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Research article

Global unique solutions for the 2-D inhomogeneous incompressible viscoelastic rate-type fluids with stress-diffusion

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Abstract: We establish the global unique solution for the 2-D inhomogeneous incompressible viscoelastic rate-type fluids with stress-diffusion by employing the standard energy method and the standard compactness arguments.

Keywords: viscoelastic rate-type fluids; energy method; global unique solutions; Friedrich's method; compactness argument
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1. Introduction

In this paper, we consider the Cauchy problem for the following two-dimensional inhomogeneous incompressible viscoelastic rate-type fluids with stress-diffusion:

$$\begin{cases} \rho_t + u\cdot\nabla\rho = 0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \\ \rho(u_t + u\cdot\nabla u) - \Delta u + \nabla P + \sigma \mathrm{div}\,(\nabla b\otimes\nabla b - \frac{1}{2}|\nabla b|^2\mathbb{I}) = 0, \\ b_t + u\cdot\nabla b + \frac{1}{\nu}(e'(b) - \sigma\Delta b) = 0, \\ \mathrm{div}\, u = 0, \\ (\rho, u, b)(t,x)|_{t=0} = (\rho_0, u_0, b_0)(x), \end{cases} \tag{1.1}$$

where the unknowns $\rho = \rho(x,t)$, $u = (u^1(x,t), u^2(x,t))$ and $b = b(x,t)$ stand for the density, velocity of the fluid and the spherical part of the elastic strain, respectively. P is a scalar pressure function, which guarantees the divergence-free condition of the velocity field. The coefficients ν and σ are two positive constants. In addition, we suppose that $e(\cdot)$ is a smooth convex function about b and $e(0)\le 0$,

$e'(0) = 0$, $e''(b) \leq C_0$, where C_0 is a positive constant depending on the initial data. The class of fluids is the elastic response described by a spherical strain [3]. Compared with [3], we have added the divergence-free condition to investigate the effect of density on viscoelastic rate-type fluids, while the divergence-free condition is for computational convenience.

It is easy to observe that for $\sigma = 0$, the system (1.1) degenerates two distinct systems involving the inhomogeneous Navier-Stokes equation for the fluid and a transport equation with damped $e'(b)$. Numerous researchers have extensively studied the well-posedness concern regarding the inhomogeneous Navier-Stokes equations; see [1, 7–9, 11, 14] and elsewhere. However, the transport equation has a greater effect on the regularity of density than on that of velocity. Additionally, due to the presence of the damped term $e'(b)$, the initial elasticity in system (1.1) exhibits higher regularity compared to the initial velocity.

In the case where $\sigma > 0$, system (1.1) resembles the inhomogeneous magnetohydrodynamic (MHD) equations, with b as a scalar function in (1.1) that does not satisfy the divergence condition found in MHD equations. It is essential to highlight that the system (1.1) represents a simplified model, deviating from standard viscoelastic rate-type fluid models with stress-diffusion to facilitate mathematical calculations. Related studies on system (1.1) can be found in [3, 4, 15]. In particular, Bulíček, Málek, and Rodriguez in [5] established the well-posedness of a 2D homogeneous system (1.1) in Sobolev space. Our contribution lies in incorporating the density equation into this established framework.

Inspired by [11, 18], we initially establish a priori estimates for the system (1.1). Subsequently, by using a Friedrich's method and the compactness argument, we obtain the existence and uniqueness of the solutions. Our main result is as follows:

Theorem 1.1. *Let the initial data* (ρ_0, u_0, b_0) *satisfy*

$$0 < m < \rho_0(x) < M < \infty, \quad (u_0, b_0) \in H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2), \ e(b_0) \in L^1(\mathbb{R}^2), \tag{1.2}$$

where m, M *are two given positive constants with* $m < M$. *Then system* (1.1) *has a global solution* (ρ, u, b) *such that, for any given* $T > 0$, $(t, x) \in [0, T) \times \mathbb{R}^2$,

$$\begin{aligned} & m < \rho(t, x) < M, \\ & u \in L^\infty(0, T; H^1(\mathbb{R}^2)) \cap L^2(0, T; H^2(\mathbb{R}^2)), \quad \partial_t u \in L^\infty(0, T; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2)), \\ & b \in L^\infty(0, T; H^2(\mathbb{R}^2)) \cap L^2(0, T; H^3(\mathbb{R}^2)), \quad \partial_t b \in L^\infty(0, T; H^1(\mathbb{R}^2)) \cap L^2(0, T; H^2(\mathbb{R}^2)). \end{aligned}$$

Moreover, if $\nabla\rho_0 \in L^4(\mathbb{R}^2)$, *then the solution is unique.*

Remark 1.1. *Compared to the non-homogeneous MHD equations, handling the damping term* $e'(b)$ *poses a challenge, so that we cannot obtain the time-weighted energy of the velocity field. To explore the uniqueness of the solution, it is necessary to improve the regularity of the initial density data.*

The key issue to prove the global existence part of Theorem 1.1 is establishing the a priori $L^\infty(0, T; H^1(\mathbb{R}^2))$ estimate on $(u, \nabla b)$ for any positive time T. We cannot directly estimate the L^2 estimate of (u, b), which mainly occurs in the velocity term $\operatorname{div}(\nabla b \otimes \nabla b - \frac{1}{2}|\nabla b|^2 \mathbb{I})$. Therefore, we need to estimate the L^2 of the ∇b equation. Afterwards, the L^2 estimation of equation b was affected by a damping term $e'(b)$, so we made an L^2 estimation of equation $e'(b)$. Finally, to show the $L^\infty(0, T; H^1(\mathbb{R}^2))$ of u, we also need an estimate of the second derivative of b. In summary, we found that the initial value of the b equation needs to be one derivative higher than the initial value of the u equation.

Concerning the uniqueness of the strong solutions, a common approach is to consider the difference equations between two solutions and subsequently derive some energy estimates for the resulting system differences based on the fundamental natural energy of the system. However, for system (1.1), the presence of a damping term $e'(b)$ of the equation b and density equation prevents the calculation of the time-weighted energy of the velocity field. To research the solution's uniqueness, we need to enhance the regularity of the initial density data.

The paper is structured as follows: Section 2 presents prior estimates for system (1.1). In Section 3, we establish the existence and uniqueness of Theorem 1.1.

2. A priori estimates

Proposition 2.1. *Assume that m, M are two given positive constants and $0 < m \leq M < \infty$, the initial data ρ_0 satisfies $0 < m \leq \rho_0 \leq M < +\infty$, and the initial data $(\sqrt{\rho_0}u_0, \nabla b_0) \in L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. Let (ρ, u, b) be a smooth solution of system (1.1), then there holds for any $t > 0$,*

$$0 < m \leq \rho(t) \leq M < +\infty, \tag{2.1}$$

$$\|(\sqrt{\rho}u, \nabla b, u)(t)\|_{L^2}^2 + \int_0^t \|(\nabla u, \nabla^2 b)\|_{L^2}^2 d\tau \leq C\|(\sqrt{\rho_0}u_0, \nabla b_0)\|_{L^2}^2, \tag{2.2}$$

where C is a constant depending only on σ,ν.

Proof. First, any Lebesgue norm of ρ_0 is preserved through the evolution, and $0 < m \leq \rho(t) \leq M < +\infty$.

To prove (2.2), taking the L^2 inner product of the second equation of (1.1) with u and integrating by parts, then we obtain

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = -\sigma \int_{\mathbb{R}^2} \Delta b \nabla b \cdot u dx, \tag{2.3}$$

where we used the fact that

$$\operatorname{div}(\nabla b \otimes \nabla b - \frac{1}{2}|\nabla b|^2 \mathbb{I}) = \Delta b \nabla b.$$

Multiplying the third equation of (1.1) by $-\sigma\Delta b$ and integrating by parts, we obtain

$$\frac{\sigma}{2}\frac{d}{dt}\|\nabla b\|_{L^2}^2 + \frac{\sigma^2}{\nu}\|\Delta b\|_{L^2}^2 - \frac{\sigma^2}{\nu}\int_{\mathbb{R}^2} e'(b)\Delta b dx = \sigma \int_{\mathbb{R}^2} u \cdot \nabla b \Delta b dx. \tag{2.4}$$

Thanks to the convexity of $e(b)$, we know

$$-\frac{\sigma^2}{\nu}\int_{\mathbb{R}^2} e'(b)\Delta b dx = \frac{\sigma^2}{\nu}\int_{\mathbb{R}^2} \nabla(e'(b))\nabla b dx = \frac{\sigma^2}{\nu}\int_{\mathbb{R}^2} e''(b)|\nabla b|^2 dx \geq 0. \tag{2.5}$$

By inserting (2.5) into (2.4), combining the result with (2.3), one yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{\rho}u,\nabla b)\|_{L^2}^2+\|\nabla u\|_{L^2}^2+C\|\Delta b\|_{L^2}^2\le 0.$$

Integrating it with respect to time, we have

$$\|(\sqrt{\rho}u,\nabla b)(t)\|_{L^2}^2+\int_0^t\|(\nabla u,\nabla^2 b)\|_{L^2}^2\mathrm{d}\tau\le C\|(\sqrt{\rho_0}u_0,\nabla b_0)\|_{L^2}^2. \tag{2.6}$$

On the other hand, applying $0<m\le\rho\le M<+\infty$, which together with (2.6) implies

$$\|u\|_{L^\infty(L^2)}^2\le m^{-1}\|\sqrt{\rho}u\|_{L^\infty(L^2)}^2\le C\|(\sqrt{\rho_0}u_0,\nabla b_0)\|_{L^2}^2,$$

which, along with inequality (2.6), yields (2.2). □

Proposition 2.2. *Under the assumptions of Proposition 2.1, the corresponding solution* (ρ,u,b) *of the system* (1.1) *admits the following bound for any* $t>0$*:*

$$\|(\nabla u,\nabla^2 b,\nabla b,b)\|_{L^2}^2+\int_0^t\|(\nabla^2 u,\nabla^3 b,\sqrt{\rho}u_\tau,b_\tau,u_\tau)\|_{L^2}^2\mathrm{d}\tau\le C, \tag{2.7}$$

where C *is a positive constant depending on* m*,* M*,* u_0*,* ρ_0*, and* ∇b_0*.*

Proof. First, we obtain by taking L^2 inner product of $(1.1)_3$ with $e'(b)$ that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}e(b)\mathrm{d}x+\frac{1}{\nu}\|e'(b)\|_{L^2}^2\\
&\le\|u\cdot\nabla b\|_{L^2}\|e'(b)\|_{L^2}+\frac{1}{4\nu}\|e'(b)\|_{L^2}^2+C\|\Delta b\|_{L^2}^2\\
&\le\frac{1}{2\nu}\|e'(b)\|_{L^2}^2+C\|u\|_{L^4}^2\|\nabla b\|_{L^4}^2+C\|\Delta b\|_{L^2}^2\\
&\le\frac{1}{2\nu}\|e'(b)\|_{L^2}^2+C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}+C\|\Delta b\|_{L^2}^2\\
&\le\frac{1}{2\nu}\|e'(b)\|_{L^2}^2+C(\|\nabla u\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2).
\end{aligned}$$

Integrating with respect to time, we obtain

$$\|e(b)\|_{L^\infty(L^1)}+\|e'(b)\|_{L^2(L^2)}^2\le\|(\sqrt{\rho_0}u_0,\nabla b_0)\|_{L^2}^2+\|e(b_0)\|_{L^1}. \tag{2.8}$$

Similarly, multiplying $(1.1)_3$ by b, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2\le C(\|b\|_{L^2}^2+\|e'(b)\|_{L^2}^2),$$

after using (2.8) and Grönwall's inequality, we obtain

$$b\in L^\infty(0,t;L^2(\mathbb{R}^2))\cap L^2(0,t;H^1(\mathbb{R}^2)). \tag{2.9}$$

In the following, applying Laplace operator Δ to $(1.1)_3$ and multiplying the resulting equation by Δb; additionally, multiplying $(1.1)_2$ by u_t and $(1.1)_3$ by b_t, respectively, then integrating them on $\mathbb{R}^2$ and adding up all these results together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}(|\Delta b|^2+|\nabla u|^2+\frac{\sigma}{\nu}|\nabla b|^2)\mathrm{d}x+\int_{\mathbb{R}^2}(\rho|u_t|^2+|b_t|^2+\frac{\sigma}{\nu}|\nabla^3 b|^2)\mathrm{d}x\\
&=-\int_{\mathbb{R}^2}\rho u\cdot\nabla u\cdot u_t\mathrm{d}x-\sigma\int_{\mathbb{R}^2}\Delta b\nabla b\cdot u_t\mathrm{d}x-\int_{\mathbb{R}^2}u\cdot\nabla b b_t\mathrm{d}x\\
&\quad-\frac{1}{\nu}\int_{\mathbb{R}^2}e'(b)b_t\mathrm{d}x-\int_{\mathbb{R}^2}\Delta(u\cdot\nabla b)\cdot\Delta b\mathrm{d}x-\frac{1}{\nu}\int_{\mathbb{R}^2}\Delta e'(b)\cdot\Delta b\mathrm{d}x\\
&\triangleq\sum_{j=1}^{6}I_j. \qquad (2.10)
\end{aligned}$$

Utilizing Gagliardo-Nirenberg's, Hölder's, Young's inequalities (2.2), we estimate the first term as follows:

$$\begin{aligned}
I_1&\le\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^4}\|\nabla u\|_{L^4}\\
&\le\frac{1}{16}\|\sqrt{\rho}u_t\|_{L^2}^2+C\|u\|_{L^4}^2\|\nabla u\|_{L^4}^2\\
&\le\frac{1}{16}\|\sqrt{\rho}u_t\|_{L^2}^2+C\|u\|_{L^2}\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}\\
&\le\frac{1}{16}\|\sqrt{\rho}u_t\|_{L^2}^2+\frac{1}{16}\|\nabla^2 u\|_{L^2}^2+C\|\nabla u\|_{L^2}^4.
\end{aligned}$$

Similarly, by direct calculations, the other terms can be bounded as

$$\begin{aligned}
I_2&\le\frac{1}{16}\|\sqrt{\rho}u_t\|_{L^2}^2+\frac{\sigma}{8\nu}\|\nabla^3 b\|_{L^2}^2+C\|\nabla^2 b\|_{L^2}^4,\\
I_3&\le\frac{1}{4}\|b_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C\|\nabla b\|_{L^2}^2\|\Delta b\|_{L^2}^2,\\
I_4&\le\frac{1}{4}\|b_t\|_{L^2}^2+C\|e'(b)\|_{L^2}^2,\\
I_5&\le\frac{3\sigma}{16\nu}\|\nabla^3 b\|_{L^2}^2+\frac{1}{16}\|\nabla^2 u\|_{L^2}^2+C\|\nabla u\|_{L^2}^4+C\|\nabla^2 b\|_{L^2}^4,\\
I_6&\le\frac{3\sigma}{16\nu}\|\nabla^3 b\|_{L^2}^2+C\|\nabla b\|_{L^2}^2.
\end{aligned}$$

Next, according to the regularity theory of the Stokes system in Eq $(1.1)_2$, it follows that

$$\begin{aligned}
\|\nabla^2 u\|_{L^2}^2&\le\|\rho u_t\|_{L^2}^2+\|\rho u\cdot\nabla u\|_{L^2}^2+\sigma\|\nabla b\Delta b\|_{L^2}^2\\
&\le\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}+C\|\nabla^2 b\|_{L^2}^2\|\nabla^3 b\|_{L^2}\\
&\le\|\sqrt{\rho}u_t\|_{L^2}^2+\frac{1}{2}\|\nabla^2 u\|_{L^2}^2+\frac{\sigma}{2\nu}\|\nabla^3 b\|_{L^2}^2+C(\|\nabla u\|_{L^2}^4+\|\nabla^2 b\|_{L^2}^4),
\end{aligned}$$

after multiplying by $\frac{1}{8}$, we arrive at

$$\frac{1}{16}\|\nabla^2 u\|_{L^2}^2\le\frac{1}{8}\|\rho u_t\|_{L^2}^2+\frac{\sigma}{16\nu}\|\nabla^3 b\|_{L^2}^2+C(\|\nabla u\|_{L^2}^4+\|\nabla^2 b\|_{L^2}^4). \qquad (2.11)$$

Substituting the estimates $I_1 - I_6$ into (2.10) and combining inequality (2.11), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|(\nabla u, \nabla b, \nabla^2 b)\|_{L^2}^2 + 1) + \|(\sqrt{\rho} u_t, b_t, \nabla^3 b, \nabla^2 u)\|_{L^2}^2 \\
&\leq C(\|(\nabla u, \nabla b, \nabla^2 b)\|_{L^2}^2 + 1)\|(\nabla u, \nabla b, \nabla^2 b)\|_{L^2}^2 + C\|e'(b)\|_{L^2}^2,
\end{aligned}$$

which, along with Grönwall's inequality (2.2), (2.8), and (2.9), leads to

$$\|(\nabla u, \nabla b, \nabla^2 b)\|_{L^2}^2 + \int_0^t \|(\sqrt{\rho} u_\tau, b_\tau, \nabla^3 b, \nabla^2 u)\|_{L^2}^2 \mathrm{d}\tau \leq C, \tag{2.12}$$

which completes the proof of Proposition 2.2. □

Proposition 2.3. *Under the assumptions of Proposition 2.2, there holds*

$$\|(\sqrt{\rho} u_t, b_t, \nabla b_t, u_t)\|_{L^2}^2 + \int_0^t \|(\nabla u_\tau, \nabla b_\tau, \Delta b_\tau)\|_{L^2}^2 \mathrm{d}\tau \leq C, \tag{2.13}$$

where C is a positive constant depending on m, M, u_0, ρ_0 and b_0.

Proof. Taking the derivative of Eq $(1.1)_2$ with respect to time t, then multiplying u_t on both sides of the resulting equation and integrating by parts gives

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 =& -\int_{\mathbb{R}^2} \rho_t u_t \cdot u_t \mathrm{d}x - \int_{\mathbb{R}^2} \rho_t u \cdot \nabla u \cdot u_t \mathrm{d}x - \int_{\mathbb{R}^2} \rho u_t \cdot \nabla u \cdot u_t \mathrm{d}x \\
&- \int_{\mathbb{R}^2} \sigma \Delta b_t \nabla b \cdot u_t \mathrm{d}x - \sigma \int_{\mathbb{R}^2} \Delta b \nabla b_t \cdot u_t \mathrm{d}x.
\end{aligned} \tag{2.14}$$

Next, we compute each term on the right-hand side of the equation above one by one using estimates (2.2) and (2.7). The bound of the first term has been estimated as

$$\begin{aligned}
-\int_{\mathbb{R}^2} \rho_t u_t \cdot u_t \mathrm{d}x &= \int_{\mathbb{R}^2} \mathrm{div}\,(\rho u) u_t \cdot u_t \mathrm{d}x = -\int_{\mathbb{R}^2} 2\rho u u_t \cdot \nabla u_t \mathrm{d}x \\
&\leq C\|\rho\|_{L^\infty}\|u\|_{L^4}\|u_t\|_{L^4}\|\nabla u_t\|_{L^2} \\
&\leq C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}}\|u_t\|_{L^2}^{\frac{1}{2}} \\
&\leq \frac{1}{10}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|u_t\|_{L^2}^2.
\end{aligned}$$

By using Gagliardo-Nirenberg's, Hölder's, and Young's inequalities and (2.2), we have

$$\begin{aligned}
&-\int_{\mathbb{R}^2} \rho_t u \cdot \nabla u \cdot u_t \mathrm{d}x = \int_{\mathbb{R}^2} \nabla \cdot (\rho u) u \cdot \nabla u \cdot u_t \mathrm{d}x \\
&= -\int_{\mathbb{R}^2} \rho u \cdot \nabla u \cdot \nabla u \cdot u_t \mathrm{d}x - \int_{\mathbb{R}^2} \rho u \cdot u \cdot \nabla^2 u \cdot u_t \mathrm{d}x - \int_{\mathbb{R}^2} u \cdot \nabla u \cdot \rho u \cdot \nabla u_t \mathrm{d}x \\
&\leq \|\rho\|_{L^\infty}(\|u\|_{L^6}\|\nabla u\|_{L^3}^2\|u_t\|_{L^6} + \|u\|_{L^6}^2\|\nabla^2 u\|_{L^2}\|u_t\|_{L^6} + \|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}) \\
&\leq C\|u\|_{L^2}^{\frac{1}{3}}\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^{\frac{2}{3}}\|u_t\|_{L^2}^{\frac{1}{3}}\|\nabla u_t\|_{L^2}^{\frac{2}{3}} + C\|u\|_{L^2}^{\frac{2}{3}}\|\nabla u\|_{L^2}^{\frac{4}{3}}\|\nabla^2 u\|_{L^2}\|u_t\|_{L^2}^{\frac{1}{3}}\|\nabla u_t\|_{L^2}^{\frac{2}{3}} \\
&\quad + C\|u\|_{L^2}^{\frac{2}{3}}\|\nabla u\|_{L^2}^{\frac{5}{3}}\|\nabla^2 u\|_{L^2}^{\frac{2}{3}}\|\nabla u_t\|_{L^2}
\end{aligned}$$

$$\leq \frac{1}{10}\|\nabla u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|u_t\|_{L^2}^2.$$

Similarly,

$$-\int_{\mathbb{R}^2} \rho u_t \cdot \nabla u \cdot u_t \mathrm{d}x \leq C\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|u_t\|_{L^4}^2 \leq \frac{1}{10}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|u_t\|_{L^2}^2$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}^2} \sigma\Delta b_t \nabla b \cdot u_t \mathrm{d}x - \sigma\int_{\mathbb{R}^2} \Delta b \nabla b_t \cdot u_t \mathrm{d}x \\
&\leq C\|\Delta b_t\|_{L^2}\|\nabla b\|_{L^4}\|u_t\|_{L^4} + C\|\Delta b\|_{L^4}\|\nabla b_t\|_{L^2}\|u_t\|_{L^4} \\
&\leq C\|\Delta b_t\|_{L^2}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\Delta b\|_{L^2}^{\frac{1}{2}}\|u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{1}{2}} + C\|\Delta b\|_{L^2}^{\frac{1}{2}}\|\nabla^3 b\|_{L^2}^{\frac{1}{2}}\|\nabla b_t\|_{L^2}\|u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{1}{2}} \\
&\leq \frac{\sigma}{16\nu}\|\nabla b_t\|_{L^2}^2 + \frac{\sigma}{16\nu}\|\Delta b_t\|_{L^2}^2 + \frac{1}{10}\|\nabla u_t\|_{L^2}^2 + C(\|\nabla b\|_{L^2}^2 + \|\nabla^3 b\|_{L^2}^2)\|\nabla^2 b\|_{L^2}^2\|u_t\|_{L^2}^2.
\end{aligned}$$

Inserting these estimates into (2.14), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{8}{5}\|\nabla u_t\|_{L^2}^2 \leq\ & \frac{\sigma}{8\nu}\|(\nabla b_t, \Delta b_t)\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 \\
&+ C\|(\nabla u, \nabla^2 b, \nabla^3 b)\|_{L^2}^2\|u_t\|_{L^2}^2. \qquad (2.15)
\end{aligned}$$

Now we turn to the b equation of (1.1). Differentiating $(1.1)_3$ with respect to t, we obtain

$$b_{tt} + u_t \cdot \nabla b + u \cdot \nabla b_t + \frac{1}{\nu}(e''(b)b_t - \sigma\Delta b_t) = 0.$$

Multiplying it by b_t and $-\Delta b_t$, integrating the resulting equation, and summing up these results, due to the divergence-free condition $\operatorname{div} u = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(b_t, \nabla b_t)\|_{L^2}^2 + \frac{\sigma}{\nu}\|(\nabla b_t, \Delta b_t)\|_{L^2}^2 + \frac{1}{\nu}\int_{\mathbb{R}^2} e''(b)(b_t)^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^2} u_t \cdot b \cdot \nabla b_t \mathrm{d}x + \int_{\mathbb{R}^2} u_t \cdot \nabla b \cdot \Delta b_t \mathrm{d}x + \int_{\mathbb{R}^2} u \cdot \nabla b_t \cdot \Delta b_t \mathrm{d}x + \frac{1}{\nu}\int_{\mathbb{R}^2} e''(b) b_t \cdot \Delta b_t \mathrm{d}x \\
&\leq \|u_t\|_{L^4}\|b\|_{L^4}\|\nabla b_t\|_{L^2} + \|u_t\|_{L^4}\|\nabla b\|_{L^4}\|\Delta b_t\|_{L^2} + \|u\|_{L^4}\|\nabla b_t\|_{L^4}\|\Delta b_t\|_{L^2} + C\|b_t\|_{L^2}\|\Delta b_t\|_{L^2} \\
&\leq \frac{\sigma}{4\nu}\|\nabla b_t\|_{L^2}^2 + \|u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|b\|_{L^2}\|\nabla b\|_{L^2} + \frac{\sigma}{8\nu}\|\Delta b_t\|_{L^2}^2 + \|u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla b\|_{L^2}\|\Delta b\|_{L^2} \qquad (2.16) \\
&\quad + \frac{\sigma}{8\nu}\|\Delta b_t\|_{L^2}^2 + C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla b_t\|_{L^2}\|\nabla^2 b_t\|_{L^2} + \frac{\sigma}{8\nu}\|\Delta b_t\|_{L^2}^2 + C\|b_t\|_{L^2}^2 \\
&\leq \frac{\sigma}{4\nu}\|\nabla b_t\|_{L^2}^2 + \frac{\sigma}{2\nu}\|\Delta b_t\|_{L^2}^2 + \frac{1}{2}\|\nabla u_t\|_{L^2}^2 + C(\|\nabla b\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)\|u_t\|_{L^2}^2 + C\|b_t\|_{L^2}^2 \\
&\quad + C\|\nabla u\|_{L^2}^2\|\nabla b_t\|_{L^2}^2.
\end{aligned}$$

Summing up (2.15) and (2.16) yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{\rho}u_t, b_t, \nabla b_t)\|_{L^2}^2 + \|(\nabla u_t, \nabla b_t, \Delta b_t)\|_{L^2}^2 \leq C\|\nabla^2 u\|_{L^2}^2 + C\|b_t\|_{L^2}^2$$

$$+ C\|(\nabla u, \nabla^2 b, \nabla^3 b, \nabla b)\|_{L^2}^2 \|(\sqrt{\rho} u_t, b_t, \nabla b_t)\|_{L^2}^2.$$

Applying (2.7) and Grönwall's inequality to the above inequality, we obtain

$$\|(\sqrt{\rho} u_t, b_t, \nabla b_t)\|_{L^2}^2 + \int_0^t \|(\nabla u_\tau, \nabla b_\tau, \Delta b_\tau)\|_{L^2}^2 d\tau \le C.$$

What's more, by the same argument of $\|u\|_{L^\infty(L^2)}$ in Proposition 2.1, we have

$$\|u_t\|_{L^\infty(L^2)} \le C,$$

which completes the proof of Proposition 2.3. □

Proposition 2.4. *Under the assumption of Proposition 2.3, it holds that for any $t > 0$:*

$$\int_0^t \|\nabla u\|_{L^\infty} d\tau \le C t^{\frac{2}{3}} \tag{2.17}$$

and

$$\sup_{t>0} \|\nabla \rho(t)\|_{L^p} \le C(t). \tag{2.18}$$

Proof. Again, it follows from the regularity of the Stokes system

$$\begin{aligned}
&\|\nabla^2 u\|_{L^4} + \|\nabla P\|_{L^4} \\
&\le \|\rho u_t\|_{L^4} + \|\rho u \cdot \nabla u\|_{L^4} + \|\Delta b \nabla b\|_{L^4} \\
&\le C(\|u_t\|_{L^4} + \|u\|_{L^\infty} \|\nabla u\|_{L^4} + \|\Delta b\|_{L^4} \|\nabla b\|_{L^\infty}) \\
&\le C(\|u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} + \|\nabla^2 b\|_{L^2}^{\frac{1}{2}} \|\nabla^3 b\|_{L^2} \|\nabla b\|_{L^2}^{\frac{1}{2}}).
\end{aligned}$$

By Propositions 2.1–2.3, we obtain

$$\begin{aligned}
\int_0^t \|\nabla^2 u\|_{L^4} d\tau + \int_0^t \|\nabla P\|_{L^4} d\tau &\le C(\int_0^t \|\nabla^2 u\|_{L^4}^2 d\tau)^{\frac{1}{2}} t^{\frac{1}{2}} + C(\int_0^t \|\nabla P\|_{L^4}^2 d\tau)^{\frac{1}{2}} t^{\frac{1}{2}} \\
&\le C(\|u_t\|_{L^2(L^2)} + \|\nabla u_t\|_{L^2(L^2)} + \|u\|_{L^\infty(L^2)}^{\frac{1}{2}} \|\nabla u\|_{L^\infty(L^2)}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2(L^2)} \\
&\quad + \|\nabla b\|_{L^\infty(L^2)}^{\frac{1}{2}} \|\nabla^2 b\|_{L^\infty(L^2)}^{\frac{1}{2}} \|\nabla^3 b\|_{L^2(L^2)}) t^{\frac{1}{2}} \\
&\le C t^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \|\nabla u\|_{L^\infty} d\tau &\le \int_0^t \|\nabla u\|_{L^2}^{\frac{1}{3}} \|\nabla^2 u\|_{L^4}^{\frac{2}{3}} d\tau \le C(\int_0^t \|\nabla^2 u\|_{L^4}^2 d\tau)^{\frac{1}{3}} t^{\frac{2}{3}} \\
&\le \Big(\|u_t\|_{L^2(L^2)}^{\frac{2}{3}} + \|\nabla u_t\|_{L^2(L^2)}^{\frac{2}{3}} + \|u\|_{L^\infty(L^2)}^{\frac{2}{3}} \|\nabla u\|_{L^\infty(L^2)}^{\frac{2}{3}} \|\nabla^2 u\|_{L^2(L^2)}^{\frac{2}{3}} \\
&\quad + \|\nabla b\|_{L^\infty(L^2)}^{\frac{2}{3}} \|\nabla^2 b\|_{L^\infty(L^2)}^{\frac{2}{3}} \|\nabla^3 b\|_{L^2(L^2)}^{\frac{2}{3}}\Big) t^{\frac{2}{3}} \\
&\le C t^{\frac{2}{3}},
\end{aligned}$$

which leads to (2.17). Finally, we recall that the density ρ satisfies

$$\partial_t\rho + u\cdot\nabla\rho = 0.$$

Applying the operator ∇ to both sides of the above equation yields

$$\partial_t\nabla\rho + u\cdot\nabla(\nabla\rho) = -\nabla u\cdot\nabla\rho.$$

By applying the L^p estimate to the above equation, combined with the divergence free condition implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^p} \le \|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^p}.$$

The Grönwall's inequality implies

$$\|\nabla\rho\|_{L^p} \le \|\nabla\rho_0\|_{L^p}\exp\int_0^t \|\nabla u\|_{L^\infty}\mathrm{d}\tau \le C(t).$$

We thus complete the proof of Proposition 2.4. □

3. Proof of Theorem 1.1

The section is to prove Theorem 1.1. For any given ρ_0 and $(u_0, b_0)\in H^s(\mathbb{R}^2)\times H^{s+1}(\mathbb{R}^2)$, we define the initial data

$$\rho_0^\epsilon = \rho_0 * \eta_\epsilon,\quad u_0^\epsilon = u_0 * \eta_\epsilon,\quad b^\epsilon = b_0 * \eta_\epsilon,$$

where η_ϵ is the standard Friedrich's mollifier with $\epsilon > 0$. With the initial data $(\rho_0^\epsilon, u_0^\epsilon, b_0^\epsilon)$, the system (1.1) has a unique global smooth solution $(\rho^\epsilon, u^\epsilon, b^\epsilon)$. From Propositions 2.1 and 2.2, we obtain

$$m \le \rho^\epsilon(x,t) \le M,$$

$$\|(u^\epsilon, b^\epsilon, \nabla u^\epsilon, \nabla b^\epsilon, \nabla^2 b^\epsilon)\|_{L^2}^2 + \int_0^t \|(\sqrt{\rho}u_\tau^\epsilon, b_\tau^\epsilon, \nabla^3 b^\epsilon, \nabla^2 u^\epsilon)\|_{L^2}^2\mathrm{d}\tau \le C.$$

By standard compactness arguments and Lions-Aubin's Lemma, we can obtain a subsequence denoted again by (u^ϵ, b^ϵ), that (u^ϵ, b^ϵ) converges strongly to (u, b) in $L^2(\mathbb{R}^+; H^{s_1})\times L^2(\mathbb{R}^+; H^{s_2})$, as $\epsilon\to 0$, for $s_1 < 2$ and $s_2 < 3$. By the definition of ρ_0^ϵ and let $\epsilon\to 0$, we find that the limit ρ of ρ^ϵ satisfies $m\le\rho\le M$.

Next, we shall prove the uniqueness of the solutions. Assume that $(\rho_i, u_i, b_i)\,(i = 1,2)$ be two solutions of system (1.1), which satisfy the regularity propositions listed in Theorem 1.1. We denote

$$(\tilde\rho, \tilde u, \tilde b, \tilde P) \stackrel{def}{=} (\rho_2-\rho_1, u_2-u_1, b_2-b_1, P_2-P_1).$$

Then the system for $(\tilde\rho, \tilde u, \tilde b, \tilde P)$ reads

$$\begin{cases}\tilde\rho_t + u_2\cdot\nabla\tilde\rho = -\tilde u\cdot\nabla\rho_1,\\ \rho_2\tilde u_t + \rho_2 u_2\cdot\nabla\tilde u - \Delta\tilde u + \nabla\tilde P = \tilde F,\\ \tilde b_t + u_2\cdot\nabla\tilde b + \frac{1}{\nu}(e'(b_2) - e'(b_1) - \sigma\Delta\tilde b) = -\tilde u\cdot\nabla b_1,\\ \operatorname{div}\tilde u = 0,\\ (\tilde\rho, \tilde u, \tilde b)(t,x)|_{t=0} = (0,0,0),\end{cases}\tag{3.1}$$

where

$$\tilde{F} = -\sigma\Delta\tilde{b}\nabla b_2 - \sigma\Delta b_2\nabla\tilde{b} - \tilde{\rho}\partial_t u_1 - \tilde{\rho}u_1 \cdot \nabla u_1 - \rho_2\tilde{u} \cdot \nabla u_1.$$

Setting $\nu = \sigma = 1$ in what follows.

Step 1: Taking L^2 inner product to the second equation of (3.1) with $\tilde{u}$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_2}\tilde{u}\|_{L^2}^2 + \|\nabla\tilde{u}\|_{L^2}^2 = &-\int_{\mathbb{R}^2}\Delta\tilde{b}\nabla b_2 \cdot \tilde{u}\mathrm{d}x - \int_{\mathbb{R}^2}\Delta b_2\nabla\tilde{b} \cdot \tilde{u}\mathrm{d}x - \int_{\mathbb{R}^2}\tilde{\rho}\partial_t u_1 \cdot \tilde{u}\mathrm{d}x \\ &- \int_{\mathbb{R}^2}\tilde{\rho}u_1 \cdot \nabla u_1 \cdot \tilde{u}\mathrm{d}x - \int_{\mathbb{R}^2}\rho_2\tilde{u} \cdot \nabla u_1 \cdot \tilde{u}\mathrm{d}x. \end{aligned} \tag{3.2}$$

By Hölder's and interpolation inequalities, we have

$$\begin{aligned}&-\int_{\mathbb{R}^2}\Delta\tilde{b}\nabla b_2 \cdot \tilde{u}\mathrm{d}x - \int_{\mathbb{R}^2}\Delta b_1\nabla\tilde{b} \cdot \tilde{u}\mathrm{d}x \\ &\le C\|\Delta\tilde{b}\|_{L^2}\|\nabla b_2\|_{L^4}\|\tilde{u}\|_{L^4} + C\|\Delta b_1\|_{L^4}\|\nabla\tilde{b}\|_{L^2}\|\tilde{u}\|_{L^4} \\ &\le C\|\Delta\tilde{b}\|_{L^2}\|\nabla b_2\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b_2\|_{L^2}^{\frac{1}{2}}\|\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|\Delta b_1\|_{L^2}^{\frac{1}{2}}\|\nabla^3 b_1\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{b}\|_{L^2}\|\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{1}{8}\|(\Delta\tilde{b}, \nabla\tilde{b})\|_{L^2}^2 + \frac{1}{8}\|\nabla\tilde{u}\|_{L^2}^2 + C(\|\nabla b_2\|_{L^2}^2\|\nabla^2 b_2\|_{L^2}^2 + \|\nabla^3 b_1\|_{L^2}^2\|\nabla^2 b_1\|_{L^2}^2)\|\tilde{u}\|_{L^2}^2. \end{aligned} \tag{3.3}$$

Similarly,

$$\begin{aligned}&-\int_{\mathbb{R}^2}\tilde{\rho}\partial_t u_1 \cdot \tilde{u}\mathrm{d}x - \int_{\mathbb{R}^2}\tilde{\rho}u_1 \cdot \nabla u_1 \cdot \tilde{u}\mathrm{d}x \\ &\le \|\tilde{\rho}\|_{L^2}(\|\partial_t u_1\|_{L^4} + \|u_1 \cdot \nabla u_1\|_{L^4})\|\tilde{u}\|_{L^4} \\ &\le \|\tilde{\rho}\|_{L^2}(\|\partial_t u_1\|_{L^2} + \|\nabla\partial_t u_1\|_{L^2} + \|u_1\|_{L^\infty}\|\Delta u_1\|_{L^2} + \|u_1\|_{L^\infty}\|\nabla u_1\|_{L^2}) \\ &\quad \times (\|\tilde{u}\|_{L^2} + \|\nabla\tilde{u}\|_{L^2}) \\ &\le \frac{1}{8}\|\nabla\tilde{u}\|_{L^2}^2 + \mathcal{F}_1(t)\|\tilde{\rho}\|_{L^2}^2 + C\|\tilde{u}\|_{L^2}^2, \end{aligned} \tag{3.4}$$

where

$$\mathcal{F}_1(t) = \|\partial_t u_1\|_{L^2}^2 + \|\nabla\partial_t u_1\|_{L^2}^2 + \|u_1\|_{L^\infty}^2\|\Delta u_1\|_{L^2}^2 + \|u_1\|_{L^\infty}^2\|\nabla u_1\|_{L^2}^2.$$

Hölder's inequality implies

$$-\int_{\mathbb{R}^2}\rho_2\tilde{u} \cdot \nabla u_1 \cdot \tilde{u}\mathrm{d}x \le \|\nabla u_1\|_{L^\infty}\|\sqrt{\rho_2}\tilde{u}\|_{L^2}^2. \tag{3.5}$$

By substituting above estimates (3.3)–(3.5) into (3.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_2}\tilde{u}\|_{L^2}^2 + \|\nabla\tilde{u}\|_{L^2}^2 \le \frac{1}{4}\|\Delta\tilde{b}\|_{L^2}^2 + \frac{1}{4}\|\nabla\tilde{b}\|_{L^2}^2 + C\mathcal{F}_2(t)\|\tilde{u}\|_{L^2}^2 + \mathcal{F}_1(t)\|\tilde{\rho}\|_{L^2}^2, \tag{3.6}$$

where

$$\mathcal{F}_2(t) = \|\nabla b_2\|_{L^2}^2\|\nabla^2 b_2\|_{L^2}^2 + \|\nabla^3 b_1\|_{L^2}^2\|\nabla^2 b_1\|_{L^2}^2 + \|\nabla u_1\|_{L^\infty} + 1.$$

Step 2: Taking L^2 inner product to the third equation of (3.1) with $\tilde{b} - \Delta\tilde{b}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\tilde{b}, \nabla\tilde{b})\|_{L^2}^2 + \|(\nabla\tilde{b}, \Delta\tilde{b})\|_{L^2}^2 + \int_{\mathbb{R}^2}[e'(b_2) - e'(b_1)]\tilde{b}\mathrm{d}x \\
&= \int_{\mathbb{R}^2} u_2 \cdot \nabla\tilde{b} \cdot \Delta\tilde{b}\mathrm{d}x - \int_{\mathbb{R}^2} \tilde{u} \cdot \nabla b_1 \cdot (\tilde{b} - \Delta\tilde{b})\mathrm{d}x + \int_{\mathbb{R}^2}[e'(b_2) - e'(b_1)]\Delta\tilde{b}\mathrm{d}x. \qquad (3.7)
\end{aligned}$$

We shall estimate each term on the right-hand side of (3.7). For the first term of (3.7), using Hölder's inequality, we have

$$\int_{\mathbb{R}^2} u_2 \cdot \nabla\tilde{b} \cdot \Delta\tilde{b}\mathrm{d}x \le \|u_2\|_{L^\infty}\|\nabla\tilde{b}\|_{L^2}\|\Delta\tilde{b}\|_{L^2} \le \frac{1}{8}\|\Delta\tilde{b}\|_{L^2}^2 + C\|u_2\|_{L^\infty}^2\|\nabla\tilde{b}\|_{L^2}^2. \qquad (3.8)$$

Meanwhile, we have

$$\int_{\mathbb{R}^2}[e'(b_2) - e'(b_1)]\tilde{b}\mathrm{d}x = \int_{\mathbb{R}^2} e''(\xi)\tilde{b}^2\mathrm{d}x > 0, \qquad (3.9)$$

where ξ is a function between b_2 and b_1.
Moreover,

$$\begin{aligned}
&-\int_{\mathbb{R}^2} \tilde{u} \cdot \nabla b_1 \cdot (\tilde{b} - \Delta\tilde{b})\mathrm{d}x \le C\|\tilde{u}\|_{L^4}\|\nabla b_1\|_{L^4}(\|\tilde{b}\|_{L^2} + \|\Delta\tilde{b}\|_{L^2}) \\
&\le \frac{1}{8}\|\Delta\tilde{b}\|_{L^2}^2 + C\|\tilde{b}\|_{L^2}^2 + C\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\nabla b_1\|_{L^2}\|\Delta b_1\|_{L^2} \\
&\le \frac{1}{8}\|\Delta\tilde{b}\|_{L^2}^2 + C\|\tilde{b}\|_{L^2}^2 + \frac{1}{8}\|\nabla\tilde{u}\|_{L^2}^2 + C\|\tilde{u}\|_{L^2}^2\|\nabla b_1\|_{L^2}^2\|\Delta b_1\|_{L^2}^2, \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^2}[e'(b_2) - e'(b_1)]\Delta\tilde{b}\mathrm{d}x &= \int_{\mathbb{R}^2} e''(\xi)\tilde{b}\Delta\tilde{b}\mathrm{d}x \\
&\le C_0\|\tilde{b}\|_{L^2}\|\Delta\tilde{b}\|_{L^2} \le \frac{1}{4}\|\Delta\tilde{b}\|_{L^2}^2 + C\|\tilde{b}\|_{L^2}^2. \qquad (3.11)
\end{aligned}$$

By inserting (3.8)–(3.11) into (3.7), one yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\tilde{b}, \nabla\tilde{b})\|_{L^2}^2 + \frac{1}{2}\|(\nabla\tilde{b}, \Delta\tilde{b})\|_{L^2}^2 \\
&\le C\|u_2\|_{L^\infty}^2\|\nabla\tilde{b}\|_{L^2}^2 + C\|\tilde{b}\|_{L^2}^2 + \frac{1}{8}\|\nabla\tilde{u}\|_{L^2}^2 + C\|\tilde{u}\|_{L^2}^2\|\nabla b_1\|_{L^2}^2\|\Delta b_1\|_{L^2}^2.
\end{aligned} \qquad (3.12)$$

Step 3: We will derive the estimate of $\|\tilde{\rho}\|_{L^2}$ as follows:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\rho}\|_{L^2}^2 &\le \|\tilde{u} \cdot \nabla\rho_1\|_{L^2}\|\tilde{\rho}\|_{L^2} \\
&\le \|\tilde{u}\|_{L^4}\|\nabla\rho_1\|_{L^4}\|\tilde{\rho}\|_{L^2} \\
&\le \|\nabla\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\rho_1\|_{L^4}\|\tilde{\rho}\|_{L^2} \\
&\le \frac{1}{4}\|\nabla\tilde{u}\|_{L^2}^2 + C\|\nabla\rho_1\|_{L^4}^{\frac{4}{3}}(\|\tilde{\rho}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2). \qquad (3.13)
\end{aligned}$$

Step 4: Summing up the above estimates, that is, (3.6), (3.12), and (3.13), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\tilde{\rho},\ \sqrt{\rho_2}\tilde{u},\tilde{b},\nabla\tilde{b})\|_{L^2}^2+\|(\nabla\tilde{u},\nabla\tilde{b},\Delta\tilde{b})\|_{L^2}^2\\
&\le C\mathcal{F}_5(t)\|\nabla\tilde{b}\|_{L^2}^2+C\|\tilde{b}\|_{L^2}^2+C\mathcal{F}_4(t)\|\tilde{u}\|_{L^2}^2+\mathcal{F}_3(t)\|\tilde{\rho}\|_{L^2}^2\\
&\le C(1+\mathcal{F}_3(t)+\mathcal{F}_4(t)+\mathcal{F}_5(t))\|(\tilde{\rho},\ \sqrt{\rho_2}\tilde{u},\tilde{b},\nabla\tilde{b})\|_{L^2}^2, \qquad (3.14)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_3(t)&=\|\partial_t u_1\|_{L^2}^2+\|\nabla\partial_t u_1\|_{L^2}^2+\|u_1\|_{L^\infty}^2\|\Delta u_1\|_{L^2}^2+\|u_1\|_{L^\infty}^2\|\nabla u_1\|_{L^2}^2+\|\nabla\rho_1\|_{L^4}^{\frac{4}{3}},\\
\mathcal{F}_4(t)&=\|\nabla b_2\|_{L^2}^2\|\nabla^2 b_2\|_{L^2}^2+\|\nabla^3 b_1\|_{L^2}^2\|\nabla^2 b_1\|_{L^2}^2+\|\nabla u_1\|_{L^\infty}+\|\nabla b_1\|_{L^2}^2\|\Delta b_1\|_{L^2}^2\\
&\quad+\|\nabla\rho_1\|_{L^4}^{\frac{4}{3}}+1,\\
\mathcal{F}_5(t)&=\|u_2\|_{L^\infty}^2.
\end{aligned}$$

Noticing the fact that $\int_0^t(1+\mathcal{F}_3(\tau)+\mathcal{F}_4(\tau)+\mathcal{F}_5(\tau))\mathrm{d}\tau\le Ct+C$ and that $\|f\|_{L^\infty}^2\le\|f\|_{L^2}\|\nabla^2 f\|_{L^2}$, we can obtain that there exists a small ϵ_0 such that

$$\|(\tilde{\rho},\ \sqrt{\rho_2}\tilde{u},\tilde{b},\nabla\tilde{b})\|_{L^\infty(L^2)}=0,$$

for $t\in[0,\epsilon_0]$. Therefore, we obtain $\tilde{\rho}(t)=\tilde{u}(t)=\tilde{b}(t)\equiv0$ for any $t\in[0,\epsilon_0]$. The uniqueness of such strong solutions on the whole time interval $[0,+\infty)$ then follows by a bootstrap argument.

Moreover, the continuity with respect to the initial data may also be obtained by the same argument in the proof of the uniqueness, which ends the proof of Theorem 1.1.

4. Conclusions

This paper focuses on two-dimensional inhomogeneous incompressible viscoelastic rate-type fluids with stress-diffusion. We have established its global solution, and the uniqueness of the solution in specific situations is also proved in this paper.

Author contributions

Xi Wang and Xueli Ke: Conceptualization, methodology, validation, writing-original draft, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors have no relevant financial or non-financial interests to disclose. The authors have no competing interests to declare that are relevant to the content of this article.

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