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Research article

Mathematical exploration on control of bifurcation for a 3D predator-prey model with delay

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Abstract: In this current paper, we developed a new predator-prey model accompanying delay based on the earlier works. By applying inequality strategies, fixed point theorem, and a suitable function, we got new necessary conditions for the existence, uniqueness, nonnegativeness, and boundedness of the solution to the developed delayed predator-prey model. The bifurcation behavior and stability nature of the defined delayed predator-prey model were investigated by using stability and bifurcation theory of delayed differential equations. We have modified the Hopf bifurcation's appearance time and stability domain by building two distinct hybrid delayed feedback controllers for the delayed predator-prey model. The time of Hopf bifurcation appearance and stability domain of the model were explored. Matlab experiment diagrams were given to support the learned important results. The derived outcomes in this paper were original and have significant theoretical implications for maintaining equilibrium between the densities of the three species.

Keywords: predator-prey model; feature of solution; Hopf bifurcation; hybrid controller; delay; stability
Mathematics Subject Classification: 34C23, 34K18, 37GK15, 39A11, 92B20

1. Introduction

It is widely accepted that the delayed dynamical model is an important tool for answering many issues in biology. Numerous academics focus extensively on the development of diverse predator-prey models to elucidate the dynamics and internal structure within biological populations. We can successfully regulate the concentrations of predators and preys in the natural world by investigating the various dynamical characteristics of predator-prey models. Numerous studies on predator-prey models have been submitted and released in recent years, and several outstanding studies have been discussed. For example, Din et al. [1] studied the discrete predator-prey interactions with chaos control and bifurcation analysis. Liu and Guo [2] examined the dynamics of a predator-prey system with nonlinear prey-taxis, using the Lyapunov-Schmidt reduction approach, the fold-Hopf singularity, Hopf bifurcation, and steady state bifurcation are examined in depth. Al-Kaff et al. [3] investigated bifurcation and chaos in a discrete predator-prey system using a linked logistic map, a wide range of the system's behavior is thoroughly examined in the study. Pita et al. [4] investigated a review of some recent advances in predator-prey models. For more extensive investigations, see [5–9].

In particular, Lotka-Volterra models are vital predator-prey models in biology. They play an important role in describing the correlation between predators and prey. During the past decades, many works on this topic have been reported. For example, Hsu et al. [10] studied three different species of omnivorous Lotka-Volterra food web models. Bunin [11] studied the Lotka-Volterra model for ecological communities exhibiting symmetry and connected it to other well-known models. Wu et al. [12] investigated the Grey Lotka-Volterra model and its application, effectively analyzing the link between the two variables and forecasting their values using the gray Lotka-Volterra model. Kloppers et al. [13] estimated the parameters of the Lotka-Volterra model from empirical data. Marasco et al. [14] found the development of market share dynamics through the application of Lotka-Volterra models, and Wang et al. [15] examined the problems with free boundaries in a Lotka-Volterra competition system. Zhou [16] investigated a two-species Lotka-Volterra competition system in one-dimensional advective environments. Cherniha et al. [17] studied a review and new findings regarding the construction and application of exact solutions to the diffusive Lotka-Volterra system.

In 2022, Prabir et al. [18] suggested the subsequent Lotka-Volterra mutualistic symbiotic relationship:

$$\begin{cases} \dfrac{d\mho_1}{dT} = r_1\mho_1\left[1 - \dfrac{\mho_1}{K_1} - \dfrac{\alpha_{12}}{K_1}{\mho_2}^2\right] + \delta_{13}\mho_1\mho_3, \\ \dfrac{d\mho_2}{dT} = r_2\mho_2\left[1 - \dfrac{\mho_2}{K_2} - \dfrac{\alpha_{21}}{K_2}{\mho_1}^2\right] - \dfrac{a(1-p)\mho_2\mho_3}{b + (1-p)\mho_2}, \\ \dfrac{d\mho_3}{dT} = \mho_3\left[-e + \dfrac{ac(1-p)\mho_2}{b + (1-p)\mho_2}\right], \end{cases} \tag{1.1}$$

where $\mho_1$ and $\mho_2$ indicate the density of two logistically increasing competing species, $\mho_3$ represents the density of a predatory species of $\mho_2$, r_1, r_2 are the natural rates of growth for species $\mho_1$ and $\mho_2$, K_1, K_2 are the carrying capacity of species $\mho_1$ and $\mho_2$ respectively, α_{12}, α_{21} represent the fight between two species $\mho_1$ and $\mho_2$, δ_{13} is the commensal coefficient of $\mho_1$ over $\mho_3$ species, a is the frequency at which $\mho_3$ species attack $\mho_2$ species, p is the refuge rate of $\mho_2$ species, c refers to the rate of conservation of $\mho_2$ species, b is the steady value for half saturation for the Holling type II function, and e is the rate of extinction for species $\mho_3$. One might refer to [18] for a more precise explanation of the meaning of system (1.1).

In order to make the suggested mathematical model nondimensional, we have now added a few variables and constants as $t = r_1 T, u_1 = \frac{\mho_1}{K_1}, u_2 = \frac{\mho_2}{K_2}, u_3 = \frac{a\mho_3}{r_1 K_2}, \gamma_{12} = \frac{\alpha_{12} K_2^2}{K_1}, \gamma = \frac{\delta_{13} K_2}{a}, r = \frac{r_2}{r_1}, \gamma_{21} = \frac{\alpha_{21} K_1^2}{K_2}, v_1 = \frac{b}{K_2}, v_2 = \frac{e}{r_1}, v_3 = \frac{ac}{r_1}$.

Then model (1.1) can be simplified to the following form

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(1 - u_1(t) - \gamma_{12} u_2^2(t)) + \gamma u_1(t) u_3(t), \\ \dfrac{du_2(t)}{dt} = r u_2(t)(1 - u_2(t) - \gamma_{21} u_1^2(t)) - \dfrac{(1-p) u_2(t) u_3(t)}{v_1 + (1-p) u_2(t)}, \\ \dfrac{du_3(t)}{dt} = u_3(t)\left(-v_2 + \dfrac{v_3 (1-p) u_2(t)}{v_1 + (1-p) u_2(t)}\right), \end{cases} \tag{1.2}$$

with initial conditions $u_1(0) \geq 0, u_2(0) \geq 0, u_3(0) \geq 0$.

In this instance we want to call your attention to the truth that the evolution of species frequently depends on both historical and present times; hence, delays must be incorporated into biological models. Notice that the development of the density of two logistically increasing competing species $\mho_1$ and $\mho_2$ rely on not only the current time but also the past time. In other words, the development of the density of two logistically increasing competing species $\mho_1$ and $\mho_2$ shall have feedback delays. Based on this idea, we assume that there are two self-feedback delays in model (1.2): One is the self-feedback delay from the first species u_1 to the first species u_1, and the other the self-feedback delay from the first species u_2 to the first species u_2. The points that follow delayed Lotka-Volterra commercially available symbiosis system may then be loosely formulated:

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(1 - u_1(t-\theta) - \gamma_{12} u_2^2(t)) + \gamma u_1(t) u_3(t), \\ \dfrac{du_2(t)}{dt} = r u_2(t)(1 - u_2(t-\theta) - \gamma_{21} u_1^2(t)) - \dfrac{(1-p) u_2(t) u_3(t)}{v_1 + (1-p) u_2(t)}, \\ \dfrac{du_3(t)}{dt} = u_3(t)\left(-v_2 + \dfrac{v_3 (1-p) u_2(t)}{v_1 + (1-p) u_2(t)}\right), \end{cases} \tag{1.3}$$

where $\theta > 0$ is a time delay and all the parameters $\gamma_{12}, \gamma, r, \gamma_{21}, v_1, v_2, v_3$ are positive constants. The initial conditions of system (1.3) are given by $u_i(s) = \phi_i(s) \geq 0, i = 1, 2, 3, s \in [-\theta, 0]$ and $\phi_i(s) \in C([-\theta, 0], R^+), i = 1, 2, 3$.

Mathematically speaking, delays play a key role in determining how different differential systems behave dynamically [19–21]. Delays can lead to a variety of consequences, including changes in stability, the formation of bifurcations, and the start of chaotic behavior [22–24]. During the past decades, many authors deal with this topic, see [25–27]. Specifically, the dynamic phenomena of delays-induced Hopf bifurcation is quite significant [28–30]. From a biological perspective, delay-induced Hopf bifurcation nicely captures the equilibrium between densities of different living populations [31]. We contend that in order to understand the dynamics of interactions across various biological populations, it is critical to investigate delay-induced Hopf bifurcation in a variety of biological models. Motivated by this idea, we will concentrate on the Hopf bifurcation and its bifurcation control mechanisms. Specifically, we will address three critical questions: (1) Assess the distinctive features of the solution to system (1.3), including its nonnegativity, existence and uniqueness, and boundedness. (2) Examine the system's stability problem and the formation of the

Hopf bifurcation phenomena (1.3). (3) Build both controllers for adjusting the range of stability and the generation time of the Hopf bifurcation in system (1.3).

The following is an introduction of the study's primary luminous spot: (i) A completely new delay-independent bifurcation and stability criterion for system (1.3) is established based on the earlier studies. (ii) Using separate controllers, the realm of stability and the time of Hopf bifurcation in system (1.3) may be regulated well. (iii) This study examines how delay affects Hopf bifurcation and the stability of first and second species densities in system (1.3).

The stated arrangement of the article is as follows. The "Feature of solution" section discusses the unique characteristics of system (1.3), such as boundedness, nonnegativeness, existence, and uniqueness. The "Bifurcation analysis" section discusses the bifurcation phenomena and the system's stability (1.3). The "Bifurcation domination using extended hybrid controller I" section centers around the control issue associated with bifurcation phenomena for system (1.3) by formulating a plausible hybrid delayed feedback controller that involves parameter perturbation is accompanied by latency and state feedback. "Bifurcation domination using extended hybrid controller II" section addresses the control challenge of the bifurcation phenomena in system (1.3). Creating a suitable hybrid delayed feedback controller with parameter perturbation for delay and state feedback. The "Software experiments" section provides Matlab software simulation results to assess the veracity of the obtained main results. "Conclusions" concludes this study concisely.

2. Feature of solution

This section will examine the characteristics of the solution to system (1.3), such as its existence and uniqueness, nonnegativeness, and boundedness. To do this, we will employ fixed point theory, inequality methods, and the design of an appropriate function.

Theorem 2.1. *Let* $\Delta = \{(u_1, u_2, u_3) \in R^3 : \max\{|u_1|, |u_2|, |u_3|\} \leq M\}$*, where* $M > 0$ *denotes a constant. For every* $(u_{10}, u_{20}, u_{30}) \in \Delta$*, system (1.3) under the initial value* (u_{10}, u_{20}, u_{30}) *owns a unique solution* $U = (u_1, u_2, u_3) \in \Delta$.

Proof. Set

$$f(U) = (f_1(U), f_2(U), f_3(U)), \tag{2.1}$$

where

$$\begin{cases} f_1(U) = u_1(t)(1 - u_1(t-\theta) - \gamma_{12}u_2^2(t)) + \gamma u_1(t)u_3(t), \\ f_2(U) = ru_2(t)(1 - u_2(t-\theta) - \gamma_{21}u_1^2(t)) - \dfrac{(1-p)u_2(t)u_3(t)}{v_1 + (1-p)u_2(t)}, \\ f_3(U) = u_3(t)(-v_2 + \dfrac{v_3(1-p)u_2(t)}{v_1 + (1-p)u_2(t)}). \end{cases} \tag{2.2}$$

For arbitrary $U, \tilde{U} \in \Delta$, one gains

$$\begin{aligned} &\|f(U) - f(\tilde{U})\| \\ &= |f_1(u) - f_1(\bar{u})| + |f_2(u) - f_2(\bar{u})| + |f_3(u) - f_3(\bar{u})| \\ &= \Big|[u_1 - u_1u_1(t-\theta) - \gamma_{12}u_1u_2^2 + \gamma u_1u_3] \\ &\quad - \left[\bar{u}_1 - \bar{u}_1\bar{u}_1(t-\theta) - \gamma_{12}\bar{u}_1\bar{u}_2^2 + \gamma\bar{u}_1\bar{u}_3\right]\Big| \end{aligned}$$

$$
\begin{aligned}
&+\left\|\left[ru_2 - ru_2u_2(t-\theta) - r\gamma_{21}u_1^2u_2 - \frac{(1-p)u_2u_3}{v_1+(1-p)u_2}\right]\right.\\
&\left.-\left[r\bar{u}_2 - r\bar{u}_2\bar{u}_2(t-\theta) - r\gamma_{21}\bar{u}_1^2\bar{u}_2 - \frac{(1-p)\bar{u}_2\bar{u}_3}{v_1+(1-p)\bar{u}_2}\right]\right\|\\
&+\left\|\left[-v_2u_3 + \frac{v_3(1-p)u_2u_3}{v_1+(1-p)u_2})\right] - \left[-v_2\bar{u}_3 + \frac{v_3(1-p)\bar{u}_2\bar{u}_3}{v_1+(1-p)\bar{u}_2})\right]\right\|\\
\le{}& |u_1-\bar{u}_1| + |u_1u_1(t-\theta) - \bar{u}_1\bar{u}_1(t-\theta)| + \gamma_{12}\left|u_1u_2^2 - \bar{u}_1\bar{u}_2^2\right|\\
&+\gamma|u_1u_3 - \bar{u}_1\bar{u}_3| + r|u_2-\bar{u}_2| + r|u_2u_2(t-\theta) - \bar{u}_2\bar{u}_2(t-\theta)|\\
&+r\gamma_{21}\left|u_1^2u_2 - \bar{u}_1^2\bar{u}_2\right| + (1-p)\left|\frac{(1-p)u_2u_3}{v_1+(1-p)u_2} - \frac{(1-p)\bar{u}_2\bar{u}_3}{v_1+(1-p)\bar{u}_2}\right|\\
&+v_2|u_3-\bar{u}_3| + v_3(1-p)\left|\frac{u_2u_3}{v_1+(1-p)u_2} - \frac{\bar{u}_2\bar{u}_3}{v_1+(1-p)\bar{u}_2}\right|\\
\le{}& |u_1-\bar{u}_1| + |u_1u_1(t-\theta) - \bar{u}_1u_1(t-\theta) + \bar{u}_1u_1(t-\theta) - \bar{u}_1\bar{u}_1(t-\theta)|\\
&+\gamma_{12}\left|u_1u_2^2 - \bar{u}_1u_2^2 + \bar{u}_1u_2^2 - \bar{u}_1\bar{u}_2^2\right| + \gamma|u_1u_3 - \bar{u}_1u_3 + \bar{u}_1u_3 - \bar{u}_1\bar{u}_3|\\
&+r|u_2-\bar{u}_2| + r|u_2u_2(t-\theta) - \bar{u}_2u_2(t-\theta) + \bar{u}_2u_2(t-\theta) - \bar{u}_2\bar{u}_2(t-\theta)|\\
&+r\gamma_{21}\left|u_1^2u_2 - \bar{u}_1^2u_2 + \bar{u}_1^2u_2 - \bar{u}_1^2\bar{u}_2\right|\\
&+(v_3+1)(1-p)\left|\frac{u_2u_3[v_1+(1-p)\bar{u}_2] - \bar{u}_2\bar{u}_3[v_1+(1-p)u_2]}{[v_1+(1-p)u_2][v_1+(1-p)\bar{u}_2]}\right|\\
\le{}& |u_1-\bar{u}_1| + M|u_1-\bar{u}_1| + M|u_1(t-\theta) - \bar{u}_1(t-\theta)| + \gamma_{12}M^2|u_1-\bar{u}_1|\\
&+2\gamma_{12}M^2|u_2-\bar{u}_2| + \gamma M|u_1-\bar{u}_1| + \gamma M|u_3-\bar{u}_3| + r|u_2-\bar{u}_2| + rM|u_2-\bar{u}_2|\\
&+rM|u_2(t-\theta) - \bar{u}_2(t-\theta)| + 2r\gamma_{21}M^2|u_1-\bar{u}_1| + r\gamma_{21}M^2|u_2-\bar{u}_2|\\
&+\frac{v_1(v_3+1)}{(1-p)M}|u_2-\bar{u}_2| + \frac{v_1(v_3+1)}{(1-p)M}|u_3-\bar{u}_3| + (v_3+1)|u_3-\bar{u}_3|\\
={}& (1 + 2M + \gamma_{12}M^2 + \gamma M + 2r\gamma_{21}M^2)|u_1-\bar{u}_1| + (2\gamma_{12}M^2 + r + rM + rM\\
&+r\gamma_{21}M^2 + \frac{v_1(v_3+1)}{(1-p)M})|u_2-\bar{u}_2| + (\gamma M + \frac{v_1(v_3+1)}{(1-p)M} + v_3 + 1)|u_3-\bar{u}_3|\\
={}& \sigma_1|u_1-\bar{u}_1| + \sigma_2|u_2-\bar{u}_2| + \sigma_3|u_3-\bar{u}_3|\\
\le{}& L(|u_1-\bar{u}_1| + |u_2-\bar{u}_2| + |u_3-\bar{u}_3|),
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{cases}
\sigma_1 = 1 + 2M + \gamma_{12}M^2 + \gamma M + 2r\gamma_{21}M^2,\\
\sigma_2 = 2\gamma_{12}M^2 + r + rM + rM + r\gamma_{21}M^2 + \dfrac{v_1(v_3+1)}{(1-p)M},\\
\sigma_3 = \gamma M + \dfrac{v_1(v_3+1)}{(1-p)M} + v_3 + 1.
\end{cases}
\tag{2.4}
$$

Let

$$L = \max\{\sigma_1, \sigma_2, \sigma_3\}. \tag{2.5}$$

Then it follows from (2.3) that

$$\|f(U) - f(\tilde{U})\| \le L\|U - \tilde{U}\|. \tag{2.6}$$

Consequently, $f(U)$ adheres to the Lipschitz condition for U. By applying the fixed point theorem, it's possible to infer with ease that Theorem 2.1 is right.

Theorem 2.2. *Each solution to system (1.3) beginning with R_+^3 maintains a nonnegative value.*

Proof. Considering the initial equation of system (1.3), it's possible to achieve

$$\frac{du_1}{dt} = u_1[1 - u_1(t-\theta) - \gamma_{12}u_2^2] + \gamma u_1 u_3, \tag{2.7}$$

then

$$\frac{du_1}{u_1} = [1 - u_1(t-\theta) - \gamma_{12}u_2^2 + \gamma u_3]dt, \tag{2.8}$$

which leads to

$$\int_0^t \frac{du_1}{u_1} = \int_0^t [1 - u_1(s-\theta) - \gamma_{12}u_2^2 + \gamma u_3]ds, \tag{2.9}$$

then one gets

$$\ln\frac{u_1(t)}{u_1(0)} = \int_0^t [1 - u_1(s-\theta) - \gamma_{12}u_2^2 + \gamma u_3]ds, \tag{2.10}$$

thus

$$u_1(t) = u_1(0)\exp\{\int_0^t [1 - u_1(s-\theta) - \gamma_{12}u_2^2 + \gamma u_3]ds\} > 0. \tag{2.11}$$

In the same way, we know

$$u_2(t) = u_2(0)\exp\left\{\int_0^t [r - ru_2(s-\theta) - r\gamma_{21}u_2^2 - \frac{(1-p)u_3}{v_1 + (1-p)u_2}]ds\right\} > 0. \tag{2.12}$$

$$u_3(t) = u_3(0)\exp\left\{\int_0^t [-v_2 + \frac{v_3(1-p)u_2}{v_1 + (1-p)u_2}]ds\right\} > 0. \tag{2.13}$$

Therefore, the validity of Theorem 2.2 is affirmed.

Theorem 2.3. *Each solution to system (1.3), starting with R_+^3, is invariably restricted.*

Proof. From the second equation of system (1.3), we have

$$\frac{du_2}{dt} = ru_2(1 - u_2(t-\theta) - \gamma_{21}u_1^2) - \frac{(1-p)u_2u_3}{v_1 + (1-p)u_2}. \tag{2.14}$$

Then

$$\frac{du_2}{dt} \le ru_2. \tag{2.15}$$

Integrating from $(t-\theta)$ to t on both sides of the Eq (2.15) leads to

$$\int_{t-\theta}^t \frac{du_2}{u_2} \le \int_{t-\theta}^t rdt. \tag{2.16}$$

We get

$$u_2(t) \le u_2(t-\theta)e^{r\theta}, \tag{2.17}$$

then

$$u_2(t-\theta) \geq u_2(t)e^{-r\theta}, \tag{2.18}$$

from (2.14), we can get

$$\frac{du_2}{dt} \leq ru_2(t)(1-u_2(t-\theta)) \leq ru_2(t)(1-u_2(t)e^{-r\theta}). \tag{2.19}$$

So

$$\frac{du_2}{dt} \leq u_2(t)(r - re^{-r\theta}u_2(t)). \tag{2.20}$$

Hence

$$u_2(t) \leq e^{r\theta}. \tag{2.21}$$

Now, we define the following function

$$W_1 = u_2 + \frac{1}{v_3}u_3. \tag{2.22}$$

Then

$$\begin{aligned}
\frac{dW_1}{dt} &= \frac{du_2}{dt} + \frac{1}{v_3}\frac{du_3}{dt} \\
&= \left[ru_2(1-u_2(t-\theta)-\gamma_{21}u_1^2) - \frac{(1-p)u_2u_3}{v_1+(1-p)u_2}\right] + \left[\frac{u_3}{v_3}(-v_2 + \frac{v_3(1-p)u_2}{v_1+(1-p)u_2})\right] \\
&= ru_2(1-u_2(t-\theta)-\gamma_{21}u_1^2) - \frac{v_2u_3}{v_3} \\
&\leq ru_2[1-u_2(t-\theta)] - \frac{v_2u_3}{v_3} \\
&= -v_2\left(u_2 + \frac{1}{v_3}u_3\right) + v_2u_2 + ru_2[1-u_2(t-\theta)] \\
&\leq -v_2\left(u_2 + \frac{1}{v_3}u_3\right) + (v_2+r)u_2 \\
&\leq -v_2\left(u_2 + \frac{1}{v_3}u_3\right) + (v_2+r)e^{r\theta} \\
&= -v_2W_1(t) + (v_2+r)e^{r\theta}.
\end{aligned} \tag{2.23}$$

Thus

$$W_1(t) \leq \frac{(v_2+r)e^{r\theta}}{v_2}. \tag{2.24}$$

Next we define another function W_2 as follows

$$W_2 = u_1 + u_2 + \frac{1}{v_3}u_3. \tag{2.25}$$

Then, we get

$$\frac{dW_2}{dt} = \frac{du_1}{dt} + \frac{du_2}{dt} + \frac{1}{v_3}\frac{du_3}{dt}$$

$$
\begin{aligned}
&= \left[u_1(1-u_1(t-\theta)-\gamma_{12}u_2^2)+\gamma u_1u_3\right] \\
&\quad+\left[ru_2(1-u_2(t-\theta)-\gamma_{21}u_1^2)-\frac{(1-p)u_2u_3}{v_1+(1-p)u_2}\right] \\
&\quad+\left[\frac{u_3}{v_3}(-v_2+\frac{v_3(1-p)u_2}{v_1+(1-p)u_2})\right] \\
&= \left[u_1(1-u_1(t-\theta)-\gamma_{12}u_2^2)+\gamma u_1u_3\right] \\
&\quad+\left[u_1(1-u_1(t-\theta)-\gamma_{12}u_2^2)+\gamma u_1u_3\right] \\
&\le -v_2\left(u_1+u_2+\frac{1}{v_3}u_3\right)+v_2u_1+v_2u_2+u_1[1-u_1(t-\theta)] \\
&\quad+\gamma u_1u_3+ru_2[1-u_2(t-\theta)-\gamma_{21}u_1^2] \\
&= -v_2\left(u_1+u_2+\frac{1}{v_3}u_3\right)-\gamma_{21}ru_2u_1^2+(v_2+1+\gamma u_3)u_1+(v_2+r)u_2 \\
&\le -v_2\left(u_1+u_2+\frac{1}{v_3}u_3\right)-\gamma_{21}ru_2u_1^2+(v_2+1+\gamma u_3)u_1+(v_2+r)u_2 \\
&\le -v_2\left(u_1+u_2+\frac{1}{v_3}u_3\right)-\gamma_{21}rM_1u_1^2+(v_2+1+\gamma M_2)u_1+(v_2+r)M_1 \\
&\le -v_2\left(u_1+u_2+\frac{1}{v_3}u_3\right)+Q,
\end{aligned} \tag{2.26}
$$

where

$$
\begin{cases}
M_1=\mathrm{e}^{r\theta}, \\
M_2=\dfrac{(v_2+r)e^{r\theta}}{v_2}, \\
Q=\dfrac{-4\gamma_{21}rM_1{}^2(v_2+r)-(v_2+1+\gamma M_2)^2}{-4\gamma_{21}rM_1}.
\end{cases}
$$

Stemming from Eq (2.26), it is deduced that

$$
W_2(t)\le\frac{Q}{v_2}. \tag{2.27}
$$

By Eq (2.28), we obtain

$$
W_2(t)\to\frac{Q}{v_2},\ \text{when}\quad t\to\infty. \tag{2.28}
$$

Consequently, every solution to the system (1.3) is consistently limited.

3. Bifurcation analysis

This part delves into examining the bifurcation and stability aspects of model (1.3). Initially, our assumption is that $E(u_{1\star},u_{2\star},u_{3\star})$ represents the balance point in model (1.3); subsequently,

$u_{1\star}, u_{2\star}, u_{3\star}$ adhere to this stipulation:

$$\begin{cases} u_{1\star}[1 - u_{1\star} - \gamma_{12}u_{2\star}^2] + \gamma u_{1\star}u_{3\star} = 0, \\ ru_{2\star}[1 - u_{2\star} - \gamma_{21}u_{1\star}^2] - \dfrac{(1-p)u_{2\star}u_{3\star}}{v_1 + (1-p)u_{2\star}} = 0, \\ u_{3\star}\left[-v_2 + \dfrac{v_3(1-p)u_{2\star}}{v_1 + (1-p)u_{2\star}}\right] = 0. \end{cases} \tag{3.1}$$

Let

$$\begin{cases} \bar{u}_1(t) = u_1(t) - u_{1\star}, \\ \bar{u}_2(t) = u_2(t) - u_{2\star}, \\ \bar{u}_3(t) = u_3(t) - u_{3\star}. \end{cases} \tag{3.2}$$

Substituting system (3.2) into system (1.3), we achieve the linear configuration of model (1.3) at $E(u_{1\star}, u_{2\star}, u_{3\star})$ (denote $\bar{u}_1$ as u_1, $\bar{u}_2$ as u_2,$\bar{u}_3$ as u_3)

$$\begin{cases} \dfrac{du_1}{dt} = b_1u_1 + b_2u_2 + b_3u_3 + b_4u_1(t-\theta), \\ \dfrac{du_2}{dt} = b_5u_1 + b_6u_2 + b_7u_3 + b_8u_2(t-\theta), \\ \dfrac{du_3}{dt} = b_9u_2 + b_{10}u_3, \end{cases} \tag{3.3}$$

where

$$\begin{cases} b_1 = 1 - u_{1\star} - \gamma_{12}u_{2\star}^2 - \gamma u_{3\star}, \\ b_2 = -2\gamma_{12}u_{1\star}u_{2\star}, \\ b_3 = -\gamma u_{1\star}, \\ b_4 = -u_{1\star}, \\ b_5 = -2r\gamma_{21}u_{1\star}u_{2\star}, \\ b_6 = r - ru_{2\star} - r\gamma_{21}u_{1\star}^2 - \dfrac{(1-p)u_{3\star}}{v_1 + (1-p)u_{2\star}} + \dfrac{(1-p)^2u_{2\star}u_{3\star}}{[v_1 + (1-p)u_{2\star}]^2}, \\ b_7 = -\dfrac{(1-p)u_{2\star}}{v_1 + (1-p)u_{2\star}}, \\ b_8 = -ru_{2\star}, \\ b_9 = \dfrac{v_3(1-p)u_{3\star}}{v_1 + (1-p)u_{2\star}} - \dfrac{v_3(1-p)^2u_{2\star}u_{3\star}}{[v_1 + (1-p)u_{2\star}]^2}, \\ b_{10} = \dfrac{v_3(1-p)u_{2\star}}{v_1 + (1-p)u_{2\star}} - v_2. \end{cases} \tag{3.4}$$

The characteristic Eq (3.3) of the system owns the following expressions:

$$\det\begin{bmatrix} \lambda - b_1 - b_4e^{-\lambda\theta} & -b_2 & -b_3 \\ -b_5 & \lambda - b_6 - b_8e^{-\lambda\theta} & -b_7 \\ 0 & -b_9 & \lambda - b_{10} \end{bmatrix} = 0, \tag{3.5}$$

which leads to

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 + (c_4\lambda^2 + c_5\lambda + c_6)e^{-\lambda\theta} + (c_7\lambda + c_8)e^{-2\lambda\theta} = 0, \tag{3.6}$$

where

$$\begin{cases} c_1 = -b_1 - b_6 - b_{10}, \\ c_2 = b_1 b_6 + b_1 b_{10} - b_2 b_5 + b_6 b_{10} - b_7 b_9, \\ c_3 = b_1 b_7 b_9 - b_1 b_6 b_{10} + b_2 b_5 b_{10} - b_3 b_5 b_9, \\ c_4 = -b_4 - b_8, \\ c_5 = b_1 b_8 + b_4 b_6 + b_4 b_{10} + b_8 b_{10}, \\ c_6 = b_4 b_7 b_9 - b_1 b_8 b_{10} - b_4 b_6 b_{10}, \\ c_7 = b_4 b_8, \\ c_8 = -b_4 b_8 b_{10}. \end{cases} \tag{3.7}$$

If $\theta = 0$, then Eq (3.6) becomes

$$\lambda^3 + (c_1 + c_4)\lambda^2 + (c_2 + c_5 + c_7)\lambda + (c_3 + c_6 + c_8) = 0. \tag{3.8}$$

If

$$(\mathcal{A}_1) \begin{cases} c_1 + c_4 > 0, \\ c_3 + c_6 + c_8 > 0, \\ (c_1 + c_4)(c_2 + c_5 + c_7) > c_3 + c_6 + c_8, \end{cases}$$

is fulfilled, then the three roots $\lambda_1, \lambda_2, \lambda_3$ of Eq (3.8) have negative real parts. Thus, the equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ of system (1.3) with $\theta = 0$ is locally asymptotically stable.

Assume that $\lambda = i\varepsilon$ is the root of Eq (3.6), then Eq (3.6) becomes

$$-c_4\varepsilon^2 + c_5 i\varepsilon + c_6 + (-\varepsilon^3 i - c_1\varepsilon^2 - c_2 i\varepsilon + c_3)e^{i\varepsilon\theta} + (c_7 i\varepsilon + c_8)e^{-i\varepsilon\theta} = 0. \tag{3.9}$$

It follows from (3.9) that

$$\begin{cases} F_1 \sin(\varepsilon\theta) + F_2 \cos(\varepsilon\theta) = F_3, \\ F_4 \sin(\varepsilon\theta) + F_5 \cos(\varepsilon\theta) = F_6, \end{cases} \tag{3.10}$$

where

$$\begin{cases} F_1 = \varepsilon^3 + g_1\varepsilon, \\ F_2 = -c_1\varepsilon^2 + g_2, \\ F_3 = c_4\varepsilon^2 - c_6, \\ F_4 = -c_1\varepsilon^2 + g_3, \\ F_5 = -\varepsilon^3 + g_4\varepsilon, \\ F_6 = -c_5\varepsilon, \end{cases} \tag{3.11}$$

and

$$\begin{cases} g_1 = c_7 - c_2, \\ g_2 = c_3 + c_8, \\ g_3 = c_3 - c_8, \\ g_4 = c_2 + c_7. \end{cases} \tag{3.12}$$

It follows from (3.10) that

$$\begin{cases} \sin(\varepsilon\theta) = \dfrac{F_2F_6 - F_3F_5}{F_2F_4 - F_1F_5}, \\ \cos(\varepsilon\theta) = \dfrac{F_3F_4 - F_1F_6}{F_2F_4 - F_1F_5}. \end{cases} \tag{3.13}$$

Because of $\sin^2(\varepsilon\theta) + \cos^2(\varepsilon\theta) = 1$, we can get

$$\left[\frac{F_2F_6 - F_3F_5}{F_2F_4 - F_1F_5}\right]^2 + \left[\frac{F_3F_4 - F_1F_6}{F_2F_4 - F_1F_5}\right]^2 = 1. \tag{3.14}$$

So,

$$\begin{aligned} &F_3^2F_5^2 + F_2^2F_6^2 + F_1^2F_6^2 + F_3^2F_4^2 - F_1^2F_5^2 - F_2^2F_4^2 \\ &-2F_2F_3F_5F_6 - 2F_1F_3F_4F_6 + 2F_1F_2F_4F_5 = 0. \end{aligned} \tag{3.15}$$

Using (3.11) and (3.15), we know

$$-\varepsilon^{12} + D_1\varepsilon^{10} + D_2\varepsilon^8 + D_3\varepsilon^6 + D_4\varepsilon^4 + D_5\varepsilon^2 + D_6 = 0. \tag{3.16}$$

Therefore, the results can be obtained as follows:

$$\varepsilon^{12} - D_1\varepsilon^{10} - D_2\varepsilon^8 - D_3\varepsilon^6 - D_4\varepsilon^4 - D_5\varepsilon^2 - D_6 = 0, \tag{3.17}$$

where

$$\begin{cases} D_1 = c_4^2 - 2g_4 - 2g_1 - 2c_1^2, \\ D_2 = -2c_4^2g_4 - 2c_4c_6 + c_5^2 + c_1^2c_4^2 - g_4^2 + 4g_1g_4 \\ \quad - g_1^2 - c_1^4 + 2c_1^2g_4 + 2c_1g_3 + 2c_1g_2 - 2c_1^2g_1, \\ D_3 = c_4^2g_4^2 + 4c_4c_6g_4 + c_6^2 + c_1^2c_5^2 + 2c_5^2g_1 - 2c_1c_4^2g_3 - 2c_1^2c_4c_6 - 2g_1g_4^2 \\ \quad + 2g_1^2g_4 + 2c_1^3g_3 + 2c_1^3g_2 - 2c_1c_4c_5g_4 - 2c_4c_5g_2 - 2c_1c_4c_5g_1 + 2c_4c_5g_3 \\ \quad - 2c_1g_3g_4 + 2(g_2 - c_1g_1)(-c_1g_4 - g_3) + 2c_1g_1g_2, \\ D_4 = -2c_4c_6g_4^2 - 2c_6^2g_4 - 2c_1c_5^2g_2 + c_5^2g_1^2 + c_4^2g_3^2 + 4c_1c_4c_6g_3 + c_1^2c_6^2 - g_1^2g_4^2 \\ \quad - c_1^2g_3^2 - 4c_1^2g_2g_3 - c_1^2g_2^2 + 2c_5g_4(c_1c_6 + c_4g_2) + 2c_5c_6g_2 + 2c_1c_5c_6g_1 \\ \quad + 2c_5g_3(c_4g_1 - c_6) + 2g_3g_4(g_2 - c_1g_1) + 2g_1g_2(-c_1g_4 - g_3), \\ D_5 = c_6^2g_4^2 + c_5^2g_2^2 - 2c_4c_6g_3^2 - 2c_1c_6^2g_3 + 2c_1g_2g_3^2 + 2c_1g_2^2g_3 - 2c_5c_6g_2g_4 \\ \quad - 2c_5c_6g_1g_3 + 2g_1g_2g_3g_4, \\ D_6 = c_6^2g_3^2 - g_2^2g_3^2. \end{cases} \tag{3.18}$$

Let

$$\Re_1(\varepsilon) = \varepsilon^{12} - D_1\varepsilon^{10} - D_2\varepsilon^8 - D_3\varepsilon^6 - D_4\varepsilon^4 - D_5\varepsilon^2 - D_6. \tag{3.19}$$

Assume that

$$(\mathcal{A}_2)\ \ |c_6| > |g_2|.$$

By virtue of $(\mathcal{A}_2)$, we know $\Re_1(0) = -(c_6^2g_3^2 - g_2^2g_2^2) < 0$, and since $\lim_{\varepsilon\to\infty} \Re_1(\varepsilon) > 0$, then we will know Eq (3.17) has at least one positive real root. Therefore Eq (3.6) has at least one pair of pure roots. Without loss of generality, we can assume that Eq (3.17) has twelve positive real roots (say, ε_j, $j = 1, 2, 3, \cdots, 12$). Relying on (3.12), we know

$$\theta_j^{(n)} = \frac{1}{\varepsilon_j}\left[\arccos\left(\frac{F_3^\star F_4^\star - F_1^\star F_6^\star}{F_2^\star F_4^\star - F_1^\star F_5^\star}\right) + 2n\pi\right], \tag{3.20}$$

where $j = 1, 2, \cdots, 12; n = 0, 1, 2, \cdots;$

$$\begin{cases} F_1^\star = \varepsilon_j^3 + g_1\varepsilon_j, \\ F_2^\star = -c_1\varepsilon_j^2 + g_2, \\ F_3^\star = c_4\varepsilon_j^2 - c_6, \\ F_4^\star = -c_1\varepsilon_j^2 + g_3, \\ F_5^\star = -\varepsilon_j^3 + g_4\varepsilon_j, \\ F_6^\star = -c_5\varepsilon_j. \end{cases} \tag{3.21}$$

Assume $\theta_0 = \min_{\{j=1,2,\cdots,12;n=0,1,2,\cdots\}}\{\theta_j^{(n)}\}$ and suppose that when $\theta = \theta_0$, Eq (3.6) has a pair of imaginary roots $\pm i\varepsilon_0$. Next we present the following assumption:

$$(\mathcal{A}_3)\ \ G_{1R}G_{2R} + G_{1I}G_{2I} > 0,$$

where

$$\begin{cases} G_{1R} = -2c_1\varepsilon_0\sin(\varepsilon_0\theta_0) - (3\varepsilon_0^2 - c_2 - c_7)\cos(\varepsilon_0\theta_0), \\ G_{1I} = (-3\varepsilon_0^2 + c_2 - c_7)\sin(\varepsilon_0\theta_0) + 2c_1\varepsilon_0\cos(\varepsilon_0\theta_0), \\ G_{2R} = (-c_1\varepsilon_0^3 + c_3\varepsilon_0 + c_8\varepsilon_0)\sin(\varepsilon_0\theta_0) + (-\varepsilon_0^4 + c_2\varepsilon_0^2 - c_7\varepsilon_0^2)\cos(\varepsilon_0\theta_0), \\ G_{2I} = (-\varepsilon_0^4 + c_2\varepsilon_0^2 + c_7\varepsilon_0^2)\sin(\varepsilon_0\theta_0) + (c_1\varepsilon_0^3 - c_3\varepsilon_0 + c_8\varepsilon_0)\cos(\varepsilon_0\theta_0). \end{cases} \tag{3.22}$$

Lemma 3.1. *Suppose that $\lambda(\theta) = \epsilon_1(\theta) + i\epsilon_2(\theta)$ is the root of Eq (3.6) at $\theta = \theta_0$ such that $\epsilon_1(\theta_0) = 0$, $\epsilon_2(\theta_0) = \varepsilon_0$, then $Re\left(\frac{d\lambda}{d\theta}\right)\Big|_{\theta=\theta_0,\varepsilon=\varepsilon_0} > 0$.*

Proof. By Eq (3.6), we can get

$$\begin{aligned} &(3\lambda^2 + 2c_1\lambda + c_2)e^{\lambda\theta}\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} + (\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3)e^{\lambda\theta}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta + \lambda\right) \\ &+c_7e^{-\lambda\theta}\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} - (c_7\lambda + c_8)e^{-\lambda\theta}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta + \lambda\right) + (2c_4\lambda + c_5)\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} = 0. \end{aligned} \tag{3.23}$$

It means that

$$\left(\frac{d\lambda}{d\theta}\right)^{-1} = \frac{G_1(\lambda)}{G_2(\lambda)} - \frac{\theta}{\lambda}, \tag{3.24}$$

where

$$\begin{cases} G_1(\lambda) = (3\lambda^2 + 2c_1\lambda + c_2)e^{\lambda\theta} + c_7e^{-\lambda\theta}, \\ G_2(\lambda) = (c_7\lambda + c_8)\lambda e^{-\lambda\theta} - (\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3)\lambda e^{\lambda\theta}. \end{cases} \tag{3.25}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_0,\varepsilon=\varepsilon_0} = \mathrm{Re}\left[\frac{G_1(\lambda)}{G_2(\lambda)}\right]_{\theta=\theta_0,\varepsilon=\varepsilon_0} = \frac{G_{1R}G_{2R} + G_{1I}G_{2I}}{G_{2R}^2 + G_{2I}^2}. \tag{3.26}$$

By the assumption $(\mathcal{A}_3)$, we get

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_0,\varepsilon=\varepsilon_0} > 0. \tag{3.27}$$

This concludes the proof. According to the preceding reasoning, the following result is simply deduced.

Theorem 3.1. *Assume that $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ hold, then the equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ of model (1.3) holds a locally asymptotically stable state if $\theta \in [0, \theta_0)$ and model (1.3) generates Hopf bifurcations around the equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ when $\theta = \theta_0$.*

4. Bifurcation domination using extended hybrid controller I

Motivated by the works of [30,32–34], we design the hybrid controller in Sections 4 and 5. In fact, we add a perturbation by adjusting the rate of change of population and then we can check the controllability via theoretical analysis and computer simulations. In this part, we will look at the Hopf bifurcation control issue in system (1.3) using a suitable hybrid controller that combines state feedback and parameter perturbation with delay. Using the ideas from [30,32], we achieve the following controlled 3D Lotka-Volterra commensal symbiosis system:

$$\begin{cases} \dfrac{du_1(t)}{dt} = \delta_1[u_1(t)(1-u_1(t-\theta)-\gamma_{12}u_2^2(t))+\gamma u_1(t)u_3(t)]+\delta_2[u_1(t)-u_1(t-\theta)], \\ \dfrac{du_2}{dt} = ru_2(t)[1-u_2(t-\theta)-\gamma_{21}u_1^2(t)]-\dfrac{(1-p)u_2(t)u_3(t)}{v_1+(1-p)u_2(t)}, \\ \dfrac{du_3}{dt} = u_3(t)\left[-v_2+\dfrac{v_3(1-p)u_2(t)}{v_1+(1-p)u_2(t)}\right], \end{cases} \tag{4.1}$$

where δ_1, δ_2 stands for feedback gain parameters. Systems (4.1) and (1.3) own the same equilibrium points $E(u_{1\star}, u_{2\star}, u_{3\star})$. Let

$$\begin{cases} u_1(t) = \bar{u}_1(t) + u_{1\star}, \\ u_2(t) = \bar{u}_2(t) + u_{2\star}, \\ u_3(t) = \bar{u}_3(t) + u_{3\star}, \end{cases} \tag{4.2}$$

denote $\bar{u}_1$ as u_1, $\bar{u}_2$ as u_2,$\bar{u}_3$ as u_3, and the linear system of system (4.1) around $E(u_{1\star}, u_{2\star}, u_{3\star})$ takes the following expression:

$$\begin{cases} \dfrac{du_1}{dt} = d_1u_1 + d_2u_2 + d_3u_3 + d_4u_1(t-\theta), \\ \dfrac{d\bar{u}_2}{dt} = d_5u_1 + d_6u_2 + d_7u_3 + d_8u_2(t-\theta), \\ \dfrac{du_3}{dt} = d_9u_2 + d_{10}u_3, \end{cases} \tag{4.3}$$

where

$$\begin{cases} d_1 = (1-u_{1\star}-\gamma_{12}u_{2\star}^2-\gamma u_{3\star})\delta_1+\delta_2, \\ d_2 = -2\gamma_{12}u_{1\star}u_{2\star}\delta_1, \\ d_3 = -\gamma u_{1\star}\delta_1, \\ d_4 = -\delta_1 u_{1\star}-\delta_2, \\ d_5 = -2r\gamma_{21}u_{1\star}u_{2\star}, \\ d_6 = r - ru_{2\star} - r\gamma_{21}u_{1\star}^2 - \dfrac{(1-p)u_{3\star}}{v_1+(1-p)u_{2\star}} + \dfrac{(1-p)^2u_{2\star}u_{3\star}}{[v_1+(1-p)u_{2\star}]^2}, \\ d_7 = -\dfrac{(1-p)u_{2\star}}{v_1+(1-p)u_{2\star}}, \\ d_8 = -ru_{2\star}, \\ d_9 = \dfrac{v_3(1-p)u_{3\star}}{v_1+(1-p)u_{2\star}} - \dfrac{v_3(1-p)^2u_{2\star}u_{3\star}}{[v_1+(1-p)u_{2\star}]^2}, \\ d_{10} = \dfrac{v_3(1-p)u_{2\star}}{v_1+(1-p)u_{2\star}} - v_2. \end{cases} \tag{4.4}$$

The following expression is owned by the system (4.4)'s characteristic equation:

$$\det\begin{bmatrix} \lambda - d_1 - d_4 e^{-\lambda\theta} & -d_2 & -d_3 \\ -d_5 & \lambda - d_6 - d_8 e^{-\lambda\theta} & -d_7 \\ 0 & -d_9 & \lambda - d_{10} \end{bmatrix} = 0, \tag{4.5}$$

which leads to

$$\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 + (h_4\lambda^2 + h_5\lambda + h_6)e^{-\lambda\theta} + (h_7\lambda + h_8)e^{-2\lambda\theta} = 0, \tag{4.6}$$

where

$$\begin{cases} h_1 = -d_1 - d_6 - d_{10}, \\ h_2 = d_1 d_6 + d_1 d_{10} - d_2 d_5 + d_6 d_{10} - d_7 d_9, \\ h_3 = d_1 d_7 d_9 - d_1 d_6 d_{10} + d_2 d_5 d_{10} - d_3 d_5 d_9, \\ h_4 = -d_4 - d_8, \\ h_5 = d_1 d_8 + d_4 d_6 + d_4 d_{10} + d_8 d_{10}, \\ h_6 = d_4 d_7 d_9 - d_1 d_8 d_{10} - d_4 d_6 d_{10}, \\ h_7 = d_{48}, \\ h_8 = -d_4 d_8 d_{10}. \end{cases} \tag{4.7}$$

If $\theta = 0$, then Eq (4.6) reads as:

$$\lambda^3 + (h_1 + h_4)\lambda^2 + (h_2 + h_5 + h_7)\lambda + (h_3 + h_6 + h_8) = 0. \tag{4.8}$$

If

$$(\mathcal{A}_4)\begin{cases} \Delta_1 = h_1 + h_4 > 0, \\ \Delta_2 = \begin{vmatrix} h_1 + h_4 & 1 \\ h_3 + h_6 + h_8 & h_2 + h_5 + h_7 \end{vmatrix} > 0, \\ \Delta_3 = \begin{vmatrix} h_1 + h_4 & 1 & 0 \\ h_3 + h_6 + h_8 & h_2 + h_5 + h_7 & h_1 + h_4 \\ 0 & 0 & h_3 + h_6 + h_8 \end{vmatrix} > 0, \end{cases}$$

holds, then the three roots $\lambda_1, \lambda_2, \lambda_3$ of Eq (4.8) have negative real portions. Thus, the equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ of model (4.1) remains locally asymptotically stable at $\theta = 0$. From (4.6), we can get

$$h_4\lambda^2 + h_5\lambda + h_6 + (\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3)e^{\lambda\theta} + (h_7\lambda + h_8)e^{-\lambda\theta} = 0. \tag{4.9}$$

Suppose that $\lambda = i\zeta$ is the root of Eq (4.9). Then, Eq (4.9) takes

$$h_4(i\zeta)^2 + h_5(i\zeta) + h_6 + [(i\zeta)^3 + h_1(i\zeta)^2 + h_2(i\zeta) + h_3]e^{i\zeta\theta} + [h_7(i\zeta) + h_8]e^{-i\zeta\theta} = 0, \tag{4.10}$$

which results in

$$\begin{aligned} &h_4(i\zeta)^2 + h_5(i\zeta) + h_6 + [(i\zeta)^3 + h_1(i\zeta)^2 + h_2(i\zeta) + h_3][\cos(\zeta\theta) + i\sin(\zeta\theta)] \\ &+[h_7(i\zeta) + h_8][\cos(\zeta\theta) - i\sin(\zeta\theta)] = 0. \end{aligned} \tag{4.11}$$

It follows from (4.11) that

$$\begin{cases} H_1\sin(\zeta\theta) + H_2\cos(\zeta\theta) = H_3, \\ H_4\sin(\zeta\theta) + H_5\cos(\zeta\theta) = H_6, \end{cases} \tag{4.12}$$

where

$$\begin{cases} H_1 = \zeta^3 + j_1\zeta, \\ H_2 = -h_1\zeta^2 + j_2, \\ H_3 = h_4\zeta^2 - h_6, \\ H_4 = -h_1\zeta^2 + j_3, \\ H_5 = -\zeta^3 + j_4\zeta, \\ H_6 = -h_5\zeta, \end{cases} \tag{4.13}$$

and

$$\begin{cases} j_1 = h_7 - h_2, \\ j_2 = h_3 + h_8, \\ j_3 = h_3 - h_8, \\ j_4 = h_2 + h_7. \end{cases} \tag{4.14}$$

It follows from Cramers rule that

$$\begin{cases} \sin(\zeta\theta) = \dfrac{H_2H_6 - H_3H_5}{H_2H_4 - H_1H_5}, \\ \cos(\zeta\theta) = \dfrac{H_3H_4 - H_1H_6}{H_2H_4 - H_1H_5}. \end{cases} \tag{4.15}$$

In view of $\cos^2(\zeta\theta) + \sin^2(\zeta\theta) = 1$, we get

$$\left[\frac{H_2H_6 - H_3H_5}{H_2H_4 - H_1H_5}\right]^2 + \left[\frac{H_3H_4 - H_1H_6}{H_2H_4 - H_1H_5}\right]^2 = 1. \tag{4.16}$$

Then,

$$\begin{aligned} &H_3^2H_5^2 +^2_H H_6^2 + H_1^2H_6^2 + H_3^2H_4^2 - H_1^2H_5^2 - H_2^2H_4^2 \\ &-2H_2H_3H_5H_6 - 2H_1H_3H_4H_6 + 2H_1H_2H_4H_5 = 0. \end{aligned} \tag{4.17}$$

Using (4.13) and (4.17), we know

$$\zeta^{12} - I_1\zeta^{10} - I_2\zeta^8 - I_3\zeta^6 - I_4\zeta^4 - I_5\zeta^2 - I_6 = 0, \tag{4.18}$$

where

$$\begin{cases} I_1 = h_4^2 - 2j_4 - 2j_1 - 2h_1^2, \\ I_2 = -2h_4^2j_4 - 2h_4h_6 + h_5^2 + h_1^2h_4^2 - j_4^2 + 4j_1j_4 \\ \qquad - j_1^2 - h_1^4 + 2h_1^2j_4 + 2h_1j_3 + 2h_1j_2 - 2h_1^2j_1, \\ I_3 = h_4^2j_4^2 + 4h_4h_6j_4 + h_6^2 + h_1^2h_5^2 + 2h_5^2j_1 - 2h_1h_4^2j_3 - 2h_1^2h_4h_6 - 2j_1j_4^2 \\ \qquad + 2j_1^2j_4 + 2h_1^3j_3 + 2h_1^3j_2 - 2h_1h_4h_5j_4 - 2h_4h_5j_2 - 2h_1h_4h_5j_1 + 2h_4h_5j_3 \\ \qquad - 2h_1j_3j_4 + 2(j_2 - h_1j_1)(-h_1j_4 - j_3) + 2h_1j_1j_2, \\ I_4 = -2h_4h_6j_4^2 - 2h_6^2j_4 - 2h_1h_5^2j_2 + h_5^2j_1^2 + h_4^2j_3^2 + 4h_1h_4h_6j_3 + h_1^2h_6^2 - j_1^2j_4^2 \\ \qquad - h_1^2j_3^2 - 4h_1^2j_2j_3 - h_1^2j_2^2 + 2h_5j_4(h_1h_6 + h_4j_2) + 2h_5h_6j_2 + 2h_1h_5h_6j_1 \\ \qquad + 2h_5j_3(h_4j_1 - h_6) + 2j_3j_4(j_2 - h_1j_1) + 2j_1j_2(-h_1j_4 - j_3), \\ I_5 = h_6^2j_4^2 + h_5^2j_2^2 - 2h_4h_6j_3^2 - 2h_1h_6^2j_3 + 2h_1j_2j_3^2 + 2h_1j_2^2j_3 - 2h_5h_6j_2j_4 \\ \qquad - 2h_5h_6j_1j_3 + 2j_1j_2j_3j_4, \\ I_6 = h_6^2j_3^2 - j_2^2j_3^2. \end{cases} \tag{4.19}$$

Let

$$\Re_2(\zeta) = \zeta^{12} - I_1\zeta^{10} - I_2\zeta^8 - I_3\zeta^6 - I_4\zeta^4 - I_5\zeta^2 - I_6. \tag{4.20}$$

Suppose that

$$(\mathcal{A}_5)\ \ |h_6| > |j_2|$$

holds, noticing that $\lim_{\zeta\to+\infty}\Re_2(\zeta) = +\infty > 0$, then we find that Eq (4.18) owns at least one positive real root. Thus, Eq (4.6) owns at least one pair of pure roots. Without loss of generality, here we assume that Eq (4.18) admits twelve positive real roots (say, $\zeta_\iota, \iota = 1, 2, 3, \cdots, 12$). According to (4.15), one gets

$$\theta_l^{(k)} = \frac{1}{\zeta_l}\left[\arccos\left(\frac{H_3^\star(\zeta_l)H_4^\star(\zeta_l) - H_1^\star(\zeta_l)H_6^\star(\zeta_l)}{H_2^\star(\zeta_l)H_4^\star(\zeta_l) - H_1^\star(\zeta_l)H_5^\star(\zeta_l)}\right) + 2k\pi\right], \tag{4.21}$$

where $\iota = 1, 2, 3, \cdots 12; k = 0, 1, 2\cdots;$

$$\begin{cases} H_1^\star(\zeta_l) = \zeta_l^3 + j_1\zeta_l, \\ H_2^\star(\zeta_l) = -h_1\zeta_l^2 + j_2, \\ H_3^\star(\zeta_l) = h_4\zeta_l^2 - h_6, \\ H_4^\star(\zeta_l) = -h_1\zeta_l^2 + j_3, \\ H_5^\star(\zeta_l) = -\zeta_l^3 + j_4\zeta_l, \\ H_6^\star(\zeta_l) = -h_5\zeta_l. \end{cases} \tag{4.22}$$

Denote $\theta_\star = \min_{\{\iota=1,2,3,\cdots,12;k=0,1,2,\cdots\}}\{\theta_\iota^{(k)}\}$ and suppose that when $\theta = \theta_\star$, (4.6) owns a pair of imaginary roots $\pm i\zeta_0$.

Now, the following condition is presented:

$$(\mathcal{A}_6)\ \ Q_{1R}Q_{2R} + Q_{1I}Q_{2I} > 0,$$

where

$$\begin{cases} Q_{1R} = -2h_1\zeta_0\sin(\zeta_0\theta_\star) - (3\zeta_0^2 - h_2 - h_7)\cos(\zeta_0\theta_\star), \\ Q_{1I} = (-3\zeta_0^2 + h_2 - h_7)\sin(\zeta_1\theta_\star) + 2h_1\zeta_0\cos(\zeta_0\theta_\star), \\ Q_{2R} = (-h_1\zeta_0^3 + h_3\zeta_0 + h_8\zeta_0)\sin(\zeta_0\theta_\star) + (-\zeta_0^4 + h_2\zeta_0^2 - h_7\zeta_0^2)\cos(\zeta_0\theta_\star), \\ Q_{2I} = (-\zeta_0^4 + h_2\zeta_0^2 + h_7\zeta_0^2)\sin(\zeta_0\theta_\star) + (h_1\zeta_0^3 - h_3\zeta_0 + h_8\zeta_0)\cos(\zeta_0\theta_\star). \end{cases} \tag{4.23}$$

Lemma 4.1. *Let $\lambda(\theta) = \eta_1(\theta) + i\eta_2(\theta)$ be the root of Eq. (4.9) at $\theta = \theta_\star$ obeying $\eta_1(\theta_\star) = 0, \eta_2(\theta_\star) = \zeta_0$, then $Re\left(\frac{d\lambda}{d\theta}\right)\Big|_{\theta=\theta_\star,\zeta=\zeta_0} > 0$.*

$$\begin{aligned} &(3\lambda^2 + 2h_1\lambda + h_2)e^{\lambda\theta}\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} + (\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3)e^{\lambda\theta}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta + \lambda\right) \\ &+h_7e^{-\lambda\theta}\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} - (h_7\lambda + h_8)e^{-\lambda\theta}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta + \lambda\right) + (2h_4\lambda + h_5)\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} = 0, \end{aligned} \tag{4.24}$$

which leads to

$$\left(\frac{d\lambda}{d\theta}\right)^{-1} = \frac{Q_1(\lambda)}{Q_2(\lambda)} - \frac{\theta}{\lambda}, \tag{4.25}$$

where

$$\begin{cases} Q_1(\lambda) = (3\lambda^2 + 2h_1\lambda + h_2)e^{\lambda\theta} + h_7e^{-\lambda\theta}, \\ Q_2(\lambda) = (h_7\lambda + h_8)\lambda e^{-\lambda\theta} - (\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3)\lambda e^{\lambda\theta}. \end{cases} \tag{4.26}$$

Hence,

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_\star,\zeta=\zeta_0} = \mathrm{Re}\left[\frac{Q_1(\lambda)}{Q_2(\lambda)}\right]_{\theta=\theta_\star,\zeta=\zeta_0} = \frac{Q_{1R}Q_{2R} + Q_{1I}Q_{2I}}{Q_{2R}^2 + Q_{2I}^2}. \tag{4.27}$$

By ($\mathcal{A}_6$), one gets

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_\star,\zeta=\zeta_0} > 0, \tag{4.28}$$

which completes the proof.

Based on the study above, the following conclusion is lightly acquired.

Theorem 4.1. *Suppose that* ($\mathcal{A}_4$)–($\mathcal{A}_6$) *hold, then the equilibrium point* $E(u_{1\star}, u_{2\star}, u_{3\star})$ *of model (4.1) holds locally asymptotically stable if* $\theta \in [0, \theta_\star)$ *and model (4.1) produces Hopf bifurcations at the equilibrium point* $E(u_{1\star}, u_{2\star}, u_{3\star})$ *when* $\theta = \theta_\star$.

5. Bifurcation domination using extended hybrid controller II

In this part, we will investigate the Hopf bifurcation problem of system (1.3) using a suitable extended delayed feedback controller consisting of parameter perturbation with delay. In accordance with [33,34], we propose the following controlled predator-prey model:

$$\begin{cases} \dfrac{du_1(t)}{dt} = \rho_1[u_1(t)(1 - u_1(t-\theta) - \gamma_{12}u_2^2(t)) + \gamma u_1(t)u_3(t)] + \rho_2[u_1(t) - u_1(t-\theta)], \\ \dfrac{du_2}{dt} = ru_2(t)[1 - u_2(t-\theta) - \gamma_{21}u_1^2] - \dfrac{(1-p)u_2u_3}{v_1 + (1-p)u_2}, \\ \dfrac{du_3}{dt} = \rho_3u_3[-v_2 + \dfrac{v_3(1-p)u_2}{v_1 + (1-p)u_2}] + \rho_4[u_3(t) - u_3(t-\theta)], \end{cases} \tag{5.1}$$

where $\rho_1 - \rho_4$ stands for feedback gain parameters. System (5.1) owns the same equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ as that of system (1.3). Let

$$\begin{cases} u_1(t) = \bar{u}_1(t) + u_{1\star}, \\ u_2(t) = \bar{u}_2(t) + u_{2\star}, \\ u_3(t) = \bar{u}_3(t) + u_{3\star}, \end{cases} \tag{5.2}$$

denote $\bar{u}_1$ as u_1, $\bar{u}_2$ as u_2,$\bar{u}_3$ as u_3, and the linear system of system (5.1) around $E(u_{1\star}, u_{2\star}, u_{3\star})$ takes the following expression:

$$\begin{cases} \dfrac{du_1}{dt} = f_1u_1 + f_2u_2 + f_3u_3 + f_4u_1(t-\theta), \\ \dfrac{du_2}{dt} = f_5u_1 + f_6u_2 + f_7u_3 + f_8u_2(t-\theta), \\ \dfrac{du_3}{dt} = f_9u_2 + f_{10}u_3 + f_{11}u_3(t-\theta), \end{cases} \tag{5.3}$$

where

$$\begin{cases} f_1 = (1 - u_{1\star} - \gamma_{12}u_{2\star}^2 - \gamma u_{3\star})\rho_1 + \rho_2, \\ f_2 = -2\gamma_{12}u_{1\star}u_{2\star}\rho_1, \\ f_3 = -\gamma u_{1\star}\rho_1, \\ f_4 = -\rho_1 u_{1\star} - \rho_2, \\ f_5 = -2r\gamma_{21}u_{1\star}u_{2\star}, \\ f_6 = r - ru_{2\star} - r\gamma_{21}u_{1\star}^2 - \dfrac{(1-p)u_{3\star}}{v_1 + (1-p)u_{2\star}} + \dfrac{(1-p)^2 u_{2\star}u_{3\star}}{[v_1 + (1-p)u_{2\star}]^2}, \\ f_7 = -\dfrac{(1-p)u_{2\star}}{v_1 + (1-p)u_{2\star}}, \\ f_8 = -ru_{2\star}, \\ f_9 = \left(\dfrac{v_3(1-p)u_{3\star}}{v_1 + (1-p)u_{2\star}} - \dfrac{v_3(1-p)^2 u_{2\star}u_{3\star}}{[v_1 + (1-p)u_{2\star}]^2}\right)\rho_3, \\ f_{10} = \left(\rho_4 - v_2 + \dfrac{v_3(1-p)u_{2\star}}{v_1 + (1-p)u_{2\star}}\right)\rho_3, \\ f_{11} = -\rho_4. \end{cases} \tag{5.4}$$

The characteristic equation of system (5.3) owns the following expression:

$$\det\begin{bmatrix} \lambda - f_1 - d_4 e^{-\lambda\theta} & -f_2 & -f_3 \\ -f_5 & \lambda - f_6 - f_8 e^{-\lambda\theta} & -f_7 \\ 0 & -f_9 & \lambda - f_{10} - f_{11}e^{-\lambda\theta} \end{bmatrix} = 0, \tag{5.5}$$

which leads to

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + (m_4\lambda^2 + m_5\lambda + m_6)e^{-\lambda\theta} + (m_7\lambda + m_8)e^{-2\lambda\theta} + m_9 e^{-3\lambda\theta} = 0, \tag{5.6}$$

that is,

$$(\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3)e^{2\lambda\theta} + (m_4\lambda^2 + m_5\lambda + m_6)e^{\lambda\theta} + m_9 e^{-\lambda\theta} + m_7\lambda + m_8 = 0, \tag{5.7}$$

where

$$\begin{cases} m_1 = -f_1 - f_6 - f_{10}, \\ m_2 = f_1f_6 + f_1f_{10} - f_2f_5 + f_6f_{10} - f_7f_9, \\ m_3 = f_1f_7f_9 - f_1f_6f_{10} + f_2f_5f_{10} - f_3f_5f_9, \\ m_4 = -f_4 - f_8 - f_{11}, \\ m_5 = f_1f_8 + f_4f_6 + f_4f_{10} + f_8f_{10} + f_1f_{11} + f_6f_{11}, \\ m_6 = f_4f_7f_9 - f_1f_8f_{10} - f_4f_6f_{10} + f_1f_6f_{11} + f_2f_5f_{11}, \\ m_7 = f_4f_8 + f_4f_{11} + f_8f_{11}, \\ m_8 = -f_4f_8f_{10} - f_4f_6f_{10} - f_4f_6f_{11} - f_1f_8f_{11}, \\ m_9 = -f_4f_8f_{11}. \end{cases} \tag{5.8}$$

If $\delta = 0$, then Eq (5.6) reads as:

$$\lambda^3 + (m_1 + m_4)\lambda^2 + m_3 + (m_2 + m_5 + m_7) + m_3 + m_6 + m_8 + m_9 = 0. \tag{5.9}$$

If

$$(\mathcal{A}_7)\begin{cases} \nabla_1 = m_1 + m_4 > 0, \\ \nabla_2 = \begin{vmatrix} m_1 + m_4 & 1 \\ m_3 + m_6 + m_8 + m_9 & m_2 + m_5 + m_7 \end{vmatrix} > 0, \\ \nabla_3 = \begin{vmatrix} m_1 + m_4 & 1 & 0 \\ m_3 + m_6 + m_8 + m_9 & m_2 + m_5 + m_7 & m_1 + m_4 \\ 0 & 0 & m_3 + m_6 + m_8 + m_9 \end{vmatrix} > 0, \end{cases}$$

is fulfilled, the three roots of Eq (5.6), $\lambda_1, \lambda_2, \lambda_3$, have negative real components. Therefore, the equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ of system (5.1) with $\theta = 0$ is locally asymptotically stable.

Suppose that $\lambda = i\omega$ is the root of Eq (5.7), then Eq (5.7) becomes:

$$\begin{aligned} &[(i\omega)^3 + m_1(i\omega)^2 + m_2(i\omega) + m_3]e^{2i\omega\theta} + [m_4(i\omega)^2 + m_5(i\omega) + m_6)]e^{i\omega\theta} \\ &+m_9e^{-i\omega\theta} + m_7(i\omega) + m_8 = 0. \end{aligned} \tag{5.10}$$

By (5.10), we have

$$\begin{cases} \left(\omega^3 - m_2\omega\right)\sin(2\omega\theta) + \left(-m_1\omega^2 + m_3\right)\cos(2\omega\theta) - m_5\omega\sin(\omega\theta) \\ +(m_6 + m_9 - m_4\omega^2)\cos(\omega\theta) + m_8 = 0, \\ \left(m_2\omega - \omega^3\right)\cos(2\omega\theta) + \left(-m_1\omega^2 + m_3\right)\sin(2\omega\theta) + m_5\omega\cos(\omega\theta) \\ +(m_6 - m_9 - m_4\omega^2)\sin(\omega\theta) + m_7\omega = 0, \end{cases} \tag{5.11}$$

and from (5.11), we can get

$$M_1\cos^2(\omega\theta) + M_2\cos(\omega\theta) + M_3 = (M_4 + M_5\cos(\omega\theta))\sqrt{1 - \cos^2(\omega\theta)}, \tag{5.12}$$

where

$$\begin{cases} M_1 = -2m_1\omega^2 + 2m_3, \\ M_2 = m_6 + m_9 - m_4\omega^2, \\ M_3 = m_1\omega^2 - m_3 + m_8, \\ M_4 = m_5\omega, \\ M_5 = 2m_2\omega - 2\omega^3. \end{cases} \tag{5.13}$$

So, we can get

$$N_1\cos^4(\omega\theta) + N_2\cos^3(\omega\theta) + N_3\cos^2(\omega\theta) + N_4\cos(\omega\theta) + N_5 = 0, \tag{5.14}$$

where

$$\begin{cases} N_1 = M_1^2 + M_5^2, \\ N_2 = 2M_1M_2 + 2M_4M_5, \\ N_3 = 2M_1M_3 + M_2^2 + M_4^2 - M_5^2, \\ N_4 = 2M_1M_3 + 2M_4M_5, \\ N_5 = M_3^2 + M_4^2. \end{cases} \tag{5.15}$$

From (5.14), we can suppose that $\cos(\omega\theta) = y$, and we have

$$N_1y^4 + N_2y^3 + N_3y^2 + N_4y + N_5 = 0. \tag{5.16}$$

According to the computer software, we can get

$$\cos(\omega_i\theta_i) = y_i (i = 1, 2, 3, 4). \tag{5.17}$$

It follows from (5.17) that

$$\theta_t^{(n)} = \frac{1}{\omega_t}[\arccos y_t + 2n\pi], \tag{5.18}$$

where $t = 1, 2, 3, 4$; $n = 0, 1, 2, 3, \cdots$.

Let $\theta^\star = \min_{\{t=1,2,3,4;n=1,2,3,\cdots\}}\{\theta_t^{(n)}\}$, and assume that when $\theta = \theta^\star$, Eq (5.6) has at least one pair of pure real roots $\pm i\omega_0$. Next, the following assumption is needed:

$$(\mathcal{A}_8)\ \ T_{1R}T_{2R} + T_{1I}T_{2I} > 0,$$

where

$$\begin{cases} T_{1R} = (-3\omega_0^2 + m_2)\cos(2\omega_0\theta^\star) - 2m_1\omega_0\sin(2\omega_0\theta^\star) \\ \qquad + m_5\cos(\omega_0\theta^\star) - 2m_4\omega_0\sin(\omega_0\theta^\star) + m_7, \\ T_{1I} = (-3\omega_0^2 + m_2)\sin(2\omega_0\theta^\star) + 2m_1\omega_0\cos(2\omega_0\theta^\star) \\ \qquad + m_5\sin(\omega_0\theta^\star) + 2m_4\omega_0\cos(\omega_0\theta^\star), \\ T_{2R} = (-2\omega_0^4 + 2m_2\omega_0^2)\cos(2\omega_0\theta^\star) - (2m_1\omega_0^3 - 2m_3\omega_0)\sin(2\omega_0\theta^\star) \\ \qquad + m_5\omega_0^2\cos(\omega_0\theta^\star) + (m_9\omega_0 - m_4\omega_0^3 + m_6\omega_0)\sin(\omega_0\theta^\star), \\ T_{2I} = (-2\omega_0^4 + 2m_2\omega_0^2)\sin(2\omega_0\theta^\star) + (2m_1\omega_0^3 - 2m_3\omega_0)\cos(2\omega_0\theta^\star) \\ \qquad + m_5\omega_0^2\sin(\omega_0\theta^\star) + (m_9\omega_0 + m_4\omega_0^3 - m_6\omega_0)\cos(\omega_0\theta^\star). \end{cases} \tag{5.19}$$

Lemma 5.1. *Suppose that $\lambda(\theta) = \psi_1(\theta) + i\psi_2(\theta)$ is the root of Eq (5.7) at $\theta = \theta^\star$ such that $\psi_1(\theta^\star) = 0$,$\psi_2(\theta^\star) = \omega_0$, then $Re\left(\frac{d\lambda}{d\theta}\right)\Big|_{\theta=\theta^\star,\omega=\omega_0} > 0$.*

Proof. By Eq (5.7), one gets

$$\begin{aligned} &(3\lambda^2 + 2m_1\lambda + m_2)e^{2\lambda\theta}\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} + (\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3)e^{2\lambda\theta}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}2\theta + 2\lambda\right) \\ &+(2m_4\lambda + m_5)e^{\lambda\theta}\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} + (m_4\lambda^2 + m_5\lambda + m_6)e^{\lambda\theta}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta + \lambda\right) + m_7\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} \\ &-m_9e^{-\lambda\theta}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\theta + \lambda\right) = 0, \end{aligned} \tag{5.20}$$

which implies

$$\left(\frac{d\lambda}{d\theta}\right)^{-1} = \frac{T_1(\lambda)}{T_2(\lambda)} - \frac{\theta}{\lambda}, \tag{5.21}$$

where

$$\begin{cases} T_1(\lambda) = (3\lambda^2 + 2m_1\lambda + m_2)e^{2\lambda\theta} + (2m_4\lambda + m_5)e^{\lambda\theta} + m_7, \\ T_2(\lambda) = -\lambda[2(\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3)e^{2\lambda\theta} + (m_4\lambda^2 + m_5\lambda + m_6)e^{\lambda\theta} - m_9e^{-\lambda\theta}]. \end{cases} \tag{5.22}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta^\star,\omega=\omega_0} = \mathrm{Re}\left[\frac{T_1(\lambda)}{T_2(\lambda)}\right]_{\theta=\theta^\star,\omega=\omega_0} = \frac{T_{1R}T_{2R} + T_{1I}T_{2I}}{T_{2R}^2 + T_{2I}^2}. \tag{5.23}$$

By ($\mathcal{A}_8$), we have

$$\text{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta^\star,\omega=\omega_0} > 0, \tag{5.24}$$

which concludes the proof.

Based on the research presented above, the following conclusion is loosely drawn.

Theorem 5.1. *Suppose that* $(\mathcal{A}_7)-(\mathcal{A}_8)$ *hold, then the equilibrium point* $E(u_{1\star}, u_{2\star}, u_{3\star})$ *of model (5.1) is locally asymptotically stable if* $\theta \in [0, \theta^\star)$ *and model (5.1) generates Hopf bifurcations at the equilibrium point* $E(u_{1\star}, u_{2\star}, u_{3\star})$ *when* $\theta = \theta^\star$.

Remark 5.1. *In Section 4, the controller is called the hybrid controller that includes state feedback and parameter perturbation with delay. This controller is only added to the first equation of system (1.3). In Section 5, the controller is called the extended hybrid controller that includes state feedback and parameter perturbation with delay. This controller is added to the first equation and the third equation of system (1.3). Hybrid controller II owns more control parameters than those in hybrid controller I and has greater adjustment flexibility in controlling the stability domain and the onset of Hopf bifurcation of system (1.3).*

6. Software experiments

Example 6.1. Think about the following Lotka-Volterra commensal symbiosis system:

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(1 - u_1(t-\theta) - 1.1u_2^2(t)) + 0.04u_1(t)u_3(t), \\ \dfrac{du_2(t)}{dt} = 0.5u_2(t)(1 - u_2(t-\theta) - 0.3u_1^2(t)) - \dfrac{(1-0.1)u_2(t)u_3(t)}{0.2 + (1-0.1)u_2(t)}, \\ \dfrac{du_3(t)}{dt} = u_3(t)(-0.35 + \dfrac{0.5(1-0.1)u_2(t)}{0.2 + (1-0.1)u_2(t)}). \end{cases} \tag{6.1}$$

It is straightforward to see that system (6.1) has a single positive equilibrium point $E(0.7092, 0.5185, 0.1224)$. One can easily verify that the conditions ($\mathcal{A}_1$)–($\mathcal{A}_3$) of Theorem 3.1 hold true. Using Matlab software, we can obtain $\theta_0 \approx 1.3$. To validate the accuracy of Theorem 3.1, we use two distinct delay values: $\theta = 0.8$ and $\theta = 1.9$. For $\theta = 0.8 < \theta_0 \approx 1.3$, simulation graphs are provided in Figure 1. Figure 1 shows that $u_1 \to 0.7092, u_2 \to 0.5185, u_3 \to 0.1224$ as $t \to +\infty$. In this case, the equilibrium point $E(0.7092, 0.5185, 0.1224)$ of model (6.1) has a locally asymptotically stable state. For $\theta = 1.9 > \theta_0 \approx 1.3$, we receive simulation graphs, as shown in Figure 2. Figure 2 shows that u_1 maintains a periodic vibrating level around 0.7092, whereas u_2 maintains a level around 0.5185 and u_3 maintains a periodic vibrating level around 0.1224. That is, a set of periodic solutions (known as Hopf bifurcations) arise at the equilibrium point $E(0.7092, 0.5185, 0.1224)$.

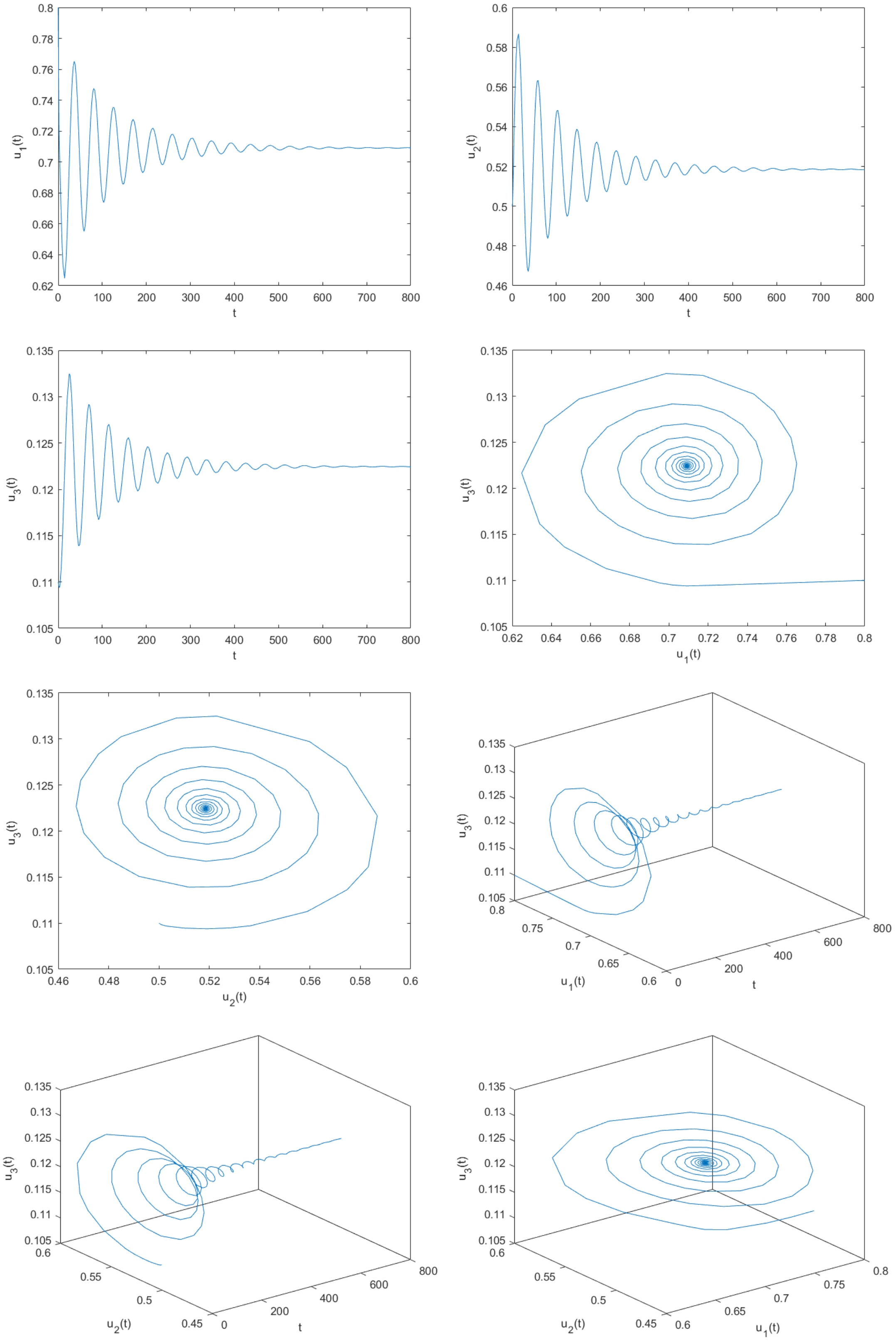

Figure 1. Matlab simulation figures of system (6.1) under the delay $\theta = 0.8 < \theta_0 \approx 1.3$, and the equilibrium point $E(u_{1*}, u_{2*}, u_{3*}) = E(0.7092, 0.5185, 0.1224)$ holds a locally asymptotically stable level.

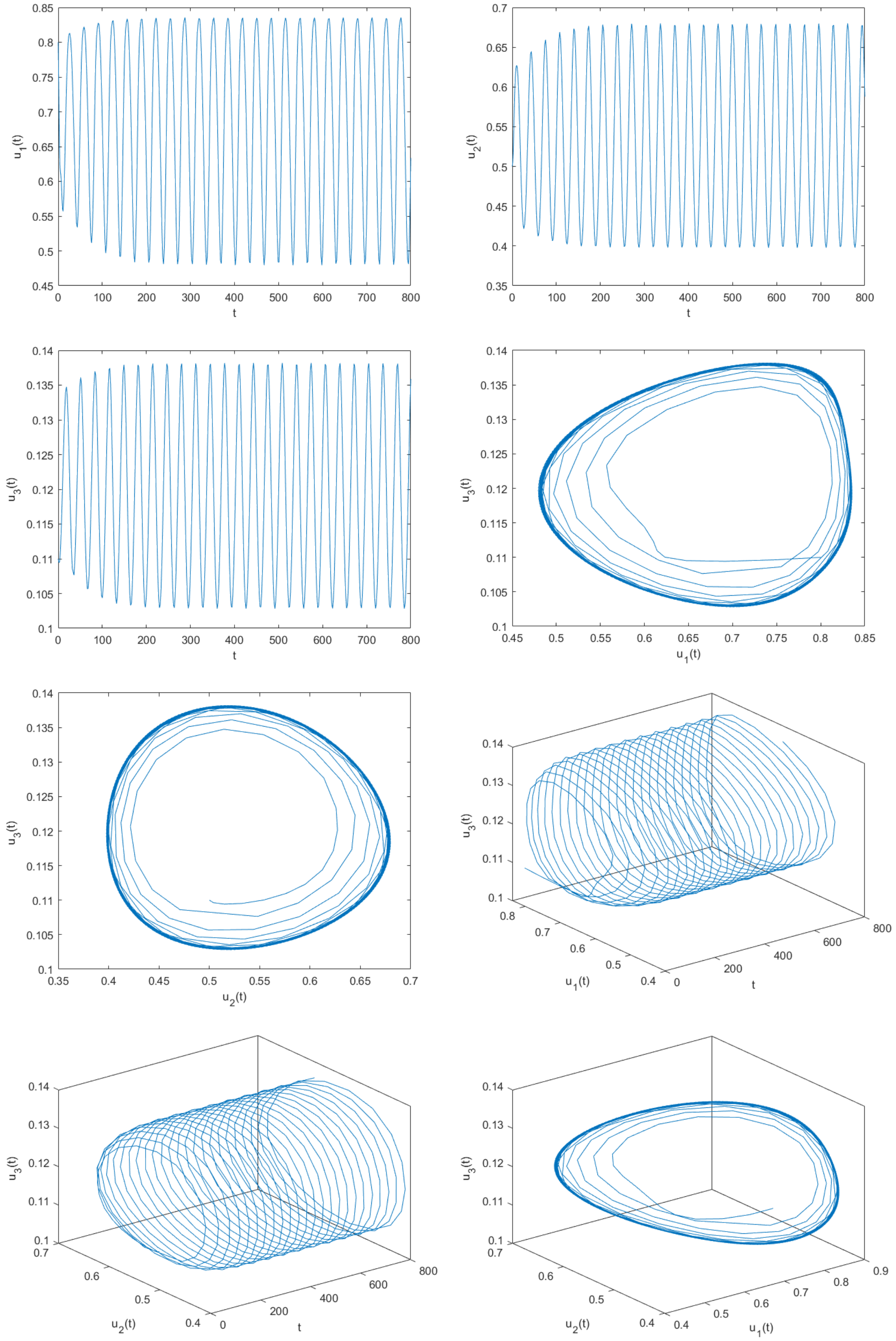

Figure 2. Matlab simulation figures of system (6.1) with delay $\theta = 1.9 > \theta_0 \approx 1.3$, and a cluster of periodic solutions (i.e., Hopf bifurcations) arise around the equilibrium point $E(E(u_{1*}, u_{2*}, u_{3*}) = E(0.7092, 0.5185, 0.1224)$.

Example 6.2. Think about the following controlled Lotka-Volterra commensal symbiosis system:

$$\begin{cases} \dfrac{du_1(t)}{dt} = \delta_1[u_1(t)(1 - u_1(t-\theta) - 1.1u_2^2(t)) + 0.04u_1(t)u_3(t)] + \delta_2[u_1(t) - u_1(t-\theta)], \\ \dfrac{du_2}{dt} = 0.5u_2(t)[1 - u_2(t-\theta) - 0.3u_1^2(t)] - \dfrac{(1-0.1)u_2(t)u_3(t)}{0.2 + (1-0.1)u_2(t)}, \\ \dfrac{du_3}{dt} = u_3(t)\left[-0.35 + \dfrac{0.5(1-0.1)u_2(t)}{0.2 + (1-0.1)u_2(t)}\right]. \end{cases} \tag{6.2}$$

It is straightforward to see that system (6.2) has a single positive equilibrium point $E(0.7092, 0.5185, 0.1224)$. Let $\delta_1 = 0.5, \delta_2 = -0.1$. One can easily verify that the conditions $(\mathcal{A}_4)$–$(\mathcal{A}_6)$ of Theorem 4.1 hold true. Using Matlab software, we can obtain $\theta_\star \approx 1.8$. To validate the correctness of the acquired assertions of Theorem 4.1, we use two distinct delay values: $\theta = 1.6$ and $\theta = 2.25$. For $\theta = 1.6 < \theta_\star \approx 1.8$, we get simulation diagrams which are presented in Figure 3. Based on Figure 3, we find that $u_1 \to 0.7092, u_2 \to 0.5185, u_3 \to 0.1224$ when $t \to +\infty$. In other words, the equilibrium point $E(0.7092, 0.5185, 0.1224)$ of model (6.2) holds a locally asymptotically stable state. For $\theta = 2.25 > \theta_\star \approx 1.8$, we get simulation diagrams which are presented in Figure 4. Based on Figure 4, we find that u_1 maintains a periodic vibrating level around 0.7092, whereas u_2 maintains a level around 0.5185 and u_3 maintains a periodic vibrating level around 0.1224. That is to say, a family of periodic solutions (namely, Hopf bifurcations) appear near the equilibrium point $E(0.7092, 0.5185, 0.1224)$.

Example 6.3. Think about the following controlled Lotka-Volterra commensal symbiosis system:

$$\begin{cases} \dfrac{du_1(t)}{dt} = \rho_1[u_1(t)(1 - u_1(t-\theta) - 1.1u_2^2(t)) + 0.04u_1(t)u_3(t)] + \rho_2[u_1(t) - u_1(t-\theta)], \\ \dfrac{du_2}{dt} = 0.5u_2(t)[1 - u_2(t-\theta) - 0.3u_1^2] - \dfrac{(1-0.1)u_2u_3}{0.2 + (1-0.1)u_2}, \\ \dfrac{du_3}{dt} = \rho_3u_3\left[-0.35 + \dfrac{0.5(1-0.1)u_2}{0.2 + (1-0.1)u_2}\right] + \rho_4[u_3(t) + u_3(t-\theta)]. \end{cases} \tag{6.3}$$

It is straightforward to see that system (6.3) has a single positive equilibrium point $E(0.7092, 0.5185, 0.1224)$. Let $\rho_1 = 0.5, \rho_2 = -0.1, \rho_3 = 0.6, \rho_4 = -0.1$. One can easily verify that the conditions $(\mathcal{A}_7)$ and $(\mathcal{A}_8)$ of Theorem 5.1 hold true. By applying Matlab software, one can get $\theta^\star \approx 2.20$. To validate the correctness of the acquired assertions of Theorem 5.1, we choose both different delay values: $\theta = 2.00$ and $\theta = 2.75$. For $\theta = 2.00 < \theta^\star \approx 2.20$, we get simulation diagrams which are presented in Figure 5. Based on Figure 5, we find that $u_1 \to 0.7092, u_2 \to 0.5185, u_3 \to 0.1224$ when $t \to +\infty$. In other words, the equilibrium point $E(0.7092, 0.5185, 0.1224)$ of model (6.3) holds a locally asymptotically stable state. For $\theta = 2.75 > \theta^\star \approx 2.20$, we get simulation diagrams which are presented in Figure 6. Based on Figure 6, we find that u_1 maintains a periodic vibrating level around 0.7092, whereas u_2 maintains a level around 0.5185 and u_3 maintains a periodic vibrating level around 0.1224. That is to say, a family of periodic solutions (namely, Hopf bifurcations) appear near the equilibrium point $E(0.7092, 0.5185, 0.1224)$.

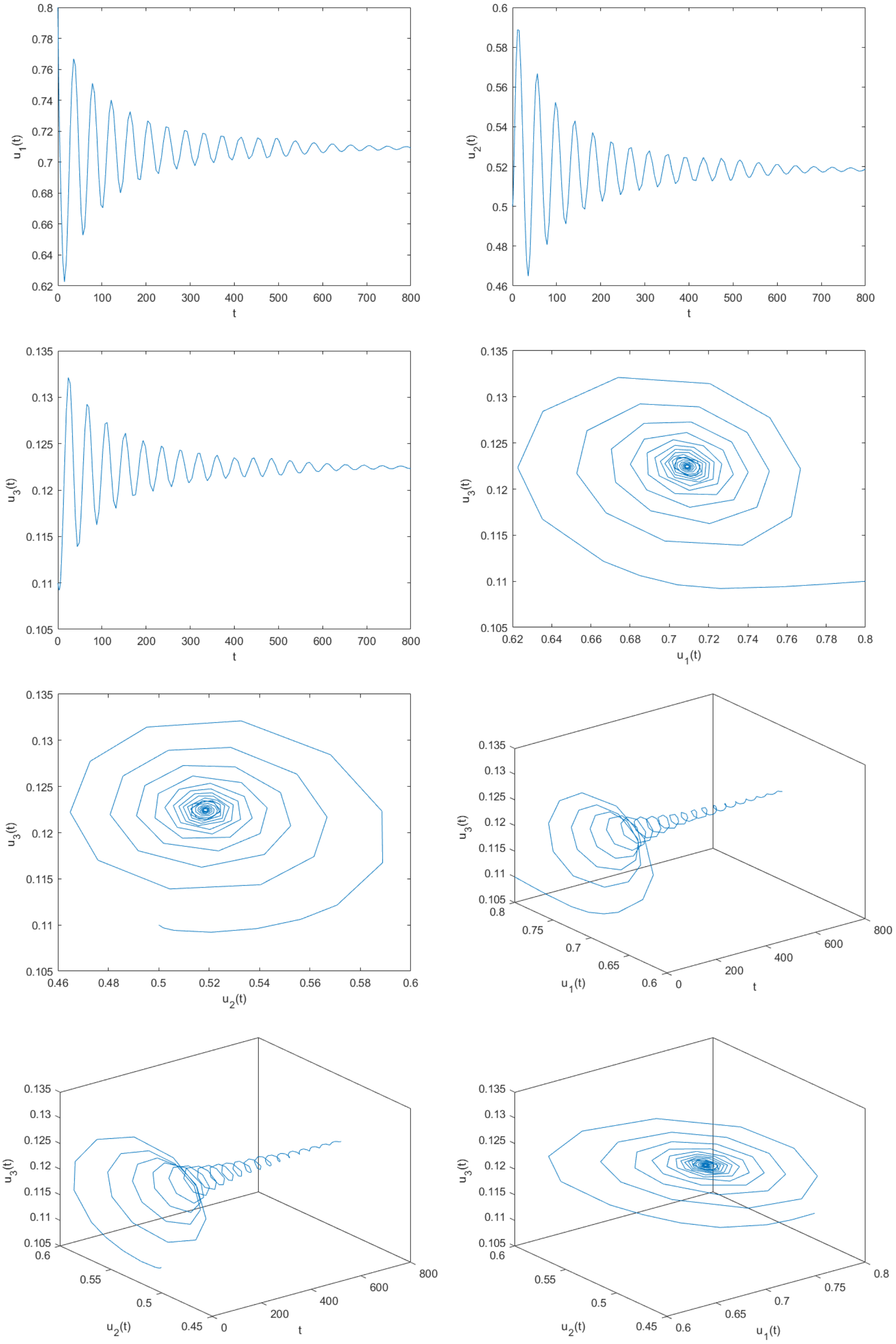

Figure 3. Matlab simulation figures of system (6.2) under the delay $\theta = 1.6 < \theta_{\star} \approx 1.8$, and the equilibrium point $E(u_{1*}, u_{2*}, u_{3*}) = E(0.7092, 0.5185, 0.1224)$ holds a locally asymptotically stable level.

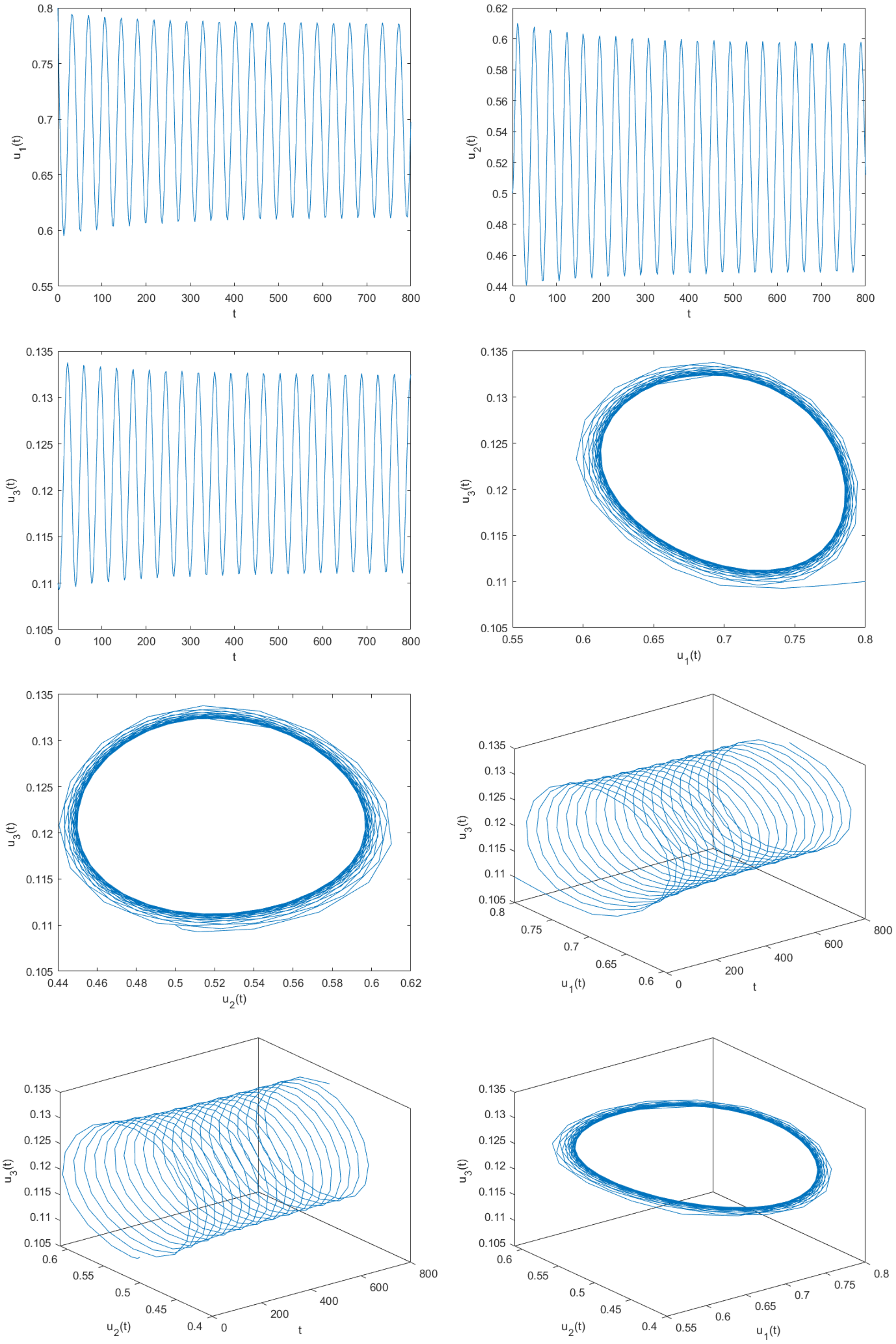

Figure 4. Matlab simulation figures of system (6.2) under the delay $\theta = 2.25 > \theta_\star \approx 1.8$, and a cluster of periodic solutions (i.e., Hopf bifurcations) arise around the equilibrium point $E(u_{1*}, u_{2*}, u_{3*}) = E(0.7092, 0.5185, 0.1224)$.

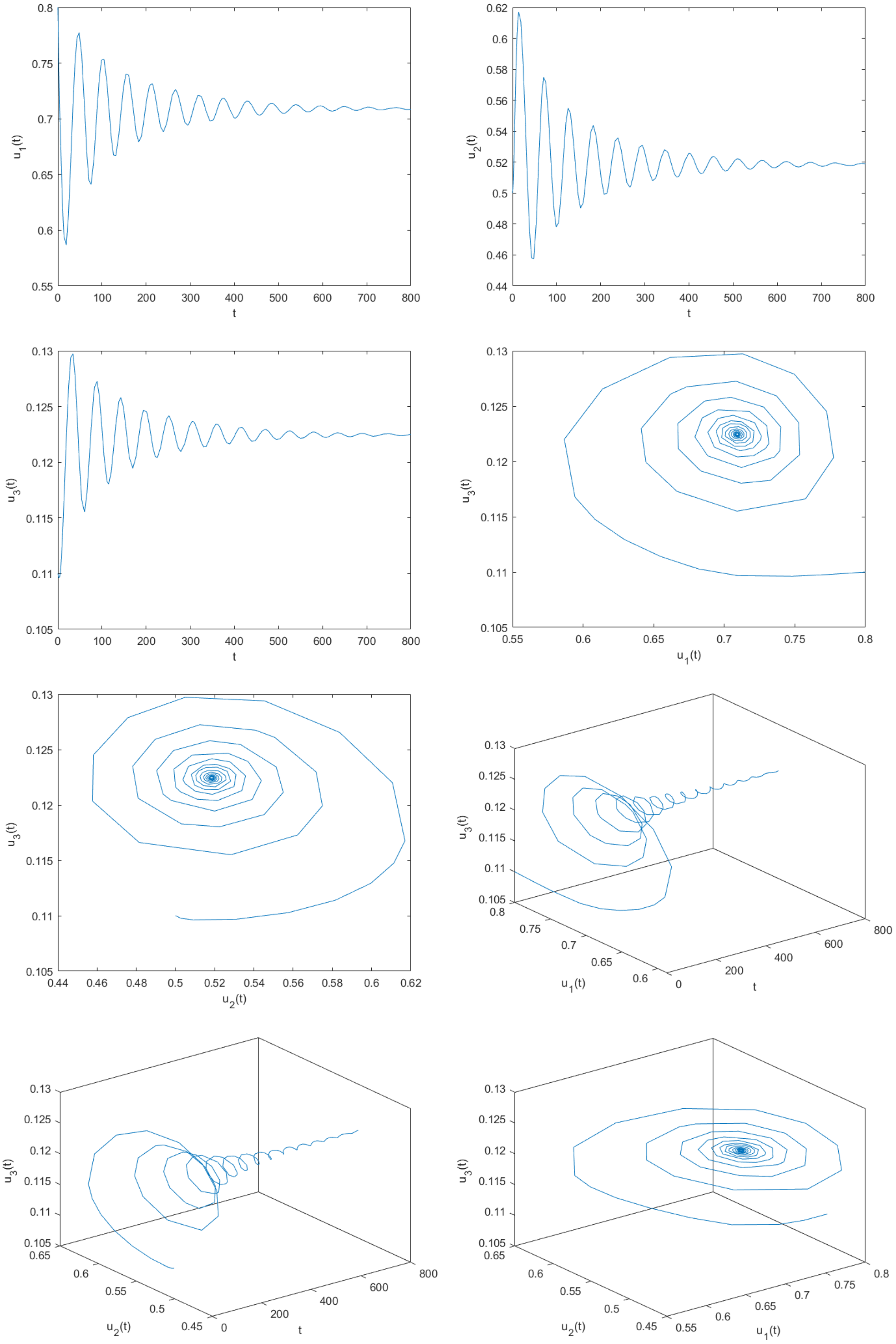

Figure 5. Matlab simulation figures of system (6.2) under the delay $\theta = 2.00 < \theta^{\star} \approx 2.20$, and the equilibrium point $E(u_{1*}, u_{2*}, u_{3*}) = E(0.7092, 0.5185, 0.1224)$ holds a locally asymptotically stable level.

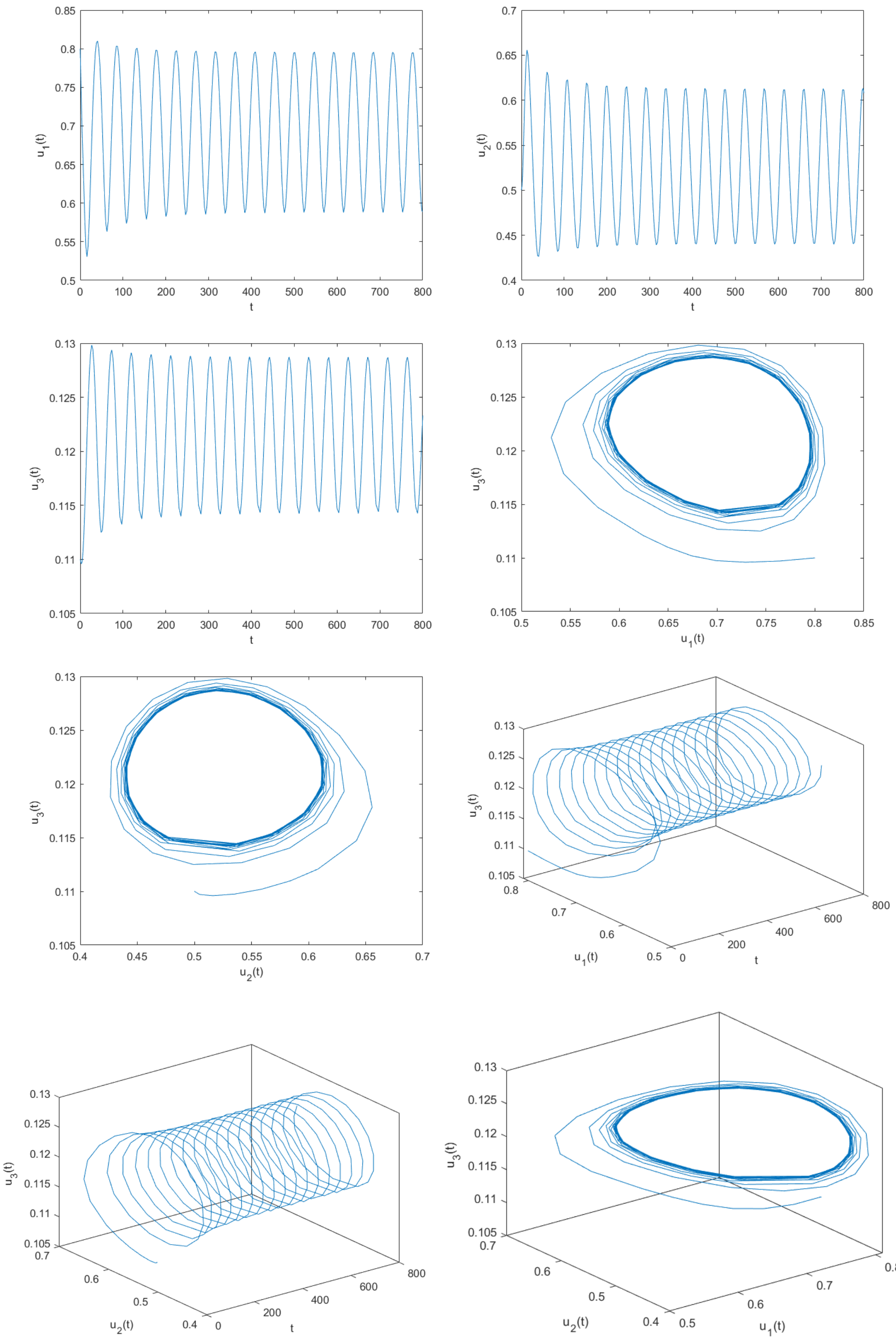

Figure 6. Matlab simulation figures of system (6.3) under the delay $\theta = 2.75 > \theta^\star \approx 2.20$, and a cluster of periodic solutions (i.e., Hopf bifurcations) arise around the equilibrium point $E(u_{1*}, u_{2*}, u_{3*}) = E(0.7092, 0.5185, 0.1224)$.

Remark 6.1. *It follows from the Matlab simulation results of Examples 6.1–6.3 that one can know that the bifurcation value of system (6.1) is equal to* 1.3*, the bifurcation value of system (6.2) is equal to* 1.8 *and the bifurcation value of system (6.3) is equal to* 2.20*, which indicates that we can expand the domain of stability of system (6.1) and postpone the time of emergence of Hopf bifurcation of system (6.1) via the formulated two hybrid delayed feedback controllers.*

7. Conclusions

It is generally recognized that the delayed dynamical model is an important tool for understanding the interactions of many biological populations in the natural environment [35–37]. Many studies on predator-prey models have conducted and yielded numerous results over the last few decades [38–40]. In this study, we provide a novel delayed Lotka-Volterra commensal symbiosis model. This paper discusses the uniqueness, nonnegativeness, and boundedness of the delayed Lotka-Volterra commensal symbiosis solution. The Hopf bifurcation issue is addressed. Then, the critical delay value θ_0 is retrieved. In order to modify the domain of stability and the time of the bifurcation phenomenon in this model, we have successfully developed two distinct hybrid delayed feedback controllers. Two critical delay values, $\theta_\star, \theta^\star$, are acquired. In these two controllers, the role of delay is displayed. Theoretically, the exploration fruits are very useful for managing and balancing the populations of two species. Furthermore, the exploratory concepts may be used for other fractional-order and integer-order dynamical systems in a wide range of disciplines to dominate the bifurcation phenomena, stability, and chaos [41–43]. During the past decades, many works on this topis is explored, see [44–46]. In this paper, we only deal with the Hopf bifurcation onset and Hopf bifurcation control in this paper. We leave the stability and direction of Hopf bifurcation periodic solutions for future work and we will refer to the works in [47–49]. In addition, we will explore the Hopf bifurcation of fractional-order dynamical models [50–52].

Author contributions

Yingyan Zhao: Conceptualization, formal analysis, investigation, methodology, software, writing-original draft, writing-review & editing; Changjin Xu: Conceptualization, formal analysis, investigation, methodology, software, writing-original draft, writing-review & editing; Yiya Xu: Formal analysis, investigation, software, writing-review & editing; Jinting Lin: Conceptualization, formal analysis, investigation, methodology, writing-original draft, writing-review & editing; Yicheng Pang: Conceptualization, investigation, methodology, writing-review & editing; Zixin Liu: Conceptualization, investigation, methodology, writing-review & editing; Jianwei Shen: Conceptualization, investigation, methodology, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflict of interest.

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