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Research article

Strong consistency properties of the variance change point estimator based on strong-mixing samples

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Abstract: In this paper, our primary attention was centered on the issue of detecting the variance change point for strong-mixing samples. We delved into the cumulative sum (CUSUM) estimator of variance change model and established the strong convergence rate of the variance change point estimation. Furthermore, to corroborate the effectiveness of the CUSUM based methodology, we have conducted a series of simulations, the outcomes of which underscored its validity.

Keywords: variance change point; CUSUM; strong-mixing sequence; strong consistency
Mathematics Subject Classification: 62F12

1. Introduction

The problem of change point detection originates from practical application research such as product quality control and risk management. Its application fields are extremely broad, spanning signal processing [1], finance [2], ecology [3], disease outbreak monitoring [4], and neuroscience [5,6], and it has been extensively studied over the past few decades. To detect change points and estimate their locations, numerous methods have emerged, including least squares (LS, [7]), Bayesian methods [8], maximum likelihood methods [9], and some nonparametric methods [10, 11]. Among them, the cumulative sum (CUSUM) method based on LS estimation stands out as an attractive approach for detecting variance changes in sequences, as it avoids certain assumptions about the underlying error distribution function and is computationally simple [12].

We consider the following variance change point model:

$$Y_t = \begin{cases} \mu + \sigma_1 e_t, & 1 \le t \le k_0, \\ \mu + \sigma_2 e_t, & k_0 + 1 \le t \le n, \end{cases} \tag{1.1}$$

where μ and $\sigma_1 \neq \sigma_2$ are parameters, and k_0 is the unknown location of a change point with $k_0 = [\tau_0 n]$. Throughout the remainder of this paper, we operate under the assumption that $0 < \gamma_1 < \tau_0 < \gamma_2 < 1$, where γ_1 and γ_2 are constants and independent of n.

In prior works, [13] as well as [14] tackled the issue of identifying scale shifts in infinite order moving average processes. [15] introduced a novel class of weighted difference statistics aimed at detecting and estimating variance change points in time series featuring weakly dependent blocks and dependent panel data; and [16] derived the strong convergence rate for variance change estimators in linear processes. More recently, [17] proposed a weighted sum of variance powers test for variance shifts in data sequences; and [18] established the weak convergence rate for multiple variance change estimation in linear processes under negatively super-additive dependence and so on. Building upon these foundations, the present paper delves into the realm of variance change point estimation for strong-mixing (or α-mixing) dependent random variables.

Now, let us revisit the notion of strong-mixing (or α-mixing) random variables, which is defined as follows.

Definition 1.1. *For a sequence $\{X_n, n \geq 1\}$ of random variables or random vectors, the α-mixing coefficient $\alpha(n)$ is defined as*

$$\alpha(n) := \sup_{k\geq 1} \sup\{|P(AB) - P(A)P(B)| : A \in \mathcal{F}_{n+k}^{\infty}, B \in \mathcal{F}_1^k\},$$

where $\mathcal{F}_n^m = \sigma(X_i : n \leq i \leq m)$ denotes the σ-algebra generated by $X_n, \cdots, X_m$ with $n \leq m$. Then, the sequence $\{X_n, n \geq 1\}$ is said to be strong-mixing (or α-mixing), if $\alpha(n) \to 0$ as $n \to \infty$.

The strong-mixing is weaker than most other mixing conditions. One can refer to [19], for instance. There are several examples of α-mixing sequences, including the AR (Autoregressive) model and many time series models, which have many applications in practice. For more details about the examples of time series with strong mixing properties, one can refer to [20] and [21], among others.

For the change point model under strong-mixing, [22] got the consistency estimators for mean and covariance functions and the limit distribution of the CUSUM estimator under strong-mixing sequences; and [23] considered the mean-variance model with strong-mixing and described a combination test for the mean shift and variance change. So far, no scholars have discussed the strong consistency of the CUSUM estimator of the variance change point model under strong mixing conditions, which arouses the author's interest. The objective of this research endeavor is to identify variance change point in strong-mixing samples. To this end, we derive the strong convergence rate of the variance change point estimator. Furthermore, we demonstrate the efficacy of the CUSUM-type estimator through simulation and the analysis of real data, utilizing the R software package.

The structure of this paper is outlined as follows: In Section 2, we introduce the CUSUM-type variance change point estimator and establish its strong convergence rate. Section 3 presents simulation studies designed to showcase the performance of the proposed estimator. Lastly, Section 4 provides the proofs of our main results.

Throughout the paper, $a_n = O(b_n)$ denotes that there exists a positive constant C such that $|a_n| \leq C|b_n|$, while $a_n = o(b_n)$ means $a_n/b_n \to 0$, as $n \to \infty$. Let C, C_1, C_2, $\cdots$ denote positive constants whose values may vary at each occurrence. All limits are taken as the sample size n tends to ∞, unless otherwise specified.

2. Estimation and main results

Let $\tau_0 = k_0/n$. Assuming the parameter μ is known, without compromising the essence of generality, we postulate that $\mu = 0$.

We give the CUSUM estimator of the variance change point k_0:

$$\hat{k}_n = \arg\max_{1\le k\le n}|U_k|, \text{ and } \hat{\tau}_n = \hat{k}_n/n, \tag{2.1}$$

with

$$U_k = \left[\frac{k\,(n-k)}{n}\right]^{1-\alpha}\left|\frac{1}{k}\sum_{t=1}^{k} Y_t^2 - \frac{1}{n-k}\sum_{t=k+1}^{n} Y_t^2\right|,$$

where $0 \le \alpha < 1$. We list some assumptions as follows.

Assumption 1. $\{e_t, t = 1, 2, \ldots\}$ is a second order stationary sequence of α-mixing random variables with $Ee_t = 0$ and $Var(e_t) = \sigma^2 < \infty$.

Assumption 2. $E|e_t|^{2(r+\delta)} < \infty, r > 2, \delta > 0$, and $\alpha(n) = O(n^{-\gamma})$ for some $\gamma > r(r + \delta)/(2\delta)$.

Assumption 3. There is a positive constant c and a random variable e, for which, given any e_t, it holds true that

$$P(|e_t| \ge a) \le cP(|e| \ge a).$$

Remark 2.1. If the $\{e_t, t \ge 1\}$ are identically distributed, then Assumption 3 is satisfied obviously. In addition, let $p > 0$, $P(e_t = 0) = 1 - \frac{1}{t}, P(e_t = t^{1/p}) = \frac{1}{t}, t \ge 1$. According to [24], we can see that the sequence $\{e_t, t \ge 1\}$ is stochastically dominated by a nonnegative random variable e with distribution function $F(x) = 1 - \sup_{t\ge1} P(e_t > x), x \in \mathbb{R}$. That is to say, Assumption 3 is satisfied.

For the variance change points estimator $\hat{\tau}_n$, we demonstrate a result of strong consistency in the following theorem.

Theorem 2.1. In the model (1.1), let $\hat{\tau}_n$ be the variance change point estimator defined by (2.1). If the assumptions (A1)–(A3) hold, then

$$\hat{\tau}_n \to \tau_0 \; a.s., \; n \to \infty.$$

Under the premises stated in Theorem 2.1, we proceed to derive the strong convergence rate of the variance change point estimator $\hat{\tau}_n$ in the following theorem.

Theorem 2.2. In the model (1.1), let $\hat{\tau}_n$ be the variance change point estimator defined by (2.1). If the assumptions (A1)–(A3) hold, then

$$\hat{\tau}_n - \tau_0 = o(M(n)/n) \; a.s.$$

for any $M(n)$ satisfying that $M(n) \uparrow \infty$.

3. Numerical analyses

3.1. Simulation

In this section, we undertake a series of straightforward simulations to validate the finite sample behavior of the CUSUM estimator of the variance change point for strong-mixing sequence.

We commence by generating strong mixing data. We define an AR(1) process $e_n = 0.5e_{n-1} + \xi_n, n \geq 1$, where $\xi_n \sim^{i.i.d.} N(0,1)$ and $e_0 \sim N(0, \frac{4}{3})$. It is straightforward to observe that $\{e_n, n \geq 1\}$ is a mean zero sequence of α-mixing random variables, as established by [20].

Take $\tau_0 = 0.25, 0.5, \alpha = 0, 0.1, 0.5$ and $\sigma_1 = 1, \sigma_2 = \delta = 3, -0.8$, respectively. We use the R software to compute τ_n with $n = 200, 400, 800, 1600$ for 1000 times. The results of these simulations are visually represented through the construction of box-plots, which are presented in Figure 1.

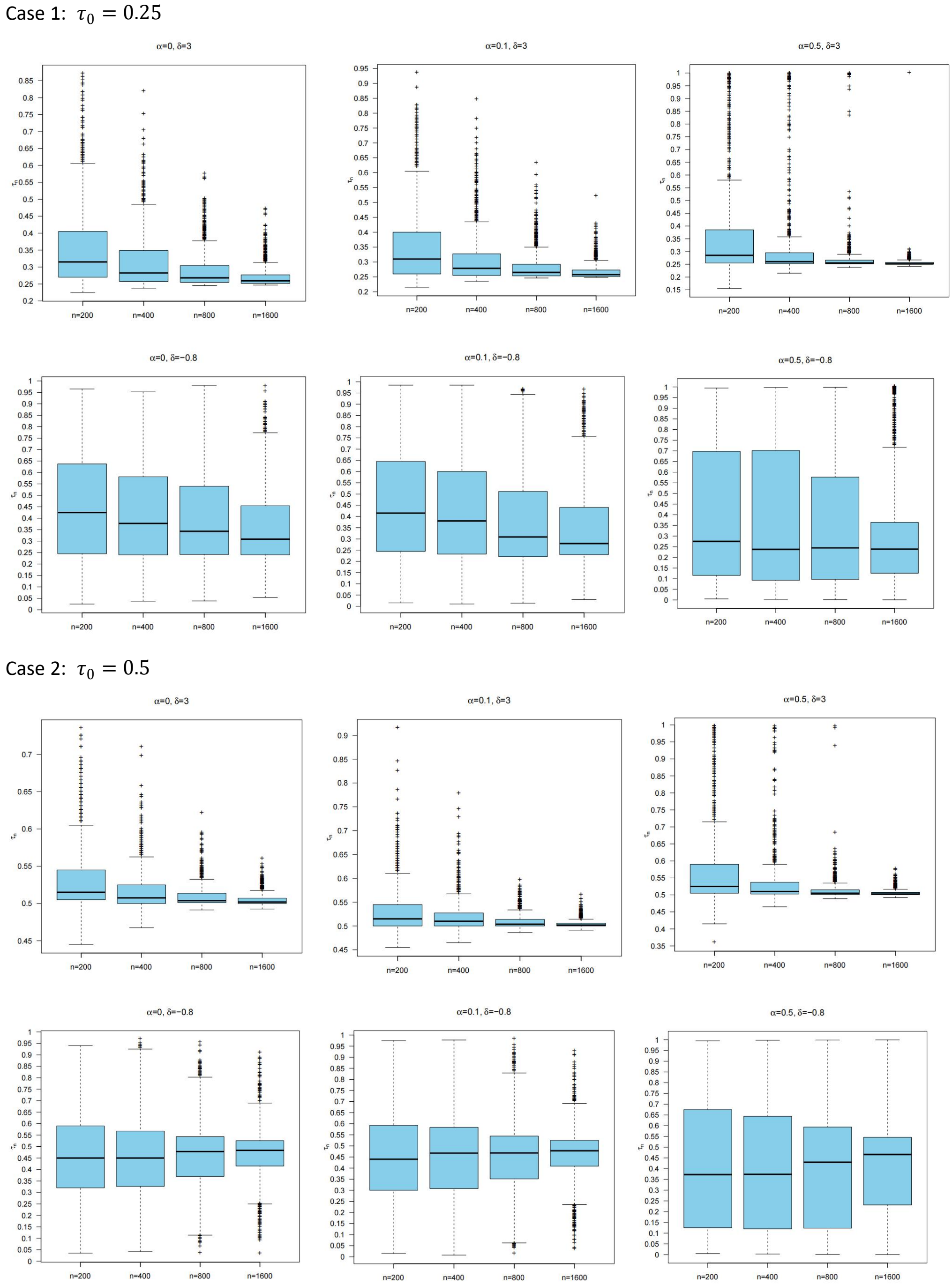

Figure 1. Box-plots of τ_n with $n = 200, 400, 800, 1600$.

As evident from Figure 1, for each specified value of α and δ, the CUSUM estimator τ_n converges to the true parameter τ_0 uniformly as n increases. Furthermore, as can be seen from Figure 1, the optimal choice of α varies with different variances, which needs to be determined according to specific situations.

3.2. Real data analysis

We will utilize variance change point estimator to conduct research on the volatility changes of stock returns of the AMD (Advanced Micro Devices) Semiconductor Company in the United States. Through the Yahoo Finance website, we downloaded a total of 212 stock price datasets for AMD stock, spanning from March 3, 2008 to December 31, 2008. Let P_t represent the closing price of AMD stock, and the return rate is defined as $r_t = \log P_t - \log P_{t-1}$, for $1 \leq t \leq 211$, with $P_0 = 1$. The ACF (Autocorrelation Function) graph of AMD stock returns is shown in Figure 2 and the AMD stock returns are depicted in Figure 3.

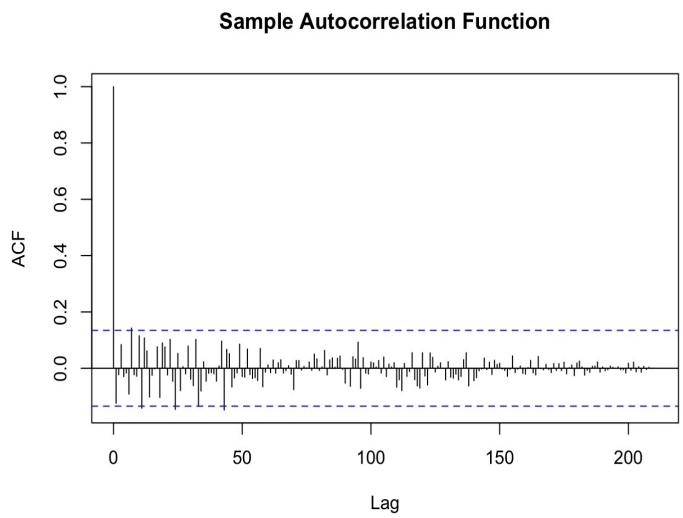

Figure 2. The ACF of AMD stock returns.

From Figure 2, as the lag increases, the sample autocorrelation function gradually tends to 0, indicating that the AMD stock return data r_t, for $1 \leq t \leq 211$, satisfies the ρ-mixing property, so the real data is α-mixing.

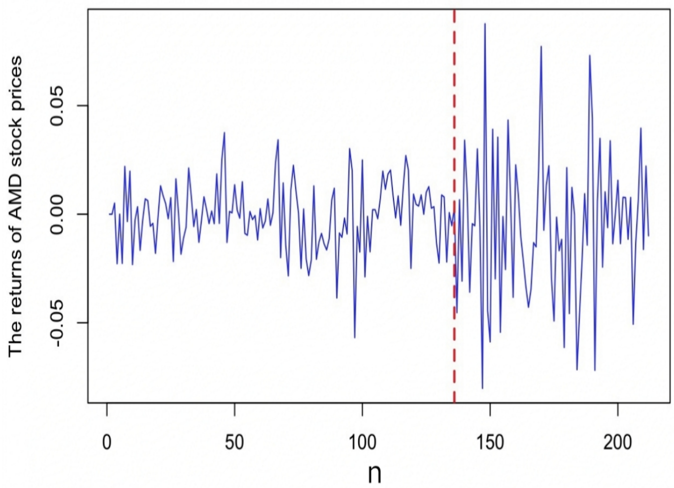

Figure 3. The AMD stock returns from March 3, 2008 to December 31, 2008.

As shown in Figure 3, the mean of AMD stock returns is around zero with no significant changes, but there is a significant change in variance. Using the CUSUM type statistic for variance change point, as seen in Eq (2.1), we can detect that the location of the variance change point is at 136 (corresponding to the date of September 12, 2008). In fact, the reason for the fluctuation in AMD stock is the announcement of the bankruptcy of the American company Lehman Brothers Holdings, Inc. on September 15, 2008, which undoubtedly increased the risk in the stock market.

4. Proof of the theorems

To substantiate the primary theorems of this paper, we rely on several pivotal lemmas. In this section, let $\{e_i, i \geq 1\}$ be a sequence of strong mixing random variables with mixing coefficients $\{\alpha(n), n \geq 1\}$.

The first one is the Rosenthal type maximum inequality and the Rosenthal type inequality for strong-mixing random variables.

Lemma 4.1. (i) (cf. [25]). *Let $r > 2, \delta > 0, Ee_i = 0$, and $E|e_i|^{r+\delta} < \infty$. Suppose that $\alpha(n) = O(n^{-\gamma})$ for some $\gamma > r(r+\delta)/(2\delta)$. Then, for any $\varepsilon > 0$, there exists a positive constant $C = C(\tau, r, \delta, \gamma)$ such that*

$$E\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m} e_i\right|^r\right) \leq C\left\{n^\tau \sum_{i=1}^{n} E|e_i|^r + \left(\sum_{i=1}^{n} \|e_i\|_{r+\delta}^2\right)^{r/2}\right\}.$$

(ii) (cf. [26]) *If $Ee_i = 0$ and $E|e_i|^{2+\delta} < \infty$ for some $\delta > 0$, then*

$$E\left(\sum_{i=1}^{n} e_i\right)^2 \leq \left[1 + 16\sum_{l=1}^{n} \alpha^{\delta/(2+\delta)}(l)\right] \sum_{i=1}^{n} \|e_i\|_{2+\delta}^2.$$

The second one is about the probabilistic inequalities for strong-mixing random variables, which is used to prove Theorem 2.1.

Lemma 4.2. *Let $r > 2, \delta > 0, Ee_i = 0$, and $E|e_i|^{r+\delta} < \infty$. Suppose that $\alpha(n) = O(n^{-\gamma})$ for some $\gamma > r(r+\delta)/(2\delta)$. Then, we establish, for large n,*

$$(i)\ P\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}\frac{e_i}{i^{\alpha}}\right| > n^{1-\alpha}\varepsilon\right) \le \frac{C}{n^{\frac{2}{r}}\varepsilon^r},$$

$$(ii)\ P\left(\max_{k\geqslant n}\frac{1}{k}\left|\sum_{i=1}^{k}e_i\right| > \varepsilon\right) \le \frac{C}{n^{\frac{2}{r}}\varepsilon^r}.$$

Proof. It is readily apparent that

$$P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}e_i\right| > \varepsilon\right) \le P\left(\bigcup_{i=1}^{n}\{|e_i| > n\}\right) + P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}e_iI\left(|e_i| \le n\right)\right| > \varepsilon\right).$$

Since $E\left|e_i\right|^{r+\delta} < \infty, r > 2$, and $Ee_i = 0$, we have

$$P\left(\bigcup_{i=1}^{n}\{|e_i| > n\}\right) \le \sum_{i=1}^{n}P\left(|e_i| > n\right) \le \frac{C}{n^{r-1}}, \tag{4.1}$$

$$\frac{1}{n^{1-\alpha}}\max_{1\le k\le n}\left|\sum_{i=1}^{k}E\frac{e_iI\left(|e_i| \le n\right)}{i}\right| \le \frac{C}{n^{r-1}}\sum_{i=1}^{n}\frac{1}{i^{\alpha}} \le \frac{C}{n^{r-1}}. \tag{4.2}$$

Utilizing the Markov inequality in conjunction with Lemma 4.1(a), along with the given conditions in (4.2) and $E|e_i|^{r+\delta} < \infty$, we can deduce that for sufficiently large values of n and $\tau = 1 - r\alpha$,

$$\begin{aligned}
&P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{e_iI\left(|e_i| \le n\right) - Ee_iI\left(|e_i| \le n\right)}{i^{\alpha}}\right| > n^{1-\alpha}\varepsilon\right)\\
&\le \frac{C}{n^{r-r\alpha}\varepsilon^r}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{e_iI\left(|e_i| \le n\right) - Ee_iI\left(|e_i| \le n\right)}{i^{\alpha}}\right|\right)^r\\
&\le \frac{C}{n^{r-r\alpha}\varepsilon^r}\left\{n^{\tau}\sum_{i=1}^{n}E\left|\frac{e_iI\left(|e_i| \le n\right) - Ee_iI\left(|e_i| \le n\right)}{i^{\alpha}}\right|^r\right.\\
&\quad\left.+\left(\sum_{i=1}^{n}\left\|\frac{e_iI\left(|e_i| \le n\right) - Ee_iI\left(|e_i| \le n\right)}{i^{\alpha}}\right\|_{r+\delta}^2\right)^{r/2}\right\}\\
&\le \frac{C}{n^{r-r\alpha}\varepsilon^r}(n^{\tau} + C + n^{r/2-r\alpha}) \le \frac{C}{n^{\frac{2}{r}}\varepsilon^r}.
\end{aligned} \tag{4.3}$$

Combining this with the Eq (4.1), we can deduce that

$$P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\frac{e_i}{i^{\alpha}}\right| > n^{1-\alpha}\varepsilon\right) \le \frac{C}{n^{r-1}} + \frac{C}{n^{\frac{2}{r}}\varepsilon^r} \le \frac{C}{n^{\frac{2}{r}}\varepsilon^r}.$$

For (ii), we split the set $\{k \geq n\}$ into a union of subsets of the form $\left\{(1+\ell)^j n \leq k < (1+\ell)^{j+1} n\right\}$ for $\ell > 0$ and $j = 0, 1, 2, \cdots$. Leveraging the result from (i), we proceed with the verification of (ii) as follows:

$$
\begin{aligned}
P\left(\max_{k\geq n} \frac{1}{k}\left|\sum_{i=1}^{k} e_i\right| > \varepsilon\right) &\leq \sum_{j=0}^{\infty} P\left(\max_{(1+\ell)^j n \leq k < (1+\ell)^{j+1} n} \frac{1}{k}\left|\sum_{i=1}^{k} e_i\right| > \varepsilon\right) \\
&\leq \sum_{j=0}^{\infty} P\left(\max_{1\leq k<(1+l)^{j+1} n}\left|\sum_{i=1}^{k} \frac{e_i}{i}\right| > (1+l)^j n\varepsilon\right) \\
&\leq \sum_{j=0}^{\infty} \frac{C}{\left[(1+l)^{j+1} n\right]^{r/2} \varepsilon^r} \leq \frac{C}{n^{\frac{2}{r}} \varepsilon^r}.
\end{aligned}
$$

The proof has been thoroughly established. □

Proof of Theorem 2.1. Define $\beta = \left(\sigma_1^2 - \sigma_2^2\right)$. When $k \leq k_0$, we get

$$
\begin{aligned}
&EU_k \\
&= \left[\frac{k(n-k)}{n}\right]^{1-\alpha} E\left\{\frac{1}{k}\sum_{t=1}^{k} Y_t^2 - \frac{1}{n-k}\sum_{t=k+1}^{n} Y_t^2\right\} \\
&= \left[\frac{k(n-k)}{n}\right]^{1-\alpha} E\left\{\frac{1}{k}\sum_{t=1}^{k} Y_t^2 - \frac{1}{n-k}\sum_{t=k+1}^{k_0} Y_t^2 - \frac{1}{n-k}\sum_{t=k_0+1}^{n} Y_t^2\right\} \\
&= \left[\frac{k(n-k)}{n}\right]^{1-\alpha} \left(\sigma_1^2 - \frac{k_0-k}{n-k}\sigma_1^2 - \frac{n-k_0}{n-k}\sigma_2^2\right) \\
&= \left[\frac{k(n-k)}{n}\right]^{1-\alpha} \frac{n-k_0}{n-k}|\beta|.
\end{aligned}
$$

Similarly, when $k > k_0$, we have

$$
\begin{aligned}
&EU_k \\
&= \left[\frac{k(n-k)}{n}\right]^{1-\alpha} E\left\{\frac{1}{k}\sum_{t=1}^{k_0} Y_t^2 - \frac{1}{k}\sum_{t=k_0+1}^{k} Y_t^2 - \frac{1}{n-k}\sum_{t=k+1}^{n} Y_t^2\right\} \\
&= \left[\frac{k(n-k)}{n}\right]^{1-\alpha} \left|\frac{k_0}{k}\sigma_1^2 + \frac{k-k_0}{k}\sigma_2^2 - \sigma_2^2\right| \\
&= \left[\frac{k(n-k)}{n}\right]^{1-\alpha} \frac{k_0}{k}|\beta|.
\end{aligned}
$$

It's easy to see that EU_k is increasing when $k \leq k_0$ and it is decreasing when $k > k_0$, then EU_k achieves the maximum at k_0, thus,

$$
\left|EU_{k_0}\right| = \left[\frac{(n-k_0)\,k_0}{n}\right]^{1-\alpha} |\beta|,
$$

which implies that

$$
\left|EU_{k_0}\right| - |EU_k| \geq \frac{k_0 \wedge (n-k_0)}{n^{1-\alpha}} |k-k_0|\,|\beta| \geq \frac{C_1}{n^{1-\alpha}} |k-k_0| = C_1 n^{\alpha-1}\left|\tau_k - \tau_{k_0}\right|.
$$

Note that

$$|U_k| - |U_{k_0}| \leq |U_k - EU_k| + |U_{k_0} - EU_{k_0}| + |EU_k| - |EU_{k_0}|$$
$$\leq 2 \max_{1\leq k\leq n} |U_k - EU_k| + |EU_k| - |EU_{k_0}|.$$

According to the findings presented in Kokoszka and Leipus (1998), we obtain

$$|\beta|\gamma|\hat{\tau}_n - \tau_0| \leq 2n^{\alpha-1} \max_{1\leq k\leq n} |U_k - EU_k|,$$

where $\gamma \doteq (1-\alpha)\tau_0^{-\alpha}(1-\tau_0)^{-\alpha}\min\{\tau_0, 1-\tau_0\}$. Hence, to prove Theorem 2.1, our primary objective is to demonstrate that

$$n^{\alpha-1} \max_{1\leq k\leq n} |U_k - EU_k| \to 0 \;\; a.s. \;\; n \to \infty. \tag{4.4}$$

From the definition of EU_k, we get

$$\begin{aligned} & n^{\alpha-1} \max_{1\leq k\leq n} |U_k - EU_k| \\ &= n^{\alpha-1} \max_{1\leq k\leq n} \left| \left\{ \frac{1}{k^\alpha} \sum_{t=1}^{k} (Y_t^2 - EY_t^2) - \frac{1}{(n-k)^\alpha} \sum_{t=k+1}^{n} (Y_t^2 - EY_t^2) \right\} \right| \\ &\leq n^{\alpha-1} \max_{1\leq k\leq n} \frac{1}{k^\alpha} \left| \sum_{t=1}^{k} (Y_t^2 - EY_t^2) \right| + \max_{1\leq k\leq n} \frac{1}{(n-k)^\alpha} \left| \sum_{t=k+1}^{n} (Y_t^2 - EY_t^2) \right|. \end{aligned} \tag{4.5}$$

Define $\gamma = C_1\varepsilon/2$. From (4.4) and (4.5), the Theorem 2.1 is followed by showing

$$P\left(\bigcup_{n=h}^{\infty} \left\{ \max_{1\leq k\leq n} \frac{1}{n^{1-\alpha}k^\alpha} \left| \sum_{t=1}^{k} (Y_t^2 - EY_t^2) \right| > \gamma \right\} \right) \to 0, \;\; \text{as} \;\; h \to \infty \tag{4.6}$$

and

$$P\left(\bigcup_{n=h}^{\infty} \left\{ \max_{1\leq k\leq n} \frac{1}{(n-k)^\alpha} \left| \sum_{t=k+1}^{n} (Y_t^2 - EY_t^2) \right| > \gamma \right\} \right) \to 0, \;\; \text{as} \;\; h \to \infty. \tag{4.7}$$

We consider the Eq (4.6), for $n \to \infty$ and $h \to \infty$. Noting that $\{Y_t^2 - EY_t^2\}$ are α-mixing random variables with mean zero, we have by Lemma 4.2 that

$$\begin{aligned} & P\left(\bigcup_{n=h}^{\infty} \left\{ \max_{1\leq k\leq n} \frac{1}{n^{1-\alpha}k^\alpha} \left| \sum_{t=1}^{k} (Y_t^2 - EY_t^2) \right| > \gamma \right\} \right) \\ =& P\left(\bigcup_{n=h}^{\infty} \bigcup_{k=1}^{n} \left\{ \frac{1}{n^{1-\alpha}k^\alpha} \left| \sum_{t=1}^{k} (Y_t^2 - EY_t^2) \right| > \gamma \right\} \right) \\ \leq& P\left(\bigcup_{k=1}^{h} \left\{ \frac{1}{h^{1-\alpha}k^\alpha} \left| \sum_{t=1}^{k} (Y_t^2 - EY_t^2) \right| > \gamma \right\} \right) + P\left(\bigcup_{k=h}^{\infty} \left\{ \frac{1}{k} \left| \sum_{t=1}^{k} (Y_t^2 - EY_t^2) \right| > \gamma \right\} \right) \\ \leq& P\left(\max_{1\leq k\leq h} \frac{1}{h^{1-\alpha}} \left| \sum_{t=1}^{k} \frac{(Y_t^2 - EY_t^2)}{i^\alpha} \right| > \gamma \right) + P\left(\max_{k\geq h} \frac{1}{k} \left| \sum_{t=1}^{k} (Y_t^2 - EY_t^2) \right| > \gamma \right) \\ \leq& \frac{C}{n^{\frac{2}{r}}\varepsilon^r}. \end{aligned}$$

Similarly, Eq (4.7) can be proved by Lemma 4.2. With the above derivation, we have successfully completed the proof of Theorem 2.1. □

Proof of Theorem 2.2. It suffices to demonstrate for any $\varepsilon > 0$ that

$$\lim_{m\to\infty} P\left(\bigcup_{n=m}^{\infty}\{|\hat{\tau}_n - \tau_0| > \varepsilon M(n)/n\}\right) = 0.$$

Since $\tau_0 \in (\gamma_1, \gamma_2)$, it is readily apparent that

$$\begin{aligned}
&P\left(\bigcup_{n=m}^{\infty}\{|\hat{\tau}_n - \tau_0| > \varepsilon M(n)/n\}\right)\\
&\leq P\left(\bigcup_{n=m}^{\infty}\{\hat{\tau}_n \notin (\gamma_1, \gamma_2)\}\right) + P\left(\bigcup_{n=m}^{\infty}\{|\hat{\tau}_n - \tau_0| > \varepsilon M(n)/n, \hat{\tau}_n \in (\gamma_1, \gamma_2)\}\right)\\
&\doteq J_1 + J_2.
\end{aligned}$$

By Theorem 2.1, it follows that $J_1 \to 0$ as $m \to \infty$. Next, we prove $J_2 \to 0$.

Write $H_{n,M(n)} = \{k : n\gamma_1 < k < n\gamma_2, |k - k_0| > \varepsilon M(n)\}$. It becomes evident upon observation that for $\beta < 0$,

$$\begin{aligned}
\{|\hat{\tau}_n - \tau_0| > \varepsilon M(n)/n, \hat{\tau}_n \in (\gamma_1, \gamma_2)\} &\subseteq \left\{\max_{k\in H_{n,M(n)}} |U_k| \geq \left|U_{k_0}\right|\right\}\\
&\subseteq \left\{\max_{k\in H_{n,M(n)}} U_k - U_{k_0} \geq 0\right\} \bigcup \left(\bigcup_{k\in H_{n,M(n)}} \{U_k < 0\}\right).
\end{aligned}$$

Analogously, we can establish that for $\beta > 0$,

$$\begin{aligned}
\{|\hat{\tau}_n - \tau_0| > \varepsilon M(n)/n, \hat{\tau}_n \in (\gamma_1, \gamma_2)\} &\subseteq \left\{\max_{k\in H_{n,M(n)}} |U_k| \geq \left|U_{k_0}\right|\right\}\\
&\subseteq \left\{\max_{k\in H_{n,M(n)}} -U_k + U_{k_0} \geq 0\right\} \bigcup \left(\bigcup_{k\in H_{n,M(n)}} \{U_k \geq 0\}\right).
\end{aligned}$$

Without compromising the generality of the argument, we postulate that $\beta < 0$. To prove the theorem, our sole objective is to establish that as $n \to \infty$,

$$\lim_{m\to\infty} P\left(\bigcup_{n=m}^{\infty}\left\{\max_{k\in H_{n,M(n)}} U_k - U_{k_0} \geq 0\right\}\right) \to 0 \tag{4.8}$$

and

$$P\left(\bigcup_{n=m}^{\infty}\bigcup_{k\in H_{n,M(n)}} \{U_k < 0\}\right) \to 0. \tag{4.9}$$

It is readily observable that

$$
\begin{aligned}
&P\left(\bigcup_{n=m}^{\infty}\left\{\max_{k\in H_{n,M(n)}} U_k - U_{k_0} \geq 0\right\}\right)\\
&\leq P\left(\bigcup_{n=m}^{\infty}\left\{\max_{k\in H_{n,M(n)},k<k_0} U_k - U_{k_0} \geq 0\right\}\right) + P\left(\bigcup_{n=m}^{\infty}\left\{\max_{k\in H_{n,M(n)},k\geq k_0} U_k - U_{k_0} \geq 0\right\}\right)\\
&\doteq K_1 + K_2.
\end{aligned}
$$

Since the proof that $K_2 \to 0$ as $n \to \infty$ mimics the proof that $K_1 \to 0$ as $n \to \infty$, we shall solely focus on proving the latter case. We have

$$
\begin{aligned}
&P\left(\bigcup_{n=m}^{\infty}\left\{\max_{k\in H_{n,M(n)},k<k_0} U_k - U_{k_0} \geq 0\right\}\right)\\
&\leq P\left(\bigcup_{n=m}^{\infty}\bigcup_{k\in H_{n,M(n)},k<k_0}\left\{\left|U_k - EU_k - (U_{k_0} - EU_{k_0})\right| \geq EU_{k_0} - EU_k\right\}\right).
\end{aligned}
\tag{4.10}
$$

For $\beta < 0, EU_k \geq 0, 1 \leq k \leq n-1$, we have

$$
EU_{k_0} - EU_k = \left|EU_{k_0}\right| - |EU_k| \geq Cn^{-\alpha} |\beta| |k - k_0| . \tag{4.11}
$$

Furthermore, for $k \in H_{n,M(n)}, k < k_0$, we can derive that

$$
\begin{aligned}
&\left|U_k - EU_k - (U_{k_0} - EU_{k_0})\right|\\
&\leq \frac{C_1 |k-k_0|}{n^{1+\alpha}}\left|\sum_{t=1}^{k}(Y_t^2 - EY_t^2)\right| + \frac{C_2 |k-k_0|}{n^{1+\alpha}}\left|\sum_{t=k_0+1}^{n}(Y_t^2 - EY_t^2)\right|\\
&+ \frac{C_3}{n^{\alpha}}\left|\sum_{t=1}^{k_0}(Y_t^2 - EY_t^2)\right| + \frac{C_4}{n^{\alpha}}\left|\sum_{t=1}^{k_0}(Y_t^2 - EY_t^2)\right|.
\end{aligned}
\tag{4.12}
$$

By (4.10)–(4.12), it follows that

$$
\begin{aligned}
&P\left(\bigcup_{n=m}^{\infty}\left\{\max_{k\in H_{n,M(n)},k<k_0} U_k - U_{k_0} \geq 0\right\}\right)\\
&\leq P\left(\bigcup_{n=m}^{\infty}\left\{\max_{1\leq k<n}\frac{1}{n}\left|\sum_{t=1}^{k}(Y_t^2 - EY_t^2)\right| \geq C_1\right\}\right) + P\left(\bigcup_{n=m}^{\infty}\left\{\max_{1\leq k<n}\frac{1}{n}\left|\sum_{t=k_0+1}^{n}(Y_t^2 - EY_t^2)\right| \geq C_2\right\}\right)\\
&\quad + P\left(\bigcup_{n=m}^{\infty}\left\{\max_{k\in H_{n,M(n)},k<k_0}\frac{1}{k_0-k}\left|\sum_{t=k+1}^{k_0}(Y_t^2 - EY_t^2)\right| \geq C_3\right\}\right)\\
&+ P\left(\bigcup_{n=m}^{\infty}\left\{\max_{k\in H_{n,M(n)},k<k_0}\frac{1}{k_0-k}\left|\sum_{t=k+1}^{k_0}(Y_t^2 - EY_t^2)\right| \geq C_4\right\}\right)\\
&\doteq P_1 + P_2 + P_3 + P_4.
\end{aligned}
$$

By (4.6), it follows that $P_1 \to 0$ as $m \to \infty$. Analogously, by leveraging Eq (4.7), it can be demonstrated that $P_2 \to 0$ as $m \to \infty$. Furthermore, in accordance with Lemma 4.2, it follows logically that $P_3 \to 0$ and $P_4 \to 0$ since $M(n) \to \infty$.

Next, we prove (4.9). Drawing upon the proof of Theorem 2.1, it follows that for $k \in H_{n,M(n)}$,

$$\begin{aligned} P\left(\bigcup_{n=m}^{\infty}\bigcup_{k\in H_{n,M(n)}}\{U_k < 0\}\right) &\leq P\left(\bigcup_{n=m}^{\infty}\bigcup_{k\in H_{n,M(n)}}\{|U_k - EU_k| > C_5 n\bar{\tau}\}\right) \\ &\leq P\left(\bigcup_{n=m}^{\infty}\left\{\max_{1\leq k<n}|U_k - EU_k| > C_5 n\bar{\tau}\right\}\right), \end{aligned}$$

where $\bar{\tau} = \min\{\tau_0, 1-\tau_0\}$, which tends to zero as $n \to \infty$.

This concludes the proof of Theorem 2.2, thereby establishing the desired result.□

5. Conclusions

This article primarily utilizes the Rosenthal type inequalities and probabilistic inequalities for strong-mixing sequences to investigate the strong consistency of variance change point estimation under strong-mixing sequences. It fills a research gap concerning variance change points in dependent sequences. While the classic CUSUM estimator is a well-known approach for change point models, there are other types of change point models and estimators that merit further exploration by readers.

Author contributions

Mengmei Xi: Formal analysis and wrote the original main manuscript; Xuejun Wang: Methodology, review and editing; Yi Wu: Figures. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interest.

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