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Research article

Characterizations of generalized Lie n-higher derivations on certain triangular algebras

He Yuan*, **Qian Zhang and Zhendi Gu**

College of Mathematics and Computer, Jilin Normal University, Siping 136000, China

* **Correspondence:** Email: yuanhe1983@126.com.

Abstract: The aim of this paper was to provide a characterization of nonlinear generalized Lie n-higher derivations for a certain class of triangular algebras. It was shown that, under some mild conditions, each component G_r of a nonlinear generalized Lie n-higher derivation $\{G_r\}_{r\in N}$ of the triangular algebra $\mathcal{U}$ could be expressed as the sum of an additive generalized higher derivation and a nonlinear mapping vanishing on all $(n-1)$-th commutators on $\mathcal{U}$.

Keywords: triangular algebra; generalized Lie n-higher derivation; generalized higher derivation; Lie n-higher derivation; generalized Lie n-derivation
Mathematics Subject Classification: 16W25, 15A78

1. Introduction

Let R be a commutative ring with unity and A be an algebra over R. Let N be the set of nonnegative integers and $L = \{L_r\}_{r\in N}$ be a sequence of R-linear mappings $L_r : A \to A$ such that $L_0 = Id_A$. If $L_r(xy) = \sum\limits_{i+j=r} L_i(x)L_j(y)$ holds for all $x, y \in A$, then L is said to be a higher derivation. If $L_r([x, y]) = \sum\limits_{i+j=r} [L_i(x), L_j(y)]$ holds for all $x, y \in A$, then L is said to be a Lie higher derivation. If $L_r(P_n(x_1, x_2, \ldots, x_n)) = \sum\limits_{i_1+i_2+\cdots+i_n=r} P_n(L_{i_1}(x_1), L_{i_2}(x_2), \ldots, L_{i_n}(x_n))$ holds for all $x_1, x_2, \ldots, x_n \in A$, then L is said to be a Lie n-higher derivation, where $p_n(x_1, x_2, \ldots, x_n)$ is the $(n-1)$-th commutator.

In 2003, Cheung [1] studied Lie derivations on triangular algebras and provided sufficient conditions under which every Lie derivation is the sum of a derivation and a linear mapping into its center. Yu and Zhang [2] extended the above result to nonlinear Lie derivations on triangular algebras in 2010. In 2012, Xiao and Wei [3] studied nonlinear Lie higher derivations $L = \{L_r\}_{r\in N}$ on triangular algebras and obtained, under certain conditions, that each component L_r can be expressed as a sum of an additive higher derivation and a nonlinear mapping that vanishes on all commutators. Qi [4] characterized Lie higher derivations on triangular algebras by acting on zero products and acting on idempotent products. In 2023, Ashraf et al. [5] described Lie-type higher derivations of triangular rings by acting on zero products. In 2011, Benkovič [6] studied generalized Lie derivations on triangular algebras. Subsequently, Lin [7] investigated generalized Lie n-derivations on triangular algebras based on the above results. Benkovič [8] revisited the study of generalized Lie n-derivations by employing commuting and centralizing mappings on triangular algebras. Ashraf and Jabeen [9] considered the nonlinear generalized Lie triple higher derivation $L = \{L_r\}_{r\in N}$ on triangular algebras and proved that, under certain assumptions, each component L_r can be expressed through an additive generalized higher derivation and a nonlinear mapping that annihilates on all second commutators.

These observations motivate us to investigate the nonlinear generalized Lie n-higher derivations on triangular algebras.

2. Preliminaries

Let A and B be unital algebras, and let M be a unital (A, B)-bi-module, which is faithful as a left A-module and also as a right B-module. The set

$$\mathcal{U} = \text{Tri}(A, M, B) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| a \in A, m \in M, b \in B \right\}$$

is an associative algebra under the usual matrix operations. Let 1_A and 1_B be identities of the algebras A and B, respectively, then $I = \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$ is the identity of the triangular algebra $\mathcal{U}$. Throughout this paper, we shall use the following notation:

$$P = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}.$$

Accordingly, $\mathcal{U}$ can be written as $\mathcal{U} = P\mathcal{U}P \oplus P\mathcal{U}Q \oplus Q\mathcal{U}Q$, where $P\mathcal{U}P$ is a sub-algebra of $\mathcal{U}$ isomorphic to A, $Q\mathcal{U}Q$ is a sub-algebra of $\mathcal{U}$ isomorphic to B, and $P\mathcal{U}Q$ is a $(P\mathcal{U}P, Q\mathcal{U}Q)$-bi-module isomorphic to the bi-module M. In order to simplify, we are going to employ the following symbols $\mathcal{U}_{11} = P\mathcal{U}P$, $\mathcal{U}_{12} = P\mathcal{U}Q$, $\mathcal{U}_{22} = Q\mathcal{U}Q$. Define two natural projections $\pi_{\mathcal{U}_{11}} : \mathcal{U} \to \mathcal{U}_{11}$ and $\pi_{\mathcal{U}_{22}} : \mathcal{U} \to \mathcal{U}_{22}$ by $\pi_{\mathcal{U}_{11}}(U_{11} + U_{12} + U_{22}) = U_{11}$ and $\pi_{\mathcal{U}_{22}}(U_{11} + U_{12} + U_{22}) = U_{22}$, where $U_{11} \in \mathcal{U}_{11}, U_{12} \in \mathcal{U}_{12}, U_{22} \in \mathcal{U}_{22}$. It is easy to see that $\pi_{\mathcal{U}_{11}}(Z(\mathcal{U}))$ is a sub-algebra of $Z(\mathcal{U}_{11})$ and that $\pi_{\mathcal{U}_{22}}(Z(\mathcal{U}))$ is a sub-algebra of $Z(\mathcal{U}_{22})$. According to [10, Proposition 3], there exists a unique algebraic isomorphism $\tau : \pi_{\mathcal{U}_{11}}(Z(\mathcal{U})) \to \pi_{\mathcal{U}_{22}}(Z(\mathcal{U}))$ such that $U_{11}U_{12} = U_{12}\tau(U_{11})$ for all $U_{11} \in \pi_{\mathcal{U}_{11}}(Z(\mathcal{U})), U_{12} \in \mathcal{U}_{12}$. Benkovič [11] proposed the following condition for studying Lie n-derivations on triangular algebras.

Suppose that A is an associative algebra such that for each $a \in A$,

$$[[a, A], A] = 0 \Rightarrow [a, A] = 0. \tag{2.1}$$

The above condition (2.1) is very important for studying mappings on triangular algebras, and at the same time he provided the following two remarks.

Remark 2.1. *[11, Remark 2.1] Let $\mathcal{U}$ be a triangular ring. For any $U \in \mathcal{U}$ and for any integer $n \geq 2$, we have*

$$p_n(U, P, \ldots, P) = (-1)^{n-1}PUQ \qquad \text{and} \qquad p_n(U, Q, \ldots, Q) = PUQ.$$

Remark 2.2. *[11, Remark 2.2] Let $U \in \mathcal{U}$. If $[U, P\mathcal{U}Q] = 0$, then $PUP + QUQ \in Z(\mathcal{U})$.*

The concepts outlined below are necessary for this paper.

Let A be an associative algebra and $Z(A)$ be the center of A. A mapping $F : A \to A$ is said to be additive modulo $Z(A)$ if $F(x + y) - F(x) - F(y) \in Z(A)$ for all $x, y \in A$. If $nx = 0$ implies $x = 0$, for some positive integer $n \in N$ and arbitrary $x \in A$, then A is called n-torsion free.

We denote the following sequence of polynomials:

$$\begin{aligned} p_1(x_1) &= x_1, \\ p_2(x_1, x_2) &= [p_1(x_1), x_2] = [x_1, x_2], \\ p_3(x_1, x_2, x_3) &= [p_2(x_1, x_2), x_3] = [[x_1, x_2], x_3], \\ &\cdots\cdots \\ p_n(x_1, x_2, \ldots, x_n) &= [p_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n]. \end{aligned}$$

The polynomial $p_n(x_1, x_2, \ldots, x_n)$ is said to be an $(n-1)$*-th commutator* $(n \geq 2)$.

Let N be the set of nonnegative integers and $G = \{G_r\}_{r\in N}$ be a sequence of nonlinear mappings $G_r : \mathcal{U} \to \mathcal{U}$ such that $G_0 = Id_{\mathcal{U}}$. Then, for all $x_1, x_2, \ldots, x_n \in \mathcal{U}$, G is said to be a:

(i) Nonlinear generalized higher derivation on $\mathcal{U}$ if there exists a higher derivation $L = \{L_r\}_{r\in N}$ such that $L_0 = Id_{\mathcal{U}}$ and

$$G_r(x_1x_2) = \sum_{i+j=r} G_i(x_1)L_j(x_2).$$

(ii) Nonlinear generalized Lie higher derivation on $\mathcal{U}$ if there exists a Lie higher derivation $L = \{L_r\}_{r\in N}$ such that $L_0 = Id_{\mathcal{U}}$ and

$$G_r([x_1, x_2]) = \sum_{i+j=r} [G_i(x_1), L_j(x_2)].$$

(iii) Nonlinear generalized Lie n-higher derivation on $\mathcal{U}$ if there exists a Lie n-higher derivation $L = \{L_r\}_{r\in N}$ such that $L_0 = Id_{\mathcal{U}}$ and

$$G_r(P_n(x_1, x_2, \ldots, x_n)) = \sum_{i_1+i_2+\cdots+i_n=r} P_n(G_{i_1}(x_1), L_{i_2}(x_2), \ldots, L_{i_n}(x_n)).$$

If G is a nonlinear generalized Lie n-higher derivation, then when $r = 1$, G_1 is in fact a nonlinear generalized Lie n-derivation, i.e., there exists a Lie n-derivation L_1 such that

$$G_1(P_n(x_1, x_2, \ldots, x_n)) = P_n(G_1(x_1), x_2, \ldots, x_n) + \sum_{i=2}^{n} P_n(x_1, x_2, \ldots, L_1(x_i), \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in \mathcal{U}$. In this case, according to [5, Theorem 3.1], there exist a derivation $d_1 : \mathcal{U} \to \mathcal{U}$ and a mapping $\tau_1 : \mathcal{U} \to Z(\mathcal{U})$ satisfying $\tau_1(P_n(x_1, x_2, \ldots, x_n)) = 0$ such that $L_1(x) = d_1(x) + \tau_1(x)$ for all $x \in \mathcal{U}$. Moreover, L_1 and d_1 satisfy the following properties:

$$\begin{cases} L_1(P) \in \mathcal{U}_{12} + Z(\mathcal{U}), & L_1(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, \\ L_1(Q) \in \mathcal{U}_{12} + Z(\mathcal{U}), & L_1(I) \in Z(\mathcal{U}), \\ L_1(\mathcal{U}_{11}) \subseteq \mathcal{U}_{11} + \mathcal{U}_{12} + Z(\mathcal{U}), & L_1(\mathcal{U}_{22}) \subseteq \mathcal{U}_{22} + \mathcal{U}_{12} + Z(\mathcal{U}), \\ d_1(P), d_1(Q) \in \mathcal{U}_{12}, & d_1(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, \\ d_1(\mathcal{U}_{11}) \subseteq \mathcal{U}_{11} + \mathcal{U}_{12}, & d_1(\mathcal{U}_{22}) \subseteq \mathcal{U}_{22} + \mathcal{U}_{12}. \end{cases}$$

Although Lin also used the condition $PG(x)Q = 0$ to describe the nonlinear generalized Lie n-derivations on triangular algebras, we will characterize the nonlinear generalized Lie n-derivations in a new way using the above result. Furthermore, this paper will use the method of induction on r to study the generalized Lie n-higher derivations, and the generalized Lie n-derivations are precisely the case of $r = 1$.

In Section 2, we present the preliminaries and tools that are necessary. Section 3 investigates the generalized Lie n-derivations in a new manner. Subsequently, the generalized Lie n-higher derivations were studied through induction on r in Section 4.

3. Nonlinear generalized Lie n-derivations

This section will prove that a nonlinear generalized Lie n-derivation G_1 of the triangular algebras $\mathcal{U}$ can be expressed as the sum of an additive generalized derivation and a nonlinear mapping vanishing on all $(n-1)$-th commutators on $\mathcal{U}$. To start, we will provide the following lemma.

Lemma 3.1. *Let $\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{22}$ be a triangular algebra. Suppose that $\mathcal{U}$ satisfies the following conditions:*
(i) $\pi_{\mathcal{U}_{11}}(Z(\mathcal{U})) = Z(\mathcal{U}_{11})$ and $\pi_{\mathcal{U}_{22}}(Z(\mathcal{U})) = Z(\mathcal{U}_{22})$;
(ii) $\mathcal{U}_{11}$ or $\mathcal{U}_{22}$ satisfies (2.1).
If G_1 is a nonlinear generalized Lie n-derivation on $\mathcal{U}$, then $G_1 = \omega_1 + f_1$, where $\omega_1 : \mathcal{U} \to \mathcal{U}$ is additive modulo $Z(\mathcal{U})$ and $f_1 : \mathcal{U} \to Z(\mathcal{U})$ satisfies $f_1(\mathcal{U}_{12}) = \{0\}$.

Proof. The definition of G_1 implies that

$$\begin{aligned} G_1(0) &= G_1\left(P_n(0,0,\ldots,0)\right) \\ &= P_n\left(G_1(0),0,\ldots,0\right) + P_n\left(0,L_1(0),\ldots,0\right) + \cdots + P_n\left(0,0,\ldots,L_1(0)\right) \\ &= 0. \end{aligned}$$

Therefore,

$$G_1(0) = 0. \tag{3.1}$$

Since $P_n(U_{11}, U_{22}, U_{12}, Q, \ldots, Q) = 0$, it follows that

$$\begin{aligned} 0 =& G_1\left(P_n(U_{11}, U_{22}, U_{12}, Q, \ldots, Q)\right) \\ =& P_n\left(G_1(U_{11}), U_{22}, U_{12}, Q, \ldots, Q\right) + P_n\left(U_{11}, L_1(U_{22}), U_{12}, Q, \ldots, Q\right) \\ &+ \cdots + P_n\left(U_{11}, U_{22}, U_{12}, Q, \ldots, L_1(Q)\right) \\ =& \left[\left[G_1(U_{11}), U_{22}\right], U_{12}\right] + \left[\left[U_{11}, L_1(U_{22})\right], U_{12}\right]. \end{aligned}$$

It follows from $L_1(\mathcal{U}_{22}) \subseteq \mathcal{U}_{22} + \mathcal{U}_{12} + Z(\mathcal{U})$ that $[[G_1(U_{11}), U_{22}], U_{12}] = 0$ for all $U_{11} \in \mathcal{U}_{11}, U_{22} \in \mathcal{U}_{22}, U_{12} \in \mathcal{U}_{12}$. In view of Remark 2.2, we have $QG_1(U_{11})Q \in Z(\mathcal{U}_{22})$. Likewise, $PG_1(U_{22})P \in Z(\mathcal{U}_{11})$ can be derived from $0 = G_1\left(P_n(U_{22}, U_{11}, U_{12}, Q, \ldots, Q)\right)$ for all $U_{22} \in \mathcal{U}_{22}$. In conclusion,

$$\begin{aligned} G_1(U_{11}) =& PG_1(U_{11})P - \tau^{-1}(QG_1(U_{11})Q) + PG_1(U_{11})Q + \tau^{-1}(QG_1(U_{11})Q) + QG_1(U_{11})Q \\ \in& \mathcal{U}_{11} + \mathcal{U}_{12} + Z(\mathcal{U}), \\ G_1(U_{22}) =& PG_1(U_{22})P + \tau(PG_1(U_{22})P) + PG_1(U_{22})Q + QG_1(U_{22})Q - \tau(PG_1(U_{22})P) \end{aligned}$$

$$\in \mathcal{U}_{22} + \mathcal{U}_{12} + Z(\mathcal{U}).$$

It follows that

$$G_1(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12} + Z(\mathcal{U}), i \in \{1, 2\}. \tag{3.2}$$

Set

$$f_1(U) = QG_1(PUP)Q + \tau^{-1}(QG_1(PUP)Q) + PG_1(QUQ)P + \tau(PG_1(QUQ)P)$$

for all $U \in \mathcal{U}$. Obviously, $f_1(\mathcal{U}) \subseteq Z(\mathcal{U})$, $f_1(\mathcal{U}_{12}) = \{0\}$ and $f_1(P_n(U_1, U_2, \dots, U_n)) = 0$ for all $U_1, U_2, \dots, U_n \in \mathcal{U}$. Define a mapping $\omega_1(U) = G_1(U) - f_1(U)$ for all $U \in \mathcal{U}$. Next, we will prove that $\omega_1 : \mathcal{U} \to \mathcal{U}$ is additive modulo $Z(\mathcal{U})$.

By $U_{12} = P_n(U_{12}, Q, \dots, Q)$, we obtain

$$\begin{aligned} G_1(U_{12}) =& G_1(P_n(U_{12}, Q, \dots, Q)) \\ =& P_n(G_1(U_{12}), Q, \dots, Q) + P_n(U_{12}, L_1(Q), \dots, Q) + \cdots + P_n(U_{12}, Q, \dots, L_1(Q)) \\ =& PG_1(U_{12})Q \end{aligned}$$

for all $U_{12} \in \mathcal{U}_{12}$. Therefore,

$$G_1(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}. \tag{3.3}$$

According to (3.1)–(3.3), we arrive at

$$\omega_1(0) = 0, \quad \omega_1(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, \quad \omega_1(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12}. \tag{3.4}$$

For any $U_{11} \in \mathcal{U}_{11}$, $U_{12} \in \mathcal{U}_{12}$, on the one hand, it follows from $d_1(Q) \in \mathcal{U}_{12}$ that

$$\begin{aligned} & G_1(P_n(U_{11} + U_{12}, Q, \dots, Q)) \\ =& P_n(G_1(U_{11} + U_{12}), Q, \dots, Q) + P_n(U_{11} + U_{12}, L_1(Q), \dots, Q) \\ & + \cdots + P_n(U_{11} + U_{12}, Q, \dots, L_1(Q)) \\ =& P_n(\omega_1(U_{11} + U_{12}), Q, \dots, Q) + P_n(U_{11} + U_{12}, d_1(Q), \dots, Q) \\ & + \cdots + P_n(U_{11} + U_{12}, Q, \dots, d_1(Q)) \\ =& P_n(\omega_1(U_{11} + U_{12}), Q, \dots, Q) + P_n(U_{11} + U_{12}, d_1(Q), \dots, Q), \end{aligned}$$

and on the other hand, by (3.1), we get

$$\begin{aligned} & G_1(P_n(U_{11} + U_{12}, Q, \dots, Q)) \\ =& G_1(P_n(U_{11}, Q, \dots, Q)) + G_1(P_n(U_{12}, Q, \dots, Q)) \\ =& P_n(G_1(U_{11}), Q, \dots, Q) + P_n(U_{11}, L_1(Q), \dots, Q) + \cdots + P_n(U_{11}, Q, \dots, L_1(Q)) \\ & + P_n(G_1(U_{12}), Q, \dots, Q) + P_n(U_{12}, L_1(Q), \dots, Q) + \cdots + P_n(U_{12}, Q, \dots, L_1(Q)) \\ =& P_n(\omega_1(U_{11}), Q, \dots, Q) + P_n(U_{11}, d_1(Q), \dots, Q) + \cdots + P_n(U_{11}, Q, \dots, d_1(Q)) \\ & + P_n(\omega_1(U_{12}), Q, \dots, Q) + P_n(U_{12}, d_1(Q), \dots, Q) + \cdots + P_n(U_{12}, Q, \dots, d_1(Q)) \\ =& P_n(\omega_1(U_{11}), Q, \dots, Q) + P_n(U_{11}, d_1(Q), \dots, Q) \\ & + P_n(\omega_1(U_{12}), Q, \dots, Q) + P_n(U_{12}, d_1(Q), \dots, Q). \end{aligned}$$

Comparing the above two relations, we obtain

$$
\begin{aligned}
0 =&P_n(\omega_1(U_{11}+U_{12})-\omega_1(U_{11})-\omega_1(U_{12}),Q,\ldots,Q)\\
=&P(\omega_1(U_{11}+U_{12})-\omega_1(U_{11})-\omega_1(U_{12}))Q.
\end{aligned}
$$

Therefore,

$$\omega_1(U_{11}+U_{12})-\omega_1(U_{11})-\omega_1(U_{12})\in\mathcal{U}_{11}+\mathcal{U}_{22}$$

for all $U_{11}\in\mathcal{U}_{11}, U_{12}\in\mathcal{U}_{12}$. Based on

$$
\begin{aligned}
&G_1(P_n(U_{11}+U_{12},V_{12},Q,\ldots,Q))\\
=&P_n(\omega_1(U_{11}+U_{12}),V_{12},Q,\ldots,Q)+P_n(U_{11}+U_{12},d_1(V_{12}),Q,\ldots,Q)\\
&+\cdots+P_n(U_{11}+U_{12},V_{12},\ldots,d_1(Q))\\
=&P_n(\omega_1(U_{11}+U_{12}),V_{12},Q,\ldots,Q)+P_n(U_{11}+U_{12},d_1(V_{12}),Q,\ldots,Q)
\end{aligned}
$$

and

$$
\begin{aligned}
&G_1(P_n(U_{11}+U_{12},V_{12},Q,\ldots,Q))\\
=&G_1(P_n(U_{11},V_{12},Q,\ldots,Q))+G_1(P_n(U_{12},V_{12},Q,\ldots,Q))\\
=&P_n(\omega_1(U_{11}),V_{12},Q,\ldots,Q)+P_n(U_{11},d_1(V_{12}),Q,\ldots,Q)+\cdots+P_n(U_{11},V_{12},Q,\ldots,d_1(Q))\\
&+P_n(\omega_1(U_{12}),V_{12},Q,\ldots,Q)+P_n(U_{12},d_1(V_{12}),Q,\ldots,Q)+\cdots+P_n(U_{12},V_{12},Q,\ldots,d_1(Q))\\
=&P_n(\omega_1(U_{11}),V_{12},Q,\ldots,Q)+P_n(U_{11},d_1(V_{12}),Q,\ldots,Q)\\
&+P_n(\omega_1(U_{12}),V_{12},Q,\ldots,Q)+P_n(U_{12},d_1(V_{12}),Q,\ldots,Q),
\end{aligned}
$$

and comparing the above two equations, we arrive at

$$
\begin{aligned}
0 =&P_n(\omega_1(U_{11}+U_{12})-\omega_1(U_{11})-\omega_1(U_{12}),V_{12},Q,\ldots,Q)\\
=&[\omega_1(U_{11}+U_{12})-\omega_1(U_{11})-\omega_1(U_{12}),V_{12}]
\end{aligned}
$$

for all $U_{11}\in\mathcal{U}_{11}$, $U_{12},V_{12}\in\mathcal{U}_{12}$. It follows from Remark 2.2 that

$$\omega_1(U_{11}+U_{12})-\omega_1(U_{11})-\omega_1(U_{12})\in Z(\mathcal{U}). \tag{3.5}$$

Similarly, we have

$$\omega_1(U_{22}+U_{12})-\omega_1(U_{22})-\omega_1(U_{12})\in Z(\mathcal{U})$$

for all $U_{22}\in\mathcal{U}_{22}, U_{12}\in\mathcal{U}_{12}$.

In view of (3.5) and $d_1(V_{12})\in\mathcal{U}_{12}$, we obtain

$$
\begin{aligned}
\omega_1(U_{12}+V_{12}) =&G_1\left(P_n(P+U_{12},Q+V_{12},Q,\ldots,Q)\right)\\
=&P_n\left(\omega_1(P+U_{12}),Q+V_{12},Q,\ldots,Q\right)+P_n\left(P+U_{12},d_1(Q+V_{12}),Q,\ldots,Q\right)\\
&+\cdots+P_n\left(P+U_{12},Q+V_{12},Q,\ldots,d_1(Q)\right)\\
=&P_n\left(\omega_1(P)+\omega_1(U_{12}),Q+V_{12},Q,\ldots,Q\right)\\
&+P_n\left(P+U_{12},d_1(Q)+d_1(V_{12}),Q,\ldots,Q\right)\\
=&\omega_1(P)V_{12}+\omega_1(P)Q+\omega_1(U_{12})+d_1(Q)+d_1(V_{12})
\end{aligned}
\tag{3.6}
$$

for all $U_{12}, V_{12} \in \mathcal{U}_{12}$. Since

$$\begin{aligned}\omega_1(V_{12}) =& G_1\left(P_n(P, V_{12}, Q, \ldots, Q)\right)\\ =& P_n\left(G_1(P), V_{12}, Q, \ldots, Q\right) + P_n\left(P, L_1(V_{12}), Q, \ldots, Q\right) + \cdots + P_n\left(P, V_{12}, Q, \ldots, L_1(Q)\right)\\ =& \omega_1(P)V_{12} + d_1(V_{12})\end{aligned}$$

and

$$\begin{aligned}0 &= G_1(P_n(P, Q, \ldots, Q))\\ &= P_n\left(G_1(P), Q, \ldots, Q\right) + P_n\left(P, L_1(Q), \ldots, Q\right) + \cdots + P_n\left(P, Q, \ldots, L_1(Q)\right)\\ &= \omega_1(P)Q + d_1(Q),\end{aligned}$$

it follows from (3.6) that

$$\omega_1(U_{12} + V_{12}) = \omega_1(U_{12}) + \omega_1(V_{12}) \tag{3.7}$$

for all $U_{12}, V_{12} \in \mathcal{U}_{12}$. On the one hand,

$$\begin{aligned}&G_1(P_n(U_{11} + V_{11}, U_{12}, Q, \ldots, Q))\\ =& P_n(G_1(U_{11} + V_{11}), U_{12}, Q, \ldots, Q) + P_n(U_{11} + V_{11}, L_1(U_{12}), Q, \ldots, Q)\\ &+ \cdots + P_n(U_{11} + V_{11}, U_{12}, Q, \ldots, L_1(Q))\\ =& P_n(\omega_1(U_{11} + V_{11}), U_{12}, Q, \ldots, Q) + P_n(U_{11} + V_{11}, d_1(U_{12}), Q, \ldots, Q),\end{aligned}$$

and on the other hand, it follows from (3.7) that

$$\begin{aligned}&G_1(P_n(U_{11} + V_{11}, U_{12}, Q, \ldots, Q))\\ =& G_1(P_n(U_{11}, U_{12}, Q, \ldots, Q)) + G_1(P_n(V_{11}, U_{12}, Q, \ldots, Q))\\ =& P_n(G_1(U_{11}), U_{12}, Q, \ldots, Q) + P_n(U_{11}, L_1(U_{12}), Q, \ldots, Q)\\ &+ \cdots + P_n(U_{11}, U_{12}, Q, \ldots, L_1(Q)) + P_n(G_1(V_{11}), U_{12}, Q, \ldots, Q)\\ &+ P_n(V_{11}, L_1(U_{12}), Q, \ldots, Q) + \cdots + P_n(V_{11}, U_{12}, Q, \ldots, L_1(Q))\\ =& P_n(\omega_1(U_{11}), U_{12}, Q, \ldots, Q) + P_n(U_{11}, d_1(U_{12}), Q, \ldots, Q)\\ &+ P_n(\omega_1(V_{11}), U_{12}, Q, \ldots, Q) + P_n(V_{11}, d_1(U_{12}), Q, \ldots, Q).\end{aligned}$$

Comparing the above two relations, we have

$$\begin{aligned}0 =& P_n(\omega_1(U_{11} + V_{11}) - \omega_1(U_{11}) - \omega_1(V_{11}), U_{12}, Q, \ldots, Q)\\ =& (\omega_1(U_{11} + V_{11}) - \omega_1(U_{11}) - \omega_1(V_{11}))U_{12}.\end{aligned}$$

Since $\mathcal{U}_{12}$ is faithful as a left $\mathcal{U}_{11}$-module, we get

$$\omega_1(U_{11} + V_{11})P = \omega_1(U_{11})P + \omega_1(V_{11})P \tag{3.8}$$

for all $U_{11}, V_{11} \in \mathcal{U}_{11}$. Since $P_n(U_{11}, Q, \ldots, Q) = 0$, we arrive at

$$\begin{aligned}0 =& G_1\left(P_n(U_{11}, Q, \ldots, Q)\right)\\ =& P_n\left(G_1(U_{11}), Q, \ldots, Q\right) + P_n\left(U_{11}, L_1(Q), \ldots, Q\right) + \cdots + P_n\left(U_{11}, Q, \ldots, L_1(Q)\right)\\ =& P_n\left(\omega_1(U_{11}), Q, \ldots, Q\right) + P_n\left(U_{11}, d_1(Q), \ldots, Q\right)\\ =& \omega_1(U_{11})Q + U_{11}d_1(Q).\end{aligned}$$

That is,

$$\omega_1(U_{11})Q + U_{11}d_1(Q) = 0 \tag{3.9}$$

for all $U_{11} \in \mathcal{U}_{11}$. Substituting $U_{11} + V_{11}$ for U_{11} in (3.9), we obtain

$$\omega_1(U_{11} + V_{11})Q + (U_{11} + V_{11})d_1(Q) = 0. \tag{3.10}$$

Using (3.9) and (3.10), we arrive at

$$\omega_1(U_{11} + V_{11})Q - \omega_1(U_{11})Q - \omega_1(V_{11})Q = 0. \tag{3.11}$$

According to (3.8) and (3.11), we find

$$\omega_1(U_{11} + V_{11}) = \omega_1(U_{11}) + \omega_1(V_{11}) \tag{3.12}$$

for all $U_{11}, V_{11} \in \mathcal{U}_{11}$. Using an analogous manner, we show

$$\omega_1(U_{22} + V_{22}) = \omega_1(U_{22}) + \omega_1(V_{22}) \tag{3.13}$$

for all $U_{22}, V_{22} \in \mathcal{U}_{22}$.

For any $U_{11} \in \mathcal{U}_{11}$, $U_{12} \in \mathcal{U}_{12}$, $U_{22} \in \mathcal{U}_{22}$, on the one hand, we have

$$\begin{aligned}
&G_1(P_n(U_{11} + U_{12} + U_{22}, Q, \ldots, Q))\\
&=P_n(\omega_1(U_{11} + U_{12} + U_{22}), Q, \ldots, Q) + P_n(U_{11} + U_{12} + U_{22}, d_1(Q), \ldots, Q)\\
&\quad + \cdots + P_n(U_{11} + U_{12} + U_{22}, Q, \ldots, d_1(Q))\\
&=P_n(\omega_1(U_{11} + U_{12} + U_{22}), Q, \ldots, Q) + P_n(U_{11} + U_{12} + U_{22}, d_1(Q), \ldots, Q).
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
&G_1(P_n(U_{11} + U_{12} + U_{22}, Q, \ldots, Q))\\
&=G_1(P_n(U_{11}, Q, \ldots, Q)) + G_1(P_n(U_{12}, Q, \ldots, Q)) + G_1(P_n(U_{22}, Q, \ldots, Q))\\
&=P_n(\omega_1(U_{11}), Q, \ldots, Q) + P_n(U_{11}, d_1(Q), \ldots, Q) + \cdots + P_n(U_{11}, Q, \ldots, d_1(Q))\\
&\quad + P_n(\omega_1(U_{12}), Q, \ldots, Q) + P_n(U_{12}, d_1(Q), \ldots, Q) + \cdots + P_n(U_{12}, Q, \ldots, d_1(Q))\\
&\quad + P_n(\omega_1(U_{22}), Q, \ldots, Q) + P_n(U_{22}, d_1(Q), \ldots, Q) + \cdots + P_n(U_{22}, Q, \ldots, d_1(Q))\\
&=P_n(\omega_1(U_{11}), Q, \ldots, Q) + P_n(U_{11}, d_1(Q), \ldots, Q) + P_n(\omega_1(U_{12}), Q, \ldots, Q)\\
&\quad + P_n(U_{12}, d_1(Q), \ldots, Q) + P_n(\omega_1(U_{22}), Q, \ldots, Q) + P_n(U_{22}, d_1(Q), \ldots, Q).
\end{aligned}$$

Comparing the above two relations, we obtain

$$\begin{aligned}
0 &=P_n(\omega_1(U_{11} + U_{12} + U_{22}) - \omega_1(U_{11}) - \omega_1(U_{12}) - \omega_1(U_{22}), Q, \ldots, Q)\\
&=P(\omega_1(U_{11} + U_{12} + U_{22}) - \omega_1(U_{11}) - \omega_1(U_{12}) - \omega_1(U_{22}))Q.
\end{aligned}$$

Therefore,

$$\omega_1(U_{11} + U_{12} + U_{22}) - \omega_1(U_{11}) - \omega_1(U_{12}) - \omega_1(U_{22}) \in \mathcal{U}_{11} + \mathcal{U}_{22}.$$

We have

$$
\begin{aligned}
&G_1(P_n(U_{11} + U_{12} + U_{22}, V_{12}, \dots, Q)) \\
&= P_n(\omega_1(U_{11} + U_{12} + U_{22}), V_{12}, \dots, Q) + P_n(U_{11} + U_{12} + U_{22}, d_1(V_{12}), \dots, Q) \\
&\quad + \cdots + P_n(U_{11} + U_{12} + U_{22}, V_{12}, \dots, d_1(Q)) \\
&= P_n(\omega_1(U_{11} + U_{12} + U_{22}), V_{12}, \dots, Q) + P_n(U_{11} + U_{12} + U_{22}, d_1(V_{12}), \dots, Q)
\end{aligned}
$$

and

$$
\begin{aligned}
&G_1(P_n(U_{11} + U_{12} + U_{22}, V_{12}, \dots, Q)) \\
&= G_1(P_n(U_{11}, V_{12}, \dots, Q)) + G_1(P_n(U_{12}, V_{12}, \dots, Q)) + G_1(P_n(U_{22}, V_{12}, \dots, Q)) \\
&= P_n(\omega_1(U_{11}), V_{12}, \dots, Q) + P_n(U_{11}, d_1(V_{12}), \dots, Q) + \cdots + P_n(U_{11}, V_{12}, \dots, d_1(Q)) \\
&\quad + P_n(\omega_1(U_{12}), V_{12}, \dots, Q) + P_n(U_{12}, d_1(V_{12}), \dots, Q) + \cdots + P_n(U_{12}, V_{12}, \dots, d_1(Q)) \\
&\quad + P_n(\omega_1(U_{22}), V_{12}, \dots, Q) + P_n(U_{22}, d_1(V_{12}), \dots, Q) + \cdots + P_n(U_{22}, V_{12}, \dots, d_1(Q)) \\
&= P_n(\omega_1(U_{11}), V_{12}, \dots, Q) + P_n(U_{11}, d_1(V_{12}), \dots, Q) + P_n(\omega_1(U_{12}), V_{12}, \dots, Q) \\
&\quad + P_n(U_{12}, d_1(V_{12}), \dots, Q) + P_n(\omega_1(U_{22}), V_{12}, \dots, Q) + P_n(U_{22}, d_1(V_{12}), \dots, Q).
\end{aligned}
$$

Comparing the above two equations, we arrive at

$$
\begin{aligned}
0 &= P_n(\omega_1(U_{11} + U_{12} + U_{22}) - \omega_1(U_{11}) - \omega_1(U_{12}) - \omega_1(U_{22}), V_{12}, \dots, Q) \\
&= [\omega_1(U_{11} + U_{12} + U_{22}) - \omega_1(U_{11}) - \omega_1(U_{12}) - \omega_1(U_{22}), V_{12}]
\end{aligned}
$$

for all $U_{11} \in \mathcal{U}_{11}$, $U_{12}, V_{12} \in \mathcal{U}_{12}$, $U_{22} \in \mathcal{U}_{22}$. It follows that

$$
\omega_1(U_{11} + U_{12} + U_{22}) - \omega_1(U_{11}) - \omega_1(U_{12}) - \omega_1(U_{22}) \in Z(\mathcal{U}) \tag{3.14}
$$

for all $U_{11} \in \mathcal{U}_{11}$, $U_{12} \in \mathcal{U}_{12}$, $U_{22} \in \mathcal{U}_{22}$.

Let $U = U_{11} + U_{12} + U_{22}, V = V_{11} + V_{12} + V_{22}$ be two elements in $\mathcal{U}$, where $U_{ij}, V_{ij} \in \mathcal{U}_{ij}$, $1 \le i \le j \le 2$. Then, it follows from (3.7) and (3.12)–(3.14) that

$$
\begin{aligned}
\omega_1(U + V) &= \omega_1\left((U_{11} + U_{12} + U_{22}) + (V_{11} + V_{12} + V_{22})\right) \\
&= \omega_1\left((U_{11} + V_{11}) + (U_{12} + V_{12}) + (U_{22} + V_{22})\right) \\
&= \omega_1(U_{11} + V_{11}) + \omega_1(U_{12} + V_{12}) + \omega_1(U_{22} + V_{22}) + \alpha \\
&= \omega_1(U_{11}) + \omega_1(V_{11}) + \omega_1(U_{12}) + \omega_1(V_{12}) + \omega_1(U_{22}) + \omega_1(V_{22}) + \alpha \\
&= \omega_1(U_{11} + U_{12} + U_{22}) + \omega_1(V_{11} + V_{12} + V_{22}) + \beta \\
&= \omega_1(U) + \omega_1(V) + \beta,
\end{aligned} \tag{3.15}
$$

where $\alpha, \beta \in Z(\mathcal{U})$, that is, ω_1 is additive module $Z(\mathcal{U})$. □

Using Lemma 3.1, we present the main theorem of this section.

Theorem 3.2. *Let $\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{22}$ be a triangular algebra. Suppose that $\mathcal{U}$ satisfies the following conditions:*
(i) $\pi_{\mathcal{U}_{11}}(Z(\mathcal{U})) = Z(\mathcal{U}_{11})$ and $\pi_{\mathcal{U}_{22}}(Z(\mathcal{U})) = Z(\mathcal{U}_{22})$;

(ii) $\mathcal{U}_{11}$ or $\mathcal{U}_{22}$ satisfies (2.1).
If G_1 is a nonlinear generalized Lie n-derivation on $\mathcal{U}$, then there exists a generalized derivation χ_1 and a nonlinear mapping h_1 satisfying $h_1(P_n(U_1, U_2, \ldots, U_n)) = 0$ such that $G_1(U) = \chi_1(U) + h_1(U)$ for all $U, U_1, U_2, \ldots, U_n \in \mathcal{U}$. Additionally, the mappings G_1 and χ_1 fulfill the following properties:

$$\begin{cases} G_1(0) = 0, & G_1(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, \quad G_1(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12} + Z(\mathcal{U}), \\ \chi_1(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, & \chi_1(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12}. \end{cases}$$

Proof. According to Lemma 3.1, we define two mappings $g_1 : \mathcal{U} \to Z(\mathcal{U})$ and $\chi_1 : \mathcal{U} \to \mathcal{U}$ by

$$g_1(U) = \omega_1(U) - \omega_1(U_{11}) - \omega_1(U_{12}) - \omega(U_{22})$$

and

$$\chi_1(U) = \omega_1(U) - g_1(U)$$

for all $U = U_{11} + U_{12} + U_{22} \in \mathcal{U}$. It follows that

$$\begin{aligned} g_1(P_n(U_1, U_2, \ldots, U_n)) &= 0, \\ \chi_1(U_{11} + U_{12} + U_{22}) &= \omega_1(U_{11}) + \omega_1(U_{12}) + \omega_1(U_{22}), \\ \chi_1(U_{11}) = \omega_1(U_{11}) - g_1(U_{11}) &= \omega_1(U_{11}) - \omega_1(U_{11}) + \omega_1(U_{11}) = \omega_1(U_{11}) \end{aligned}$$

for all $U_1, U_2, \ldots, U_n \in \mathcal{U}, U_{11} \in \mathcal{U}_{11}, U_{12} \in \mathcal{U}_{12}, U_{22} \in \mathcal{U}_{22}$. Similarly, we have $\chi_1(U_{12}) = \omega_1(U_{12})$ and $\chi_1(U_{22}) = \omega_1(U_{22})$. Therefore,

$$\chi_1(U_{11} + U_{12} + U_{22}) = \chi_1(U_{11}) + \chi_1(U_{12}) + \chi_1(U_{22}) \tag{3.16}$$

for all $U_{11} \in \mathcal{U}_{11}, U_{12} \in \mathcal{U}_{12}, U_{22} \in \mathcal{U}_{22}$. In addition, the following properties can be obtained from (3.1)–(3.4):

$$\begin{cases} G_1(0) = 0, & G_1(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, \quad G_1(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12} + Z(\mathcal{U}), \\ \chi_1(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, & \chi_1(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12}. \end{cases}$$

Let $h_1(U) = g_1(U) + f_1(U)$, then $G_1(U) = \omega_1(U) + f_1(U) = \chi_1(U) + h_1(U)$ for all $U \in \mathcal{U}$. Since $g_1(P_n(U_1, U_2, \ldots, U_n)) = f_1(P_n(U_1, U_2, \ldots, U_n)) = 0$, we have $h_1(P_n(U_1, U_2, \ldots, U_n)) = 0$ for all $U_1, U_2, \ldots, U_n \in \mathcal{U}$. Let $U = U_{11} + U_{12} + U_{22}, V = V_{11} + V_{12} + V_{22}$ be two elements in $\mathcal{U}$, where $U_{ij}, V_{ij} \in \mathcal{U}_{ij}$, $1 \leq i \leq j \leq 2$. Then, it can be inferred from (3.15) and (3.16) that χ_1 is additive. Next, we will prove that χ_1 is a generalized derivation.

According to $U_{11}U_{12} = P_n(U_{11}, U_{12}, Q, \ldots, Q)$, we obtain

$$\begin{aligned} \chi_1(U_{11}U_{12}) =& G_1(P_n(U_{11}, U_{12}, Q, \ldots, Q)) \\ =& P_n(G_1(U_{11}), U_{12}, Q, \ldots, Q) + P_n(U_{11}, L_1(U_{12}), Q, \ldots, Q) \\ & + \cdots + P_n(U_{11}, U_{12}, Q, \ldots, L_1(Q)) \\ =& P_n(\chi_1(U_{11}), U_{12}, Q, \ldots, Q) + P_n(U_{11}, d_1(U_{12}), Q, \ldots, Q) \\ & + \cdots + P_n(U_{11}, U_{12}, Q, \ldots, d_1(Q)) \\ =& \chi_1(U_{11})U_{12} + U_{11}d_1(U_{12}) \end{aligned} \tag{3.17}$$

for all $U_{11} \in \mathcal{U}_{11}, U_{12} \in \mathcal{U}_{12}$. Similarly, it follows from $U_{12}U_{22} = P_n(U_{12}, U_{22}, Q, \ldots, Q)$ that

$$\chi_1(U_{12}U_{22}) = \chi_1(U_{12})U_{22} + U_{12}d_1(U_{22}) \tag{3.18}$$

for all $U_{12} \in \mathcal{U}_{12}, U_{22} \in \mathcal{U}_{22}$.

In view of (3.17), we get

$$\chi_1(U_{11}V_{11}U_{12}) = \chi_1\left((U_{11}V_{11})U_{12}\right) = \chi_1(U_{11}V_{11})U_{12} + U_{11}V_{11}d_1(U_{12})$$

and

$$\begin{aligned}\chi_1\left(U_{11}V_{11}U_{12}\right) &= \chi_1\left(U_{11}(V_{11}U_{12})\right)\\ &= \chi_1\left(U_{11}\right)V_{11}U_{12} + U_{11}d_1\left(V_{11}U_{12}\right)\\ &= \chi_1\left(U_{11}\right)V_{11}U_{12} + U_{11}d_1\left(V_{11}\right)U_{12} + U_{11}V_{11}d_1\left(U_{12}\right)\end{aligned}$$

for all $U_{11}, V_{11} \in \mathcal{U}_{11}, U_{12} \in \mathcal{U}_{12}$. Comparing the above equations, we obtain

$$\left(\chi_1(U_{11}V_{11}) - \chi_1(U_{11})V_{11} - U_{11}d_1(V_{11})\right)U_{12} = 0$$

for all $U_{11}, V_{11} \in \mathcal{U}_{11}, U_{12} \in \mathcal{U}_{12}$. Since $\mathcal{U}_{12}$ is faithful as a left $\mathcal{U}_{11}$-module, we conclude

$$\chi_1(U_{11}V_{11})P = \chi_1(U_{11})V_{11}P + U_{11}d_1(V_{11})P \tag{3.19}$$

for all $U_{11}, V_{11} \in \mathcal{U}_{11}$. Replacing U_{11} by $U_{11}V_{11}$ in (3.9), we get

$$\chi_1(U_{11}V_{11})Q + U_{11}V_{11}d_1(Q) = 0. \tag{3.20}$$

Since L_1 is a Lie n-derivation, it follows that

$$\begin{aligned}0 =& L_1\left(P_n(V_{11}, Q, \ldots, Q)\right)\\ =& P_n\left(L_1(V_{11}), Q, \ldots, Q\right) + P_n(V_{11}, L_1(Q), \ldots, Q) + \cdots + P_n\left(V_{11}, Q, \ldots, L_1(Q)\right)\\ =& P_n\left(d_1(V_{11}), Q, \ldots, Q\right) + P_n\left(V_{11}, d_1(Q), \ldots, Q\right) + \cdots + P_n\left(V_{11}, Q, \ldots, d_1(Q)\right)\\ =& d_1(V_{11})Q + V_{11}d_1(Q).\end{aligned}$$

This implies that

$$d_1(V_{11})Q + V_{11}d_1(Q) = 0 \tag{3.21}$$

for all $V_{11} \in \mathcal{U}_{11}$. Multiplying the left side of (3.21) by U_{11} and then combining this with (3.20) yields

$$\chi_1(U_{11}V_{11})Q = \chi_1(U_{11})V_{11}Q + U_{11}d_1(V_{11})Q \tag{3.22}$$

for all $U_{11}, V_{11} \in \mathcal{U}_{11}$. Adding (3.19) to (3.22) gives

$$\chi_1(U_{11}V_{11}) = \chi_1(U_{11})V_{11} + U_{11}d_1(V_{11}) \tag{3.23}$$

for all $U_{11}, V_{11} \in \mathcal{U}_{11}$. In a similar manner, we get

$$\chi_1(U_{22}V_{22}) = \chi_1(U_{22})V_{22} + U_{22}d_1(V_{22}) \tag{3.24}$$

for all $U_{22}, V_{22} \in \mathcal{U}_{22}$.

Considering $\chi_1(U_{11})V_{22} + U_{11}d_1(V_{22}) = G_1(P_n(U_{11}, V_{22}, Q, \ldots, Q)) = 0$ and (3.17), (3.18), (3.23), (3.24), we can conclude that

$$\begin{aligned}\chi_1(UV) =& \chi_1\left((U_{11} + U_{12} + U_{22})(V_{11} + V_{12} + V_{22})\right)\\ =& \chi_1(U_{11}V_{11} + U_{11}V_{12} + U_{12}V_{22} + U_{22}V_{22})\\ =& \chi_1(U_{11}V_{11}) + \chi_1(U_{11}V_{12}) + \chi_1(U_{12}V_{22}) + \chi_1(U_{22}V_{22})\\ =& \chi_1(U_{11})V_{11} + U_{11}d_1(V_{11}) + \chi_1(U_{11})V_{12} + U_{11}d_1(V_{12})\\ &+ \chi_1(U_{12})V_{22} + U_{12}d_1(V_{22}) + \chi_1(U_{22})V_{22} + U_{22}d_1(V_{22})\\ =& \left(\chi_1(U_{11}) + \chi_1(U_{12}) + \chi_1(U_{22})\right)(V_{11} + V_{12} + V_{22})\\ &+ (U_{11} + U_{12} + U_{22})\left(d_1(V_{11}) + d_1(V_{12}) + d_1(V_{22})\right)\\ =& \chi_1(U_{11} + U_{12} + U_{22})V + Ud_1(V_{11} + V_{12} + V_{22})\\ =& \chi_1(U)V + Ud_1(V)\end{aligned}$$

for all $U = U_{11} + U_{12} + U_{22}, V = V_{11} + V_{12} + V_{22} \in \mathcal{U}$. □

4. Nonlinear generalized Lie n-higher derivations

This section will study the nonlinear generalized Lie n-higher derivations on triangular algebras. First, we present the following result.

Lemma 4.1. *[5, Theorem 4.1] Let $\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{22}$ be an $(n-1)$-torsion free triangular ring such that*
(i) $\pi_{\mathcal{U}_{11}}(Z(\mathcal{U})) = Z(\mathcal{U}_{11})$ and $\pi_{\mathcal{U}_{22}}(Z(\mathcal{U})) = Z(\mathcal{U}_{22})$;
(ii) $\mathcal{U}_{11}$ or $\mathcal{U}_{22}$ satisfies (2.1).
Let $\{\xi_r\}_{r\in N}$ be a family of additive mappings $\xi_r : \mathcal{U} \to \mathcal{U}$ such that for each $r \in N$,

$$\xi_r(P_n(U_1, U_2, \ldots, U_n)) = \sum_{i_1+i_2+\cdots+i_n=r} P_n(\xi_{i_1}(U_1), \xi_{i_2}(U_2), \ldots, \xi_{i_n}(U_n))$$

for all $U_1, U_2, \ldots, U_n \in \mathcal{U}$ with $U_1U_2\cdots U_n = 0$. Then, for each $r \in N, \xi_r = d_r + h_r$, where $\{d_r\}_{r\in N}$ is an additive higher derivation on $\mathcal{U}$ and $\{h_r\}_{r\in N}$ is a family of additive mappings $h_r : \mathcal{U} \to Z(\mathcal{U})$ vanishing at every $(n-1)$-th commutator $P_n(U_1, U_2, \ldots, U_n)$ with $U_1U_2\cdots U_n = 0$. Moreover, ξ_r and d_r satisfy the following properties:

$$\begin{cases}\xi_r(P) \in \mathcal{U}_{12} + Z(\mathcal{U}), & \xi_r(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12},\\ \xi_r(Q) \in \mathcal{U}_{12} + Z(\mathcal{U}), & \xi_r(I) \in Z(\mathcal{U}),\\ \xi_r(\mathcal{U}_{11}) \subseteq \mathcal{U}_{11} + \mathcal{U}_{12} + Z(\mathcal{U}), & \xi_r(\mathcal{U}_{22}) \subseteq \mathcal{U}_{22} + \mathcal{U}_{12} + Z(\mathcal{U}),\\ d_r(P), d_r(Q) \in \mathcal{U}_{12}, & d_r(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12},\\ d_r(\mathcal{U}_{11}) \subseteq \mathcal{U}_{11} + \mathcal{U}_{12}, & d_r(\mathcal{U}_{22}) \subseteq \mathcal{U}_{22} + \mathcal{U}_{12}.\end{cases}$$

The main theorem of this section is presented below.

Theorem 4.2. *Let $\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{22}$ be an $(n-1)$-torsion free triangular algebra satisfying*
(i) $\pi_{\mathcal{U}_{11}}(Z(\mathcal{U})) = Z(\mathcal{U}_{11})$ and $\pi_{\mathcal{U}_{22}}(Z(\mathcal{U})) = Z(\mathcal{U}_{22})$;
(ii) $\mathcal{U}_{11}$ or $\mathcal{U}_{22}$ satisfies (2.1).
Let $G = \{G_r\}_{r\in N}$ be a nonlinear generalized Lie n-higher derivation on $\mathcal{U}$, then $G_r = \chi_r + h_r$, $r \in N$, where $\{\chi_r\}_{r\in N}$ is an additive generalized higher derivation on $\mathcal{U}$ and $\{h_r\}_{r\in N}$ is a family of nonlinear mappings $h_r : \mathcal{U} \to Z(\mathcal{U})$ such that $h_r(P_n(U_1, U_2, \ldots, U_n)) = 0$ for all $U_1, U_2, \ldots, U_n \in \mathcal{U}$.

To prove the theorem, we will employ an inductive approach with respect to the component index r. If $r = 1$, then the mapping $G_r : \mathcal{U} \to \mathcal{U}$ is a nonlinear generalized Lie n-derivation. According to Theorem 3.2, this implies the existence of a generalized derivation χ_1 and a nonlinear mapping h_1 such that $h_1(P_n(U_1, U_2, \ldots, U_n)) = 0$ and $G_1(U) = \chi_1(U) + h_1(U)$ for all $U, U_1, U_2, \ldots, U_n \in \mathcal{U}$. Furthermore, G_1 and χ_1 satisfy the following properties:

$$C_1 : \begin{cases} G_1(0) = 0, \quad G_1(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, \quad G_1(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12} + Z(\mathcal{U}), \\ \chi_1(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, \quad \chi_1(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12}. \end{cases}$$

We now assume that Theorem 4.2 holds for all $1 < t < r$, $r \in N$. That is, $G_t(U) = \chi_t(U) + h_t(U)$ for all $U \in \mathcal{U}$ and $1 < t < r$, where $\chi_t : \mathcal{U} \to \mathcal{U}$ is an additive mapping such that $\chi_t(VW) = \sum_{i+j=t} \chi_i(V)d_j(W)$, and $h_t : \mathcal{U} \to Z(\mathcal{U})$ is a nonlinear mapping such that $h_t(P_n(U_1, U_2, \ldots, U_n)) = 0$ for all $V, W, U_1, U_2, \ldots, U_n \in \mathcal{U}$, where $\{d_j\}_{j\in N}$ is an additive higher derivation on $\mathcal{U}$. In addition, G_t and χ_t satisfy the following properties:

$$C_t : \begin{cases} G_t(0) = 0, \quad G_t(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, \quad G_t(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12} + Z(\mathcal{U}), \\ \chi_t(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, \quad \chi_t(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12}. \end{cases}$$

We will prove that the above properties also hold for r. That is, $G_r(U) = \chi_r(U) + h_r(U)$, where χ_r is an additive mapping satisfying $\chi_r(VW) = \sum_{i+j=r} \chi_i(V)d_j(W)$, and $h_r : \mathcal{U} \to Z(\mathcal{U})$ is a nonlinear mapping satisfying $h_r(P_n(U_1, U_2, \ldots, U_n)) = 0$ for all $U, V, W, U_1, U_2, \ldots, U_n \in \mathcal{U}$. To begin, we will introduce the following lemma.

Lemma 4.3. *Let $\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{22}$ be a triangular algebra. Suppose that $\mathcal{U}$ satisfies the following conditions:*
(i) $\pi_{\mathcal{U}_{11}}(Z(\mathcal{U})) = Z(\mathcal{U}_{11})$ and $\pi_{\mathcal{U}_{22}}(Z(\mathcal{U})) = Z(\mathcal{U}_{22})$;
(ii) $\mathcal{U}_{11}$ or $\mathcal{U}_{22}$ satisfies (2.1),
then the nonlinear mapping G_r mentioned above satisfies $G_r = \omega_r + f'_1$, where $\omega_r : \mathcal{U} \to \mathcal{U}$ is additive modulo $Z(\mathcal{U})$ and $f'_1 : \mathcal{U} \to Z(\mathcal{U})$ satisfies $f'_1(\mathcal{U}_{12}) = \{0\}$.

Proof. Applying the property C_t, we get

$$\begin{aligned} G_r(0) =& G_r(P_n(0, 0, \ldots, 0)) \\ =& P_n(G_r(0), 0, \ldots, 0) + P_n(0, L_r(0), \ldots, 0) + \cdots + P_n(0, 0, \ldots, L_r(0)) \\ &+ \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1, i_2, \ldots, i_n < r}} P_n(G_{i_1}(0), L_{i_2}(0), \ldots, L_{i_n}(0)) \\ =& 0. \end{aligned}$$

Again, according to the property C_t and Lemma 4.1, we have

$$\begin{aligned}G_r(U_{12}) =&G_r(P_n(U_{12}, Q, \ldots, Q))\\ =&P_n(G_r(U_{12}), Q, \ldots, Q) + P_n(U_{12}, L_r(Q), \ldots, Q) + \cdots + P_n(U_{12}, Q, \ldots, L_r(Q))\\ &+ \sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}} P_n(G_{i_1}(U_{12}), L_{i_2}(Q), \ldots, L_{i_n}(Q))\\ =&PG_r(U_{12})Q\end{aligned}$$

for all $U_{12} \in \mathcal{U}_{12}$. Therefore, $G_r(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}$.

Using the property C_t and Lemma 4.1, we arrive at

$$\begin{aligned}0 =&G_r(P_n(U_{11}, U_{22}, U_{12}, Q, \ldots, Q))\\ =&P_n(G_r(U_{11}), U_{22}, U_{12}, Q, \ldots, Q) + P_n(U_{11}, L_r(U_{22}), U_{12}, Q, \ldots, Q)\\ &+ \cdots + P_n(U_{11}, U_{22}, U_{12}, Q, \ldots, L_r(Q))\\ &+ \sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}} P_n(G_{i_1}(U_{11}), L_{i_2}(U_{22}), L_{i_3}(U_{12}), \ldots, L_{i_n}(Q))\\ =&[[G_r(U_{11}), U_{22}], U_{12}].\end{aligned}$$

It follows from Remark 2.2 that $QG_r(U_{11})Q \in Z(\mathcal{U}_{22})$. Similarly, we can obtain $PG_r(U_{22})P \in Z(\mathcal{U}_{11})$ from $0 = G_r(P_n(U_{22}, U_{11}, U_{12}, Q, \ldots, Q))$. Hence,

$$\begin{aligned}G_r(U_{11}) =&PG_r(U_{11})P - \tau^{-1}(QG_r(U_{11})Q) + PG_r(U_{11})Q + (\tau^{-1}(QG_r(U_{11})Q) + QG_r(U_{11})Q)\\ \in&\mathcal{U}_{11} + \mathcal{U}_{12} + Z(\mathcal{U}),\\ G_r(U_{22}) =&(PG_r(U_{22})P + \tau(PG_r(U_{22})P)) + PG_r(U_{22})Q + QG_r(U_{22})Q - \tau(PG_r(U_{22})P)\\ \in&\mathcal{U}_{22} + \mathcal{U}_{12} + Z(\mathcal{U})\end{aligned}$$

for all $U_{11} \in \mathcal{U}_{11}, U_{22} \in \mathcal{U}_{22}$, that is, $G_r(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12} + Z(\mathcal{U})$ with $i \in \{1, 2\}$. Set

$$f_1'(U) = QG_r(PUP)Q + \tau^{-1}(QG_r(PUP)Q) + PG_r(QUQ)P + \tau(PG_r(QUQ)P)$$

for all $U \in \mathcal{U}$. Clearly, $f_1'(\mathcal{U}_{12}) = \{0\}$, $f_1'(U) \in Z(\mathcal{U})$, and $f_1'(P_n(U_1, U_2, \ldots, U_n)) = 0$ for all $U, U_1, U_2, \ldots, U_n \in \mathcal{U}$. Define a mapping $\omega_r(U) = G_r(U) - f_1'(U)$ for all $U \in \mathcal{U}$. It is easy to obtain the following relations:

$$\omega_r(0) = 0, \quad \omega_r(\mathcal{U}_{12}) \subseteq \mathcal{U}_{12}, \quad \omega_r(\mathcal{U}_{ii}) \subseteq \mathcal{U}_{ii} + \mathcal{U}_{12}.$$

Next, we will prove that $\omega_r : \mathcal{U} \to \mathcal{U}$ is additive modulo $Z(\mathcal{U})$.

On the one hand, since $d_{i_2}(Q), \ldots, d_{i_n}(Q)) \in \mathcal{U}_{12}$, we get

$$\begin{aligned}&G_r(P_n(U_{11} + U_{12}, Q, \ldots, Q))\\ =&P_n(G_r(U_{11} + U_{12}), Q, \ldots, Q) + P_n(U_{11} + U_{12}, L_r(Q), \ldots, Q)\\ &+ \cdots + P_n(U_{11} + U_{12}, Q, \ldots, L_r(Q))\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}} P_n(G_{i_1}(U_{11}+U_{12}),L_{i_2}(Q),\ldots,L_{i_n}(Q))\\
=&P_n(\omega_r(U_{11}+U_{12}),Q,\ldots,Q)+P_n(U_{11}+U_{12},d_r(Q),\ldots,Q)\\
&+\cdots+P_n(U_{11}+U_{12},Q,\ldots,d_r(Q))\\
&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}} P_n(\chi_{i_1}(U_{11}+U_{12}),d_{i_2}(Q),\ldots,d_{i_n}(Q))\\
=&P_n(\omega_r(U_{11}+U_{12}),Q,\ldots,Q)+P_n(U_{11}+U_{12},d_r(Q),\ldots,Q)\\
&+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} [\chi_{i_1}(U_{11}+U_{12}),d_{i_2}(Q)]
\end{aligned}
$$

for any $U_{11}\in\mathcal{U}_{11}$, $U_{12}\in\mathcal{U}_{12}$. On the other hand, according to $G_r(0)=0$, we arrive at

$$
\begin{aligned}
&G_r(P_n(U_{11}+U_{12},Q,\ldots,Q))\\
=&G_r(P_n(U_{11},Q,\ldots,Q))+G_r(P_n(U_{12},Q,\ldots,Q))\\
=&P_n(G_r(U_{11}),Q,\ldots,Q)+P_n(U_{11},L_r(Q),\ldots,Q)+\cdots+P_n(U_{11},Q,\ldots,L_r(Q))\\
&+P_n(G_r(U_{12}),Q,\ldots,Q)+P_n(U_{12},L_r(Q),\ldots,Q)+\cdots+P_n(U_{12},Q,\ldots,L_r(Q))\\
&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}} P_n(G_{i_1}(U_{11}),L_{i_2}(Q),\ldots,L_{i_n}(Q))+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}} P_n(G_{i_1}(U_{12}),L_{i_2}(Q),\ldots,L_{i_n}(Q))\\
=&P_n(\omega_r(U_{11}),Q,\ldots,Q)+P_n(U_{11},d_r(Q),\ldots,Q)+\cdots+P_n(U_{11},Q,\ldots,d_r(Q))\\
&+P_n(\omega_r(U_{12}),Q,\ldots,Q)+P_n(U_{12},d_r(Q),\ldots,Q)+\cdots+P_n(U_{12},Q,\ldots,d_r(Q))\\
&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}} P_n(\chi_{i_1}(U_{11}),d_{i_2}(Q),\ldots,d_{i_n}(Q))+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}} P_n(\chi_{i_1}(U_{12}),d_{i_2}(Q),\ldots,d_{i_n}(Q))\\
=&P_n(\omega_r(U_{11}),Q,\ldots,Q)+P_n(U_{11},d_r(Q),\ldots,Q)+P_n(\omega_r(U_{12}),Q,\ldots,Q)\\
&+P_n(U_{12},d_r(Q),\ldots,Q)+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} [\chi_{i_1}(U_{11}),d_{i_2}(Q)]+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} [\chi_{i_1}(U_{12}),d_{i_2}(Q)].
\end{aligned}
$$

Comparing the above two relations, we obtain

$$
\begin{aligned}
0=&P_n(\omega_r(U_{11}+U_{12})-\omega_r(U_{11})-\omega_r(U_{12}),Q,\ldots,Q)\\
&+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} [\chi_{i_1}(U_{11}+U_{12})-\chi_{i_1}(U_{11})-\chi_{i_1}(U_{12}),d_{i_2}(Q)].
\end{aligned}
$$

Based on the fact that χ_{i_1} is additive, we get

$$
P(\omega_r(U_{11}+U_{12})-\omega_r(U_{11})-\omega_r(U_{12}))Q=0.
$$

Therefore,

$$
\omega_r(U_{11}+U_{12})-\omega_r(U_{11})-\omega_r(U_{12})\in\mathcal{U}_{11}+\mathcal{U}_{22}.
$$

In the same way, we can obtain

$$\begin{aligned}
&G_r(P_n(U_{11}+U_{12},V_{12},Q,\ldots,Q))\\
&=P_n(\omega_r(U_{11}+U_{12}),V_{12},Q,\ldots,Q)+P_n(U_{11}+U_{12},d_r(V_{12}),Q,\ldots,Q)\\
&\quad+\cdots+P_n(U_{11}+U_{12},V_{12},Q,\ldots,d_r(Q))\\
&\quad+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}}P_n(\chi_{i_1}(U_{11}+U_{12}),d_{i_2}(V_{12}),\ldots,d_{i_n}(Q))\\
&=P_n(\omega_r(U_{11}+U_{12}),V_{12},Q,\ldots,Q)+P_n(U_{11}+U_{12},d_r(V_{12}),Q,\ldots,Q)\\
&\quad+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}}[\chi_{i_1}(U_{11}+U_{12}),d_{i_2}(V_{12})]
\end{aligned}$$

and

$$\begin{aligned}
&G_r(P_n(U_{11}+U_{12},V_{12},Q,\ldots,Q))\\
&=G_r(P_n(U_{11},V_{12},Q,\ldots,Q))+G_r(P_n(U_{12},V_{12},Q,\ldots,Q))\\
&=P_n(\omega_r(U_{11}),V_{12},Q,\ldots,Q)+P_n(U_{11},d_r(V_{12}),Q,\ldots,Q)+\cdots+P_n(U_{11},V_{12},Q,\ldots,d_r(Q))\\
&\quad+P_n(\omega_r(U_{12}),V_{12},Q,\ldots,Q)+P_n(U_{12},d_r(V_{12}),Q,\ldots,Q)+\cdots+P_n(U_{12},V_{12},Q,\ldots,d_r(Q))\\
&\quad+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}}P_n(\chi_{i_1}(U_{11}),d_{i_2}(V_{12}),\ldots,d_{i_n}(Q))+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}}P_n(\chi_{i_1}(U_{12}),d_{i_2}(V_{12}),\ldots,d_{i_n}(Q))\\
&=P_n(\omega_r(U_{11}),V_{12},Q,\ldots,Q)+P_n(U_{11},d_r(V_{12}),Q,\ldots,Q)\\
&\quad+P_n(\omega_r(U_{12}),V_{12},Q,\ldots,Q)+P_n(U_{12},d_r(V_{12}),Q,\ldots,Q)\\
&\quad+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}}[\chi_{i_1}(U_{11}),d_{i_2}(V_{12})]+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}}[\chi_{i_1}(U_{12}),d_{i_2}(V_{12})].
\end{aligned}$$

Comparing the above two equations, we have

$$[\omega_r(U_{11}+U_{12})-\omega_r(U_{11})-\omega_r(U_{12}),V_{12}]=0$$

for all $U_{11}\in\mathcal{U}_{11}$, $U_{12},V_{12}\in\mathcal{U}_{12}$. Consequently,

$$\omega_r(U_{11}+U_{12})-\omega_r(U_{11})-\omega_r(U_{12})\in Z(\mathcal{U}).$$

Analogously, we have

$$\omega_r(U_{22}+U_{12})-\omega_r(U_{22})-\omega_r(U_{12})\in Z(\mathcal{U}).$$

Considering

$$U_{12}+V_{12}=P_n(P+U_{12},Q+V_{12},Q,\ldots,Q)$$

and

$$d_{i_2}(V_{12})\in\mathcal{U}_{12},$$

we have

$$
\begin{aligned}
\omega_r(U_{12}+V_{12}) =& G_r(P_n(P+U_{12}, Q+V_{12}, Q, \ldots, Q)) \\
=& P_n(\omega_r(P+U_{12}), Q+V_{12}, Q, \ldots, Q) + P_n(P+U_{12}, d_r(Q+V_{12}), Q, \ldots, Q) \\
& + \cdots + P_n(P+U_{12}, Q+V_{12}, Q, \ldots, d_r(Q)) \\
& + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\ldots,i_n<r}} P_n(\chi_{i_1}(P+U_{12}), d_{i_2}(Q+V_{12}), \ldots, d_{i_n}(Q)) \\
=& \omega_r(P)Q + \omega_r(P)V_{12} + \omega_r(U_{12}) + d_r(Q) + d_r(V_{12}) \\
& + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} \chi_{i_1}(P)d_{i_2}(Q) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} \chi_{i_1}(P)d_{i_2}(V_{12}).
\end{aligned}
\tag{4.1}
$$

Since

$$
\begin{aligned}
\omega_r(V_{12}) =& G_r(P_n(P, V_{12}, Q, \ldots, Q)) \\
=& P_n(G_r(P), V_{12}, Q, \ldots, Q) + P_n(P, L_r(V_{12}), Q, \ldots, Q) + \cdots + P_n(P, V_{12}, Q, \ldots, L_r(Q)) \\
& + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\ldots,i_n<r}} P_n(G_{i_1}(P), L_{i_2}(V_{12}), \ldots, L_{i_n}(Q)) \\
=& \omega_r(P)V_{12} + d_r(V_{12}) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} \chi_{i_1}(P)d_{i_2}(V_{12})
\end{aligned}
$$

and

$$
\begin{aligned}
0 =& G_r(P_n(P, Q, \ldots, Q)) \\
=& P_n(G_r(P), Q, \ldots, Q) + P_n(P, L_r(Q), \ldots, Q) + \cdots + P_n(P, Q, \ldots, L_r(Q)) \\
& + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\ldots,i_n<r}} P_n(G_{i_1}(P), L_{i_2}(Q), \ldots, L_{i_n}(Q)) \\
=& \omega_r(P)Q + d_r(Q) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} \chi_{i_1}(P)d_{i_2}(Q),
\end{aligned}
$$

we can derive from (4.1) that

$$
\omega_r(U_{12}+V_{12}) = \omega_r(U_{12}) + \omega_r(V_{12}) \tag{4.2}
$$

for all $U_{12}, V_{12} \in \mathcal{U}_{12}$. On the one hand,

$$
\begin{aligned}
& G_r(P_n(U_{11}+V_{11}, U_{12}, Q, \ldots, Q)) \\
=& P_n(G_r(U_{11}+V_{11}), U_{12}, Q, \ldots, Q) + P_n(U_{11}+V_{11}, L_r(U_{12}), Q, \ldots, Q) \\
& + \cdots + P_n(U_{11}+V_{11}, U_{12}, Q, \ldots, L_r(Q)) \\
& + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\ldots,i_n<r}} P_n(G_{i_1}(U_{11}+V_{11}), L_{i_2}(U_{12}), \ldots, L_{i_n}(Q))
\end{aligned}
$$

$$
\begin{aligned}
=&P_n(\omega_r(U_{11}+V_{11}),U_{12},Q,\ldots,Q)+P_n(U_{11}+V_{11},d_r(U_{12}),Q,\ldots,Q)\\
&+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}}[\chi_{i_1}(U_{11}+V_{11}),d_{i_2}(U_{12})],
\end{aligned}
$$

and on the other hand, in view of (4.2), we have

$$
\begin{aligned}
&G_r(P_n(U_{11}+V_{11},U_{12},Q,\ldots,Q))\\
&=G_r(P_n(U_{11},U_{12},Q,\ldots,Q))+G_r(P_n(V_{11},U_{12},Q,\ldots,Q))\\
&=P_n(G_r(U_{11}),U_{12},Q,\ldots,Q)+P_n(U_{11},L_r(U_{12}),Q,\ldots,Q)+\cdots+P_n(U_{11},U_{12},Q,\ldots,L_r(Q))\\
&\quad+P_n(G_r(V_{11}),U_{12},Q,\ldots,Q)+P_n(V_{11},L_r(U_{12}),Q,\ldots,Q)+\cdots+P_n(V_{11},U_{12},Q,\ldots,L_r(Q))\\
&\quad+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\i_1,i_2,\ldots,i_n<r}}P_n(G_{i_1}(U_{11}),L_{i_2}(U_{12}),\ldots,L_{i_n}(Q))+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\i_1,i_2,\ldots,i_n<r}}P_n(G_{i_1}(V_{11}),L_{i_2}(U_{12}),\ldots,L_{i_n}(Q))\\
&=P_n(\omega_r(U_{11}),U_{12},Q,\ldots,Q)+P_n(U_{11},d_r(U_{12}),Q,\ldots,Q)\\
&\quad+P_n(\omega_r(V_{11}),U_{12},Q,\ldots,Q)+P_n(V_{11},d_r(U_{12}),Q,\ldots,Q)\\
&\quad+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}}[\chi_{i_1}(U_{11}),d_{i_2}(U_{12})]+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}}[\chi_{i_1}(V_{11}),d_{i_2}(U_{12})].
\end{aligned}
$$

As χ_{i_1} is additive, comparing the two equations above leads us to obtain

$$(\omega_r(U_{11}+V_{11})-\omega_r(U_{11})-\omega_r(V_{11}))U_{12}=0.$$

Since $\mathcal{U}_{12}$ is faithful as a left $\mathcal{U}_{11}$-module, we can derive from the above expression that

$$\omega_r(U_{11}+V_{11})P=\omega_r(U_{11})P+\omega_r(V_{11})P \tag{4.3}$$

for all $U_{11},V_{11}\in\mathcal{U}_{11}$. Using $P_n(U_{11},Q,\ldots,Q)=0$, we deduce

$$
\begin{aligned}
0=&G_r(P_n(U_{11},Q,\ldots,Q))\\
=&P_n(G_r(U_{11}),Q,\ldots,Q)+P_n(U_{11},L_r(Q),\ldots,Q)+\cdots+P_n(U_{11},Q,\ldots,L_r(Q))\\
&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\i_1,i_2,\ldots,i_n<r}}P_n(G_{i_1}(U_{11}),L_{i_2}(Q),\ldots,L_{i_n}(Q))\\
=&P_n(\omega_r(U_{11}),Q,\ldots,Q)+P_n(U_{11},d_r(Q),\ldots,Q)+\cdots+P_n(U_{11},Q,\ldots,d_r(Q))\\
&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\i_1,i_2,\ldots,i_n<r}}P_n(\chi_{i_1}(U_{11}),d_{i_2}(Q),\ldots,d_{i_n}(Q))\\
=&\omega_r(U_{11})Q+U_{11}d_r(Q)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}}\chi_{i_1}(U_{11})d_{i_2}(Q).
\end{aligned}
\tag{4.4}
$$

Replacing U_{11} with $U_{11}+V_{11}$ in (4.4), we arrive at

$$\omega_r(U_{11}+V_{11})Q+(U_{11}+V_{11})d_r(Q)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}}\chi_{i_1}(U_{11}+V_{11})d_{i_2}(Q)=0. \tag{4.5}$$

Again, using (4.4) and (4.5), we obtain

$$\omega_r(U_{11}+V_{11})Q=\omega_r(U_{11})Q+\omega_r(V_{11})Q. \tag{4.6}$$

Combining (4.3) and (4.6), we find

$$\omega_r(U_{11}+V_{11})=\omega_r(U_{11})+\omega_r(V_{11}) \tag{4.7}$$

for all $U_{11},V_{11}\in\mathcal{U}_{11}$. In a similar way, we get

$$\omega_r(U_{22}+V_{22})=\omega_r(U_{22})+\omega_r(V_{22}) \tag{4.8}$$

for all $U_{22},V_{22}\in\mathcal{U}_{22}$.

On the one hand, we get

$$\begin{aligned}
&G_r(P_n(U_{11}+U_{12}+U_{22},Q,\dots,Q))\\
=&P_n(\omega_r(U_{11}+U_{12}+U_{22}),Q,\dots,Q)+P_n(U_{11}+U_{12}+U_{22},d_r(Q),\dots,Q)\\
&+\cdots+P_n(U_{11}+U_{12}+U_{22},Q,\dots,d_r(Q))\\
&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\dots,i_n<r}}P_n(\chi_{i_1}(U_{11}+U_{12}+U_{22}),d_{i_2}(Q),\dots,d_{i_n}(Q))\\
=&P_n(\omega_r(U_{11}+U_{12}+U_{22}),Q,\dots,Q)+P_n(U_{11}+U_{12}+U_{22},d_r(Q),\dots,Q)\\
&+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}}[\chi_{i_1}(U_{11}+U_{12}+U_{22}),d_{i_2}(Q)],
\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
&G_r(P_n(U_{11}+U_{12}+U_{22},Q,\dots,Q))\\
=&G_r(P_n(U_{11},Q,\dots,Q))+G_r(P_n(U_{12},Q,\dots,Q))+G_r(P_n(U_{22},Q,\dots,Q))\\
=&P_n(\omega_r(U_{11}),Q,\dots,Q)+P_n(U_{11},d_r(Q),\dots,Q)+\cdots+P_n(U_{11},Q,\dots,d_r(Q))\\
&+P_n(\omega_r(U_{12}),Q,\dots,Q)+P_n(U_{12},d_r(Q),\dots,Q)+\cdots+P_n(U_{12},Q,\dots,d_r(Q))\\
&+P_n(\omega_r(U_{22}),Q,\dots,Q)+P_n(U_{22},d_r(Q),\dots,Q)+\cdots+P_n(U_{22},Q,\dots,d_r(Q))\\
&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\dots,i_n<r}}P_n(\chi_{i_1}(U_{11}),d_{i_2}(Q),\dots,d_{i_n}(Q))+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\dots,i_n<r}}P_n(\chi_{i_1}(U_{12}),d_{i_2}(Q),\dots,d_{i_n}(Q))\\
&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\dots,i_n<r}}P_n(\chi_{i_1}(U_{22}),d_{i_2}(Q),\dots,d_{i_n}(Q))\\
=&P_n(\omega_r(U_{11}),Q,\dots,Q)+P_n(U_{11},d_r(Q),\dots,Q)+P_n(\omega_r(U_{12}),Q,\dots,Q)\\
&+P_n(U_{12},d_r(Q),\dots,Q)+P_n(\omega_r(U_{22}),Q,\dots,Q)+P_n(U_{22},d_r(Q),\dots,Q)\\
&+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}}[\chi_{i_1}(U_{11}),d_{i_2}(Q)]+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}}[\chi_{i_1}(U_{12}),d_{i_2}(Q)]+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}}[\chi_{i_1}(U_{22}),d_{i_2}(Q)]
\end{aligned}$$

for all $U_{11}\in\mathcal{U}_{11}$, $U_{12}\in\mathcal{U}_{12}$, $U_{22}\in\mathcal{U}_{22}$. Comparing the above two relations, we arrive at

$$P(\omega_r(U_{11}+U_{12}+U_{22})-\omega_r(U_{11})-\omega_r(U_{12})-\omega_r(U_{22}))Q=0.$$

Hence,

$$\omega_r(U_{11} + U_{12} + U_{22}) - \omega_r(U_{11}) - \omega_r(U_{12}) - \omega_r(U_{22}) \in \mathcal{U}_{11} + \mathcal{U}_{22}.$$

We have

$$\begin{aligned}
&G_r(P_n(U_{11} + U_{12} + U_{22}, V_{12}, \dots, Q)) \\
=&P_n(\omega_r(U_{11} + U_{12} + U_{22}), V_{12}, \dots, Q) + P_n(U_{11} + U_{12} + U_{22}, d_r(V_{12}), \dots, Q) \\
&+ \cdots + P_n(U_{11} + U_{12} + U_{22}, V_{12}, \dots, d_r(Q)) \\
&+ \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\dots,i_n<r}} P_n(\chi_{i_1}(U_{11} + U_{12} + U_{22}), d_{i_2}(V_{12}), \dots, d_{i_n}(Q)) \\
=&P_n(\omega_r(U_{11} + U_{12} + U_{22}), V_{12}, \dots, Q) + P_n(U_{11} + U_{12} + U_{22}, d_r(V_{12}), \dots, Q) \\
&+ \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} [\chi_{i_1}(U_{11} + U_{12} + U_{22}), d_{i_2}(V_{12})]
\end{aligned}$$

and

$$\begin{aligned}
&G_r(P_n(U_{11} + U_{12} + U_{22}, V_{12}, \dots, Q)) \\
=&G_r(P_n(U_{11}, V_{12}, \dots, Q)) + G_r(P_n(U_{12}, V_{12}, \dots, Q)) + G_r(P_n(U_{22}, V_{12}, \dots, Q)) \\
=&P_n(\omega_r(U_{11}), V_{12}, \dots, Q) + P_n(U_{11}, d_r(V_{12}), \dots, Q) + \cdots + P_n(U_{11}, V_{12}, \dots, d_r(Q)) \\
&+ P_n(\omega_r(U_{12}), V_{12}, \dots, Q) + P_n(U_{12}, d_r(V_{12}), \dots, Q) + \cdots + P_n(U_{12}, V_{12}, \dots, d_r(Q)) \\
&+ P_n(\omega_r(U_{22}), V_{12}, \dots, Q) + P_n(U_{22}, d_r(V_{12}), \dots, Q) + \cdots + P_n(U_{22}, V_{12}, \dots, d_r(Q)) \\
&+ \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\dots,i_n<r}} P_n(\chi_{i_1}(U_{11}), d_{i_2}(V_{12}), \dots, d_{i_n}(Q)) + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\dots,i_n<r}} P_n(\chi_{i_1}(U_{12}), d_{i_2}(V_{12}), \dots, d_{i_n}(Q)) \\
&+ \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\dots,i_n<r}} P_n(\chi_{i_1}(U_{22}), d_{i_2}(V_{12}), \dots, d_{i_n}(Q)) \\
=&P_n(\omega_r(U_{11}), V_{12}, \dots, Q) + P_n(U_{11}, d_r(V_{12}), \dots, Q) + P_n(\omega_r(U_{12}), V_{12}, \dots, Q) \\
&+ P_n(U_{12}, d_r(V_{12}), \dots, Q) + P_n(\omega_r(U_{22}), V_{12}, \dots, Q) + P_n(U_{22}, d_r(V_{12}), \dots, Q) \\
&+ \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} [\chi_{i_1}(U_{11}), d_{i_2}(V_{12})] + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} [\chi_{i_1}(U_{12}), d_{i_2}(V_{12})] + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} [\chi_{i_1}(U_{22}), d_{i_2}(V_{12})].
\end{aligned}$$

Comparing the above two equations, we arrive at

$$[\omega_r(U_{11} + U_{12} + U_{22}) - \omega_r(U_{11}) - \omega_r(U_{12}) - \omega_r(U_{22}), V_{12}] = 0$$

for all $U_{11} \in \mathcal{U}_{11}$, $U_{12}, V_{12} \in \mathcal{U}_{12}$, $U_{22} \in \mathcal{U}_{22}$. It follows that

$$\omega_r(U_{11} + U_{12} + U_{22}) - \omega_r(U_{11}) - \omega_r(U_{12}) - \omega_r(U_{22}) \in Z(\mathcal{U})$$

for all $U_{11} \in \mathcal{U}_{11}$, $U_{12} \in \mathcal{U}_{12}$, $U_{22} \in \mathcal{U}_{22}$. Similar to (3.15), it can be concluded that ω_r is additive modulo $Z(\mathcal{U})$. □

Proof of Theorem 4.2. Similar to the definitions of g_1 and χ_1, the mappings g_r and χ_r can be defined as

$$g_r(U) = \omega_r(U) - \omega_r(U_{11}) - \omega_r(U_{12}) - \omega_r(U_{22})$$

and

$$\chi_r(U) = \omega_r(U) - g_r(U),$$

and it can be obtained that

$$\begin{aligned}\chi_r(U_{11}) = \omega_r(U_{11}), \quad \chi_r(U_{12}) = \omega_r(U_{12}), \quad \chi_r(U_{22}) = \omega_r(U_{22}),\\ \chi_r(U) = \chi_r(U_{11} + U_{12} + U_{22}) = \chi_r(U_{11}) + \chi_r(U_{12}) + \chi_r(U_{22}),\end{aligned}$$

for all $U = U_{11} + U_{12} + U_{22} \in \mathcal{U}$. Let $h_r(U) = g_r(U) + f_1'(U)$, then $G_r(U) = \chi_r(U) + h_r(U)$ and $h_r(P_n(U_1, U_2, \ldots, U_n)) = 0$ for all $U, U_1, U_2, \ldots, U_n \in \mathcal{U}$. Additionally, we can conclude from Lemma 4.3 that χ_r is additive. Next, we will prove that $\chi_r(UV) = \sum_{p+q=r} \chi_p(U)d_q(V)$ for all $U, V \in \mathcal{U}$, where d_q is mentioned in Lemma 4.1.

Given $U_{11}U_{12} = P_n(U_{11}, U_{12}, Q, \ldots, Q)$ and considering the property C_t along with Lemma 4.1, we deduce

$$\begin{aligned}\chi_r(U_{11}U_{12}) =& G_r(P_n(U_{11}, U_{12}, Q, \ldots, Q))\\ =& P_n(G_r(U_{11}), U_{12}, Q, \ldots, Q) + P_n(U_{11}, L_r(U_{12}), Q, \ldots, Q)\\ &+ \cdots + P_n(U_{11}, U_{12}, Q, \ldots, L_r(Q))\\ &+ \sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}} P_n(G_{i_1}(U_{11}), L_{i_2}(U_{12}), L_{i_3}(Q), \ldots, L_{i_n}(Q))\\ =& P_n(\chi_r(U_{11}), U_{12}, Q, \ldots, Q) + P_n(U_{11}, d_r(U_{12}), Q, \ldots, Q)\\ &+ \cdots + P_n(U_{11}, U_{12}, Q, \ldots, d_r(Q))\\ &+ \sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\ldots,i_n<r}} P_n(\chi_{i_1}(U_{11}), d_{i_2}(U_{12}), d_{i_3}(Q), \ldots, d_{i_n}(Q))\\ =& \chi_r(U_{11})U_{12} + U_{11}d_r(U_{12}) + \sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} \chi_{i_1}(U_{11})d_{i_2}(U_{12})\end{aligned} \tag{4.9}$$

for all $U_{11} \in \mathcal{U}_{11}, U_{12} \in \mathcal{U}_{12}$. In a similar manner, we get

$$\chi_r(U_{12}U_{22}) = \chi_r(U_{12})U_{22} + U_{12}d_r(U_{22}) + \sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} \chi_{i_1}(U_{12})d_{i_2}(U_{22}) \tag{4.10}$$

for all $U_{12} \in \mathcal{U}_{12}, U_{22} \in \mathcal{U}_{22}$.

According to (4.9) and the property C_t, we get

$$\begin{aligned}\chi_r(U_{11}V_{11}U_{12}) =& \chi_r((U_{11}V_{11})U_{12})\\ =& \chi_r(U_{11}V_{11})U_{12} + U_{11}V_{11}d_r(U_{12}) + \sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} \chi_{i_1}(U_{11}V_{11})d_{i_2}(U_{12})\end{aligned}$$

$$=\chi_r(U_{11}V_{11})U_{12}+U_{11}V_{11}d_r(U_{12})+\sum_{\substack{p+q+s=r\\0<s<r}}\chi_p(U_{11})d_q(V_{11})d_s(U_{12})$$

and

$$\begin{aligned}\chi_r(U_{11}V_{11}U_{12})=&\chi_r(U_{11}(V_{11}U_{12}))\\=&\chi_r(U_{11})V_{11}U_{12}+U_{11}d_r(V_{11}U_{12})+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}}\chi_{i_1}(U_{11})d_{i_2}(V_{11}U_{12})\\=&\chi_r(U_{11})V_{11}U_{12}+U_{11}d_r(V_{11})U_{12}+U_{11}V_{11}d_r(U_{12})\\&+\sum_{\substack{q+s=r\\0<q,s<r}}U_{11}d_q(V_{11})d_s(U_{12})+\sum_{\substack{p+q+s=r\\0<p<r}}\chi_p(U_{11})d_q(V_{11})d_s(U_{12})\\=&\chi_r(U_{11})V_{11}U_{12}+U_{11}d_r(V_{11})U_{12}+U_{11}V_{11}d_r(U_{12})\\&+\sum_{\substack{p+q+s=r\\0<s<r}}\chi_p(U_{11})d_q(V_{11})d_s(U_{12})+\sum_{\substack{p+q=r\\0<p,q<r}}\chi_p(U_{11})d_q(V_{11})U_{12}.\end{aligned}$$

Combining the above two relations, we obtain

$$\chi_r(U_{11}V_{11})U_{12}=\chi_r(U_{11})V_{11}U_{12}+U_{11}d_r(V_{11})U_{12}+\sum_{\substack{p+q=r\\0<p,q<r}}\chi_p(U_{11})d_q(V_{11})U_{12}.$$

Since $\mathcal{U}_{12}$ is faithful as a left $\mathcal{U}_{11}$-module, it follows that

$$\chi_r(U_{11}V_{11})P=\chi_r(U_{11})V_{11}P+U_{11}d_r(V_{11})P+\sum_{\substack{p+q=r\\0<p,q<r}}\chi_p(U_{11})d_q(V_{11})P \tag{4.11}$$

for all $U_{11},V_{11}\in\mathcal{U}_{11}$. Using (4.4), we obtain

$$\chi_r(U_{11}V_{11})Q+U_{11}V_{11}d_r(Q)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}}\chi_{i_1}(U_{11}V_{11})d_{i_2}(Q)=0. \tag{4.12}$$

Taking into account Lemma 4.1, we conclude

$$\begin{aligned}0=&L_r(P_n(V_{11},Q,\ldots,Q))\\=&P_n(L_r(V_{11}),Q,\ldots,Q)+P_n(V_{11},L_r(Q),\ldots,Q)+\cdots+P_n(V_{11},Q,\ldots,L_r(Q))\\&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\i_1,i_2,\ldots,i_n<r}}P_n(L_{i_1}(V_{11}),L_{i_2}(Q),\ldots,L_{i_n}(Q))\\=&P_n(d_r(V_{11}),Q,\ldots,Q)+P_n(V_{11},d_r(Q),\ldots,Q)+\cdots+P_n(V_{11},Q,\ldots,d_r(Q))\\&+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\i_1,i_2,\ldots,i_n<r}}P_n(d_{i_1}(V_{11}),d_{i_2}(Q),\ldots,d_{i_n}(Q))\\=&d_r(V_{11})Q+V_{11}d_r(Q)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}}d_{i_1}(V_{11})d_{i_2}(Q).\end{aligned} \tag{4.13}$$

Multiplying the left-hand side of (4.13) by U_{11} and combining it with (4.12), we get

$$\chi_r(U_{11}V_{11})Q + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} \chi_{i_1}(U_{11}V_{11})d_{i_2}(Q) = U_{11}d_r(V_{11})Q + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} U_{11}d_{i_1}(V_{11})d_{i_2}(Q),$$

that is,

$$\begin{aligned}&\chi_r(U_{11}V_{11})Q + \sum_{\substack{q+s=r\\0<q,s<r}} U_{11}d_q(V_{11})d_s(Q) + \sum_{\substack{p+q+s=r\\0<p,s<r}} \chi_p(U_{11})d_q(V_{11})d_s(Q)\\ &= U_{11}d_r(V_{11})Q + \sum_{\substack{q+s=r\\0<q,s<r}} U_{11}d_q(V_{11})d_s(Q).\end{aligned}$$

Hence,

$$\chi_r(U_{11}V_{11})Q + \sum_{\substack{p+q+s=r\\0<p,s<r}} \chi_p(U_{11})d_q(V_{11})d_s(Q) = U_{11}d_r(V_{11})Q \tag{4.14}$$

for all $U_{11}, V_{11} \in \mathcal{U}_{11}$. Based on

$$0 = d_t(V_{11}Q) = \sum_{q+s=t} d_q(V_{11})d_s(Q),$$

we deduce

$$\sum_{\substack{p+q+s=r\\0<p,s<r}} \chi_p(U_{11})d_q(V_{11})d_s(Q) = -\sum_{\substack{p+q=r\\0<p,q<r}} \chi_p(U_{11})d_q(V_{11})Q. \tag{4.15}$$

Combining (4.14) and (4.15), and then using

$$\chi_r(U_{11})V_{11}Q = 0,$$

we obtain

$$\chi_r(U_{11}V_{11})Q = (\chi_r(U_{11})V_{11} + U_{11}d_r(V_{11}) + \sum_{\substack{p+q=r\\0<p,q<r}} \chi_p(U_{11})d_q(V_{11}))Q.$$

Applying (4.11) yields that

$$\chi_r(U_{11}V_{11}) = \chi_r(U_{11})V_{11} + U_{11}d_r(V_{11}) + \sum_{\substack{p+q=r\\0<p,q<r}} \chi_p(U_{11})d_q(V_{11}) \tag{4.16}$$

for all $U_{11}, V_{11} \in \mathcal{U}_{11}$. Analogously,

$$\chi_r(U_{22}V_{22}) = \chi_r(U_{22})V_{22} + U_{22}d_r(V_{22}) + \sum_{\substack{p+q=r\\0<p,q<r}} \chi_p(U_{22})d_q(V_{22}) \tag{4.17}$$

for all $U_{22}, V_{22} \in \mathcal{U}_{22}$.

Based on the definition of G_r, it is clear that

$$\begin{aligned}0 =&G_r(P_n(U_{11}, V_{22}, Q, \dots, Q))\\ =&P_n(G_r(U_{11}), V_{22}, Q, \dots, Q) + P_n(U_{11}, L_r(V_{22}), Q, \dots, Q) + \cdots + P_n(U_{11}, V_{22}, Q, \dots, L_r(Q))\\ &+ \sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1,i_2,\dots,i_n<r}} P_n(G_{i_1}(U_{11}), L_{i_2}(V_{22}), L_{i_3}(Q), \dots, L_{i_n}(Q))\\ =&P_n(\chi_r(U_{11}), V_{22}, Q, \dots, Q) + P_n(U_{11}, d_r(V_{22}), Q, \dots, Q) + \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{11})d_q(V_{22})\\ =&\chi_r(U_{11})V_{22} + U_{11}d_r(V_{22}) + \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{11})d_q(V_{22}).\end{aligned}$$

Therefore, it follows from the property C_t and (4.9), (4.10), (4.16), (4.17) that

$$\begin{aligned}&\chi_r(U)V + Ud_r(V) + \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U)d_q(V)\\ =&(\chi_r(U_{11}) + \chi_r(U_{12}) + \chi_r(U_{22}))(V_{11} + V_{12} + V_{22})\\ &+ (U_{11} + U_{12} + U_{22})d_r(V_{11} + V_{12} + V_{22}) + \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{11})d_q(V_{11})\\ &+ \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{11})d_q(V_{12}) + \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{11})d_q(V_{22})\\ &+ \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{12})d_q(V_{22}) + \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{22})d_q(V_{22})\\ =&\chi_r(U_{11})V_{11} + \chi_r(U_{11})V_{12} + \chi_r(U_{11})V_{22} + \chi_r(U_{12})V_{22} + \chi_r(U_{22})V_{22}\\ &+ U_{11}d_r(V_{11}) + U_{11}d_r(V_{12}) + U_{11}d_r(V_{22}) + U_{12}d_r(V_{22}) + U_{22}d_r(V_{22})\\ &+ \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{11})d_q(V_{11}) + \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{11})d_q(V_{12}) + \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{11})d_q(V_{22})\\ &+ \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{12})d_q(V_{22}) + \sum_{\substack{p+q=r\\ 0<p,q<r}} \chi_p(U_{22})d_q(V_{22})\\ =&\chi_r(U_{11}V_{11}) + \chi_r(U_{11}V_{12}) + \chi_r(U_{12}V_{22}) + \chi_r(U_{22}V_{22})\\ =&\chi_r(U_{11}V_{11} + U_{11}V_{12} + U_{12}V_{22} + U_{22}V_{22})\\ =&\chi_r((U_{11} + U_{12} + U_{22})(V_{11} + V_{12} + V_{22}))\\ =&\chi_r(UV)\end{aligned}$$

for all $U = U_{11} + U_{12} + U_{22}, V = V_{11} + V_{12} + V_{22} \in \mathcal{U}$.

The proof of the theorem is completed. □

5. Conclusions

We used Ashraf et al.'s results on Lie n-derivations to study the nonlinear generalized Lie n-derivations, although Lin also used the condition $PG(x)Q = 0$ to describe the nonlinear generalized Lie n-derivations on triangular algebras. Based on this, we used an inductive method to describe the generalized Lie n-higher derivations. It is shown that, under some mild conditions, each component G_r of a nonlinear generalized Lie n-higher derivation $\{G_r\}_{r\in N}$ of the triangular algebra $\mathcal{U}$ can be expressed as the sum of an additive generalized higher derivation and a nonlinear mapping vanishing on all $(n-1)$-th commutators on $\mathcal{U}$.

Author contributions

He Yuan: Conceptualization, Writing-original draft; Qian Zhang: Formal analysis, Writing-original draft; Zhendi Gu: Editing, Writing-original draft. All the contributors have perused and consented to the publishable draft of the manuscript.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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