z-graph G, which is weighted with positive Z-number values, and the required lemmas for this theorem are as follows.

Lemma 3.1. *Let x be a vertex and let $\rho(x) = (u = v_0, v_1, ..., v_t = x)$ be a shortest path from u to x. Then for every integer j with $0 < j < t$, $(v_0, v_1, ..., v_j)$ is a shortest path from u to v_j and $(v_j, v_{j+1}, ..., v_t)$ is a shortest path from v_j to v_t.*

Proof. Assuming there is an alternative path $P^*(u, v_j)$ that differs from $\rho(v_j)$ where $Rank\left(\delta^*(v_j)\right) < Rank\left(\delta(v_j)\right)$, and the shortest path from u to v_j becomes $\rho^*(v_j)$. Additionally, the shortest path from u to v_t is found in $P^*(u, v_t)$, which differs from $\rho(v_t)$. This creates a contradiction, thus $\rho(v_j) = P(u, v_j)$ is the shortest path from u to v_j. Similarly, it can be demonstrated that the shortest path from v_j to v_t is $P(u, v_j)$. Therefore, the lemma is correct. □

Lemma 3.2. *At the end of the algorithm, let $\rho(v_n) = (v_1, v_2, ..., v_n)$ be the sequence of vertices of the shortest path from v_1 to v_n. Then*

$$Rank(\delta(v_1)) \leq Rank(\delta(v_2)) \leq \cdots \leq Rank(\delta(v_n)). \tag{3.1}$$

Proof. For $n = 1$, the accuracy is obvious. Assuming that inequality (3.1) is true for $n = k$, we need to prove that it is true for $n = k + 1$. In this case,

$$Rank(\delta(v_1)) \leq Rank(\delta(v_2)) \leq \cdots \leq Rank(\delta(v_k)) \leq Rank(\delta(v_k) + w(v_k v_{k+1})) = Rank(\delta(v_{k+1}))$$

due to $w(v_k v_{k+1})$, which has a positive Z-number weight. Therefore, the inequality (3.1) is satisfied. Thus, $\rho(v_{k+1}) = (v_1, v_2, ..., v_{k+1})$, and the lemma holds true. □

Theorem 3.1. *Dijkstra's algorithm computes the shortest path for each vertex x in the Z-graph G and determines the Z-cost value of this path. When the algorithm is complete,* $\delta(x)$ *is the minimum Z-cost from u to x, and* $\rho(x) = P(u, x)$ *is the shortest path from u to x.*

Proof. The correctness of the theorem is obvious: When $x = u$, then $\delta(x) = 0$ and $\rho(x) = (u)$. Let us consider the case where $x \neq u$. To establish the existence of $\rho(x)$, which is the shortest path from u to x by induction, and $\delta(x)$, which is the Z-cost value of this path, let us assume that the shortest path from u to x has the minimum number n of edges.

If $n = 1$, then the shortest path from u to x, according to Step 2*, is $\rho(x) = P(u, x)$, i.e., (u, x), and the Z-cost value of this path is $\delta(x) = w(ux)$.

Assuming k is a positive integer greater than 1, if the number of edges on the shortest path from u to x is $n = k$, let us accept that $\delta(x) = \delta(v_k)$ is the Z-cost value from $u = v_0$ to $x = v_k$, and $\rho(x) = P(u, v_k)$ is the shortest path from $u = v_0$ to $x = v_k$. As shown in Figure 2, it is worth noting that $P(u, v_k)$ may not be the same as the Q path. However, if they differ, their Z-cost values will be the same, i.e., $\delta(v_k)$.

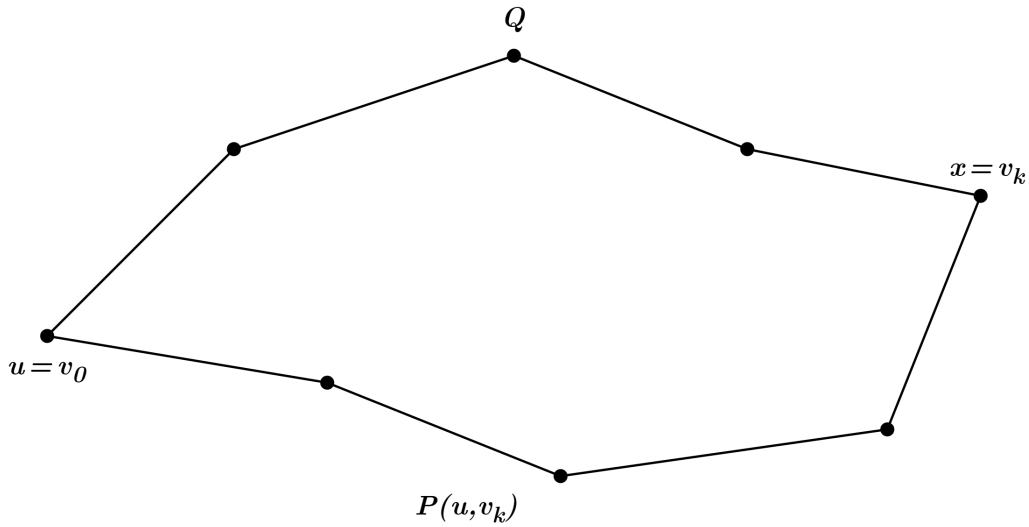


Figure 2. Shortest path from $u = v_0$ to $x = v_k$.

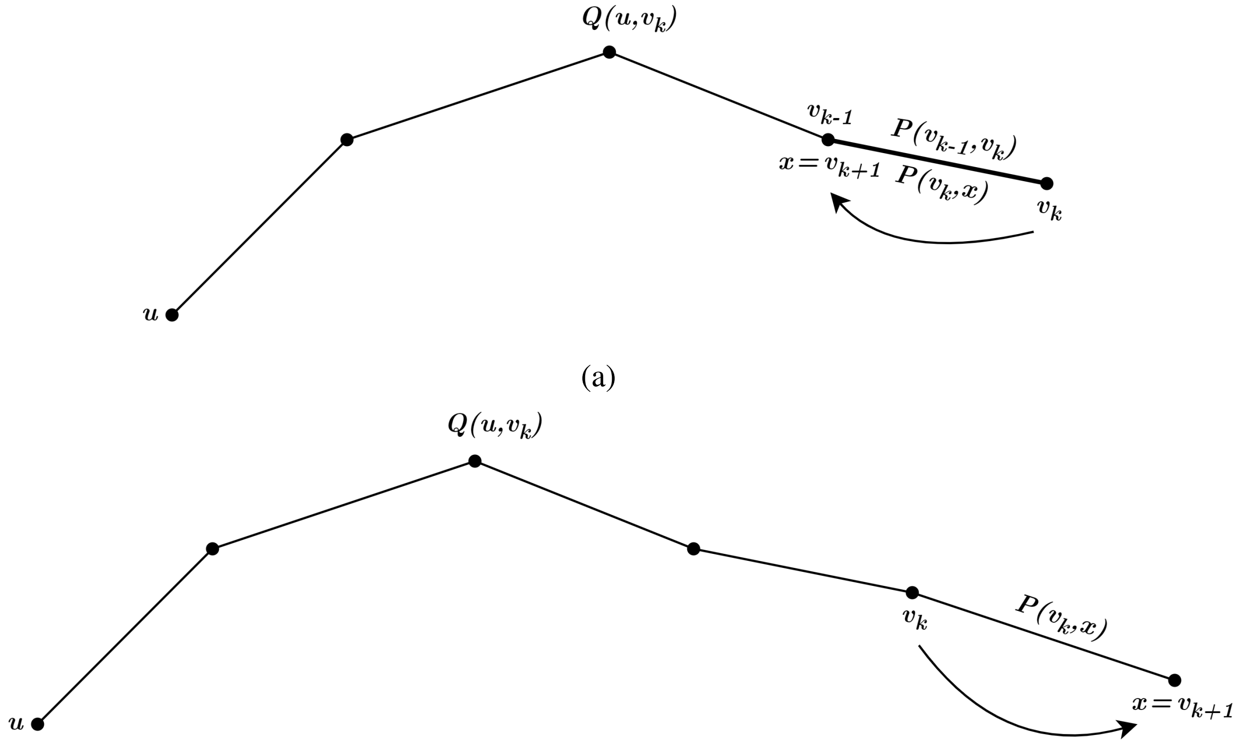
Now we will demonstrate the theorem's correctness in the event that the number of edges on the shortest path from u to x is $n = k + 1$.

Let $v_k = v_i$ and $x = v_j$, where i and j are unique integers.

If $j < i$, then

$$Rank(\delta(v_{k-1})) \leq Rank(\delta(v_k)) \leq Rank(\delta(v_k) + w(v_k x)) = Rank(\delta(x))$$

due to Lemma 3.2 and the positive Z-number weight of $w(v_k x)$ (see Figure 3(a)). Since a value greater than the current $Rank(\delta(v_{k-1}))$ value is found for vertex v_{k-1}, no updates are made as per Step 3*. Therefore, $\delta(x) = \delta(v_{k-1})$ and $\rho(x) = P(u, v_{k-1}) = (v_0, v_1, v_2, ..., v_{k-1})$.



(b)

Figure 3. Paths from u to $x = v_{k+1}$.

On the other hand, let $j > i$. If x is a vertex that has not been visited before, then

$$Rank(\delta(x)) = Rank(\delta(v_k) + w(v_k x)) < \infty.$$

In this case, due to Steps 2* and 3*, $\delta(x) = \delta(v_{k+1})$ and $\rho(x) = P(u, v_{k+1}) = (v_0, v_1, v_2, ..., v_{k+1})$ (see Figure 3(b)). If x is a vertex that has been visited before, then $\delta(x) = \min\{Rank(\delta(v_{k+1})), Rank(\delta(x^*))\}$ due to Step 3* in this case, where $Rank(\delta(x^*))$ is the previous Z-cost value of the vertex v_{k+1}. Moreover, $\rho(x) = P(u, x)$ is the shortest path with a Z-cost value $\delta(x)$. □

4. Numerical example

In this section, the method proposed in 3.3 is implemented on an illustrative example. The solution and obtained results of this problem are given as follows.

The Z-graph of a communication network with alternative paths from source node a to destination node i is shown in Figure 4. In this network, the nodes a, b, c, d, e, f, g, h, and i represent the routers and the vertices of the Z-graph. The links between neighboring nodes, which are the edges of the Z-graph, are represented by Z-numbers in the form $Z = (A, B)$, indicating their Z-cost values. The metric of the first component of these values, A, is RTT and they are triangular fuzzy numbers in seconds.

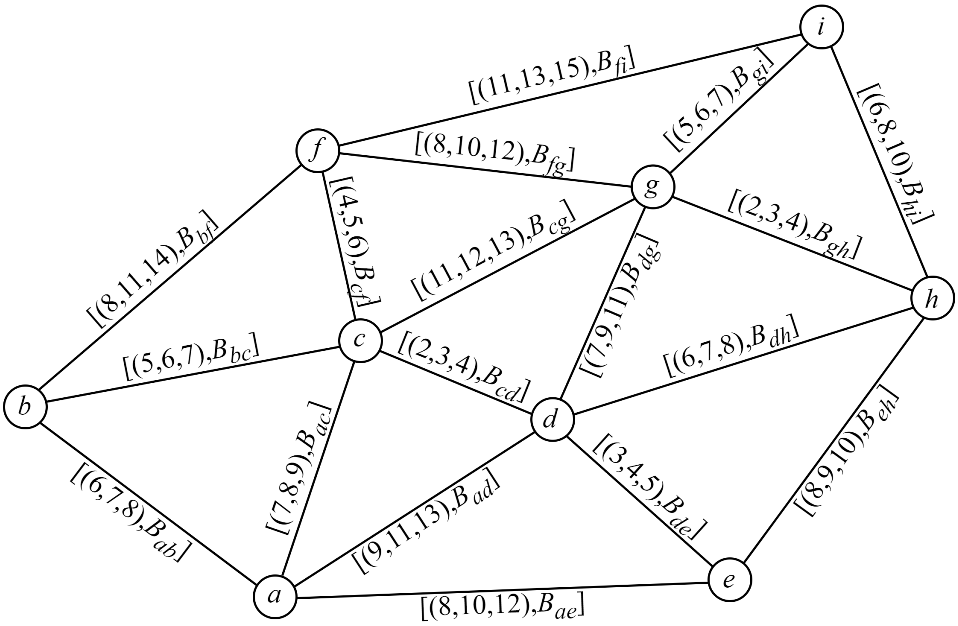
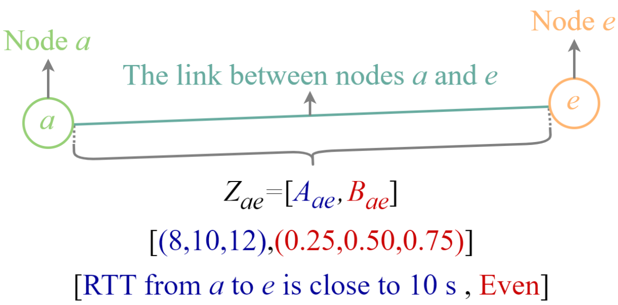


Figure 4. Nodes and links in the Z-graph of a communication network.

B, the second component of the Z-cost values, is the probability measure of the reliability of the triangular fuzzy number A, which shows the RTT value between two nodes.

The symbols for the connection between nodes a and e are displayed in Figure 5.



- Z_{ae} is a Z-number that represents the Z-cost value of the link between nodes a and e.
- A triangular fuzzy number, A_{ae} , measured in seconds, indicates the round-trip time of a data packet from node a to node e.
- B_{ae} is a triangular fuzzy number representing the probability that the RTT value between nodes a and e is equal to A_{ae}.

Figure 5. Illustrative figure of representations between two nodes.

The link between nodes a and e are weighted according to the Z-number $Z_{ae} = (A_{ae}, B_{ae})$. The triangular fuzzy number $A_{ae} = (8, 10, 12)$ indicates that a data packet sent from node a to node e completes the round trip in close to 10 seconds. The triangular fuzzy number $B_{ae} = (0.25, 0.50, 0.75)$ can be regarded as a restriction probability measure, indicating that the reliability of representing this path with A_{ae} is evenly 0.50.

Fuzzy linguistic scales are widely used to represent real-life situations of uncertainty. Erkayman et al. used Chen's fuzzy language scale to calculate fuzzy TOPSIS when solving a multiple criteria decision-making problem in [35]. In this study, the fuzzy scale "Linguistic Values and Fuzzy Values of Probability" was used, which is presented in Table 1, that was prepared by M. Ebrat and R. Ghodsi [36]. This scale was used to represent the second component, B, of the Z-numbers. The scale was adjusted for ease of operation in the solution steps by replacing 0 with 0.01 in B_1 and 1 with 0.99 in B_5.

Table 1. Linguistic expressions and fuzzy values of the probability.

Variable	Linguistic Expression	Triangular Fuzzy Number
B_1	Very Unlikely	$[0.01, 0.125, 0.25]$
B_2	Unlikely	$[0.05, 0.275, 0.5]$
B_3	Even	$[0.25, 0.5, 0.75]$
B_4	Likely	$[0.5, 0.725, 0.95]$
B_5	Very Likely	$[0.75, 0.875, 0.99]$

Table 2 shows the B values for the scenarios based on four different expressions. Each scenario was created by changing a single B value in the first scenario while keeping the A values of the Z-numbers constant.

Table 2. B values used for each scenario.

	B_{ab}	B_{ac}	B_{ad}	B_{ae}	B_{bc}	B_{bf}	B_{cd}	B_{cf}	B_{cg}	B_{de}	B_{dg}	B_{dh}	B_{eh}	B_{fg}	B_{fi}	B_{gh}	B_{gi}	B_{hi}
Scenario 1	B_2	B_3	B_2	B_3	B_4	B_5	B_1	B_2	B_5	B_1	B_4	B_3	B_4	B_3	B_4	B_1	B_2	B_5
Scenario 2	B_2	B_3	B_2	B_3	B_4	B_5	B_1	B_2	B_5	B_1	B_4	B_3	B_4	B_3	B_2	B_1	B_2	B_5
Scenario 3	B_2	B_4	B_2	B_3	B_4	B_5	B_1	B_2	B_5	B_1	B_4	B_3	B_4	B_3	B_4	B_1	B_2	B_5
Scenario 4	B_2	B_3	B_2	B_3	B_4	B_5	B_1	B_2	B_5	B_1	B_4	B_3	B_4	B_3	B_4	B_3	B_2	B_5

Note that the Z-cost values in Figure 4 and Table 2 are determined based on assumptions.

The objective is to determine the shortest path and its weight for a data packet transmitted from node a to node i.

Solution 4.1. *The steps to solve Scenario 1 using the method proposed in Section 3.3 are presented below.*

Step 1*. Set $L(a) = (-, 0)$. Set $L(x) = (-, \infty)$ for each $x \in V - \{a\}$ (see Table 3(i)). Mark node a (see Figure 6 (i)).

Table 3. Labeling for the solution steps.

(i)	(ii)	(iii)
	u=a	u=b
$N=\{\}$	$N=\{b,c,d,e\}$	$N=\{c,d,e,f\}$
$L(a)=(-,0)$	$L(a)=(-,0)$	$L(a)=(-,0)$
$L(b)=(-,\infty)$	$L(b)=(a,[(6,7,8),(0.05,0.275,0.5)])$	$L(b)=(a,[(6,7,8),(0.05,0.275,0.5)])$
$L(c)=(-,\infty)$	$L(c)=(a,[(7,8,9),(0.25,0.5,0.75)])$	$L(c)=(a,[(7,8,9),(0.25,0.5,0.75)])$
$L(d)=(-,\infty)$	$L(d)=(a,[(9,11,13),(0.05,0.275,0.5)])$	$L(d)=(a,[(9,11,13),(0.05,0.275,0.5)])$
$L(e)=(-,\infty)$	$L(e)=(a,[(8,10,12),(0.25,0.5,0.75)])$	$L(e)=(a,[(8,10,12),(0.25,0.5,0.75)])$
$L(f)=(-,\infty)$	$L(f)=(-,\infty)$	$L(f)=(b,[(14,18,22),(0.20,0.71,0.86)])$
$L(g)=(-,\infty)$	$L(g)=(-,\infty)$	$L(g)=(-,\infty)$
$L(h)=(-,\infty)$	$L(h)=(-,\infty)$	$L(h)=(-,\infty)$
$L(i)=(-,\infty)$	$L(i)=(-,\infty)$	$L(i)=(-,\infty)$
(iv)	(v)	(vi)
u=c	u=d	u=e
$N=\{d,e,f,g\}$	$N=\{e,f,g,h\}$	$N=\{f,g,h\}$
$L(a)=(-,0)$	$L(a)=(-,0)$	$L(a)=(-,0)$
$L(b)=(a,[(6,7,8),(0.05,0.275,0.5)])$	$L(b)=(a,[(6,7,8),(0.05,0.275,0.5)])$	$L(b)=(a,[(6,7,8),(0.05,0.275,0.5)])$
$L(c)=(a,[(7,8,9),(0.25,0.5,0.75)])$	$L(c)=(a,[(7,8,9),(0.25,0.5,0.75)])$	$L(c)=(a,[(7,8,9),(0.25,0.5,0.75)])$
$L(d)=(c,[(9,11,13),(0.02,0.24,0.45)])$	$L(d)=(c,[(9,11,13),(0.02,0.24,0.45)])$	$L(d)=(c,[(9,11,13),(0.02,0.24,0.45)])$
$L(e)=(a,[(8,10,12),(0.25,0.5,0.75)])$	$L(e)=(a,[(8,10,12),(0.25,0.5,0.75)])$	$L(e)=(a,[(8,10,12),(0.25,0.5,0.75)])$
$L(f)=(c,[(11,13,15),(0.10,0.45,0.70)])$	$L(f)=(c,[(11,13,15),(0.10,0.45,0.70)])$	$L(f)=(c,[(11,13,15),(0.10,0.45,0.70)])$
$L(g)=(c,[(18,20,22),(0.45,0.72,0.87)])$	$L(g)=(d,[(16,20,24),(0.04,0.43,0.69)])$	$L(g)=(d,[(16,20,24),(0.04,0.43,0.69)])$
$L(h)=(-,\infty)$	$L(h)=(d,[(15,18,21),(0.03,0.34,0.59)])$	$L(h)=(d,[(15,18,21),(0.03,0.34,0.59)])$
$L(i)=(-,\infty)$	$L(i)=(-,\infty)$	$L(i)=(-,\infty)$
(vii)	(viii)	(ix)
u=f	u=h	u=g
$N=\{g,h,i\}$	$N=\{g,i\}$	$N=\{i\}$
$L(a)=(-,0)$	$L(a)=(-,0)$	$L(a)=(-,0)$
$L(b)=(a,[(6,7,8),(0.05,0.275,0.5)])$	$L(b)=(a,[(6,7,8),(0.05,0.275,0.5)])$	$L(b)=(a,[(6,7,8),(0.05,0.275,0.5)])$
$L(c)=(a,[(7,8,9),(0.25,0.5,0.75)])$	$L(c)=(a,[(7,8,9),(0.25,0.5,0.75)])$	$L(c)=(a,[(7,8,9),(0.25,0.5,0.75)])$
$L(d)=(c,[(9,11,13),(0.02,0.24,0.45)])$	$L(d)=(c,[(9,11,13),(0.02,0.24,0.45)])$	$L(d)=(c,[(9,11,13),(0.02,0.24,0.45)])$
$L(e)=(a,[(8,10,12),(0.25,0.5,0.75)])$	$L(e)=(a,[(8,10,12),(0.25,0.5,0.75)])$	$L(e)=(a,[(8,10,12),(0.25,0.5,0.75)])$
$L(f)=(c,[(11,13,15),(0.10,0.45,0.70)])$	$L(f)=(c,[(11,13,15),(0.10,0.45,0.70)])$	$L(f)=(c,[(11,13,15),(0.10,0.45,0.70)])$
$L(g)=(d,[(16,20,24),(0.04,0.43,0.69)])$	$L(g)=(h,[(17,21,25),(0.03,0.33,0.57)])$	$L(g)=(h,[(17,21,25),(0.03,0.33,0.57)])$
$L(h)=(d,[(15,18,21),(0.03,0.34,0.59)])$	$L(h)=(d,[(15,18,21),(0.03,0.34,0.59)])$	$L(h)=(d,[(15,18,21),(0.03,0.34,0.59)])$
$L(i)=(f,[(22,26,30),(0.19,0.66,0.85)])$	$L(i)=(h,[(21,26,31),(0.05,0.51,0.74)])$	$L(i)=(g,[(22,27,32),(0.03,0.38,0.63)])$

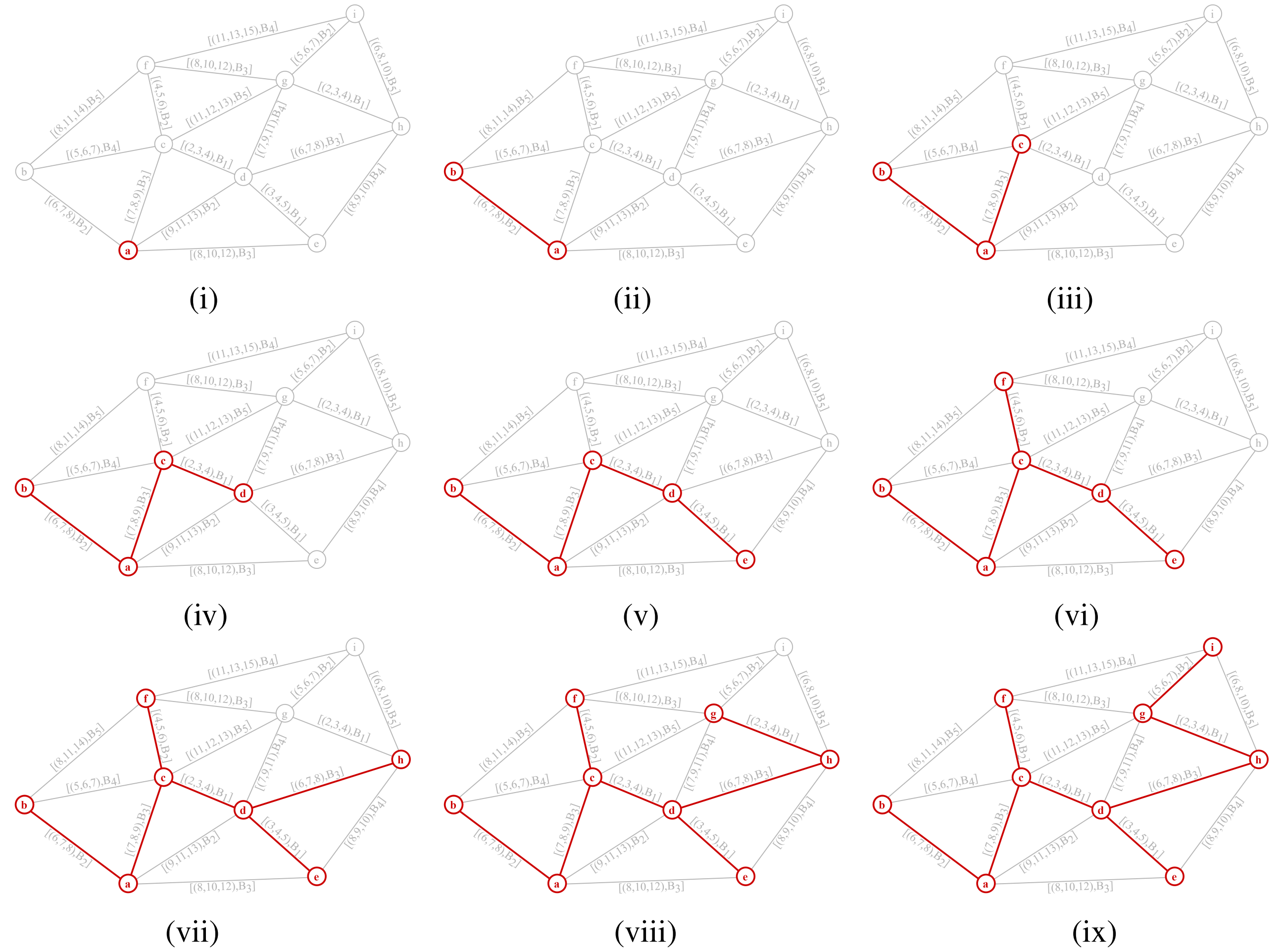


Figure 6. Solution step markings.

Step 2*. Let $u = a$. Then the neighbors of node a are $N = \{b, c, d, e\}$.

Update,

$L(b) = (a, [(6, 7, 8), (0.05, 0.275, 0.5)])$,

$L(c) = (a, [(7, 8, 9), (0.25, 0.5, 0.75)])$,

$L(d) = (a, [(9, 11, 13), (0.05, 0.275, 0.5)])$, and

$L(e) = (a, [(8, 10, 12), (0.25, 0.5, 0.75)])$ (see Table 3(ii)).

The rank values $\delta(b)$, $\delta(c)$, $\delta(d)$, and $\delta(e)$ are 0.95890, 0.99465, 0.99583, and 0.99942, respectively. Since the $Rank(\delta(b))$ value is minimal, link ab and node b are marked (see Figure 6(ii)).

Step 3*. Let $u = b$. Then $N = \{c, d, e, f\}$. Compute the Z-addition $\delta(b) + w(bv)$ and the value of $Rank(\delta(v))$ for each $v \in N$ that is a neighbor of node b in N.

$Rank(\delta(b) + w(bc)) = Rank([(11, 13, 15), (0.10, 0.48, 0.73)]) = 0.99984 \nless 0.99465 = Rank(\delta(c))$

$Rank(\delta(b) + w(bf)) = Rank([(14, 18, 22), (0.20, 0.71, 0.86)]) = 0.9999998 < \infty$

Update, $L(f) = (b, [(14, 18, 22), (0.20, 0.71, 0.86)])$ (see Table 3(iii)).

Step 4*. Since the $Rank(\delta(c))$ value is minimal, link ac and node c are marked (see Figure 6(iii)).

Step 5*. (Step 3*-Iteration 2) Let $u = c$. Then $N = \{d, e, f, g\}$. Compute the Z-addition $\delta(c) + w(cv)$ and the value of $Rank(\delta(v))$ for each $v \in N$ that is a neighbor of node c in N.

$Rank(\delta(c) + w(cd)) = Rank([(9, 11, 13), (0.02, 0.24, 0.45)]) = 0.99359 < 0.99583 = Rank(\delta(d))$

$Rank(\delta(c) + w(cf)) = Rank([(11, 13, 15), (0.10, 0.45, 0.70)]) = 0.99962 < 0.9999998 = Rank(\delta(f))$

$Rank\,(\delta(c) + w(cg)) = Rank\,([(18, 20, 22), (0.45, 0.72, 0.87)]) = 0.999999897 < \infty$

Update,

$L(d) = (c, [(9, 11, 13), (0.02, 0.24, 0.45)])$,

$L(f) = (c, [(11, 13, 15), (0.10, 0.45, 0.70)])$, and

$L(g) = (c, [(18, 20, 22), (0.45, 0.72, 0.87)])$. (see Table 3(iv)).

Step 5*. (Step 4*-Iteration 2) Since the $Rank(\delta(d))$ value is minimal, link cd and node d are marked (see Figure 6(iv)).

Step 5*. (Step 3*-Iteration 3) Let $u = d$. Then $N = \{e, f, g, h\}$. Compute the Z-addition $\delta(d) + w(dv)$ and the value of $Rank(\delta(v))$ for each $v \in N$ that is a neighbor of node d in N.

$Rank(\delta(d) + w(de)) = Rank([(12, 15, 18), (0.03, 0.26, 0.47)]) = 0.99953 \nless 0.99942 = Rank(\delta(e))$

$Rank\,(\delta(d) + w(dg)) = Rank\,([(16, 20, 24), (0.04, 0.43, 0.69)]) = 0.9999988 < 0.999999897 = Rank\,(\delta(g))$

$Rank(\delta(d) + w(dh)) = Rank([(15, 18, 21), (0.03, 0.34, 0.59)]) = 0.99996 < \infty$

Update,

$L(g) = (d, [(16, 20, 24), (0.04, 0.43, 0.69)])$ and

$L(h) = (d, [(15, 18, 21), (0.03, 0.34, 0.59)])$. (see Table 3(v)).

Step 5*. (Step 4*-Iteration 3) Since the $Rank(\delta(e))$ value is minimal, link de and node e are marked (see Figure 6(v)).

Step 5*. (Step 3*-Iteration 4) Let $u = e$. Then $N = \{f, g, h\}$. Compute the Z-addition $\delta(e) + w(ev)$ and the value of $Rank(\delta(v))$ for each $v \in N$ that is a neighbor of node e in N.

$Rank(\delta(e) + w(eh)) = Rank([(20, 24, 28), (0.03, 0.33, 0.58)]) = 0.9999998 \nless 0.99996 = Rank(\delta(h))$

Do not update (see Table 3(vi)).

Step 5*. (Step 4*-Iteration 4) Since the $Rank(\delta(f))$ value is minimal, link cf and node f are marked (see Figure 6(vi)).

Step 5*. (Step 3*-Iteration 5) Let $u = f$. Then $N = \{g, h, i\}$. Compute the Z-addition $\delta(f) + w(fv)$ and the value of $Rank(\delta(v))$ for each $v \in N$ that is a neighbor of node f in N.

$Rank(\delta(f) + w(fg)) = Rank([(19, 23, 27), (0.18, 0.60, 0.81)]) = 0.999999997 \nless 0.9999988 = Rank(\delta(g))$

$Rank(\delta(f) + w(fi)) = Rank([(22, 26, 30), (0.19, 0.66, 0.85)]) = 0.999999997 < \infty$

Update, $L(i) = (f, [(22, 26, 30), (0.19, 0.66, 0.85)])$ (see Table 3(vii)).

Step 5*. (Step 4*-Iteration 5) Since the $Rank(\delta(h))$ value is minimal, link dh and node h are marked (see Figure 6(vii)).

Step 5*. (Step 3*-Iteration 6) Let $u = h$. Then $N = \{g, i\}$. Compute the Z-addition $\delta(h) + w(hv)$ and the value of $Rank(\delta(v))$ for each $v \in N$ that is a neighbor of node h in N.

$Rank(\delta(h) + w(hg)) = Rank([(17, 21, 25), (0.03, 0.33, 0.57)]) = 0.999989 < 0.9999988 = Rank(\delta(g))$

$Rank(\delta(h) + w(hi)) = Rank([(21, 26, 31), (0.05, 0.51, 0.74)]) = 0.99999998 < 0.999999997 = Rank(\delta(i))$

Update,

$L(g) = (h, [(17, 21, 25), (0.03, 0.33, 0.57)])$ and

$L(i) = (h, [(21, 26, 31), (0.05, 0.51, 0.74)])$ (see Table 3(viii)).

Step 5*. (Step 4*-Iteration 6) Since the $Rank(\delta(g))$ value is minimal, link hg and node g are marked (see Figure 6(viii)).

Step 5*. (Step 3*-Iteration 7) Let $u = g$. Then $N = \{i\}$. Compute the Z-addition $\delta(g) + w(gv)$ and the value of $Rank(\delta(v))$ for each $v \in N$ that is a neighbor of node g in N.

$Rank(\delta(g) + w(gi)) = Rank([(22, 27, 32), (0.03, 0.38, 0.63)]) = 0.9999998 < 0.99999998 = Rank(\delta(i))$

Update, $L(i) = (g, [(22, 27, 32), (0.03, 0.38, 0.63)])$ (see Table 3(ix)).

Step 5*. (Step 4*-Iteration 7) Since the $Rank(\delta(i))$ value is minimal, link gi and node i are marked (see Figure 6(ix)).

Step 6*. The algorithmic process is complete since all nodes have been visited. By considering the first components of the labels, the shortest path $\rho(i) = (a, c, d, h, g, i)$ from the starting node a to the target node i is determined. Additionally, the Z-cost value of this path is $Z_{\rho(i)} = \delta(i) = [(22, 27, 32), (0.03, 0.38, 0.63)]$.

Table 4 presents the results obtained by applying the same steps to the other scenarios listed in Table 2.

Table 4. Optimal routes obtained for each scenario.

	$\rho(i)$	$Z_{\rho(i)}$
Scenario 1	(a, c, d, h, g, i)	$[(22, 27, 32), (0.03, 0.38, 0.63)]$
Scenario 2	(a, c, f, i)	$[(22, 26, 30), (0.09, 0.44, 0.69)]$
Scenario 3	(a, d, h, g, i)	$[(22, 27, 32), (0.05, 0.40, 0.65)]$
Scenario 4	(a, c, d, g, i)	$[(21, 26, 31), (0.05, 0.48, 0.73)]$

Figure 7 shows the shortest paths from node a to node i in the graph for each scenario.

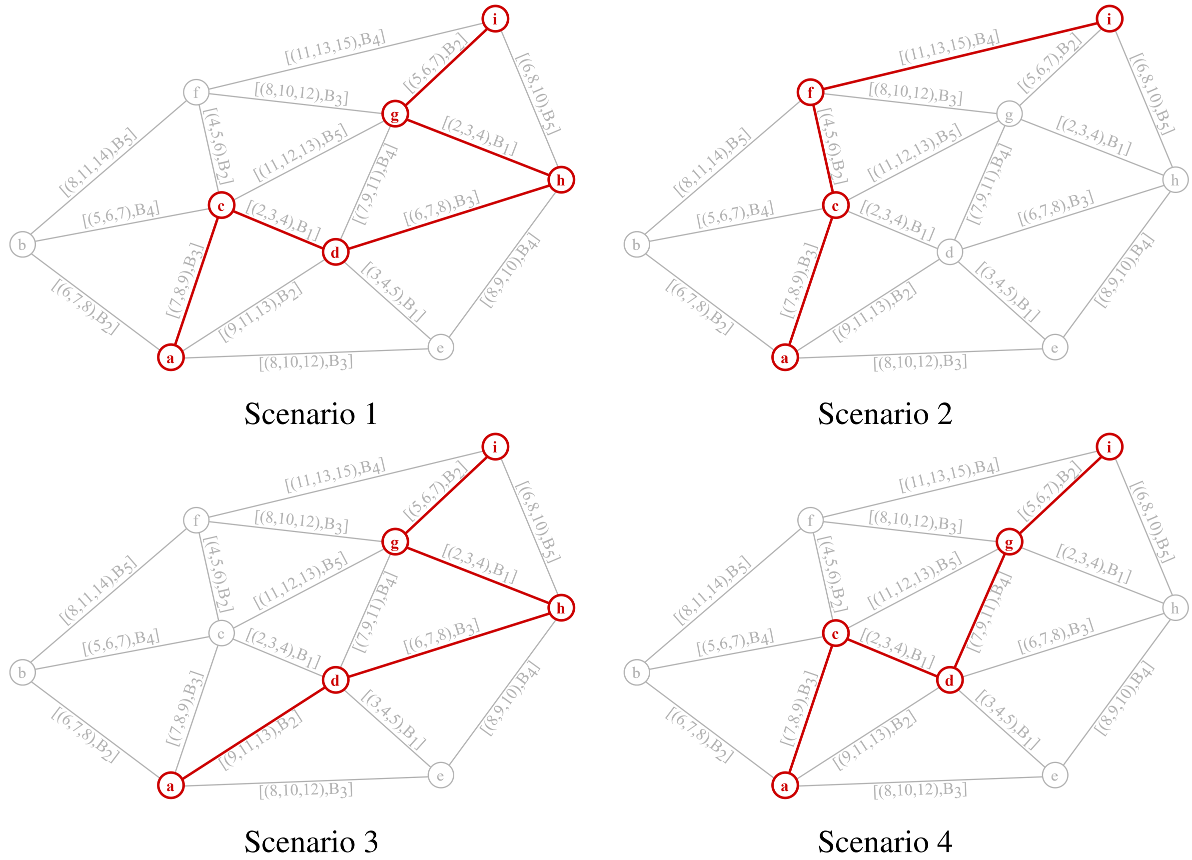


Figure 7. Shortest paths obtained for each scenario.

5. Results

In this study, Dijkstra's algorithm has been restructured to include Z-numbers. The algorithm of which steps and verification are given in 3.3 has been used in order to find the shortest path between the a and i nodes of the graph shown in Figure 4. The graph consisting of 9 nodes and 18 links has been designed for 4 different scenarios. For each scenario, the first components of the Z-cost values, which represent the RTT metric between links, have been kept constant. The scenarios have been differentiated by changing only one of the B values representing reliability in Scenario 1. Scenario 2 has been obtained by replacing the B_{fi} value with B_2, Scenario 3 by replacing the B_{ac} value with B_4, and Scenario 4 by replacing the B_{gh} value with B_3.

The results of addition operations with Z-numbers have been presented with two decimal digits after the comma and Z-ranking values have been presented with five decimal digits after the comma. If the decimal part of the Z-ranking value consists of 9 consecutive digits, then the decimal part is rounded to contain a number other than 9.

Table 3 presents the algorithmic step values of the first scenario. The nodes and paths followed for each step have been shown in Figure 6.

As a result, the shortest paths obtained for each scenario and their weights are presented in Table 4, and these paths are shown in Figure 7.

6. Conclusions

In this article, it has been shown that the shortest path can be determined more precisely by representing the uncertainties that may be encountered in the data transfer processes between routers in communication networks with Z-numbers. When the results obtained are taken into consideration, it is seen that the method put forward allows the uncertainty situations to be taken into account by blending.

By considering the central values of the first components of the Z-numbers representing the path weights of the graph shown in Figure 4, these weights can be considered as crisp numbers. In this case, the value of the shortest path from node a to node i is found to be 26 in seconds. There are a total of six paths with this value. These are: (a, c, f, i), (a, c, g, i), (a, c, d, g, i), (a, c, d, h, i), (a, d, g, i), and (a, d, h, i). Again, considering only the first components of the Z-numbers indicating the path weights, these weights can be considered as fuzzy numbers. In this case, considering the order of the fuzzy numbers, even if some of the shortest paths with equal weight are eliminated, paths with the same fuzzy numbers are encountered. For example, the fuzzy cost value of the paths (a, c, d, g, i), (a, c, d, h, i), (a, d, g, i), and (a, d, h, i) is $(21, 26, 31)$. Therefore, it is not enough to show the weights of the paths between the links with fuzzy numbers. In this study, uncertainties were included in the calculation in a more precise way using Z-numbers and B in Scenario 1 was found to be the optimal way with fuzzy number values (a, c, d, h, g, i). The value of this path is also $[(22, 27, 32), (0.03, 0.38, 0.63)]$.

In this study, the Z-number ranking algorithm according to a sigmoid function based on convex combination is used for Z-ranking and the validity of the results can be increased by selecting the most appropriate method for the problem from methods in the literature.

The Z-cost values of the connections between the nodes presented in Table 2 are hypothetical. In line with expert opinions, it is expected that the method proposed in 3.3 with Z-cost values to be obtained from real data will provide a more realistic solution to the shortest path optimization problems.

The complexity of operations involving Z-numbers renders the applicability of the proposed method increasingly challenging in real-world situations involving big data sets. Furthermore, in problems of a large dimension, it is considered that the effect of component B is reduced as the number of additions increases, which in turn results in a weakening of the overall impact of the proposed method.

The approach set out in this article can be applied to various fields such as multi-criteria decision-making, optimization, logistics, traveling salesman, and vehicle routing problems. In addition, the method set out in this article can be extended to cover different algorithms and methods.

Author contributions

Nurdoğan Güner: Conceptualization, Formal analysis, Methodology, Software, Validation, Visualization, Writing-original draft, Writing-review and editing; Halit Orhan: Conceptualization, Project administration, Supervision, Writing-review and editing; Tofigh Allahviranloo: Conceptualization, Project administration, Supervision, Writing-review and editing; Bilal Usanmaz: Conceptualization, Formal analysis, Methodology, Software, Validation, Visualization, Writing-review and editing. All the authors have read and approved the final version of the manuscript for publication.

Conflict of interest

The authors declare no conflict of interest.

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