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Research article

Dark and bright soliton phenomena of the generalized time-space fractional equation with gas bubbles

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Abstract: The objective of this work is to provide the method of getting the closed-form solitary wave solution of the fractional (3 + 1)-generalized nonlinear wave equation that characterizes the behavior of liquids with gas bubbles. The same phenomena are evident in science, engineering, and even in the field of physics. This is done by employing the Riccati-Bernoulli sub-ode in a systematic manner as applied to the Bäcklund transformation in the study of this model. New soliton solutions, in the forms of soliton, are derived in the hyperbolic and trigonometric functions. The used software is the computational software Maple, which makes it possible to perform all the necessary calculations and the check of given solutions. The result of such calculations is graphical illustrations of the steady-state characteristics of the system and its dynamics concerning waves and the inter-relationships between the parameters. Moreover, the contour plots and the three-dimensional figures describe the essential features, helping readers understand the physical nature of the model introduced in this work.

Keywords: fractional generalized non-linear wave equation; solitary wave solutions
Mathematics Subject Classification: 34G20, 35A20, 35A22, 35R11

1. Introduction

The fractional differential equations have been the focus of many contributions, taking into account their application in various areas such as finance, engineering, biology and physics, control theory, system identification, and signal processing [1–5]. Their applications include; social sciences, dietary supplements, climate, and economics, among others [6–11]. Hence, the exact solutions to fractional differential equations are important. To solve these equations, numerous analytical and numerical approaches have been reported, in the literature such as the modified F-expansion systematic [12], the extended tanh-coth method [13], the mapping method [14], the (G'/G)-expansion technique [15], etc. Rayleigh [16] did the first work on bubble dynamics. Since many subjects and industries deal with bubbly liquids, including medical science and engineering, bubbly liquids have received significant attention. This collection of recent research articles covers a broad range of topics in mathematical physics, nonlinear dynamics, and applied mathematics. They delve into Gaussian traveling wave solutions for a Schrödinger equation with logarithmic nonlinearity [17], while their subsequent work explores exact solutions for a nonlinear fourth-order time-fractional partial differential equation [18]. Focus on soliton solutions to the (2+1) dimensional Chaffee-Infante equation [19]. Investigate synchronization patterns in memristive neuron maps [20], alongside studies of ADHD brain networks and chimera states in FitzHugh-Nagumo oscillators [21, 22]. The present digital integrators are based on trigonometric quadrature rules, contributing to advancements in industrial electronics [23]. It has been found out that the propagation of linear acoustic waves in isothermal bubbly liquids, assuming that the radius of the bubbles is uniform, is governed by a fourth-order linear partial differential equation. A generalized (3+1)-dimensional nonlinear wave equation is one such model used to characterize a liquid containing gas bubbles:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t} + (p_1)\frac{\partial f}{\partial t} + (p_2)\frac{\partial^3 f}{\partial x^3} + (p_3) f \frac{\partial f}{\partial x}\right) + (p_4)\frac{\partial^2 f}{\partial z^2} + (p_5)\frac{\partial^2 f}{\partial y^2} = 0. \tag{1.1}$$

In the context of the equation, $f(x, y, z, t)$ stands for the amplitude of a wave, where p_1 is the bubble liquid viscosity and p_2 is the bubble liquid dispersion, p_3 is the bubble liquid nonlinearity, p_4 is the z transverse perturbation, and p_5 is the transverse perturbation. Equation (1.1) can be used for the description of different nonlinear processes occurring in liquids with gas bubbles. Therefore, this equation has received a lot of attention in research. For example, within the frame of linear superposition, soliton solutions of N-soliton waves have been obtained [24]. Furthermore, lump-stripe solitons and rogue wave-stripe solutions have been built [25], and lump, stripe periodic, and multi solitons also have been found [26]. Soliton solutions have been obtained by using the technique that is now known as the Hirota bilinear form and has resulted in a new form of a generalized exponential rational function of Hirota [27, 28]. Additionally, the use of the modified Kudryashov method and Nuccis reduction has been made to investigate the behavior of solitary waves [29]. However, in the past studies, the consideration of fractional derivatives on Eq (1.1) has not been thoroughly investigated, which hinders the ability to simulate the detailed kinetics of wave in systems having memory as well as non-local interactions. This has limited the modeling of wave interactions in near-neighbourhood systems with memory and non-locality. To this end, in this study, we will extend the above analysis to the generalized fractional nonlinear wave equation (GFNWE). By means of this, we obtain a more reliable and extensive model for nonlinear wave interactions that is significant for the description of various processes in fluid dynamics, plasma physics, and other complex systems. In this study, we extend this work by examining the generalized fractional nonlinear wave equation (GFNWE) as follows:

$$D_x^{\alpha}\left(D_t^{\alpha} f + (p_1) D_x^{\alpha} f + (p_2) D_x^{3\alpha} f + (p_3) f D_x^{\alpha} f\right) + (p_4) D_z^{2\alpha} f + (p_5) D_y^{2\alpha} f = 0. \tag{1.2}$$

Further, the operator integrating α-derivatives of powers agrees exactly to the idea of conformable fractional derivatives [30].

$$D_{\phi}^{\alpha} W(\phi) = \lim_{i \to 0} \frac{W\left(\phi + i(\phi)^{1-\alpha}\right) - W(\phi)}{i}, \quad 0 < \alpha \leq 1. \tag{1.3}$$

$$\begin{cases} D_{\phi}^{\alpha}\phi^{k} = k\phi^{k-\alpha}. \\ D_{\phi}^{\alpha}\left(k_1\eta(\phi) \pm k_2 t(\phi)\right) = k_1 D_{\phi}^{\alpha}(\eta(\phi)) \pm m_2 D_{\phi}^{\alpha}(t(\phi)). \\ D_{\phi}^{\alpha}\left[f \circ g\right] = \phi^{1-\alpha} g\left(\phi\right) D_{\phi}^{\alpha} f\left(g(\phi)\right). \end{cases} \tag{1.4}$$

The Riccati-Bernoulli sub-ode method is a new effective analytical tool for obtaining the solitary wave solutions of PDEs and FPDEs. This method works on the basis of altering the original FPDE into a first-order ODE, that is nonlinear. Subsequent to this transformation, there is the use of a series-form solution, which results in the derivation of non-linear algebraic equations. Solving the algebraic equations as mentioned above helps in the determination of the solitary wave solutions of the concerned FPDE only [31–36]. This approach is specifically efficient so far as it concerns other analytical techniques utilized for deriving soliton solutions. The Riccati-Bernoulli sub-ode method allows the identification of other more wide-spread and multifaceted subfamilies of soliton solutions, thus, proving its vitality in describing various and complicated aspects of solitary waves in various systems [37–39].

The second part of the paper describes the overall approach used; a detailed explanation is given in Section 3 when the solution for the generalized fractional nonlinear wave equation is discussed. The analysis of the results is provided in Section 4 along with graphical representations of the findings. Lastly, Section 5 provides the conclusion of the study where the major observations and recommendations will be made.

2. Algorithm

Since it is crucial to provide the reader with a clear understanding of the methods used in this research, a slightly deeper explanation of the process will be given below. We begin by considering a general class of nonlinear partial differential equations (PDEs) in the following form:

$$P_1\left(R_1, D_t^{\alpha}(R_1), D_{q_1}^{\alpha}(R_1), D_{q_2}^{\alpha}(R_1), R_1 D_{q_1}^{\alpha}(R_1), \ldots\right) = 0, \qquad 0 < \alpha \leq 1, \tag{2.1}$$

where $R_1 = R(t, q_1, q_2, q_3, \ldots, q_k)$ is a function of $(t, q_1, q_2, q_3, \ldots, q_k)$ and its partial derivatives. This transformation changes Eq (2.1) into a nonlinear ordinary differential equation (ODE) of the following form:

$$Q_1\left(F, F'(\phi), F''(\phi), FF'(\phi), \ldots\right) = 0. \tag{2.2}$$

Let us suppose that Eq (2.2) has the following solution:

$$G(\phi) = \sum_{j=-n}^{n} k_j g(\phi)^j, \tag{2.3}$$

where k_j are constants and $g(\phi)$ is obtained from the Bäcklund transformation, $g(\phi) = \frac{-\Omega E_2 + E_1 Z(\phi)}{E_1 + E_2 Z(\phi)}$. Where, (Ω) , (E_1) , and (E_2) are constants such that $E_2 \neq 0$ and $Z(\phi)$ are solutions of the following ODE.

$$\frac{dZ}{d\phi} = \Omega + Z(\phi)^2. \tag{2.4}$$

The Ricatti equation (2.4) possesses the following general solutions [40].

$$Z(\phi) = \begin{cases} -\sqrt{-\Omega}\tanh(\sqrt{-\Omega}\phi), & \text{as } \Omega < 0, \\ -\sqrt{-\Omega}\coth(\sqrt{-\Omega}\phi), & \text{as } \Omega < 0, \end{cases}$$
$$Z(\phi) = -\frac{1}{\psi}, \quad \text{as } \Omega = 0, \tag{2.5}$$
$$Z(\phi) = \begin{cases} \sqrt{\Omega}\tan(\sqrt{\Omega}\phi), & \text{as } \Omega > 0, \\ -\sqrt{\Omega}\cot(\sqrt{\Omega}\phi), & \text{as } \Omega > 0. \end{cases}$$

To eliminate the homogeneous term, which is the ratio of the largest nonlinear term and the highest order derivative on the right-hand side of Eq (2.2), a positive integer n is derived as given in Eq (2.3). Therefore, while presenting the balance number of a processor, it is pertinent to mention that it can be arrived at using the following method [41].

$$D\left[\frac{d^m F}{d\psi^m}\right] = n + m, \quad D\left[F^J \frac{d^m F}{d\psi^m}\right]^w = nJ + w(m+n). \tag{2.6}$$

Subsequently, the function obtained from Eq (2.3) is plugged into Eq (2.2) or in the expression that we obtain after integrating Eq (2.2). Subsequently, all terms with $g(\phi)$ are grouped, and the coefficients of the polynomial are set to zero. This process brings out a system of algebraic equations with the use of (k_i) and other variables applicable in the sequence. These algebraic equations are further solved using the Maple a computational software. Last of all, the solutions of Eq (1.1) are derived as single wave solutions and are displayed.

3. Problem execution

In this section, various solutions are discussed for the fractional (3+1)-dimensional generalized wave model of liquids containing gas bubbles, which has been defined in Eq (1.1). According to the Riccati-Bernoulli sub-ode technique and Bäcklund transformation process, solitary wave solutions are attained. The wave transformation employed in this context is as follows:

$$f(x,y,z,t) = F(\psi), \qquad \text{where} \quad \psi = \frac{\mu_1 x^\alpha}{\alpha} + \frac{\mu_2 y^\alpha}{\alpha} + \frac{\mu_3 z^\alpha}{\alpha} - \frac{\omega\, t^\alpha}{\alpha}. \tag{3.1}$$

Where, μ_1, μ_2, μ_3 and ω are unknown constants. This transformation changes Eq (1.2) into a nonlinear ordinary differential equation (ODE) of the following form:

$$\mu_1^4 p_3 \left(\frac{d^2 F}{d\psi^2}\right) + \mu_1^2 p_1 \left(F(\psi)^2\right) + F(\psi)\left(\mu_1^2 p_3 - \mu_1\omega + \mu_2^2 p_4 + \mu_3^2 p_5\right) = 0. \tag{3.2}$$

This makes it possible for us to reduce and solve the system balancing equations and other multiple wave structures that are complex in a new approach that is adopted. Used in a systematic way, such integration allows extracting particular traits of the primary constituents of waves, next to which conventional concepts are employed. By this distinct integration, the characteristics of interactions and behaviors of waves can be observed. With a view to furthering the solution, Eqs (2.3) and (2.4) are integrated into the governing Eq (3.2). Thus, we gather coefficients of the function $Z(\phi)$ in detail and obtain an analogous system of equations. This system provides a basis for the analysis of the objects and future quantitative or qualitative analysis and represents the key characteristics of the wave structures.

$$
\begin{aligned}
Z^1(\phi) =&{\mu_1}^2 p_1 {k_{-2}}^2 {E_2}^8 + 6\,{\mu_1}^4 p_3 k_{-2} {E_2}^8 \Omega^2 = 0,\\
Z^2(\phi) =&-2\,{\mu_1}^2 p_1 k_{-2} {E_2}^8 k_{-1} \Omega - 2\,{\mu_1}^4 p_3 k_{-1} {E_2}^8 \Omega^3 = 0,\\
Z^3(\phi) =&k_{-2} {E_2}^8 {\mu_2}^2 p_4 \Omega^2 + {\mu_1}^2 p_1 {k_{-1}}^2 {E_2}^8 \Omega^2 + k_{-2} {E_2}^8 {\mu_3}^2 p_5 \Omega^2 + k_{-2} {E_2}^8 {\mu_1}^2 p_3 \Omega^2 - k_{-2} {E_2}^8 \mu_1 \omega\, \Omega^2\\
&+ 8\,{\mu_1}^4 p_3 k_{-2} {E_2}^8 \Omega^3 + 2\,{\mu_1}^2 p_1 k_{-2} {E_2}^8 k_0 \Omega^2 = 0,\\
Z^4(\phi) =&-2\,{\mu_1}^2 p_1 k_{-1} {E_2}^8 k_0 \Omega^3 + k_{-1} {E_2}^8 \mu_1 \omega\, \Omega^3 - k_{-1} {E_2}^8 {\mu_2}^2 p_4 \Omega^3 - 2\,{\mu_1}^2 p_1 k_{-2} {E_2}^8 k_1 \Omega^3\\
&- 2\,{\mu_1}^4 p_3 k_{-1} {E_2}^8 \Omega^4 - k_{-1} {E_2}^8 {\mu_3}^2 p_5 \Omega^3 - k_{-1} {E_2}^8 {\mu_1}^2 p_3 \Omega^3 = 0,\\
Z^5(\phi) =&{\mu_1}^2 p_1 {k_0}^2 \Omega^4 {E_2}^8 + k_0 {\mu_3}^2 p_5 \Omega^4 {E_2}^8 + k_0 {\mu_1}^2 p_3 \Omega^4 {E_2}^8 + k_0 {\mu_2}^2 p_4 \Omega^4 {E_2}^8\\
&+ 2\,{\mu_1}^4 p_3 k_2 {E_2}^8 \Omega^6 + 2\,{\mu_1}^4 p_3 k_{-2} {E_2}^8 \Omega^4 - k_0 \mu_1 \omega\, \Omega^4 {E_2}^8 + 2\,{\mu_1}^2 p_1 k_{-2} {E_2}^8 k_2 \Omega^4\\
&+ 2\,{\mu_1}^2 p_1 k_{-1} {E_2}^8 k_1 \Omega^4 = 0,\\
Z^6(\phi) =&-2\,{\mu_1}^2 p_1 k_{-1} {E_2}^8 k_2 \Omega^5 - 2\,{\mu_1}^2 p_1 k_0 k_1 \Omega^5 {E_2}^8 + k_1 \Omega^5 {E_2}^8 \mu_1 \omega - 2\,{\mu_1}^4 p_3 k_1 {E_2}^8 \Omega^6\\
&- k_1 \Omega^5 {E_2}^8 {\mu_3}^2 p_5 - k_1 \Omega^5 {E_2}^8 {\mu_2}^2 p_4 - k_1 \Omega^5 {E_2}^8 {\mu_1}^2 p_3 = 0,\\
Z^7(\phi) =&{\mu_1}^2 p_1 {k_1}^2 \Omega^6 {E_2}^8 + k_2 \Omega^6 {E_2}^8 {\mu_1}^2 p_3 + k_2 \Omega^6 {E_2}^8 {\mu_2}^2 p_4 + k_2 \Omega^6 {E_2}^8 {\mu_3}^2 p_5 + 2\,{\mu_1}^2 p_1 k_0 k_2 \Omega^6 {E_2}^8\\
&+ 8\,{\mu_1}^4 p_3 k_2 {E_2}^8 \Omega^7 - k_2 \Omega^6 {E_2}^8 \mu_1 \omega = 0,\\
Z^8(\phi) =&-2\,{\mu_1}^2 p_1 k_1 \Omega^7 {E_2}^8 k_2 - 2\,{\mu_1}^4 p_3 k_1 {E_2}^8 \Omega^7 = 0,\\
Z^9(\phi) =&6\,{\mu_1}^4 p_3 k_2 {E_2}^8 \Omega^8 + {\mu_1}^2 p_1 {k_2}^2 \Omega^8 {E_2}^8 = 0.
\end{aligned}
\tag{3.3}
$$

This gives us the algebraic equations by setting $Z(\phi) = 0$. The solutions of this system of algebraic equations obtained from Maple are:

Set 1.

$$
\begin{aligned}
&k_0 = k_0, k_1 = 0, k_{-1} = 0, k_{-2} = -3/8\,\frac{p_1 {k_0}^2}{{\mu_1}^2 p_3}, k_2 = -6\,\frac{{\mu_1}^2 p_3}{p_1}, \Omega = 1/4\,\frac{p_1 k_0}{{\mu_1}^2 p_3}, \mu_1 = \mu_1, \mu_2 = \mu_2,\\
&\mu_3 = \sqrt{-\frac{-\mu_1 \omega + p_4 {\mu_2}^2 + {\mu_1}^2 p_3 + 4\,{\mu_1}^2 p_1 k_0}{p_5}}.
\end{aligned}
\tag{3.4}
$$

Set 2.

$$
\begin{aligned}
&k_0 = k_0, k_1 = 0, k_{-1} = 0, k_{-2} = 0, k_2 = -6\,\frac{{\mu_1}^2 p_3}{p_1}, \Omega = -1/2\,\frac{p_1 k_0}{{\mu_1}^2 p_3}, \mu_1 = \mu_1, \mu_2 = \mu_2,\\
&\mu_3 = \sqrt{-\frac{-2\,{\mu_1}^2 p_1 k_0 - \mu_1 \omega + p_4 {\mu_2}^2 + {\mu_1}^2 p_3}{p_5}}.
\end{aligned}
\tag{3.5}
$$

Solution Set 1

Set 1. For ($\Omega < 0$), $\mu_3 = \sqrt{-\frac{-\mu_1\omega+p_4{\mu_2}^2+{\mu_1}^2p_3+4{\mu_1}^2p_1k_0}{p_5}}$ and $\psi = \frac{\mu_1 x^\alpha}{\alpha} + \frac{\mu_2 y^\alpha}{\alpha} + \frac{\mu_3 z^\alpha}{\alpha} - \frac{\omega t^\alpha}{\alpha}$, the following set of solutions for Eq (2.6) are obtained.

$$f_1 = -3/8\,\frac{p_1{k_0}^2\left(E_1 - E_2\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\psi\right)\right)^2}{{\mu_1}^2p_3\left(-\Omega\,E_2 - E_1\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\psi\right)\right)^2} + k_0 - 6\,\frac{{\mu_1}^2p_3\left(-\Omega\,E_2 - E_1\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\psi\right)\right)^2}{p_1\left(E_1 - E_2\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\psi\right)\right)^2}, \tag{3.6}$$

or

$$f_2 = -3/8\,\frac{p_1{k_0}^2\left(E_1 - E_2\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\psi\right)\right)^2}{{\mu_1}^2p_3\left(-\Omega\,E_2 - E_1\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\psi\right)\right)^2} + k_0 - 6\,\frac{{\mu_1}^2p_3\left(-\Omega\,E_2 - E_1\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\psi\right)\right)^2}{p_1\left(E_1 - E_2\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\psi\right)\right)^2}. \tag{3.7}$$

Solution Set 2

Set 1. For ($\Omega > 0$), the following set of solutions for Eq (2.6) are obtained.

$$f_3 = -3/8\,\frac{p_1{k_0}^2\left(E_1 + E_2\sqrt{\Omega}\tan\left(\sqrt{\Omega}\psi\right)\right)^2}{{\mu_1}^2p_3\left(-\Omega\,E_2 + E_1\sqrt{\Omega}\tan\left(\sqrt{\Omega}\psi\right)\right)^2} + k_0 - 6\,\frac{{\mu_1}^2p_3\left(-\Omega\,E_2 + E_1\sqrt{\Omega}\tan\left(\sqrt{\Omega}\psi\right)\right)^2}{p_1\left(E_1 + E_2\sqrt{\Omega}\tan\left(\sqrt{\Omega}\psi\right)\right)^2}, \tag{3.8}$$

or

$$f_4 = -3/8\,\frac{p_1{k_0}^2\left(E_1 - E_2\sqrt{\Omega}\cot\left(\sqrt{\Omega}\psi\right)\right)^2}{{\mu_1}^2p_3\left(-\Omega\,E_2 - E_1\sqrt{\Omega}\cot\left(\sqrt{\Omega}\psi\right)\right)^2} + k_0 - 6\,\frac{{\mu_1}^2p_3\left(-\Omega\,E_2 - E_1\sqrt{\Omega}\cot\left(\sqrt{\Omega}\psi\right)\right)^2}{p_1\left(E_1 - E_2\sqrt{\Omega}\cot\left(\sqrt{\Omega}\psi\right)\right)^2}. \tag{3.9}$$

Solution Set 3

Set 2. For ($\Omega < 0$) and $\mu_3 = \sqrt{-\frac{-2{\mu_1}^2p_1k_0-\mu_1\omega+p_4{\mu_2}^2+{\mu_1}^2p_3}{p_5}}$., the following set of solutions for Eq (2.6) are obtained.

$$\begin{aligned} f_5 =& k_0 - 6{\mu_1}^2p_3\left(-\Omega\,E_2 - E_1\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\left(\frac{\mu_1 x^\alpha}{\alpha} + \frac{\mu_2 y^\alpha}{\alpha} + \frac{\mu_3 z^\alpha}{\alpha} - \frac{\omega\,t^\alpha}{\alpha}\right)\right)\right)^2 \\ & {p_1}^{-1}\left(E_1 - E_2\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\left(\frac{\mu_1 x^\alpha}{\alpha} + \frac{\mu_2 y^\alpha}{\alpha} + \frac{\mu_3 z^\alpha}{\alpha} - \frac{\omega\,t^\alpha}{\alpha}\right)\right)\right)^{-2}, \end{aligned} \tag{3.10}$$

or

$$\begin{aligned} f_6 =& k_0 - 6{\mu_1}^2p_3\left(-\Omega\,E_2 - E_1\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\left(\frac{\mu_1 x^\alpha}{\alpha} + \frac{\mu_2 y^\alpha}{\alpha} + \frac{\mu_3 z^\alpha}{\alpha} - \frac{\omega\,t^\alpha}{\alpha}\right)\right)\right)^2 \\ & {p_1}^{-1}\left(E_1 - E_2\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\left(\frac{\mu_1 x^\alpha}{\alpha} + \frac{\mu_2 y^\alpha}{\alpha} + \frac{\mu_3 z^\alpha}{\alpha} - \frac{\omega\,t^\alpha}{\alpha}\right)\right)\right)^{-2}. \end{aligned} \tag{3.11}$$

Solution Set 4

Set 2. For ($\Omega > 0$), the following set of solutions for Eq (2.6) are obtained.

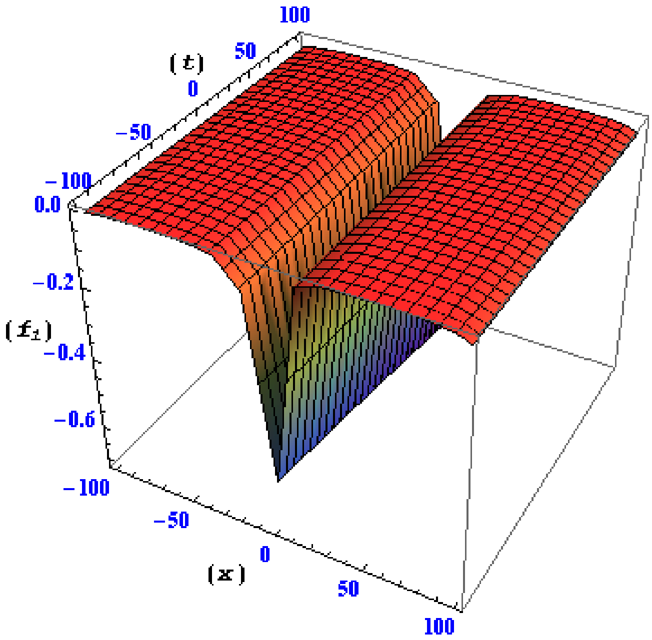
$$f_7 = k_0 - 6\mu_1{}^2 p_3 \left(-\Omega E_2 + E_1 \sqrt{\Omega}\tan\left(\sqrt{\Omega}\left(\frac{\mu_1 x^\alpha}{\alpha} + \frac{\mu_2 y^\alpha}{\alpha} + \frac{\mu_3 z^\alpha}{\alpha} - \frac{\omega t^\alpha}{\alpha}\right)\right)\right)^2 p_1{}^{-1}\left(E_1 + E_2\sqrt{\Omega}\tan\left(\sqrt{\Omega}\left(\frac{\mu_1 x^\alpha}{\alpha} + \frac{\mu_2 y^\alpha}{\alpha} + \frac{\mu_3 z^\alpha}{\alpha} - \frac{\omega t^\alpha}{\alpha}\right)\right)\right)^{-2}, \tag{3.12}$$

or

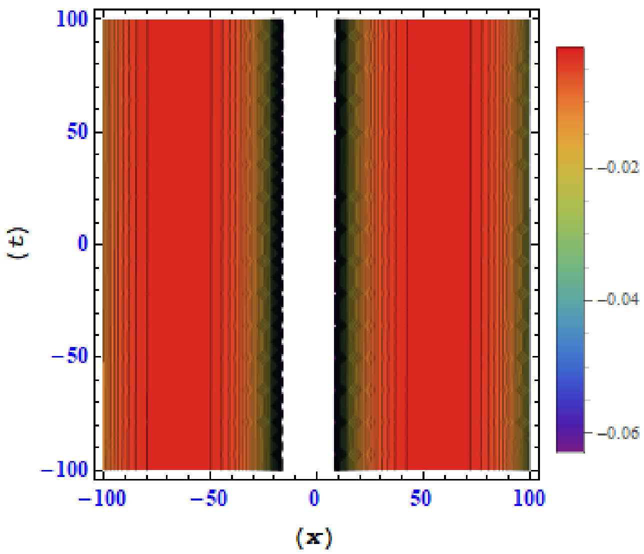
$$f_8 = k_0 - 6\mu_1{}^2 p_3 \left(-\Omega E_2 - E_1 \sqrt{\Omega}\cot\left(\sqrt{\Omega}\left(\frac{\mu_1 x^\alpha}{\alpha} + \frac{\mu_2 y^\alpha}{\alpha} + \frac{\mu_3 z^\alpha}{\alpha} - \frac{\omega t^\alpha}{\alpha}\right)\right)\right)^2 p_1{}^{-1}\left(E_1 - E_2\sqrt{\Omega}\cot\left(\sqrt{\Omega}\left(\frac{\mu_1 x^\alpha}{\alpha} + \frac{\mu_2 y^\alpha}{\alpha} + \frac{\mu_3 z^\alpha}{\alpha} - \frac{\omega t^\alpha}{\alpha}\right)\right)\right)^{-2}. \tag{3.13}$$

4. Results and discussion

The use of the Riccati-Bernoulli sub-ODE method with the Bäcklund transformation gave us exact solutions, which provided wider applicability in comparison with other approaches, such as the sine-cosine or the homotopy perturbation. It also allows for a powerful solution in handling of nonlinearity and fractional derivatives; this reduces the number of steps to be followed since FPDE is transformed into an algebraic statement. This results more stable and accurate waveforms compared to recent approaches applicable to fluid dynamics and plasma physics, as seen from the comparison. Using the figures of $3D$ surface plots and contour plots, it is easy to understand the physical changes of solitary wave solutions for the generalized fractional nonlinear wave equation with $p_1 = \omega = 0.001$, $\mu_1 = 0.05$, $\mu_2 = 0.05$, $\mu_3 = 0.0315$, $p_2 = 2$, and $p_3 = 3$. Both f_1 and f_8 solitary wave solutions depict dark solitary waves, which consist of regions of low amplitude enclosed by regions of high intensity, thereby illustrating how dark solitary waves create depressions in the media through which they travel. On the other hand, the bright solitary wave solutions f_4 and f_6 display the maximum peak in the amplitudes, and they represent enhanced regional intensity of the wave. Figure 1, $3D$ and contour representation of the solution, $f_1(x, y, z, t)$. Figure 2, $3D$ and contour representation of the solution, $f_4(x, y, z, t)$. Figure 3, $3D$ and contour representation of the solution, $f_6(x, y, z, t)$. Figure 4, $3D$ and contour representation of the solution, $f_8(x, y, z, t)$. Table 1, Comparison of the current approach with the alternative approach, specifically modified Kudryashov method [29]. These representations added emphasis on the spatial relationships as well as the changes in amplitudes which, showing how the wave behaves with its surroundings and how it advances. Taking into account these plots, one obtains more information about the actual behavior of wave solutions and some characteristics such as their stability and some features of their interactions with other solutions underlying physical processes modeled by the equations introduced above. The solutions are trigonometric and hyperbolic, and thus come with a rich structure, indicative of the wide range of physical processes that can be modeled. These types of solutions are of special importance for the description of liquid phases with gaseous bubbles, which are often met in different scientific and engineering practices. The knowledge of such waves in the field of fluid dynamics is significant to describe the multiphase flow as the key to its behavior consists of the interaction between the liquid and gaseous phases. For instance, accurate descriptions of bubbly flows, in the oil and gas industry are critical in enhancing the extraction processes and the stability of pipelines.

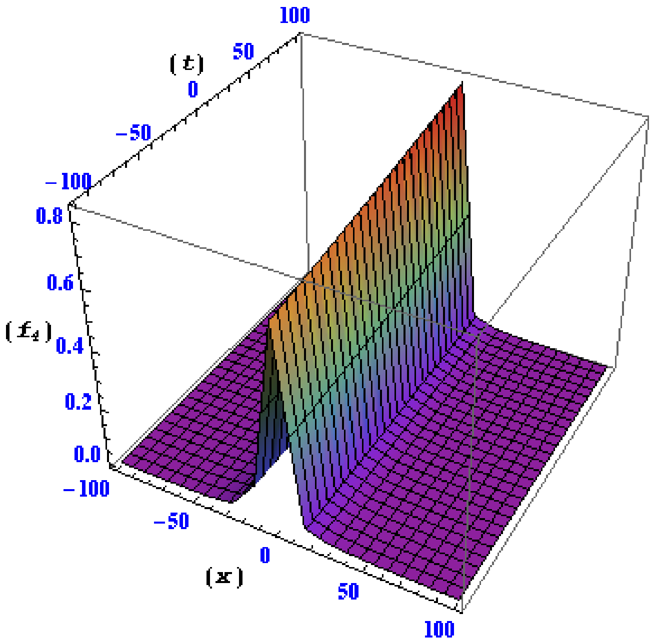


(a) $3D$ graphical representation of the solution, $f_1(x, y, z, t)$.

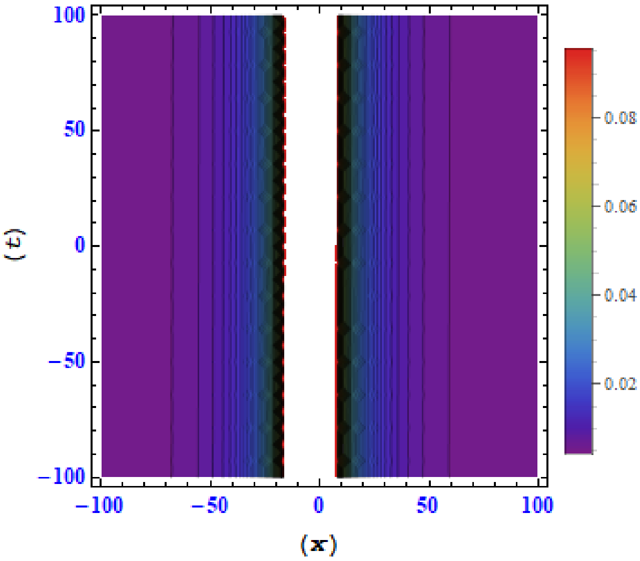


(b) Contour representation of the solution, $f_1(x, y, z, t)$.

Figure 1. $3D$ and contour representation of the solution, $f_1(x, y, z, t)$.

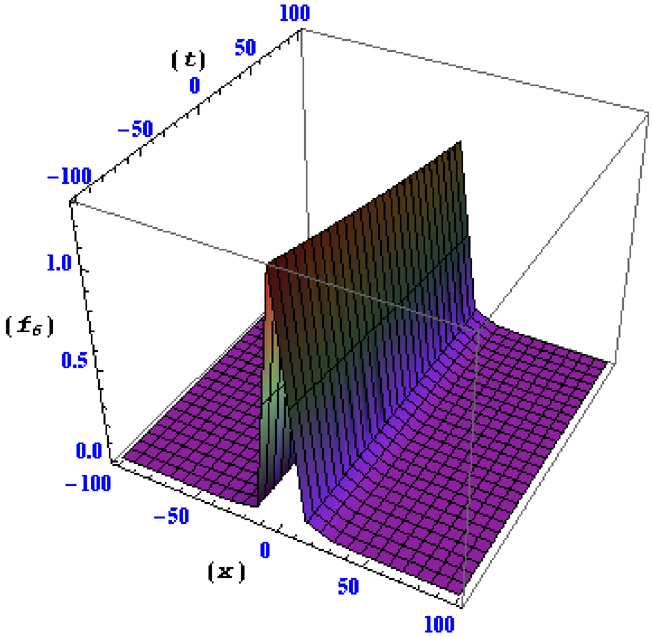


(a) 3D graphical representation of the solution, $f_4(x, y, z, t)$.

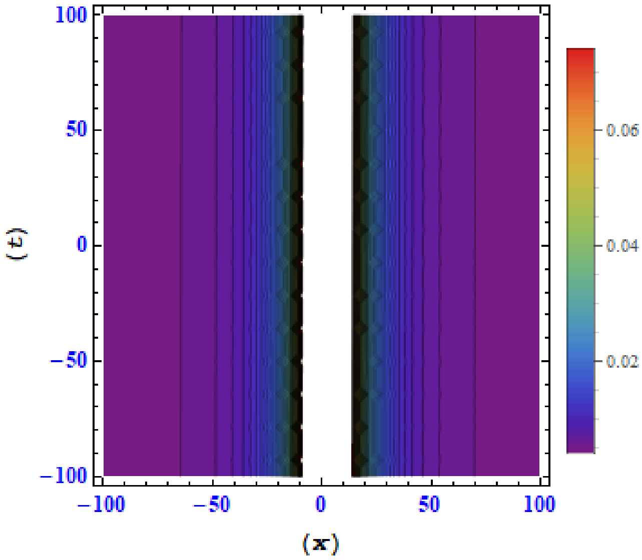


(b) Contour representation of the solution, $f_4(x, y, z, t)$.

Figure 2. 3D and contour representation of the solution, $f_4(x, y, z, t)$.

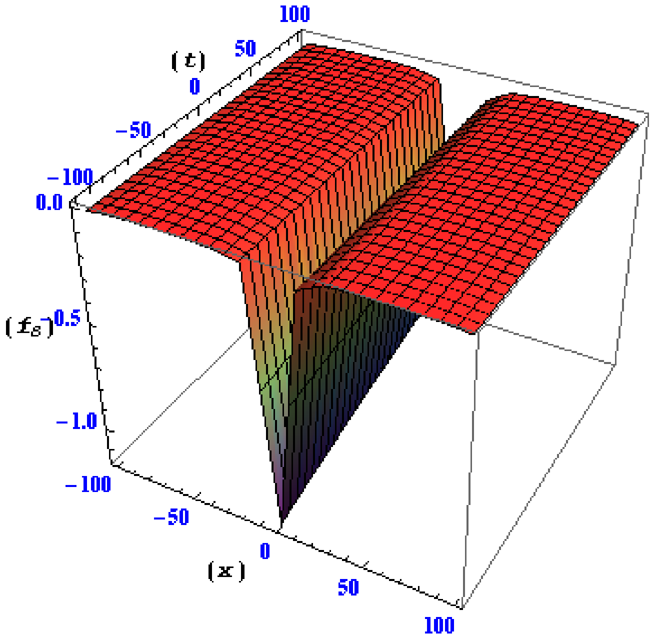


(a) 3D graphical representation of the solution, $f_6(x, y, z, t)$.

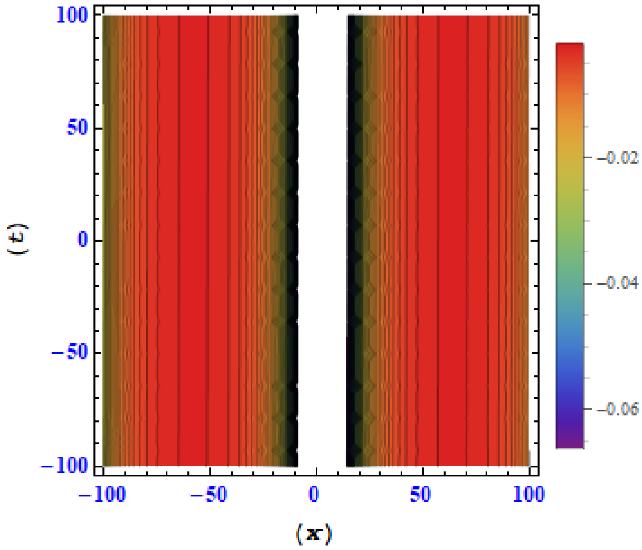


(b) Contour representation of the solution, $f_6(x, y, z, t)$.

Figure 3. 3D and contour representation of the solution, $f_6(x, y, z, t)$.



(a) $3D$ graphical representation of the solution, $f_8(x, y, z, t)$.



(b) Contour representation of the solution, $f_8(x, y, z, t)$.

Figure 4. $3D$ and contour representation of the solution, $f_8(x, y, z, t)$.

Table 1. Comparison of the current approach with the alternative approach, specifically modified Kudryashov method [29].

Case I: $\Omega < 0$ Present method
$f(x,y,z,t) = -3/8\,\frac{p_1 k_0{}^2\left(E_1 - E_2\sqrt{-\Omega}\tanh(\sqrt{-\Omega}\psi)\right)^2}{\mu_1{}^2 p_3\left(-\Omega E_2 - E_1\sqrt{-\Omega}\tanh(\sqrt{-\Omega}\psi)\right)^2} + k_0 - 6\,\frac{\mu_1{}^2 p_3\left(-\Omega E_2 - E_1\sqrt{-\Omega}\tanh(\sqrt{-\Omega}\psi)\right)^2}{p_1\left(E_1 - E_2\sqrt{-\Omega}\tanh(\sqrt{-\Omega}\psi)\right)^2}.$
Case I: $g_2\left(g_3 l_1^2 + g_4 l_2^2 + g_5 l_3^2 - l_1 v\right) < 0$ (modified Kudryashov) method
$f(x,y,z,t) = \frac{3\left(g_3 l_1^2 + g_4 l_2^2 + g_5 l_3^2 - l_1 v\right)}{g_1 l_1^2} \times \operatorname{csch}^2\left[\sqrt{\frac{-g_3 l_1^2 - g_4 l_2^2 - g_5 l_3^2 + l_1 v}{4 g_2 l_1^4}}\,(l_1 x + l_2 y + l_3 z - vt)\right].$
Case II: $\Omega > 0$ Present method
$f(x,y,z,t) = -3/8\,\frac{p_1 k_0{}^2\left(E_1 - E_2\sqrt{\Omega}\cot(\sqrt{\Omega}\psi)\right)^2}{\mu_1{}^2 p_3\left(-\Omega E_2 - E_1\sqrt{\Omega}\cot(\sqrt{\Omega}\psi)\right)^2} + k_0 - 6\,\frac{\mu_1{}^2 p_3\left(-\Omega E_2 - E_1\sqrt{\Omega}\cot(\sqrt{\Omega}\psi)\right)^2}{p_1\left(E_1 - E_2\sqrt{\Omega}\cot(\sqrt{\Omega}\psi)\right)^2}.$
Case II: $g_2\left(g_3 l_1^2 + g_4 l_2^2 + g_5 l_3^2 - l_1 v\right) > 0$ (modified Kudryashov) method
$f(x,y,z,t) = \frac{g_3 l_1^2 + g_4 l_2^2 + g_5 l_3^2 - l_1 v}{g_1 l_1^2} \times \left(-3\operatorname{csch}^2\left[\sqrt{\frac{g_3 l_1^2 + g_4 l_2^2 + g_5 l_3^2 - l_1 v}{4 g_2 l_1^4}}\,(l_1 x + l_2 y + l_3 z - vt)\right] - 2\right).$

The nonlinear wave equation derived in this work as a fractional derivative can be a useful model for liquids containing gaseous bubbles that are observed for their complicated wave interaction with a given environment. This makes the fractional derivatives important for the description of the memory and nonlocality in the fluid and necessary for the description of bubble, containing liquids. The solitary wave solutions obtained from the model reflect localized waveform structure, which can travel without loss of form or amplitude, hence resembling pressure waves in such media. These solutions offer information on wave behaviour, dispersion, and stability, all of which are critical to the behaviour of gas bubbles in diverse liquid systems, be it industrial or observed in natural occurrences. Moreover, these wave solutions are important when discussing the problem of wave carrying in the bubbly medium, a field important in the sphere of physics. Such models are employed to estimate the processes affecting sound waves in the media containing bubble gas; it is crucial in various fields, starting with underwater acoustics and ending with ultrasonography. In underwater acoustics, these models are used in sonar systems formulation so that the sound can be accurately used to find objects in bubbly water and in medical ultrasound to improve the techniques used in diagnosis.

In the next steps, potential future studies may be aimed at the further development of the described analytical approach with the inclusion of more elaborate models of a higher dimensionality and different forms of nonlinearity. In the same way, the enhancement of these theoretical models by the experimental confirmation would be the superior achievement, especially concerning the real-life applications in fluid dynamics of industrial processes, advanced ultrasounds, and diagnostics in medicine. The investigation of the above-mentioned relations between fractional order parameters and physical characteristics might help to expand the knowledge of wave processes in the media and to create more detailed models in the future. These advancements could make a breath of fresh air in the areas of engineering and technology that require the control and change of wave behavior.

5. Conclusions

In this study, considering the concept of the Bäcklund transformation integrated with the Riccati-Bernoulli sub-ode method, the solitary wave solutions for the generalized fractional nonlinear wave equation have been derived and analyzed. The mentioned approach allowed us to obtain a wide set of analytical solutions in the form of hyperboles, trigonometric functions, and rational expressions. These solutions were accurately and completely depicted in terms of space by both the $3D$ surface plot and the contour plot regarding the wave behavior. Another innovation has been where wave systems are nonlinear, and the use of $3D$ plus contour visualization has been able to provide a clearer picture of the various behaviors of the wave systems, including applications in fluid dynamics, chemical kinetics, and biological morphogenesis. The fact that the outlined methodology allows for the depiction of the diffusion reaction processes with sufficient accuracy to capture the intricate details of the interaction between the processes underlies one of the primary perceived values derived in this work the solutions obtained are practical in nature. This paper shows that both the Bäcklund transformation and the sub-ode method based on the Riccati-Bernoulli equation can be used as a powerful instruments when studying the generalized fractional nonlinear wave equation. What has been gained by reducing the equation to a form that admits an ordinary differential equation solution is an asset that is both mathematically sound to the extent that the analytical method is valid or reliable and qualitatively useful in charting the nature of solitary waves.

Author contributions

Musawa Yahya Almusawa: Conceptualization, Formal analysis, Investigation, Project administration, Validation, Visualization, Funding; Hassan Almusawa: Data curation, Resources, Validation, Software, Project administration, Writing-review & editing. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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