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Research article

On the existence of solutions for systems of generalized vector quasi-variational equilibrium problems in abstract convex spaces with applications

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Abstract: In this paper, we first introduced systems of generalized vector quasi-variational equilibrium problems as well as systems of vector quasi-variational equilibrium problems as their special cases in abstract convex spaces. Next, we established some existence theorems of solutions for systems of generalized vector quasi-variational equilibrium problems and systems of vector quasi-variational equilibrium problems in non-compact abstract convex spaces. Furthermore, we applied the obtained existence theorem of solutions for systems of vector quasi-variational equilibrium problems to solve the existence problem of Nash equilibria for noncooperative games. Then, as applications of the existence result of Nash equilibria for noncooperative games, we studied the existence of weighted Nash equilibria and Pareto Nash equilibria for multi-objective games. The results derived in this paper extended and unified the primary findings presented by some authors in the literature.

Keywords: abstract convex space; generalized abstract economy; systems of generalized vector quasi-variational equilibrium problem; Nash equilibrium
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1. Introduction and preliminaries

It is well-known that the system of vector quasi-equilibrium problems includes the system of vector quasi-variational inequalities, the system of vector quasi-optimization problems, vector quasi-saddle point problems, and Debreu-type equilibrium problems as special cases. Therefore, the system of vector quasi-equilibrium problems has been extensively and intensively studied by many authors. In 2004, Ansari et al. [1] established existence theorems of solutions for the system of vector quasi-equilibrium problems in Hausdorff topological vector spaces. As applications, they obtained existence results of Debreu-type economic equilibria. In the same year, Ansari and Khan [2] introduced the system of generalized vector quasi-equilibrium problems in the framework of Hausdorff topological vector spaces and derived existence results of solutions for these kinds of problems. Since then, many authors have studied sufficient conditions guaranteeing the existence of solutions for systems of vector quasi-equilibrium problems in the framework of topological vector spaces. Lin [3] proved an existence theorem for systems of generalized quasivariational inclusions and established existence results of solutions for a range of variational analysis and optimization problems. Lin et al. [4] introduced the concept of systems of generalized vector quasi-equilibrium problems for set-valued mappings and obtained existence results of solutions for systems of generalized vector quasi-equilibrium problems based on an existence of equilibria for generalized abstract economy. Lin and Ansari [5] studied the existence of solutions for systems of quasi-variational relations and gave various applications in the field of nonlinear analysis. Al-Homidan et al. [6] proved the existence of weak and strong solutions for systems of generalized implicit vector variational inequalities under different continuity assumptions. Patriche [7] investigated the existence of solutions for systems of generalized vector quasi-equilibrium problems by using the existence results of equilibria for generalized abstract economy. Peng et al. [8] proved the existence of solutions for systems of generalized vector quasi-equilibrium problems with set-valued mappings by means of two maximal element theorems. Peng and Wu [9] presented the equivalent relationship between the generalized Tykhonovwell-posedness of the system of vector quasi-equilibrium problems and that of the minimization problems. Lin [10] established the existence and essential components of solutions for the system of generalized vector quasi-equilibrium problems. Hou et al. [11] introduced the concept of a new system of generalized vector variational inequalities and derived exisrence results of solutions for the new system. In 2016, Farajzadeh et al. [12] obtained an existence theorem of solutions for generalized vector quasi-equilibrium problems in locally convex topological vector spaces. Further, they used the obtained existence theorem to establish an existence theorem of solutions for a system of generalized vector quasi-equilibrium problems. Recently, Hung [13] introduced two types of optimal control problems for systems characterized by the generalized bounded quasi-equilibrium problems and studied the Levitin-Polyak well-posedness for these problems.

On the other hand, some authors have investigated this topic in topological spaces without linear and convex structure. Al-Homidan and Ansari [14] obtained existence results of solutions for systems of generalized vector quasi-equilibrium problems by using existence theorems of equilibria for the generalized abstract economy in the framework of non-compact topological semilattice spaces. Plubtieng and Thammathiwat [15] proved existence theorems of solutions for systems of generalized vector quasi-equilibrium problems based on two maximal element theorems for a family of set-valued mappings on product G-convex spaces. By utilizing an existence theorem of Nash equilibria for generalized games, Ding [16] obtained existence results of solutions for several classes of systems of generalized vector quasi-equilibrium problems in locally FC-uniform spaces. Ding [17] established existence theorems of solutions for systems of generalized quasi-variational inclusion problems in non-compact FC-spaces using an existence theorem of Nash equilibria for generalized games.

In 2008, Park [18] introduced the notion of an abstract convex space that encompasses the vast majority of spaces in the existing literature for their special cases. However, to the best of our knowledge, there seems to be few related research works on systems of generalized vector quasi-variational equilibrium problems and systems of vector quasi-variational equilibrium problems in abstract convex spaces. Therefore, it is important and interesting to study such problems in abstract convex spaces under some suitable conditions. Motivated and inspired by the aforementioned works, the main purpose of this paper is to obtain some existence theorems of solutions for systems of generalized vector quasi-variational equilibrium problems and systems of vector quasi-variational equilibrium problems by means of an existence theorem of equilibria for generalized abstract economy in non-compact abstract convex spaces. As applications, we show existence theorems of Nash equilibria for noncooperative games as well as weighted Nash equilibria and Pareto Nash equilibria for multi-objective games in the framework of non-compact abstract convex spaces. At the same time, we verify the existence result of Pareto Nash equilibria for multi-objective games in the case of compact abstract convex spaces with an example.

The results obtained by this paper unify and extend many of the results in the existing literature, which not only provide a theoretical basis for proving the existence of solutions for equilibrium problems of complex systems in mathematical economics, but can also be applied to analyze and solve related problems in multi-objective decision-making scenarios with practical applications, for example, multi-objective games and conflicts in water allocation.

Now, we introduce some preliminaries that will be used in subsequent discussions.

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of the natural numbers and the set of all real numbers, respectively. For a nonempty set X, let 2^X and $\langle X\rangle$ denote the family of all subsets of X and the family of nonempty finite subsets of X, respectively. Let X and Y be two nonempty sets and $T\colon X \to 2^Y$ be a set-valued mapping. Then, the set-valued mapping $T^{-1}\colon Y \to 2^X$ is defined by

$$T^{-1}(y) = \{x \in X : y \in T(x)\}$$

for each $y \in Y$. For every $X_0 \subseteq X$, let

$$T(X_0) = \bigcup_{x\in X_0} T(x).$$

Let

$$\Delta_n = \{t = (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1}_+ : \sum_{i=0}^{n} t_i = 1\}$$

denote the standard n-dimensional simplex with vertices $\{e_0, e_1, \dots, e_n\}$, where e_i is the $(i + 1)$th unit vector in $\mathbb{R}^{n+1}$. It is easy to see that

$$\Delta_n = \mathrm{co}\{e_0, e_1, \dots, e_n\},$$

where $\mathrm{co}\{e_0, e_1, \dots, e_n\}$ denotes the convex hull of $\{e_0, e_1, \dots, e_n\}$. For any nonempty subset

$$J = \{i_0, i_1, \dots, i_k\}$$

of $\{0, 1, \dots, n\}$, let

$$\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\} = \{t \in \Delta_n : \sum_{j=0}^{k} t_{i_j} = 1\},$$

which is the face of Δ_n spanned by $\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

The following definitions can be found in Park [18].

Definition 1.1. If D is a nonempty set, E is a topological space, and Γ: $\langle D\rangle \to 2^E$ is a set-valued mapping with nonempty values

$$\Gamma_A := \Gamma(A)$$

for every $A \in \langle D\rangle$, then the family $(E, D; \Gamma)$ is called an abstract convex space. If $E = D$, then $(E; \Gamma)$ is denoted for $(E, E; \Gamma)$.

Example 1.1. We give two special cases of abstract convex spaces as follows:

(1) Any topological vector space can be viewed as a special case of an abstract convex space. Indeed, suppose that X is a topological vector space. Define a a nonempty-valued set-valued mapping

$$\Gamma : \langle X\rangle \to 2^X$$

by

$$\Gamma_A = \mathrm{co}(A)$$

for every $A \in \langle X\rangle$, where co(A) denotes the convex hull of A. It is clear that $(X; \Gamma)$ becomes an abstract convex space.

(2) A couple (X, F_A) is called an H-space if X is a topological space and $\{C_A\}$ is a given family of nonempty contractible subsets of X such that $C_A \subseteq C_B$ whenever $A \subseteq B$ (see Horvath [19]). Obviously, for any H-space (X, C_A), define a nonempty-valued set-valued mapping

$$\Gamma : \langle X\rangle \to 2^X$$

by

$$\Gamma_A = C_A$$

for every $A \in \langle X\rangle$. Then, $(X; \Gamma)$ forms an abstract convex space.

Definition 1.2. Let $(E, D; \Gamma)$ be an abstract convex space and D' be a nonempty subset of D. The Γ-convex hull of D' is defined by

$$\mathrm{co}_\Gamma(D') = \bigcup\{\Gamma_A : A \in \langle D'\rangle\}.$$

Definition 1.3. Let $(E, D; \Gamma)$ be an abstract convex space. A nonempty subset F of E is said to be a Γ-convex subset of $(E, D; \Gamma)$ relative to a nonempty subset D' of D if we have $\Gamma_N \subseteq F$ for every $N \in \langle D'\rangle$, that is, $\mathrm{co}_\Gamma(D') \subseteq F$.

Remark 1.1. Let $(E, D; \Gamma)$ be an abstract convex space. Then, by Definition 1.3, we can see that if a nonempty subset F of E is a Γ-convex subset of $(E, D; \Gamma)$ relative to a nonempty subset D' of D, then $(F, D'; \Gamma|_{\langle D'\rangle})$ itself is an abstract convex space which is called to be a subspace of $(E, D; \Gamma)$.

Definition 1.4. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a set. A set-valued mapping G: $D \to 2^E$ is said to be a Knaster-Kuratowski-Mazurkiewicz (KKM) mapping if for any $A \in \langle D\rangle$, we have $\Gamma_A \subseteq G(A)$. For a set-valued mapping H: $E \to 2^Z$ with nonempty values, if a set-valued mapping G: $D \to 2^Z$ satisfies $H(\Gamma_A) \subseteq G(A)$ for every $A \in \langle D\rangle$, then G is called a KKM mapping with respect to H. It is obvious that a KKM mapping $G\colon D \to 2^E$ is a KKM mapping with respect to the identity mapping 1_E.

Definition 1.5. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. A set-valued mapping $H\colon E \to 2^Z$ is said to be a $\mathfrak{RC}$-mapping, if for any closed-valued KKM mapping $G\colon D \to 2^Z$ with respect to H, the family $\{G(y) : y \in D\}$ has the finite intersection property. We denote

$$\mathfrak{RC}(E, Z) := \{H : E \to 2^Z | \ H \text{ is a } \mathfrak{RC}\text{-mapping}\}.$$

Example 1.2. Here, we give an example of the $\mathfrak{RC}$-mapping. A sup-semilattice X is a partially ordered set with the partial ordering denoted by $\leq$, in which any pair $(x, x') \in X \times X$ has a least upper bound $x \vee x'$ (see Horvath and Ciscar [20]). Now, suppose that $(X, \leq)$ is a sup-semilattice and $A \in \langle X \rangle$. Then, A has a least upper bound $\sup A$. If x and x' are two elements in a partially ordered set $(X, \leq)$ with the case that $x \leq x'$, then we call the set

$$[x, x'] = \{y \in X : x \leq y \leq x'\}$$

an order interval. Let $(X, \leq)$ be a sup-semilattice and $A \in \langle X \rangle$. Then, the nonempty set

$$\Delta(A) := \bigcup_{a \in A} [a, \sup A]$$

is well-defined. A topological sup-semilattice is a topological space X with a partial ordering $\leq$ for which it is a sup-semilattice with a continuous sup-operation, that is, the function $X \times X \to X$, defined by $(x, x') \mapsto x \vee x'$ for every $(x, x') \in X \times X$, is continuous. Let X be a topological sup-semilattice with path-connected intervals and let $\Gamma\colon X \to 2^X$ be a nonempty-valued set-valued mapping defined by

$$\Gamma_A := \Delta(A)$$

for every $A \in \langle X \rangle$. Then, we can see that $(X; \Gamma)$ becomes an abstract convex space. Now, we show that the identity mapping $1_X \in \mathfrak{RC}(X, X)$. In fact, let $G\colon X \to 2^X$ be a KKM mapping with closed values, which implies that for each

$$A = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle, \quad \Delta(A) = \bigcup_{a \in A} [a, \sup A] \subseteq G(A).$$

Then, by the proof of Theorem 1 in Horvath and Ciscar [20], it follows that there exists a continuous mapping $\xi\colon \Delta_n \to \Delta(A)$ such that

$$\xi(\Delta_J) \subseteq \Delta(\{x_j : j \in J\})$$

for every nonempty subset $J \subseteq \{0, 1, \ldots, n\}$. Define

$$F_i = G(x_i) \bigcap \Delta(A)$$

for every $i \in \{0, 1, \ldots, n\}$. It is obvious that each F_i is closed in $\Delta(A)$ and, thus, $\xi^{-1}(F_i)$ is closed in Δ_n. Since

$$\xi(\Delta_J) \subseteq \Delta(\{x_j : j \in J\}) \subseteq \bigcup_{i \in J} F_i$$

for every nonempty subset $J \subseteq \{0, 1, \ldots, n\}$, we have

$$\Delta_J \subseteq \bigcup_{i \in J} \xi^{-1}(F_i)$$

and so, it follows from the KKM lemma that

$$\bigcap_{i=0}^{n} \xi^{-1}(F_i) \neq \emptyset.$$

Taking any

$$z \in \bigcap_{i=0}^{n} \xi^{-1}(F_i)$$

leads to

$$\xi(z) \in \bigcap_{i=0}^{n} G(x_i),$$

which implies that the family $\{G(x) : x \in X\}$ has the finite intersection property. Thus, $1_X \in \mathfrak{RC}(X, X)$.

Lemma 1.1. [21] *Let* $\{(E_i, D_i; \Gamma_i)\}_{i \in I}$ *be any family of abstract convex spaces. Let*

$$E := \prod_{i \in I} E_i$$

be equipped with the product topology and

$$D := \prod_{i \in I} D_i.$$

For each $i \in I$, *let* π_i: $D \to D_i$ *be the projection. Define*

$$\Gamma = \prod_{i \in I} \Gamma_i : \langle D \rangle \to 2^E$$

by

$$\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A))$$

for each $A \in \langle D \rangle$. *Then,* $(E, D; \Gamma)$ *is an abstract convex space.*

Remark 1.2. For more details on abstract convex spaces, the reader may consult Park [22, 23], and the references therein.

2. Systems of generalized vector quasi-variational equilibrium problems

Let I be a finite index set. For each $i \in I$, let $(X_i; \Gamma_i)$ be an abstract convex space, Y_i be a topological space, and let

$$X = \prod_{i \in I} X_i.$$

For each $i \in I$, let

$$A_i, B_i, F_i : X \to 2^{X_i}, \quad \Psi_i : X \times X \times X_i \to 2^{Y_i}$$

and

$$C_i : X \times X \times X_i \to 2^{Y_i}$$

be set-valued mappings with nonempty values. We consider the following systems of generalized vector quasi-variational equilibrium problems (for short, SGVQVEP):

(SGVQVEP1): Find $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ and $\Psi_i(\overline{x}, \overline{y}, u_i) \subseteq C_i(\overline{x}, \overline{y}, \overline{x}_i)$ for all $u_i \in A_i(\overline{x})$.

(SGVQVEP2): Find $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ and $\Psi_i(\overline{x}, \overline{y}, u_i) \bigcap C_i(\overline{x}, \overline{y}, \overline{x}_i) \neq \emptyset$ for all $u_i \in A_i(\overline{x})$.

(SGVQVEP3): Find $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ and $\Psi_i(\overline{x}, \overline{y}, u_i) \bigcap C_i(\overline{x}, \overline{y}, \overline{x}_i) = \emptyset$ for all $u_i \in A_i(\overline{x})$.

(SGVQVEP4): Find $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ and $\Psi_i(\overline{x}, \overline{y}, u_i) \nsubseteq C_i(\overline{x}, \overline{y}, \overline{x}_i)$ for all $u_i \in A_i(\overline{x})$.

Clearly, each solution of (SGVQVEP1) (respectively, (SGVQVEP3)) is a solution of (SGVQVEP2) (respectively, (SGVQVEP4)), but the converse is not always true.

To ensure the generality of the aforementioned problems setting, we examine certain specific cases related to recent papers in the literature.

(a) For each $i \in I$, let X_i be a topological semilattice space with path-connected intervals, Y_i be a topological vector space, and let

$$C_i(x, y, u_i) = C_i(x, y)$$

or

$$C_i(x, y, u_i) = -\mathrm{int}C_i(x, y)$$

for all $(x, y, u_i) \in X \times X \times X_i$, where C_i: $X \times X \to 2^{Y_i}$ is a set-valued mapping such that each $C_i(x, y)$ is a proper closed convex cone with $\mathrm{int}C_i(x, y) \neq \emptyset$. Then, (SGVQVEP1)–(SGVQVEP4) reduces to the systems of generalized vector quasi-equilibrium problems considered by Al-Homidan and Ansari [14].

(b) For each $i \in I$, let X_i and Y_i be two topological vector spaces. If for each $i \in I$, $A_i(x) = B_i(x)$ for all $x \in X$ and

$$C_i(x, y, u_i) = C_i(x)$$

or

$$C_i(x, y, u_i) = -\mathrm{int}C_i(x)$$

for all $(x, y, u_i) \in X \times X \times X_i$, where C_i: $X \to 2^{Y_i}$ is a set-valued mapping such that each $C_i(x)$ is a proper closed convex cone with $\mathrm{int}C_i(x) \neq \emptyset$, then (SGVQVEP1)–(SGVQVEP4) reduces to the systems of generalized vector quasi-equilibrium problems introduced and studied by Lin et al. [4] and Peng et al. [8].

(c) If for each $i \in I$ and each $x \in X$,

$$A_i(x) = B_i(x) = F_i(x) = X_i$$

and for each $(x, y, u_i) \in X \times X \times X_i$,

$$\Psi_i(x, y, u_i) = \Psi_i(x, u_i)$$

and

$$C_i(x, y, u_i) = -\mathrm{int}C_i(x),$$

where C_i: $X \to 2^{Y_i}$ is a set-valued mapping such that each $C_i(x)$ is a proper closed convex cone with $\mathrm{int}C_i(x) \neq \emptyset$, then (SGVQVEP4) is considered and studied by Ansari et al. [24].

(d) Let I be a singleton and let X_i, Y_i be two topological vector spaces. If

$$A_i = B_i, \quad F_i(x) = X_i$$

for every $x \in X_i$,

$$\Psi_i(x, y, u_i) = G_i(x, u_i) + H_i(x, u_i)$$

and

$$C_i(x, y, u_i) = Y_i \setminus (-\mathrm{int}C_i)$$

for every $(x, y, u_i) \in X_i \times X_i \times X_i$, where G_i, H_i: $X_i \times X_i \to 2^{Y_i}$ are two set-valued mappings, and $C_i \subseteq Y_i$ is a proper cone, then (SGVQVEP1) collapses to the vector quasi-equilibrium problem studied by Kassay et al. [25].

(e) Let I be a singleton and let X_i, Y_i be two topological vector spaces. If

$$A_i = B_i, \quad F_i(x) = X_i$$

for every $x \in X$,

$$\Psi_i(x, y, u_i) = H_i(x, u_i)$$

and

$$C_i(x, y, u_i) = Y_i \setminus (-\mathrm{int}C_i)$$

for every $(x, y, u_i) \in X_i \times X_i \times X_i$, where

$$H_i : X_i \times X_i \to 2^{Y_i}$$

is a set-valued mapping and $C_i \subseteq Y_i$ is a proper cone, then (SGVQVEP1) becomes the vector quasi-equilibrium problem considered in Capătă [26].

(f) Let I be a singleton, X_i be a H-space, and

$$Y_i = \mathbb{R} \bigcup \{\pm\infty\}.$$

If

$$A_i = B_i, \quad \Psi_i(x, y, u_i) = f_i(x, x) - f_i(u_i, x)$$

and

$$C_i(x, y, u_i) = (-\infty, 0]$$

for every $(x, y, u_i) \in X_i \times X_i \times X_i$, where

$$f_i : X_i \times X_i \to \mathbb{R} \bigcup \{\pm\infty\}$$

is a function, then (SGVQVEP1) reduces to the quasi-equilibrium problem studied by Ding [27].

The above special cases (a)–(f) show that our (SGVQVEP1)–(SGVQVEP4) extends and unifies many kinds of systems of generalized vector quasi-equilibrium problems in the literature.

For each $i \in I$, if Ψ_i is a single-valued mapping, then (SGVQVEP1)–(SGVQVEP4) reduces to the following systems of vector quasi-variational equilibrium problems (for short, SVQVEP), respectively.

(SVQVEP1): Find $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ and $\Psi_i(\overline{x}, \overline{y}, u_i) \in C_i(\overline{x}, \overline{y}, \overline{x}_i)$ for all $u_i \in A_i(\overline{x})$.

(SVQVEP2): Find $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ and $\Psi_i(\overline{x}, \overline{y}, u_i) \notin C_i(\overline{x}, \overline{y}, \overline{x}_i)$ for all $u_i \in A_i(\overline{x})$.

Definition 2.1. Let I be a finite index set. For each $i \in I$, let $(X_i; \Gamma_i)$ be an abstract convex space, Y_i be a topological space, and let

$$X = \prod_{i \in I} X_i.$$

For each $i \in I$, let

$$\Psi_i : X \times X \times X_i \to 2^{Y_i}$$

and

$$C_i : X \times X \times X_i \to 2^{Y_i}$$

be two set-valued mappings. Then, for each $i \in I$ and each $y \in X$, Ψ_i is said to be a Γ_i-C_i-diagonally quasi-convex mapping of type(1) (respectively, type(2)–type(4)) in the third argument if for each

$$N_i = \{u_{i1}, u_{i2}, \ldots, u_{in}\} \in \langle X_i \rangle$$

and each $x \in X$ with $x_i \in \Gamma_i(N_i)$, there exists $j \in \{1, 2, \ldots, n\}$ such that $\Psi_i(x, y, u_{ij}) \subseteq C_i(x, y, x_i)$ (respectively, $\Psi_i(x, y, u_{ij}) \cap C_i(x, y, x_i) \neq \emptyset$, $\Psi_i(x, y, u_{ij}) \cap C_i(x, y, x_i) = \emptyset$, and $\Psi_i(x, y, u_{ij}) \nsubseteq C_i(x, y, x_i)$).

Remark 2.1. Definition 2.1 generalizes the corresponding definitions introduced by Peng et al. [8] from topological vector spaces to abstract convex spaces without any linear and convex structure.

If Ψ_i is a single-valued mapping for every $i \in I$, then Definition 2.1 reduces to the following definition.

Definition 2.2. Let I be a finite index set. For each $i \in I$, let $(X_i; \Gamma_i)$ be an abstract convex space, Y_i be a topological space and let

$$X = \prod_{i \in I} X_i.$$

For each $i \in I$, let

$$\Psi_i : X \times X \times X_i \to Y_i$$

be a single-valued mapping and

$$C_i : X \times X \times X_i \to 2^{Y_i}$$

be a set-valued mapping. Then, for each $i \in I$ and each $y \in X$, Ψ_i is said to be a Γ_i-SC_i-diagonally quasi-convex mapping of type(1) (respectively, type(2)) in the third argument if for each

$$N_i = \{u_{i1}, u_{i2}, \ldots, u_{in}\} \in \langle X_i \rangle$$

and each $x \in X$ with $x_i \in \Gamma_i(N_i)$, there exists $j \in \{1, 2, \ldots, n\}$ such that $\Psi_i(x, y, u_{ij}) \in C_i(x, y, x_i)$ (respectively, $\Psi_i(x, y, u_{ij}) \notin C_i(x, y, x_i)$).

For each $i \in I$, let

$$Y_i \equiv \mathbb{R}, \ \ C_i \equiv (0, +\infty)$$

and let

$$\Psi_i : X \times X \times X_i \to \mathbb{R}$$

be a real-valued function defined by

$$\Psi_i(x, y, u_i) = f_i(x_{\hat{i}}, u_i) - f_i(x)$$

for every $(x, y, u_i) \in X \times X \times X_i$, where f_i: $X \to \mathbb{R}$ is a real-valued function and

$$x_{\hat{i}} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Then, by Definition 2.2 associated to type(2), we have the following definition:

Definition 2.3. Let I be a finite index set such that $(X_i; \Gamma_i)$ is an abstract convex space for every $i \in I$. Let

$$X = \prod_{i \in I} X_i$$

and f_i: $X \to \mathbb{R}$ be a real-valued function. Then, for each $i \in I$, we say that f_i is a Γ_i-diagonally quasi-convex function if for each

$$N_i = \{u_{i1}, u_{i2}, \ldots, u_{in}\} \in \langle X_i \rangle$$

and each $x \in X$ with $x_i \in \Gamma_i(N_i)$, there exists $j \in \{1, 2, \ldots, n\}$ such that

$$f_i(x_{\hat{i}}, u_{ij}) - f_i(x) \leq 0.$$

The following lemma is a special case of Lu and Hu [28, Corollary 28].

Lemma 2.1. *Let I be a finite index set and $\{(X_i; \Gamma_i)\}_{i \in I}$ be a family of abstract convex spaces such that*

$$(X; \Gamma) := (\prod_{i \in I} X_i; \Gamma)$$

and $(X \times X; \Gamma \times \Gamma)$ are two abstract convex spaces defined as in Lemma 1.1. Let

$$\Omega = ((X_i; \Gamma_i), A_i, B_i, F_i, P_i)_{i \in I}$$

be a generalized abstract economy and K be a nonempty compact subset of $X \times X$. For each $i \in I$, assume that

(i) *For each $x \in X$, $A_i(x) \neq \emptyset$, and $\mathrm{co}_{\Gamma_i} A_i(x) \subseteq B_i(x)$;*

(ii) *For each $x \in X$, $F_i(x)$ is nonempty Γ_i-convex;*

(iii) *For each $(x, y) \in X \times X$, $x_i \notin \mathrm{co}_{\Gamma_i} P_i(x, y)$;*

(iv) *For each $u_i \in X_i$, $A_i^{-1}(u_i)$, $F_i^{-1}(u_i)$, and $P_i^{-1}(u_i)$ are open in X;*

(v) *The set* $W_i = \{(x, y) \in X \times X : P_i(x, y) \bigcap A_i(x) \neq \emptyset\}$ *is closed in* $X \times X$;

(vi) *For each* $N_{0i} \times N_{1i} \in \langle X_i \times X_i \rangle$*, there exist compact* Γ_i*-convex subsets* $L_{N_{0i}}$*,* $L_{N_{1i}}$ *of* $(X_i; \Gamma_i)$ *containing* N_{0i}*,* N_{1i}*, respectively, such that, for*

$$L := \prod_{i \in I} L_{N_{0i}} \times \prod_{i \in I} L_{N_{1i}},$$

we have

$$L \setminus K \subseteq \bigcup_{(u_i, v_i) \in L_{N_{0i}} \times L_{N_{1i}}} \Big\{ [(A_i^{-1}(u_i) \bigcap F_i^{-1}(v_i)) \times X] \bigcap [P_i^{-1}(u_i) \bigcup (X \times X \setminus W_i)] \Big\}.$$

If $(X \times X; \Gamma \times \Gamma)$ *satisfies* $1_{X \times X} \in \mathfrak{RC}(X \times X, X \times X)$*, then there exists* $(\overline{x}, \overline{y}) \in X \times X$ *such that for each* $i \in I$*,* $\overline{x}_i \in B_i(\overline{x})$*,* $\overline{y}_i \in F_i(\overline{x})$ *and*

$$P_i(\overline{x}, \overline{y}) \bigcap A_i(\overline{x}) = \emptyset.$$

By Lemma 2.1, we have the following existence theorems of solutions for (SGVQVEP1)–(SGVQVEP4) in non-compact abstract convex spaces.

Theorem 2.1. *Let* I *be a finite index set and* $\{(X_i; \Gamma_i)\}_{i \in I}$ *be a family of abstract convex spaces such that*

$$(X; \Gamma) := (\prod_{i \in I} X_i; \Gamma)$$

and $(X \times X; \Gamma \times \Gamma)$ *are two abstract convex spaces defined as in Lemma 1.1. Let* K *be a nonempty compact subset of* $X \times X$*. For each* $i \in I$*, let*

$$A_i, B_i, F_i : X \to 2^{X_i}$$

be set-valued mappings such that for each $u_i \in X_i$*,* $A_i^{-1}(u_i)$ *and* $F_i^{-1}(u_i)$ *are open in* X *and for each* $x \in X$*,* $A_i(x) \neq \emptyset$*,* $\text{co}_{\Gamma_i} A_i(x) \subseteq B_i(x)$*, and* $F_i(x)$ *is nonempty* Γ_i*-convex. For each* $i \in I$*, let* Y_i *be a topological space and let*

$$\Psi_i, C_i : X \times X \times X_i \to 2^{Y_i}$$

be two set-valued mappings such that the following conditions are satisfied:

(i) *For each* $y \in X$*,* Ψ_i *is a* Γ_i*-*C_i*-diagonally quasi-convex mapping of type(1) in the third argument*;

(ii) *For each* $u_i \in X_i$*, the set* $\{(x, y) \in X \times X : \Psi_i(x, y, u_i) \nsubseteq C_i(x, y, x_i)\}$ *is open in* $X \times X$;

(iii) *The set* $\{(x, y) \in X \times X :$ *there exists* $u_i \in A_i(x)$ *such that* $\Psi_i(x, y, u_i) \nsubseteq C_i(x, y, x_i)\}$ *is closed in* $X \times X$;

(iv) *For each* $N_{0i} \times N_{1i} \in \langle X_i \times X_i \rangle$*, there exist compact* Γ_i*-convex subsets* $L_{N_{0i}}$*,* $L_{N_{1i}}$ *of* $(X_i; \Gamma_i)$ *containing* N_{0i}*,* N_{1i}*, respectively, such that for each* $(x, y) \in L \setminus K$*, there exists* $(u_i, v_i) \in N_{0i} \times N_{1i}$ *satisfying* $u_i \in A_i(x)$*,* $v_i \in F_i(x)$*, and* $\Psi_i(x, y, u_i) \nsubseteq C_i(x, y, x_i)$*, where*

$$L := \prod_{i \in I} L_{N_{0i}} \times \prod_{i \in I} L_{N_{1i}}.$$

If $(X \times X; \Gamma \times \Gamma)$ *satisfies* $1_{X \times X} \in \mathfrak{RC}(X \times X, X \times X)$*, then (SGVQVEP1) has a solution.*

Proof. For each $i \in I$, let us define a set-valued mapping P_i: $X \times X \to 2^{X_i}$ by

$$P_i(x, y) = \{u_i \in X_i : \Psi_i(x, y, u_i) \nsubseteq C_i(x, y, x_i)\}, \ \forall (x, y) \in X \times X.$$

Step 1. Show that for each $i \in I$ and each $(x, y) \in X \times X$, we have

$$x_i \notin \mathrm{co}_{\Gamma_i} P_i(x, y). \tag{2.1}$$

Suppose to the contrary that there exist $\widehat{i} \in I$ and $(\widehat{x}, \widehat{y}) \in X \times X$ such that

$$\widehat{x_{\widehat{i}}} \in \mathrm{co}_{\Gamma_{\widehat{i}}} P_{\widehat{i}}(\widehat{x}, \widehat{y}).$$

By the definition of the convex hull in abstract convex spaces, we can see that there exists

$$N_{\widehat{i}} = \{u_{\widehat{i}1}, u_{\widehat{i}2}, \dots, u_{\widehat{i}n}\} \in \langle P_{\widehat{i}}(\widehat{x}, \widehat{y})\rangle$$

such that $\widehat{x_{\widehat{i}}} \in \Gamma_{\widehat{i}}(N_{\widehat{i}})$. Thus, we have

$$\Psi_{\widehat{i}}(\widehat{x}, \widehat{y}, u_{\widehat{i}j}) \nsubseteq C_{\widehat{i}}(\widehat{x}, \widehat{y}, \widehat{x_{\widehat{i}}}), \ \forall j \in \{1, 2, \dots, n\}. \tag{2.2}$$

By (i), there exists $\widehat{j} \in \{1, 2, \dots, n\}$ such that

$$\Psi_{\widehat{i}}(\widehat{x}, \widehat{y}, u_{\widehat{i}\widehat{j}}) \subseteq C_{\widehat{i}}(\widehat{x}, \widehat{y}, \widehat{x_{\widehat{i}}}),$$

which contradicts (2.2). Hence, (2.1) holds.

Step 2. Verify that $P_i^{-1}(u_i)$ is open in $X \times X$ for every $i \in I$ and every $u_i \in X_i$, and each

$$W_i = \{(x, y) \in X \times X : P_i(x, y) \bigcap A_i(x) \neq \emptyset\}$$

is closed in $X \times X$. Indeed, by the definition of P_i, it follows that

$$P_i^{-1}(u_i) = \{(x, y) \in X \times X : \Psi_i(x, y, u_i) \nsubseteq C_i(x, y, x_i)\}$$

for every $i \in I$ and every $u_i \in X_i$. Then, by (ii), one can conclude that $P_i^{-1}(u_i)$ is open in $X \times X$. Furthermore, by the definition of P_i again, for each $i \in I$, we have the following:

$$\begin{aligned} W_i &= \{(x, y) \in X \times X : P_i(x, y) \bigcap A_i(x) \neq \emptyset\} \\ &= \{(x, y) \in X \times X : \text{there exists } u_i \in A_i(x) \text{ such that } \Psi_i(x, y, u_i) \nsubseteq C_i(x, y, x_i)\}. \end{aligned}$$

Now, by (iii), we can see that the set

$$W_i = \{(x, y) \in X \times X : P_i(x, y) \bigcap A_i(x) \neq \emptyset\}$$

is closed in $X \times X$ for every $i \in I$.

Step 3. Prove that (iv) of Theorem 2.1 implies (vi) of Lemma 2.1. In fact, by (iv), for each $i \in I$ and each

$$N_{0i} \times N_{1i} \in \langle X_i \times X_i\rangle,$$

there exist compact Γ_i-convex subsets $L_{N_{0i}},\ L_{N_{1i}}$ of $(X_i;\Gamma_i)$ containing $N_{0i},\ N_{1i}$, respectively, such that, for

$$L := \prod_{i\in I} L_{N_{0i}} \times \prod_{i\in I} L_{N_{1i}},$$

we have the following:

$$\begin{aligned} L \setminus K &\subseteq \bigcup_{(u_i,v_i)\in L_{N_{0i}}\times L_{N_{1i}}} \left\{[(A_i^{-1}(u_i)\bigcap F_i^{-1}(v_i))\times X]\bigcap P_i^{-1}(u_i)\right\} \\ &\subseteq \bigcup_{(u_i,v_i)\in L_{N_{0i}}\times L_{N_{1i}}} \{[(A_i^{-1}(u_i)\bigcap F_i^{-1}(v_i))\times X]\bigcap [P_i^{-1}(u_i)\bigcup (X\times X\setminus W_i)]\}. \end{aligned}$$

Thus, one can see that (vi) of Lemma 2.1 is satisfied. At this point, all the conditions of Lemma 2.1 are fulfilled. Therefore, it follows from Lemma 2.1 that there exists $(\overline{x},\overline{y})\in X\times X$ such that for each $i\in I$, $\overline{x}_i\in B_i(\overline{x})$, $\overline{y}_i\in F_i(\overline{x})$, and

$$P_i(\overline{x},\overline{y})\bigcap A_i(\overline{x}) = \emptyset,$$

i.e.,

$$\Psi_i(\overline{x},\overline{y},u_i)\subseteq C_i(\overline{x},\overline{y},\overline{x}_i)$$

for all $u_i\in A_i(\overline{x})$, which implies that $(\overline{x},\overline{y})\in X\times X$ is a solution of (SGVQVEP1). This completes the proof. □

Remark 2.2. (i) of Theorem 2.1 can be replaced by the following stronger conditions:

(i)′ For each $(x,y)\in X\times X$, $\Psi_i(x,y,x_i)\subseteq C_i(x,y,x_i)$;

(i)″ For each $(x,y)\in X\times X$, the set $\{u_i\in X_i : \Psi_i(x,y,u_i)\nsubseteq C_i(x,y,x_i)\}$ is Γ_i-convex.

Indeed, for each $i\in I$ and each $(x,y)\in X\times X$, define a set-valued mapping P_i: $X\times X\to 2^{X_i}$ by

$$P_i(x,y) = \{u_i\in X_i : \Psi_i(x,y,u_i)\nsubseteq C_i(x,y,x_i)\},\ \forall (x,y)\in X\times X.$$

Suppose that (i) of Theorem 2.1 does not hold. Then, there exist $i\in I$, $\widehat{y}\in X$,

$$N_i = \{u_{i1},u_{i2},\dots,u_{in}\}\in\langle X_i\rangle,$$

and $\widehat{x}\in X$ with $\widehat{x_i}\in\Gamma_i(N_i)$ such that

$$\Psi_i(\widehat{x},\widehat{y},u_{ij})\nsubseteq C_i(\widehat{x},\widehat{y},\widehat{x_i}),\ \forall j\in\{1,2,\dots,n\}.$$

Thus, we have $N_i\subseteq P_i(\widehat{x},\widehat{y})$. By (i)″, the set

$$P_i(\widehat{x},\widehat{y}) = \{u_i\in X_i : \Psi_i(\widehat{x},\widehat{y},u_i)\nsubseteq C_i(\widehat{x},\widehat{y},\widehat{x_i})\}$$

is Γ_i-convex and so,

$$\widehat{x_i}\in\Gamma_i(N_i)\subseteq P_i(\widehat{x},\widehat{y}).$$

Therefore, we have

$$\Psi_i(\widehat{x},\widehat{y},\widehat{x_i})\nsubseteq C_i(\widehat{x},\widehat{y},\widehat{x_i}),$$

which contradicts (i)′. Hence, (i) of Theorem 2.1 holds.

Theorem 2.2. *Suppose that* I, $\{(X_i;\Gamma_i)\}_{i\in I}$, $(X;\Gamma)$, $(X\times X;\Gamma\times\Gamma)$, K, A_i, B_i, F_i *and* Y_i *are as in Theorem 2.1. For each* $i\in I$, *let*

$$\Psi_i, C_i : X\times X\times X_i \to 2^{Y_i}$$

be two set-valued mappings such that the following conditions are satisfied:

(i) *For each* $y\in X$, Ψ_i *is a* Γ_i-C_i*-diagonally quasi-convex mapping of type(2) in the third argument*;

(ii) *For each* $u_i\in X_i$, *the set* $\{(x,y)\in X\times X : \Psi_i(x,y,u_i)\bigcap C_i(x,y,x_i)=\emptyset\}$ *is open in* $X\times X$;

(iii) *The set* $\{(x,y)\in X\times X :$ *there exists* $u_i\in A_i(x)$ *such that* $\Psi_i(x,y,u_i)\subseteq Y_i\setminus C_i(x,y,x_i)\}$ *is closed in* $X\times X$;

(iv) *For each* $N_{0i}\times N_{1i}\in\langle X_i\times X_i\rangle$, *there exist compact* Γ_i*-convex subsets* $L_{N_{0i}}$, $L_{N_{1i}}$ *of* $(X_i;\Gamma_i)$ *containing* N_{0i}, N_{1i}, *respectively, such that for each* $(x,y)\in L\setminus K$, *there exists* $(u_i,v_i)\in N_{0i}\times N_{1i}$ *satisfying* $u_i\in A_i(x)$, $v_i\in F_i(x)$, *and*

$$\Psi_i(x,y,u_i)\subseteq Y_i\setminus C_i(x,y,x_i),$$

where

$$L:=\prod_{i\in I}L_{N_{0i}}\times\prod_{i\in I}L_{N_{1i}}.$$

If $(X\times X;\Gamma\times\Gamma)$ *satisfies* $1_{X\times X}\in\mathfrak{RC}(X\times X,X\times X)$, *then (SGVQVEP2) has a solution.*

Proof. For each $i\in I$, we define a set-valued mapping P_i: $X\times X\to 2^{X_i}$ by

$$P_i(x,y)=\{u_i\in X_i : \Psi_i(x,y,u_i)\bigcap C_i(x,y,x_i)=\emptyset\},\quad \forall (x,y)\in X\times X.$$

Step 1. Show that for each $i\in I$ and each $(x,y)\in X\times X$, we have

$$x_i\notin \mathrm{co}_{\Gamma_i}P_i(x,y),\ \forall i\in I \quad\text{and}\quad \forall (x,y)\in X\times X. \tag{2.3}$$

Suppose to the contrary that there exist $\widehat{i}\in I$ and $(\widehat{x},\widehat{y})\in X\times X$ such that

$$\widehat{x_{\widehat{i}}}\in \mathrm{co}_{\Gamma_{\widehat{i}}}P_{\widehat{i}}(\widehat{x},\widehat{y}).$$

The definition of the convex hull in abstract convex spaces tells us that there exists

$$N_{\widehat{i}}=\{u_{\widehat{i}1},u_{\widehat{i}2},\dots,u_{\widehat{i}n}\}\in\langle P_{\widehat{i}}(\widehat{x},\widehat{y})\rangle$$

such that $\widehat{x_{\widehat{i}}}\in\Gamma_{\widehat{i}}(N_{\widehat{i}})$. Thus, we have

$$\Psi_{\widehat{i}}(\widehat{x},\widehat{y},u_{\widehat{i}j})\bigcap C_{\widehat{i}}(\widehat{x},\widehat{y},\widehat{x_{\widehat{i}}})=\emptyset,\quad \forall j\in\{1,2,\dots,n\}. \tag{2.4}$$

By (i), there exists $\widehat{j}\in\{1,2,\dots,n\}$ such that

$$\Psi_{\widehat{i}}(\widehat{x},\widehat{y},u_{\widehat{i}\widehat{j}})\bigcap C_{\widehat{i}}(\widehat{x},\widehat{y},\widehat{x_{\widehat{i}}})\neq\emptyset,$$

which contradicts (2.4). Hence, (2.3) holds.

Step 2. Verify that $P_i^{-1}(u_i)$ is open in $X \times X$ for every $i \in I$ and every $u_i \in X_i$, and each

$$W_i = \{(x, y) \in X \times X : P_i(x, y) \bigcap A_i(x) \neq \emptyset\}$$

is closed in $X \times X$. Indeed, by the definition of P_i, it follows that

$$P_i^{-1}(u_i) = \{(x, y) \in X \times X : \Psi_i(x, y, u_i) \bigcap C_i(x, y, x_i) = \emptyset\}$$

for every $i \in I$ and every $u_i \in X_i$. Then, by (ii), one can see that $P_i^{-1}(u_i)$ is open in $X \times X$. Furthermore, by the definition of P_i again, for each $i \in I$, we have the following:

$$\begin{aligned} W_i &= \{(x, y) \in X \times X : P_i(x, y) \bigcap A_i(x) \neq \emptyset\} \\ &= \{(x, y) \in X \times X : \text{there exists } u_i \in A_i(x) \text{ such that } \Psi_i(x, y, u_i) \bigcap C_i(x, y, x_i) = \emptyset\} \\ &= \{(x, y) \in X \times X : \text{there exists } u_i \in A_i(x) \text{ such that } \Psi_i(x, y, u_i) \subseteq Y_i \setminus C_i(x, y, x_i)\}. \end{aligned}$$

Now, by (iii), we can see that the set

$$W_i = \{(x, y) \in X \times X : P_i(x, y) \bigcap A_i(x) \neq \emptyset\}$$

is closed in $X \times X$ for every $i \in I$.

Step 3. Prove that (iv) of Theorem 2.1 implies (vi) of Lemma 2.1. In fact, by (iv), for each $i \in I$ and each $N_{0i} \times N_{1i} \in \langle X_i \times X_i \rangle$, there exist compact Γ_i-convex subsets $L_{N_{0i}}$, $L_{N_{1i}}$ of $(X_i; \Gamma_i)$ containing N_{0i}, N_{1i}, respectively, such that, for

$$L := \prod_{i \in I} L_{N_{0i}} \times \prod_{i \in I} L_{N_{1i}},$$

we have the following:

$$\begin{aligned} L \setminus K &\subseteq \bigcup_{(u_i, v_i) \in L_{N_{0i}} \times L_{N_{1i}}} \left\{[(A_i^{-1}(u_i) \bigcap F_i^{-1}(v_i)) \times X] \bigcap P_i^{-1}(u_i)\right\} \\ &\subseteq \bigcup_{(u_i, v_i) \in L_{N_{0i}} \times L_{N_{1i}}} \{[(A_i^{-1}(u_i) \bigcap F_i^{-1}(v_i)) \times X] \bigcap [P_i^{-1}(u_i) \bigcup (X \times X \setminus W_i)]\}. \end{aligned}$$

Thus, one can see that (vi) of Lemma 2.1 is satisfied. At this point, all the assumptions of Lemma 2.1 are fulfilled. Therefore, according to Lemma 2.1, there exists $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$, and

$$P_i(\overline{x}, \overline{y}) \bigcap A_i(\overline{x}) = \emptyset,$$

i.e.,

$$\Psi_i(\overline{x}, \overline{y}, u_i) \bigcap C_i(\overline{x}, \overline{y}, \overline{x}_i) \neq \emptyset$$

for all $u_i \in A_i(\overline{x})$, which implies that $(\overline{x}, \overline{y}) \in X \times X$ is a solution of (SGVQVEP2). This completes the proof. □

By using the same arguments as in the proof of Theorems 2.1 and 2.2, we can obtain the following theorems of solutions for (SGVQVEP3) and (SGVQVEP4). Here, we omit their proofs.

Theorem 2.3. *Suppose that I, $\{(X_i;\Gamma_i)\}_{i\in I}$, $(X;\Gamma)$, $(X\times X;\Gamma\times\Gamma)$, K, A_i, B_i, F_i, and Y_i are as in Theorem 2.1. For each $i\in I$, let*

$$\Psi_i, C_i : X\times X\times X_i \to 2^{Y_i}$$

be two set-valued mappings such that the following conditions are satisfied:

(i) *For each $y\in X$, Ψ_i is a Γ_i-C_i-diagonally quasi-convex mapping of type(3) in the third argument*;

(ii) *For each $u_i\in X_i$, the set $\{(x,y)\in X\times X : \Psi_i(x,y,u_i)\bigcap C_i(x,y,x_i)\neq\emptyset\}$ is open in $X\times X$*;

(iii) *The set $\{(x,y)\in X\times X$: there exists $u_i\in A_i(x)$ such that $\Psi_i(x,y,u_i)\bigcap C_i(x,y,x_i)\neq\emptyset\}$ is closed in $X\times X$*;

(iv) *For each $N_{0i}\times N_{1i}\in\langle X_i\times X_i\rangle$, there exist compact Γ_i-convex subsets $L_{N_{0i}}$, $L_{N_{1i}}$ of $(X_i;\Gamma_i)$ containing N_{0i}, N_{1i}, respectively, such that for each $(x,y)\in L\setminus K$, there exists $(u_i,v_i)\in N_{0i}\times N_{1i}$ satisfying $u_i\in A_i(x)$, $v_i\in F_i(x)$, and*

$$\Psi_i(x,y,u_i)\bigcap C_i(x,y,x_i)\neq\emptyset,$$

where

$$L := \prod_{i\in I} L_{N_{0i}} \times \prod_{i\in I} L_{N_{1i}}.$$

If $(X\times X;\Gamma\times\Gamma)$ satisfies $1_{X\times X}\in\mathfrak{RC}(X\times X, X\times X)$, then (SGVQVEP3) has a solution.

Theorem 2.4. *Suppose that I, $\{(X_i;\Gamma_i)\}_{i\in I}$, $(X;\Gamma)$, $(X\times X;\Gamma\times\Gamma)$, K, A_i, B_i, F_i, and Y_i are as in Theorem 2.1. For each $i\in I$, let*

$$\Psi_i, C_i : X\times X\times X_i \to 2^{Y_i}$$

be two set-valued mappings such that the following conditions are satisfied:

(i) *For each $y\in X$, Ψ_i is a Γ_i-C_i-diagonally quasi-convex mapping of type(4) in the third argument*;

(ii) *For each $u_i\in X_i$, the set $\{(x,y)\in X\times X : \Psi_i(x,y,u_i)\subseteq C_i(x,y,x_i)\}$ is open in $X\times X$*;

(iii) *The set $\{(x,y)\in X\times X$: there exists $u_i\in A_i(x)$ such that $\Psi(x,y,u_i)\subseteq C_i(x,y,x_i)\}$ is closed in $X\times X$*;

(iv) *For each $N_{0i}\times N_{1i}\in\langle X_i\times X_i\rangle$, there exist compact Γ_i-convex subsets $L_{N_{0i}}$, $L_{N_{1i}}$ of $(X_i;\Gamma_i)$ containing N_{0i}, N_{1i}, respectively, such that for each $(x,y)\in L\setminus K$, there exists $(u_i,v_i)\in N_{0i}\times N_{1i}$ satisfying $u_i\in A_i(x)$, $v_i\in F_i(x)$, and*

$$\Psi_i(x,y,u_i)\subseteq C_i(x,y,x_i),$$

where

$$L := \prod_{i\in I} L_{N_{0i}} \times \prod_{i\in I} L_{N_{1i}}.$$

If $(X\times X;\Gamma\times\Gamma)$ satisfies $1_{X\times X}\in\mathfrak{RC}(X\times X, X\times X)$, then (SGVQVEP4) has a solution.

Remark 2.3. (1) Theorems 2.1–2.4 are new results, which partially improve and extend the Theorems 5.1–5.4 obtained by Al-Homidan and Ansari [14] in the following aspects:

(a) From topological sup-semilattice spaces with path-connected intervals to abstract convex spaces (see Example 1.1);

(b) The Hausdorff property of topological sup-semilattice spaces with path-connected intervals involved in Theorems 5.1–5.4 of Al-Homidan and Ansari [14] is necessary, while the Hausdorff property of abstract convex spaces in our Theorems 2.1–2.4 is not required;

(c) The equilibrium problems in our Theorems 2.1–2.4 are more general than those involved in Theorems 5.1–5.4 of Al-Homidan and Ansari [14];

(d) The spaces Y_i in Theorems 5.1–5.4 obtained by Al-Homidan and Ansari [14] are topological vector spaces, while the spaces Y_i in our Theorems 2.1–2.4 are general topological spaces without linear and convex structure.

(2) From the above analyses, together with Remark 5.3 in Al-Homidan and Ansari [14], we can see that Theorems 2.1–2.4 extend and generalize Theorems 3.2.1–3.2.4 by Lin et al. [4], and the Theorems 3.1–3.4 by Peng et al. [8] in a number of other ways, in addition to generalizing the Theorems 3.2.1–3.2.4 by Lin et al. [4], and Theorems 3.1–3.4 by Peng et al. [8] from topological vector spaces to abstract convex spaces. Here, from a general point of view, it is necessary to explain why an abstract convex space X satisfying property $1_X \in \mathfrak{RC}(X, X)$ contains a topological vector space as its special case. In fact, let X be a topological vector space and let $\Gamma\colon \langle X \rangle \to 2^X$ be a nonempty-valued set-valued mapping defined by

$$\Gamma_A = \mathrm{co}(A)$$

for every $A \in \langle X \rangle$, where $\mathrm{co}(A)$ denotes the convex hull of A. Then, it follows that $(X; \Gamma)$ becomes an abstract convex space. Now, let us prove that $1_X \in \mathfrak{RC}(X, X)$. Indeed, suppose that $G\colon X \to 2^X$ is a KKM mapping such that $G(x)$ is closed in X for every $x \in X$. Then, we have

$$\Gamma_A = \mathrm{co}(\{x_0, x_1, \ldots, x_n\}) \subseteq \bigcup_{i=0}^{n} G(x_i)$$

for every

$$A = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle.$$

Define a continuous mapping $\xi\colon \Delta_n \to \mathrm{co}(\{x_0, x_1, \ldots, x_n\})$ by

$$\xi(t) = \sum_{i=0}^{n} t_i x_i$$

for every

$$t = (t_0, t_1, \ldots, t_n) \in \Delta_n.$$

Let

$$C = \mathrm{co}(\{x_0, x_1, \ldots, x_n\})$$

and define

$$F_i = G(x_i) \bigcap C$$

for every $i \in \{0, 1, \ldots, n\}$. It is obvious that each F_i is closed in C. Thus, by the continuity of ξ, we can see that each $\xi^{-1}(F_i)$ is closed in Δ_n. Since

$$\xi(\Delta_J) \subseteq \mathrm{co}(\{x_j : j \in J\}) \subseteq \bigcup_{i \in J} F_i$$

for any nonempty subset $J \subseteq \{0, 1, \ldots, n\}$, we have

$$\Delta_J \subseteq \bigcup_{i\in J} \xi^{-1}(F_i).$$

Hence, by the KKM lemma, we have

$$\bigcap_{i=0}^{n} \xi^{-1}(F_i) \neq \emptyset,$$

which implies

$$\bigcap_{i=0}^{n} G(x_i) \neq \emptyset.$$

Therefore, we have $1_X \in \mathfrak{RC}(X, X)$.

If Ψ_i is a single-valued mapping for every $i \in I$, then we have the following two existence theorems of solutions for (SVQVEP1) and (SVQVEP2) from Theorems 2.1–2.4.

Theorem 2.5. *Suppose that I, $\{(X_i; \Gamma_i)\}_{i\in I}$, $(X; \Gamma)$, $(X \times X; \Gamma \times \Gamma)$, K, A_i, B_i, F_i, and Y_i are as in Theorem 2.1. For each $i \in I$, let*

$$\Psi_i : X \times X \times X_i \to Y_i$$

be a single-valued mapping and

$$C_i : X \times X \times X_i \to 2^{Y_i}$$

be a set-valued mapping such that the following conditions are satisfied:

(i) *For each $y \in X$, Ψ_i is a Γ_i-SC_i-diagonally quasi-convex mapping of type(1) in the third argument*;

(ii) *For each $u_i \in X_i$, the set $\{(x, y) \in X \times X : \Psi_i(x, y, u_i) \notin C_i(x, y, x_i)\}$ is open in $X \times X$*;

(iii) *The set $\{(x, y) \in X \times X :$ there exists $u_i \in A_i(x)$ such that $\Psi_i(x, y, u_i) \notin C_i(x, y, x_i)\}$ is closed in $X \times X$*;

(iv) *For each $N_{0i} \times N_{1i} \in \langle X_i \times X_i \rangle$, there exist compact Γ_i-convex subsets $L_{N_{0i}}$, $L_{N_{1i}}$ of $(X_i; \Gamma_i)$ containing N_{0i}, N_{1i}, respectively, such that for each $(x, y) \in L \setminus K$, there exists $(u_i, v_i) \in N_{0i} \times N_{1i}$ satisfying $u_i \in A_i(x)$, $v_i \in F_i(x)$, and*

$$\Psi_i(x, y, u_i) \notin C_i(x, y, x_i),$$

where

$$L := \prod_{i\in I} L_{N_{0i}} \times \prod_{i\in I} L_{N_{1i}}.$$

If $(X \times X; \Gamma \times \Gamma)$ satisfies $1_{X\times X} \in \mathfrak{RC}(X \times X, X \times X)$, then (SVQVEP1) has a solution.

Theorem 2.6. *Suppose that I, $\{(X_i; \Gamma_i)\}_{i\in I}$, $(X; \Gamma)$, $(X \times X; \Gamma \times \Gamma)$, K, A_i, B_i, F_i, and Y_i are as in Theorem 2.1. For each $i \in I$, let*

$$\Psi_i : X \times X \times X_i \to Y_i$$

be a single-valued mapping and

$$C_i : X \times X \times X_i \to 2^{Y_i}$$

be a set-valued mapping such that the following conditions are satisfied:

(i) *For each* $y \in X$, Ψ_i *is a* Γ_i-SC_i*-diagonally quasi-convex mapping of type(2) in the third argument*;

(ii) *For each* $u_i \in X_i$, *the set* $\{(x, y) \in X \times X : \Psi_i(x, y, u_i) \in C_i(x, y, x_i)\}$ *is open in* $X \times X$;

(iii) *The set* $\{(x, y) \in X \times X$: *there exists* $u_i \in A_i(x)$ *such that* $\Psi_i(x, y, u_i) \in C_i(x, y, x_i)\}$ *is closed in* $X \times X$;

(iv) *For each* $N_{0i} \times N_{1i} \in \langle X_i \times X_i \rangle$, *there exist compact* Γ_i*-convex subsets* $L_{N_{0i}}$, $L_{N_{1i}}$ *of* $(X_i; \Gamma_i)$ *containing* N_{0i}, N_{1i}, *respectively, such that for each* $(x, y) \in L \setminus K$, *there exists* $(u_i, v_i) \in N_{0i} \times N_{1i}$ *satisfying* $u_i \in A_i(x)$, $v_i \in F_i(x)$, *and*

$$\Psi_i(x, y, u_i) \in C_i(x, y, x_i),$$

where

$$L := \prod_{i\in I} L_{N_{0i}} \times \prod_{i\in I} L_{N_{1i}}.$$

If $(X \times X; \Gamma \times \Gamma)$ *satisfies* $1_{X\times X} \in \mathfrak{RC}(X \times X, X \times X)$, *then (SVQVEP2) has a solution.*

For each $i \in I$, let

$$A(x_i) = B(x_i) = F(x_i) \equiv X_i$$

for all $x_i \in X_i$, $Y_i = \mathbb{R}$, and let

$$C_i(x, y, x_i) \equiv (0, +\infty)$$

and

$$\Psi_i(x, y, u_i) = f_i(x_{\widehat{i}}, u_i) - f_i(x)$$

for every $(x, y, u_i) \in X \times X \times X_i$, where f_i: $X \to \mathbb{R}$ is a real-valued function. Then, by Theorem 2.6, we can derive the following existence theorem of Nash equilibria for noncooperative games in non-compact abstract convex spaces.

Theorem 2.7. *Suppose that* I, $\{(X_i; \Gamma_i)\}_{i\in I}$, *and* $(X; \Gamma)$ *are as in Theorem 2.1. Let* K *be a nonempty compact subset of* X. *For each* $i \in I$, *let* f_i: $X \to \mathbb{R}$ *be a real-valued function such that the following conditions are satisfied*:

(i) f_i *is a* Γ_i*-diagonally quasi-convex function*;

(ii) *For each* $u_i \in X_i$, *the set* $\{x \in X : f_i(x_{\widehat{i}}, u_i) > f_i(x)\}$ *is open in* X;

(iii) *The set* $\{x \in X$: *there exists* $u_i \in X_i$ *such that* $f_i(x_{\widehat{i}}, u_i) > f_i(x)\}$ *is closed in* X;

(iv) *For each* $N_i \in \langle X_i \rangle$, *there exists a compact* Γ_i*-convex subsets* L_{N_i} *of* $(X_i; \Gamma_i)$ *containing* N_i *such that for each* $x \in L \setminus K$, *there exists* $u_i \in N_i$ *satisfying*

$$f_i(x_{\widehat{i}}, u_i) > f_i(x),$$

where

$$L := \prod_{i\in I} L_{N_i}.$$

If $(X; \Gamma)$ *satisfies* $1_X \in \mathfrak{RC}(X, X)$, *then there exists one Nash equilibrium* $\overline{x} \in X$ *for the noncooperative game* $((X_i; \Gamma_i), f_i)_{i\in I}$, *i.e.,*

$$f_i(\overline{x}_{\widehat{i}}, u_i) \leq f_i(\overline{x})$$

for every $u_i \in X_i$.

Remark 2.4. If $\{(X_i;\Gamma_i)\}_{i\in I}$ is a family of Hausdorff abstract convex spaces such that X_i satisfies the first countable axiom for every $i \in I$, then (iii) of Theorem 2.7 can be replaced by the following condition:

(iii)′ For each $i \in I$, the graph of the set-valued mapping P_i: $X \to 2^{X_i}$ defined by

$$P_i(x) = \{u_i \in X_i : f_i(x_{\hat{i}}, u_i) > f_i(x)\}$$

for each $x \in X$, is closed and for each compact subset $Z \subseteq X$, the set $P_i(Z)$ is compact subset of X_i.

Indeed, by the fact that every X_i satisfies the first countable axiom, we can see that

$$X = \prod_{i\in I} X_i$$

is also a first-countable topological space. Thus, for each $i \in I$, let $\{x_n\}_{n\in\mathbb{N}}$ be a net in $\{x \in X : P_i(x) \neq \emptyset\}$ such that $x_n \to x^* \in X$. Therefore, for each $n \in \mathbb{N}$, we have $P_i(x_n) \neq \emptyset$ and there exists $u_{in} \in X_i$ such that $u_{in} \in P_i(x_n)$. Let

$$M = \{x_n\} \bigcup \{x^*\}.$$

Then, M is a compact subset of X. By (iii)′, the set

$$P_i(M) = \bigcup_{x\in M} P_i(x)$$

is a compact subset of X_i. Since

$$\{u_{in}\}_{n\in\mathbb{N}} \subseteq P_i(M),$$

it follows that $\{u_{in}\}_{n\in\mathbb{N}}$ has a convergent subnet with limit u_i^*. Without loss of generality, we assume that $u_{in} \to u_i^*$. By (iii)′ again, we have $u_i^* \in P_i(x^*)$, which implies that $x^* \in \{x \in X : P_i(x) \neq \emptyset\}$. Therefore, the set $\{x \in X : P_i(x) \neq \emptyset\}$ is closed in X, i.e., the set $\{x \in X :$ there exists $u_i \in X_i$ such that $f_i(x_{\hat{i}}, u_i) > f_i(x)\}$ is closed in X.

Corollary 2.1. *Let I be a finite index set and $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of compact abstract convex spaces such that*

$$(X;\Gamma) := (\prod_{i\in I} X_i;\Gamma)$$

is an abstract convex space defined as in Lemma 1.1. For each $i \in I$, let f_i: $X \to \mathbb{R}$ be a real-valued function such that the following conditions are satisfied:

(i) *f_i is a Γ_i-diagonally quasi-convex function*;

(ii) *For each $u_i \in X_i$, the set $\{x \in X : f_i(x_{\hat{i}}, u_i) > f_i(x)\}$ is open in X*;

(iii) *The set $\{x \in X :$ there exists $u_i \in X_i$ such that $f_i(x_{\hat{i}}, u_i) > f_i(x)\}$ is closed in X.*

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then there exists one Nash equilibrium $\overline{x} \in X$ for the noncooperative game $((X_i;\Gamma_i), f_i)_{i\in I}$, i.e., $f_i(\overline{x}_{\hat{i}}, u_i) \leq f_i(\overline{x})$ for every $u_i \in X_i$.

Proof. For each $i \in I$ and each $N_i \in \langle X_i\rangle$, let

$$L_{N_i} = X_i$$

and

$$K = L = \prod_{i\in I} L_{N_i} = \prod_{i\in I} X_i.$$

Then, it is clear that (iv) of Theorem 2.7 is satisfied automatically. Therefore, it follows from Theorem 2.7 that the conclusion of Corollary 2.1 holds. This completes the proof. □

3. Existence of equilibria for multiobjective games

Now, we give the definition of a multi-objective game in its strategy form as follows:

Definition 3.1 Let $I = \{1, 2, \ldots, n\}$ denote the set of players. The family $((X_i; \Gamma_i), V^i)_{i\in I}$ is called a multi-objective game if for each $i \in I$, X_i is the set of strategies of the ith player such that $(X_i; \Gamma_i)$ is an abstract convex space. Let

$$X = \prod_{i\in I} X_i$$

and $V^i \colon X \to \mathbb{R}^{m_i}$ be the vector payoff function of the ith player, which is defined, for each

$$x = (x_1, x_2, \ldots, x_n) \in X$$

by

$$V^i(x) := (v^i_1(x), v^i_2(x), \ldots, v^i_{m_i}(x)),$$

where $m_i \in \mathbb{N}$ and v^i_j stands for noncommensurable outcomes for every $i \in I$ and every $j \in \{1, 2, \ldots, m_i\}$.

The following definitions can be found in Patriche [29].

Definition 3.2. For a multi-objective game $((X_i; \Gamma_i), V^i)_{i\in I}$, a strategy portfolio $\overline{x} \in X$ is called a weighted Nash equilibrium with respect to the weighted vector

$$Q = (Q_i)_{i\in I}$$

such that

$$Q_i = (Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}) \in \mathbb{R}^{m_i}_+ \setminus \{0\}$$

if for each $i \in I$, we have

$$Q_i \cdot V^i(\overline{x}) \leq Q_i \cdot V^i(\overline{x}_{\hat{i}}, u_i)$$

for every $u_i \in X_i$, where

$$\mathbb{R}^{m_i}_+ := \{t = (t_1, t_2, \ldots, t_{m_i}) \in \mathbb{R}^{m_i} : t_j \geq 0, \forall j = 1, \ldots, m_i\}$$

has a nonempty interior with the topology induced by the Euclidian metric, · denotes the inner product in $\mathbb{R}^{k_i}$, and

$$\overline{x}_{\hat{i}} := (\overline{x}_1, \ldots, \overline{x}_{i-1}, \overline{x}_{i+1}, \ldots, \overline{x}_n).$$

Remark 3.1. If

$$Q_i = (Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}) \in \mathbb{R}^{m_i}_+ \setminus \{0\}$$

with

$$\sum_{j=1}^{m_i} Q_{ij} = 1$$

for every $i \in I$, then the strategy portfolio $\overline{x} \in X$ is said to be a normalized weighted Nash equilibrium with respect to the weight vector Q.

Definition 3.3. For a multi-objective game $((X_i; \Gamma_i), V^i)_{i\in I}$, a strategy $\overline{x}_i \in X_i$ of the ith player is called a Pareto efficient strategy (respectively, a weak Pareto efficient strategy) with respect to $\overline{x} \in X$ if there does not exist any $u_i \in X_i$ such that

$$V^i(\overline{x}) - V^i(\overline{x}_{\hat{i}}, u_i) \in \mathbb{R}_+^{m_i} \setminus \{0\} \ \ \text{(respectively, } V^i(\overline{x}) - V^i(\overline{x}_{\hat{i}}, u_i) \in \text{int}\mathbb{R}_+^{m_i}\text{)},$$

where $\mathbb{R}_+^{m_i}$ and $\overline{x}_{\hat{i}}$ have the same meaning as in Definition 3.2, and

$$\text{int}\mathbb{R}_+^{m_i} := \{t = (t_1, t_2, \ldots, t_{m_i}) \in \mathbb{R}^{m_i} : t_j > 0, \forall j = 1, 2, \ldots, m_i\}.$$

Definition 3.4. For a multi-objective game $((X_i; \Gamma_i), V^i)_{i\in I}$, a strategy portfolio $\overline{x} \in X$ is said to be a Pareto equilibrium (respectively, a weak Pareto equilibrium) if for each $i \in I$, the strategy $\overline{x}_i \in X_i$ of the ith player is a Pareto efficient strategy (respectively, a weak Pareto efficient strategy) with respect to $\overline{x}$.

The following lemma can be seen as a particular case of Lemma 2.1 of Wang [30].

Lemma 3.1. *For a multi-objective game* $((X_i; \Gamma_i), V^i)_{i\in I}$, *each normalized weighted Nash equilibrium with respect to a weighted vector*

$$Q = (Q_i)_{i\in I}$$

with

$$Q_i = (Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}) \in \mathbb{R}_+^{m_i} \setminus \{0\}$$

respectively,

$$Q_i = (Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}) \in \text{int}\mathbb{R}_+^{m_i}$$

and

$$\sum_{j=1}^{m_i} Q_{ij} = 1$$

is a weak Pareto equilibrium (respectively, a Pareto equilibrium).

Remark 3.2. The conclusion of Lemma 3.1 still holds for the nonnormative case, i.e., each weighted Nash equilibrium with respect to a weighted vector

$$Q = (Q_i)_{i\in I}$$

with

$$Q_i = (Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}) \in \mathbb{R}_+^{m_i} \setminus \{0\}$$

respectively,

$$Q_i = (Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}) \in \text{int}\mathbb{R}_+^{m_i})$$

is a weak Pareto equilibrium (respectively, a Pareto equilibrium).

As an application of Theorem 2.7, we can derive the following existence theorem of weighted Nash equilibria for multi-objective games in non-compact abstract convex spaces.

Theorem 3.1. *Let I be a finite index set and $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces such that*

$$(X;\Gamma) := (\prod_{i\in I} X_i;\Gamma)$$

is an abstract convex space defined as in Lemma 1.1. Let K be a nonempty compact subset of X and $((X_i;\Gamma_i), V^i)_{i\in I}$ be a multi-objective game. Suppose that there exists a weighted vector

$$Q = (Q_i)_{i\in I}$$

with

$$Q_i = (Q_{i,1}, Q_{i,2}, \dots, Q_{i,m_i}) \in \mathbb{R}^{m_i}_{+} \setminus \{0\}$$

such that for each $i \in I$, the function $X \ni x \mapsto Q_i \cdot V^i(x)$ satisfies the following conditions:

(i) *For each $N_i = \{u_{i1}, u_{i2}, \dots, u_{in}\} \in \langle X_i\rangle$ and each $x \in X$ with $x_i \in \Gamma_i(N_i)$ there exists $j \in \{1, 2, \dots, n\}$ such that $Q_i \cdot V^i(x) - Q_i \cdot V^i(x_{\hat{i}}, u_{ij}) \le 0$;*

(ii) *For each $u_i \in X_i$, the set $\{x \in X : Q_i \cdot V^i(x_{\hat{i}}, u_i) < Q_i \cdot V^i(x)\}$ is open in X;*

(iii) *$\{x \in X :$ there exists $u_i \in X_i$ such that $Q_i \cdot V^i(x_{\hat{i}}, u_i) < Q_i \cdot V^i(x)\}$ is closed in X;*

(iv) *For each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subsets L_{N_i} of $(X_i;\Gamma_i)$ containing N_i such that for each $x \in L \setminus K$, there exists $u_i \in N_i$ satisfying*

$$Q_i \cdot V^i(x_{\hat{i}}, u_i) < Q_i \cdot V^i(x),$$

where

$$L := \prod_{i\in I} L_{N_i}.$$

If $(X;\Gamma)$ satisfies $1_X \in \Re\mathfrak{C}(X, X)$, then the multi-objective game $((X_i;\Gamma_i), V^i)_{i\in I}$ possesses at least one weighted Nash equilibrium $\overline{x} \in X$ with respective to Q.

Proof. Let us construct a noncooperative game $((X_i;\Gamma_i), f_i)_{i\in I}$, where for each $i \in I$, the payoff function f_i: $X \to \mathbb{R}$ is defined by

$$f_i(x) = -Q_i \cdot V^i(x)$$

for every $x \in X$. By (i), it is easy to see that f_i is a Γ_i-diagonally quasi-convex function for every $i \in I$. It follows from (ii) that the set $\{x \in X : f_i(x_{\hat{i}}, u_i) > f_i(x)\}$ is open in X for every $i \in I$ and every $u_i \in X_i$. By (iii), for each $i \in I$, the set $\{x \in X :$ there exists $u_i \in X_i$ such that $f_i(x_{\hat{i}}, u_i) > f_i(x)\}$ is closed in X. From (iv), it follows that for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subsets L_{N_i} of $(X_i;\Gamma_i)$ containing N_i such that for each $x \in L \setminus K$, there exists $u_i \in N_i$ satisfying $f_i(x_{\hat{i}}, u_i) > f_i(x)$, where

$$L := \prod_{i\in I} L_{N_i}.$$

Therefore, by Theorem 2.7, there exists one Nash equilibrium $\overline{x} \in X$ for the noncooperative game $((X_i;\Gamma_i), f_i)_{i\in I}$, i.e.,

$$f_i(\overline{x}_{\hat{i}}, u_i) \le f_i(\overline{x})$$

for every $u_i \in X_i$, which implies

$$Q_i \cdot V^i(\overline{x}) \leq Q_i \cdot V^i(\overline{x}_{\widehat{i}}, u_i)$$

for every $u_i \in X_i$. Thus, the multi-objective game $((X_i;\Gamma_i), V^i)_{i\in I}$ has at least one weighted Nash equilibrium $\overline{x} \in X$ with respective to Q, and we complete the proof. □

Remark 3.3. (i) of Theorem 3.1 is weaker than (i) of Lu et al. [31, Corollary 4.1]. In fact, (i) of Lu et al. [31, Corollary 4.1] implies (i) of Theorem 3.1. Now, to verify this fact, we argue by contradiction. Assume that (i) of Theorem 3.1 does not hold. Then, there exist $i \in I$, $\widehat{x} \in X$,

$$N_i = \{u_{i1}, u_{i2}, \ldots, u_{in}\} \in \langle X_i \rangle$$

with $\widehat{x_i} \in \Gamma_i(N_i)$ such that

$$Q_i \cdot V^i(\widehat{x_{\widehat{i}}}, u_{ij}) < Q_i \cdot V^i(\widehat{x}), \ \forall j \in \{1, 2, \ldots, n\}.$$

Thus, we have

$$N_i \subseteq \{u_i \in X_i : Q_i \cdot V^i(\widehat{x_{\widehat{i}}}, u_i) < Q_i \cdot V^i(\widehat{x})\}.$$

By (i) of Lu et al. [31, Corollary 4.1], the set $\{u_i \in X_i : Q_i \cdot V^i(\widehat{x_{\widehat{i}}}, u_i) < Q_i \cdot V^i(\widehat{x})\}$ is Γ_i-convex and thus,

$$\widehat{x_i} \in \Gamma_i(N_i) \subseteq \{u_i \in X_i : Q_i \cdot V^i(\widehat{x_{\widehat{i}}}, u_i) < Q_i \cdot V^i(\widehat{x})\}.$$

Therefore, we have

$$Q_i \cdot V^i(\widehat{x_{\widehat{i}}}, \widehat{x_i}) = Q_i \cdot V^i(\widehat{x}) < Q_i \cdot V^i(\widehat{x}),$$

which is a contraction. Hence, (i) of Theorem 3.1 holds. Moreover, the proof of Theorem 3.1 differs from that of Corollary 4.1 of Lu et al. [31]. The proof of Theorem 3.1 is based on the existence result of Nash equilibria for noncooperative games, while the conclusion of Lu et al. [31, Corollary 4.1] essentially derives from a fixed point theorem.

Corollary 3.1. *Let I be a finite index set and $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of compact abstract convex spaces such that*

$$(X;\Gamma) := (\prod_{i\in I} X_i;\Gamma)$$

is an abstract convex space defined as in Lemma 1.1. Let $((X_i;\Gamma_i), V^i)_{i\in I}$ be a multi-objective game. Suppose that there exists a weighted vector

$$Q = (Q_i)_{i\in I}$$

with

$$Q_i = (Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}) \in \mathbb{R}_+^{m_i} \setminus \{0\}$$

such that for each $i \in I$, the function $X \ni x \mapsto Q_i \cdot V^i(x)$ satisfies the following conditions:

(i) *For each $N_i = \{u_{i1}, u_{i2}, \ldots, u_{in}\} \in \langle X_i \rangle$ and each $x \in X$ with $x_i \in \Gamma_i(N_i)$ there exists $j \in \{1, 2, \ldots, n\}$ such that $Q_i \cdot V^i(x) - Q_i \cdot V^i(x_{\widehat{i}}, u_{ij}) \leq 0$;*

(ii) *For each $u_i \in X_i$, the set $\{x \in X : Q_i \cdot V^i(x_{\widehat{i}}, u_i) < Q_i \cdot V^i(x)\}$ is open in X;*

(iii) *$\{x \in X$: there exists $u_i \in X_i$ such that $Q_i \cdot V^i(x_{\widehat{i}}, u_i) < Q_i \cdot V^i(x)\}$ is closed in X.*

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X,X)$, then the multi-objective game $((X_i;\Gamma_i),V^i)_{i\in I}$ possesses at least one weighted Nash equilibrium $\overline{x} \in X$ with respective to Q.

Proof. For each $i \in I$ and each $N_i \in \langle X_i \rangle$, let

$$L_{N_i} = X_i$$

and

$$K = L = \prod_{i\in I} L_{N_i} = \prod_{i\in I} X_i.$$

Then, it is clear that (iv) of Theorem 3.1 is satisfied automatically. Therefore, by Theorem 3.1, the multi-objective game $((X_i;\Gamma_i),V^i)_{i\in I}$ has at least one weighted Nash equilibrium $\overline{x} \in X$ with respective to Q. This completes the proof. □

Now, by using Lemma 3.1 and Theorem 3.1, we have the following existence theorem of Pareto equilibria for multi-objective games in non-compact abstract convex spaces.

Theorem 3.2. *Let I be a finite index set and $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces such that*

$$(X;\Gamma) := (\prod_{i\in I} X_i;\Gamma)$$

is an abstract convex space defined as in Lemma 1.1. Let K be a nonempty compact subset of X and $((X_i;\Gamma_i),V^i)_{i\in I}$ be a multi-objective game. Suppose that there exists a weighted vector

$$Q = (Q_i)_{i\in I}$$

with

$$Q_i = (Q_{i,1}, Q_{i,2}, \dots, Q_{i,m_i}) \in \mathbb{R}_+^{m_i} \setminus \{0\}$$

such that for each $i \in I$, the function

$$X \ni x \mapsto Q_i \cdot V^i(x)$$

satisfies the following conditions:

(i) *For each $N_i = \{u_{i1}, u_{i2}, \dots, u_{in}\} \in \langle X_i \rangle$ and each $x \in X$ with $x_i \in \Gamma_i(N_i)$ there exists $j \in \{1,2,\dots,n\}$ such that $Q_i \cdot V^i(x) - Q_i \cdot V^i(x_{\hat{i}}, u_{ij}) \le 0$;*

(ii) *For each $u_i \in X_i$, the set $\{x \in X : Q_i \cdot V^i(x_{\hat{i}}, u_i) < Q_i \cdot V^i(x)\}$ is open in X;*

(iii) *$\{x \in X :$ there exists $u_i \in X_i$ such that $Q_i \cdot V^i(x_{\hat{i}}, u_i) < Q_i \cdot V^i(x)\}$ is closed in X;*

(iv) *For each $N_i \in \langle X_i \rangle$, there exists a compact Γ_i-convex subsets L_{N_i} of $(X_i;\Gamma_i)$ containing N_i such that for each $x \in L \setminus K$, there exists $u_i \in N_i$ satisfying $Q_i \cdot V^i(x_{\hat{i}}, u_i) < Q_i \cdot V^i(x)$, where $L := \prod_{i\in I} L_{N_i}$.*

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X,X)$, then the multi-objective game $((X_i;\Gamma_i),V^i)_{i\in I}$ possesses at least one weak Pareto equilibrium $\overline{x} \in X$. Furthermore, if

$$Q_i = (Q_{i,1}, Q_{i,2}, \dots, Q_{i,m_i}) \in \mathrm{int}\mathbb{R}_+^{m_i} \setminus \{0\}$$

for every $i \in I$, then the multi-objective game $((X_i;\Gamma_i),V^i)_{i\in I}$ has at least one Pareto equilibrium $\overline{x} \in X$.

Proof. By Theorem 3.1, the multi-objective game $((X_i;\Gamma_i), V^i)_{i\in I}$ has at least one weighted Nash equilibrium $\overline{x} \in X$ with respective to Q. Now, by using Lemma 3.1 and Remark 3.2, we can see that $\overline{x} \in X$ is also a weak Pareto equilibrium of the multi-objective game $((X_i;\Gamma_i), V^i)_{i\in I}$, and a Pareto equilibrium if

$$Q_i = (Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}) \in \text{int}\mathbb{R}_+^{m_i} \setminus \{0\}$$

for every $i \in I$. This completes the proof. □

When $\{(X_i;\Gamma_i)\}_{i\in I}$ is a family of compact abstract convex spaces, by Theorem 3.2, we have the following corollary:

Corollary 3.2. *Let I be a finite index set and $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of compact abstract convex spaces such that*

$$(X;\Gamma) := (\prod_{i\in I} X_i;\Gamma)$$

is an abstract convex space defined as in Lemma 1.1. Let $((X_i;\Gamma_i), V^i)_{i\in I}$ be a multi-objective game. Suppose that there exists a weighted vector

$$Q = (Q_i)_{i\in I}$$

with

$$Q_i = (Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}) \in \mathbb{R}_+^{m_i} \setminus \{0\}$$

such that for each $i \in I$, the function $X \ni x \mapsto Q_i \cdot V^i(x)$ satisfies the following conditions:

(i) *For each $N_i = \{u_{i1}, u_{i2}, \ldots, u_{in}\} \in \langle X_i\rangle$ and each $x \in X$ with $x_i \in \Gamma_i(N_i)$ there exists $j \in \{1, 2, \ldots, n\}$ such that $Q_i \cdot V^i(x) - Q_i \cdot V^i(x_{\hat{i}}, u_{ij}) \leq 0$;*

(ii) *For each $u_i \in X_i$, the set $\{x \in X : Q_i \cdot V^i(x_{\hat{i}}, u_i) < Q_i \cdot V^i(x)\}$ is open in X;*

(iii) *$\{x \in X$: there exists $u_i \in X_i$ such that $Q_i \cdot V^i(x_{\hat{i}}, u_i) < Q_i \cdot V^i(x)\}$ is closed in X.*

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then the multi-objective game $((X_i;\Gamma_i), V^i)_{i\in I}$ possesses at least one weak Pareto equilibrium $\overline{x} \in X$. Furthermore, if

$$Q_i = (Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}) \in \text{int}\mathbb{R}_+^{m_i} \setminus \{0\}$$

for every $i \in I$, then the multi-objective game $((X_i;\Gamma_i), V^i)_{i\in I}$ has at least one Pareto equilibrium $\overline{x} \in X$.

Now, we give an example to illustrate an application of Corollary 3.2 to the problem of water use conflicts.

Example 3.1. Suppose that there is a common water body with two players (water users) that can collect water from the common water body freely to be used for certain needs. Let us assume that player 1 is located upstream of the common water body and player 2 is located downstream. Let

$$X_1 = [a, b]$$

and

$$X_2 = [c, d]$$

be the strategy spaces of player 1 and player 2, respectively, where $a, b, c, d > 0$. We call $x_i \in X_i$ the amount of water withdrawn by player i, $i = 1, 2$. Let X_1 and X_2 be endowed with Euclidean topology. It is obvious that each X_i becomes an abstract convex space when

$$\Gamma_i = \text{co},$$

where co denotes the convex hull operator on nonempty finite subsets of the real line $\mathbb{R}$. By Lemma 1.1, it follows that

$$X = X_1 \times X_2$$

is also an abstract convex space. Further, we can show that $1_X \in \mathfrak{RC}(X, X)$; for details, see the proof of Lu et al. [31, Theorem 6.3].

Suppose that the benefit generated by the unit amount of water withdrawn by player i is $\omega_i(x_1, x_2)$ and the total cost for player i is $C_i(x_i)$, where $i = 1, 2$. From a rational perspective, it is reasonable to assume that the objective of player 1, located upstream of the common water body, is to maximize its profit function $x_1\omega_1(x_1, x_2) - C_1(x_1)$ or, equivalently, to minimize the opposite $C_1(x_1) - x_1\omega_1(x_1, x_2)$ of its profit function. In order to be consistent with the multi-objective game model in this section, we take the second case. The same holds for player 2. In reality, however, downstream player 2 in the common water body often requires higher water withdrawals for reasons of equity. Therefore, we can consider giving player 2 two objectives: the first is to minimize the opposite $C_2(x_2) - x_2\omega_2(x_1, x_2)$ of its profit function, and the second is to minimize the profit function $x_1\omega_1(x_1, x_2) - C_1(x_1)$ of player 1 who is a competitor of player 2. So far, we can see that the game model of water use conflicts can be summarized as a multi-objective game model. Thus, it is necessary to consider the existence of Pareto equilibrium of this multi-objective game model. For player 2, let (λ_1, λ_2) denote the weighted vector whose components correspond to the components of its vector objective function, where

$$\lambda_1 + \lambda_2 = 1$$

and $\lambda_1, \lambda_2 > 0$. The index λ_2/λ_1 represents the size of the fairness requirements of player 2. With the increase of λ_2/λ_1, the fairness requirements of player 2 are getting higher. Now, we give the following three assumptions:

Assumption A. For each $i \in \{1, 2\}$, the functions ω_i: $X_1 \times X_2 \to \mathbb{R}$ and C_i: $X_i \to \mathbb{R}$ are continuously differentiable.

Assumption B. For each

$$x = (x_1, x_2) \in X_1 \times X_2,$$

the sets

$$\{u_1 \in X_1 : C_1(u_1) - C_1(x_1) < u_1\omega_1(u_1, x_2) - x_1\omega_1(x)\}$$

and

$$\{u_2 \in X_2 : \lambda_1(C_2(u_2) - C_2(x_2)) < \lambda_1(u_2\omega_2(x_1, u_2) - x_2\omega_2(x)) + \lambda_2 x_1(\omega_1(x) - \omega_1(x_1, u_2))\}$$

are Γ_1-convex and Γ_2-convex, respectively.

Assumption C. The sets $\{x = (x_1, x_2) \in X$: there exists $u_2 \in X_2$ such that $\lambda_1(C_2(u_2) - C_2(x_2)) < \lambda_1(u_2\omega_2(x_1, u_2) - x_2\omega_2(x)) + \lambda_2 x_1(\omega_1(x) - \omega_1(x_1, u_2))\}$ and

$$\{x = (x_1, x_2) \in X : \text{there exists } u_1 \in X_1 \text{ such that } C_1(u_1) - C_1(x_1) < u_1\omega_1(u_1, x_2) - x_1\omega_1(x)\}$$

are closed in X.

By the above assumptions, it is easy to check that all the conditions of Corollary 3.2 are fulfilled. Therefore, according to Corollary 3.2, the above multi-objective game has a Pareto equilibrium. To further analyze the underlying mechanisms of water use conflicts, we simplify and concretize the relevant functions in the above multi-objective game model as follows: for each

$$(x_1, x_2) \in X_1 \times X_2,$$

let

$$\omega_1(x_1, x_2) = \omega_2(x_1, x_2) = r - (x_1 + x_2), \quad C_1(x_1) = g_1 + h_1 x_1$$

and

$$C_2(x_2) = g_2 + h_2 x_2,$$

where $r, g_1, g_2, h_1, h_2 > 0$. If we consider only the noncooperative game model with single-objective payoff functions, then it is easy to calculate the Nash equilibrium as

$$\widehat{x_1} = \frac{r + h_2 - 2h_1}{2}$$

and

$$\widehat{x_2} = \frac{r - 3h_2 + 2h_1}{4}.$$

The corresponding Pareto equilibrium in the case of multi-objective game is computed as

$$\overline{x}_1 = \frac{\lambda_1(r + h_2 - 2h_1)}{2(\lambda_1 + \lambda_2)}$$

and

$$\overline{x}_2 = \frac{(\lambda_1 + 3\lambda_2)r - (3\lambda_1 + \lambda_2)h_2 + 2(\lambda_1 - \lambda_2)h_1}{4(\lambda_1 + \lambda_2)}.$$

When $\lambda_2 = 0$, the Pareto equilibrium of the multi-objective game model clearly coincides with the Nash equilibrium of the noncooperative game model with single-objective payoff functions. When

$$\lambda_1 + \lambda_2 = 1$$

and $\lambda_1, \lambda_2 > 0$, i.e., taking into account the equity requirements of the downstream player 2, we get the following interesting conclusions as follows:

- The equilibrium extraction of upstream player 1 under the multi-objective game scenario is clearly smaller than its equilibrium extraction under the noncooperative game model with single-objective payoff functions, i.e., $\overline{x}_1 < \widehat{x_1}$, and $\overline{x}_1$ will decrease as the equity requirements of downstream player 2 increase.
- The equilibrium extraction of downstream player 2 under the multi-objective game scenario is greater than its equilibrium extraction under the noncooperative game model with single-objective payoff functions, i.e., $\overline{x}_2 > \widehat{x_2}$, and $\overline{x}_2$ will increase as the equity requirements of player 2 increase.
- A simple calculation can lead to the conclusion that the sum of water extractions in the Pareto equilibrium of the multi-objective game model is equal to the sum of water extractions in the Nash

equilibrium of the corresponding noncooperative game model with single-objective payoff functions, i.e.,

$$\overline{x}_1 + \overline{x}_2 = \widehat{x}_1 + \widehat{x}_2.$$

From this fact, it is evident that when we consider the multi-objective game model characterized by fairness requirements, its equilibrium water extraction is simply a redistribution of the equilibrium water extraction of the corresponding noncooperative game model with single-objective payoff functions while ensuring that the total equilibrium water extraction of the noncooperative game model with single-objective payoff functions remains unchanged.

4. Conclusions

In this paper, we first obtain several existence theorems of solutions for systems of generalized vector quasi-variational equilibrium problems and systems of vector quasi-variational equilibrium problems in the framework of non-compact abstract convex spaces, by using an existence result of equilibria for generalized abstract economy. Simultaneously, we establish an existence theorem of Nash equilibria for noncooperative games with single-objective payoff functions. Next, we obtain existence theorems of weighted Nash equilibria and Pareto Nash equilibria for multi-objective games in non-compact abstract convex spaces by using the obtained existence theorem of Nash equilibria for noncooperative games. Finally, an example on the problem of water use conflicts is used to verify the existence result of Pareto Nash equilibria for multi-objective games.

It is undeniable that there are still some limitations in the results of this paper, which are mainly reflected in two aspects: First, the index set in systems of generalized vector quasi-variational equilibrium problems is a finite index set; second, the four types of systems of generalized vector quasi-variational equilibrium problems are not unified. Therefore, at least three questions can be explored in further research. The first one is to unify the four types of systems of generalized vector quasi-variational equilibrium problems under the framework of arbitrary index set and non-compact abstract convex spaces, and to study the existence of solutions for the unified system of equilibrium problems. On this basis, we can carry out further the study of the subsequent two problems, i.e., the second one is to study the generic stability of solution sets for the unified system of equilibrium problems in non-compact abstract convex spaces. Another interesting issue is to apply the existence results of solutions for the unified system of equilibrium problems to deal with the existence of Pareto Nash equilibria for robust multi-objective games with infinite players.

Author contributions

Chengqing Pan: project administration, resources, methodology, formal analysis, writing-original draft; Haishu Lu: conceptualization, methodology, formal analysis, supervision, validation, writing-original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. Q. H. Ansari, W. K. Chan, X. Q. Yang, The system of vector quasi-equilibrium problems with applications, *J. Global Optim.*, **29** (2004), 45–57. https://doi.org/10.1023/B:JOGO.0000035018.46514.ca
2. Q. H. Ansari, Z. Khan, *System of generalized vector quasi-equilibrium problems with applications*, Narosa Publishing House, 2004.
3. L. J. Lin, Systems of generalized quasivariational inclusions problems with applications to variational analysis and optimization problems, *J. Global Optim.*, **38** (2007), 21–39. https://doi.org/10.1007/s10898-006-9081-5
4. L. J. Lin, L. F. Chen, Q. H. Ansari, Generalized abstract economy and systems of generalized vector quasi-equilibrium problems, *J. Comput. Appl. Math.*, **208** (2007), 341–353. https://doi.org/10.1016/j.cam.2006.10.002
5. L. J. Lin, Q. H. Ansari, Systems of quasi-variational relations with applications, *Nonlinear Anal.*, **72** (2010), 1210–1220. https://doi.org/10.1016/j.na.2009.08.005
6. S. Al-Homidan, Q. H. Ansari, S. Schaible, Existence of solutions of systems of generalized implicit vector variational inequalities, *J. Optim. Theory Appl.*, **134** (2007), 515–531. https://doi.org/10.1007/s10957-007-9236-7
7. M. Patriche, New results on systems of generalized vector quasi-equlibrium problems, *Taiwanese J. Math.*, **19** (2015), 253–277. https://doi.org/10.11650/tjm.19.2015.4098
8. J. W. Peng, H. W. J. Lee, X. M. Yang, On system of generalized vector quasiequilibrium problems with set-valued maps, *J. Global Optim.*, **36** (2006), 139–158. https://doi.org/10.1007/s10898-006-9004-5
9. J. W. Peng, S. Y. Wu, The generalized Tykhonov well-posedness for system of vector quasi-equilibrium problems, *Optim. Lett.*, **4** (2010), 501–512. https://doi.org/10.1007/s11590-010-0179-9
10. Z. Lin, The study of the system of generalized vector quasi-equilibrium problems, *J. Global Optim.*, **36** (2006), 627–635. https://doi.org/10.1007/s10898-006-9031-2
11. S. H. Hou, H. Yu, G. Y. Chen, On system of generalized vector variational inequalities, *J. Global Optim.*, **40** (2008), 739–749. https://doi.org/10.1007/s10898-006-9112-2

12. A. Farajzadeh, B. S. Lee, S. Plubteing, On generalized quasi-vector equilibrium problems via scalarization method, *J. Optim. Theory Appl.*, **168** (2016), 584–599. https://doi.org/10.1007/s10957-015-0772-2

13. N. V. Hung, LP well-posed controlled systems for bounded quasi-equilibrium problems and their application to traffic networks, *J. Comput. Appl. Math.*, **401** (2022), 113792. https://doi.org/10.1016/j.cam.2021.113792

14. S. Al-Homidan, Q. H. Ansari, Fixed point theorems on product topological semilattice spaces, generalized abstract economies and systems of generalized vector quasi-equilibrium problems, *Taiwanese J. Math.*, **15** (2011), 307–330. https://doi.org/10.11650/twjm/1500406176

15. S. Plubtieng, T. Thammathiwat, Existence of solutions of systems of generalized implicit vector quasi-equilibrium problems in G-convex spaces, *Comput. Math. Appl.*, **62** (2011), 124–130. https://doi.org/10.1016/j.camwa.2011.04.059

16. X. P. Ding, The generalized game and the system of generalized vector quasi-equilibrium problems in locally FC-uniform spaces, *Nonlinear Anal.*, **68** (2008), 1028–1036. https://doi.org/10.1016/j.na.2006.12.003

17. X. P. Ding, Collective fixed points, generalized games and systems of generalized quasi-variational inclusion problems in topological spaces, *Nonlinear Anal.*, **73** (2010), 1834–1841. https://doi.org/10.1016/j.na.2010.05.018

18. S. Park, Elements of the KKM theory on abstract convex spaces, *J. Korean Math. Soc.*, **45** (2008), 1–27. https://doi.org/10.4134/JKMS.2008.45.1.001

19. C. D. Horvath, Contractibility and generalized convexity, *J. Math. Anal. Appl.*, **156** (1991), 341–357. https://doi.org/10.1016/0022-247X(91)90402-L

20. C. D. Horvath, J. V. L. Ciscari, Maximal elements and fixed points for binary relations on topological ordered spaces, *J. Math. Econom.*, **25** (1996), 291–306. https://doi.org/10.1016/0304-4068(95)00732-6

21. S. Park, On the von Neumann-Sion minimax theorem in KKM spaces, *Appl. Math. Lett.*, **23** (2010), 1269–1273. https://doi.org/10.1016/j.aml.2010.06.011

22. S. Park, The KKM principle in abstract convex spaces: equivalent formulations and applications, *Nonlinear Anal.*, **73** (2010), 1028–1042. https://doi.org/10.1016/j.na.2010.04.029

23. S. Park, New generalizations of basic theorems in the KKM theory, *Nonlinear Anal.*, **74** (2011), 3000–3010. https://doi.org/10.1016/j.na.2011.01.020

24. Q. H. Ansari, S. Schaible, J. C. Yao, The system of generalized vector equilibrium problems with applications, *J. Global Optim.*, **22** (2002), 3–16. https://doi.org/10.1023/A:1013857924393

25. G. Kassay, M. Miholca, N. T. Vinh, Vector quasi-equilibrium problems for the sum of two multivalued mappings, *J. Optim. Theory Appl.*, **169** (2016), 424–442. https://doi.org/10.1007/s10957-016-0919-9

26. A. Capătă, On vector quasi-equilibrium problems via a Browder-type fixed-point theorem, *Bull. Malays. Math. Sci. Soc.*, **46** (2023), 14. https://doi.org/10.1007/s40840-022-01397-8

27. X. P. Ding, Existence of Pareto equilibria for constrained multiobjective games in H-space, *Comput. Math. Appl.*, **39** (2000), 125–134. https://doi.org/10.1016/S0898-1221(00)00092-4

28. H. S. Lu, Q. W. Hu, A collectively fixed point theorem in abstract convex spaces and its applications, *J. Funct. Space*, **2013** (2013), 517469. http://doi.org/10.1155/2013/517469
29. M. Patriche, Existence of equilibrium for multiobjective games in abstract convex spaces, *Math. Rep.*, **2** (2014), 243–252.
30. S. Y. Wang, Existence of a Pareto equilibrium, *J. Optim. Theory Appl.*, **79** (1993), 373–384. https://doi.org/10.1007/BF00940586
31. H. S. Lu, X. Q. Liu, R. Li, Upper semicontinuous selections for fuzzy mappings in noncompact *WPH*-spaces with applications, *AIMS Math.*, **7** (2022), 13994–14028. http://doi.org/10.3934/math.2022773