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Research article

Strong law of large numbers for weighted sums of m-widely acceptable random variables under sub-linear expectation space

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Abstract: In this article, using the Fuk-Nagaev type inequality, we studied general strong law of large numbers for weighted sums of m-widely acceptable (m-WA, for short) random variables under sublinear expectation space with the integral condition

$$\hat{\mathbb{E}}\left(f^{-}\left(|X|\right)\right) \leq \mathrm{C}_{\mathbb{V}}\left(f^{-}\left(|X|\right)\right) < \infty$$

and *Choquet* integrals existence, respectively, where

$$f\left(x\right) = x^{1/\beta} L\left(x\right)$$

for $\beta > 1$, $L\left(x\right) > 0\ (x > 0)$ was a monotonic nondecreasing slowly varying function, and $f^{-}\left(x\right)$ was the inverse function of $f\left(x\right)$. One of the results included the Kolmogorov-type strong law of large numbers and the partial Marcinkiewicz-type strong law of large numbers for m-WA random variables under sublinear expectation space. Besides, we obtained almost surely convergence for weighted sums of m-WA random variables under sublinear expectation space. These results improved the corresponding results of Ma and Wu under sublinear expectation space.

Keywords: m-widely acceptable random variables; strong law of large numbers; sublinear expectation space; slowly varying function; almost surely convergence
Mathematics Subject Classification: 60F15

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1. Introduction

Probability limit theories are widely used in various fields of life, including statistics, finance, medicine, engineering, etc. When the mathematical model is definite, classical probability limit theories offer a convenient way to solve problems. However, in a practical situation, some phenomena exist in uncertainty, such as risk measure, super-hedging in finance, and assets pricing, which cannot be settled by classical probability limit theories. In other words, linear additivity cannot be satisfied. Therefore, to solve the limitation of the phenomena, Peng [1–3] introduced the concept of sublinear expectation and established the sublinear expectation space as an extension for classical probability limit theory. Due to the fact that classical probability space tools may not be directly applied in sublinear expectation space, Peng [4] introduced several concepts in sublinear expectation space, such as identical distribution, independence, maximum distribution, G-normal distribution, and so on. Furthermore, the theory of sublinear expectation space can be found in Peng's [1–4]. In recent years, numerous scholars have dedicated themselves to the theoretical research of sublinear expectation space. Zhang [5–7] obtained a series of major inequalities under sublinear expectation space. Dong and Tan [8] got complete convergence and complete integration convergence for arrays of row-wise m-extended negatively dependent (m-END) under sublinear expectation space. Guo and Zhang [9] studied the central limit theorem of m-dependent random variables under sublinear expectation space. In addition, we can read Zhong and Wu [10], Anna [11, 12], Liu and Zhang [13], Wu et al. [14], Feng et al. [15], and so on.

The strong law of large numbers is one of the important theorems in probability limit theories. In practical applications, especially in statistical inference and data analysis, the strong law of large numbers makes us believe that the sample mean can be used as an estimate of the population mean. Let $\{X_i,\ i \ge 1\}$ be a sequence of random variables in the probability space, and let $\{a_n,\ n \ge 1\}$ and $\{b_n,\ n \ge 1\}$ be sequences of constants with

$$0 < b_n \uparrow \infty.$$

The sequence $\{X_i,\ i \ge 1\}$ has a finite expectation $\mathrm{E}X_i$. Then, $\{a_iX_i,\ i \ge 1\}$ is said to obey the general strong law of large numbers with constant $\{b_n,\ n \ge 1\}$ if

$$\frac{1}{b_n}\sum_{i=1}^{n} a_i\left(X_i - \mathrm{E}X_i\right) \to 0 \ \ almost\ surely\ (a.s.)\ \mathrm{P} \tag{1.1}$$

holds. If

$$b_n = n,\ \ a_n = 1,$$

it is the Kolmogorov-type strong law of large numbers. If

$$b_n = n^{1/r},\ \ a_n = 1,\ \ r > 0,$$

it is the Marcinkiewicz-type strong law of large numbers. When

$$b_n = \sum_{i=1}^{n} a_i,$$

the fundamental result is obtained for the strong law of large numbers. In recent years, many results of the strong law of large numbers have been obtained in sublinear expectation space. Zhang and Lin [5, 16] established the Kolmogorov and Marcinkiewicz strong law of large numbers of independent and identical random variables under sublinear expectation space with the condition

$$\lim_{c\to\infty} \hat{\mathbb{E}}\left[(|X_1| - c)^+\right] = 0.$$

Chen [17] studied strong law of the large number of independent and identical random variables under sublinear expectation space with the condition

$$\hat{\mathbb{E}}\,|X_1|^{1+\alpha} < \infty$$

for some $\alpha \in (0, 1]$. Hu [18] obtained weak and strong laws of large numbers of independent random variables under sublinear expectation space with the condition

$$\lim_{n\to\infty} \sup_{m\geq 1} \hat{\mathbb{E}}\left[|X_m|\, I\left(|X_m| > n\right)\right] = 0.$$

Moreover, we can refer to Jiang and Wu [19], Wu and Deng [20], Ma and Wu [21], Tan et al. [22], and so on.

Recently, Wu et al. [20] obtained capacity inequalities and strong laws for m-WA (m-widely acceptable) random variables under sublinear expectation space. Ma and Wu [21] established strong law of large numbers for weighted sums of END random variables on some conditions under sublinear expectation space, which was inspired by Shen et al. [23]. Therefore, the goal of this article is to establish strong law of large numbers and almost surely convergence for weighted sums of m-WA random variables under sublinear expectation space. These results improve the corresponding results of Ma and Wu [21] under the sublinear expectation space. In addition, the main structure of this article is as follows. In the Section 2, we introduce some basic definitions and main lemmas to provide tools for proofs of main results. In the Section 3, we give the main results for strong law of large numbers and the almost surely convergence of m-WA random variables under sublinear expectation space with the condition

$$\hat{\mathbb{E}}\left(f^-\left(|X|\right)\right) \leq C_{\mathbb{V}}\left(f^-\left(|X|\right)\right) < \infty$$

and *Choquet* integrals existence. In the Section 4, corresponding proofs of main results are provided.

2. Preliminaries

We use the framework and notions of Peng [1–4]. Let $(\Omega,\ \mathcal{F})$ be a given measurable space. $\mathcal{H}$ was a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \cdots, X_n \in \mathcal{H}$, then $\varphi\left(X_1,\ X_2,\ \cdots,\ X_n\right) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}\left(\mathbb{R}^n\right)$, where $C_{l,Lip}\left(\mathbb{R}^n\right)$ denotes the linear space of local Lipschitz functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq c\left(1 + |x|^m + |y|^m\right)|x - y|,\ \ \forall x,\ y \in \mathbb{R}^n,$$

for some $c > 0$ and $m \in \mathbb{N}$ depending on φ. Therefore, $\mathcal{H}$ can be a space of random variables. In this case we denote $X \in \mathcal{H}$. We also define $C_{b,Lip}\left(\mathbb{R}^n\right)$ as the linear space bounded Lipschitz continuous functions φ fulfilling

$$|\varphi(x) - \varphi(y)| \leq c\,|x - y|,\ \ \forall x,\ y \in \mathbb{R}^n$$

for some $c > 0$.

Definition 2.1. [4] A sublinear expectation $\hat{\mathbb{E}}$ is a function $\hat{\mathbb{E}}$ on

$$\mathcal{H} : \mathcal{H} \to \bar{\mathbb{R}} := [-\infty, +\infty]$$

satisfying the following conditions: for all $X, Y \in \mathcal{H}$,

(1) Monotonicity: $\hat{\mathbb{E}}(X) \geq \hat{\mathbb{E}}(Y)$ if $X \geq Y$;

(2) Constant preserving: $\hat{\mathbb{E}}(c) = c$ for $c \in \mathbb{R}$;

(3) Sub-additivity: $\hat{\mathbb{E}}(X+Y) \leq \hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$, whenever $\hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(4) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}(X), \ \forall \lambda \geq 0$.

The triple $\left(\Omega, \ \mathcal{H}, \ \hat{\mathbb{E}}\right)$ is called a sublinear expectation space.

For a given a sublinear $\hat{\mathbb{E}}$, let's define a conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}(X) := -\hat{\mathbb{E}}(-X), \ \forall X \in \mathcal{H}.$$

From the definition, it is easily obtained that for all $X, Y \in \mathcal{H}$,

$$\hat{\varepsilon}(X) \leq \hat{\mathbb{E}}(X), \ \hat{\mathbb{E}}(X+c) = \hat{\mathbb{E}}(X) + c, \ \left|\hat{\mathbb{E}}(X-Y)\right| \leq \hat{\mathbb{E}}|X-Y|, \ \hat{\mathbb{E}}(X-Y) \geq \hat{\mathbb{E}}(X) - \hat{\mathbb{E}}(Y).$$

Definition 2.2. [5] Let $\mathcal{G} \subset \mathcal{F}$. A function $\mathrm{V}: \mathcal{G} \to [0, \ 1]$ is called a capacity satisfying

(a) $\mathrm{V}(\emptyset) = 0, \mathrm{V}(\Omega) = 1$;

(b) $\mathrm{V}(A) \leq \mathrm{V}(B), \forall A \subseteq B, A, B \in \mathcal{G}$.

It is called to be sub-additive if

$$\mathrm{V}(A \cup B) \leq \mathrm{V}(A) + \mathrm{V}(B)$$

for all $A, B \in \mathcal{G}$ with

$$A \cup B \in \mathcal{G}.$$

Let $\left(\Omega, \ \mathcal{H}, \ \hat{\mathbb{E}}\right)$ be a sublinear expectation space and $\hat{\varepsilon}$ be a conjugate expectation of $\hat{\mathbb{E}}$. We define a pair $(\mathbb{V}, \ \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\left\{\hat{\mathbb{E}}(\xi) : I_A \leq \xi, \xi \in \mathcal{H}\right\}, \ \ \mathcal{V}(A) = 1 - \mathbb{V}(A^c), \ \forall A \in \mathcal{F},$$

where A^c is the complement set of A. From the above definition,

$$\hat{\mathbb{E}}(f) \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}(g), \ \ \hat{\varepsilon}(f) \leq \mathcal{V}(A) \leq \hat{\varepsilon}(g), \ \ if \ f \leq I(A) \leq g, \ f, \ g \in \mathcal{H}.$$

For all $X \in \mathcal{H}$, $p > 0$, and $x > 0$,

$$I(|X| > x) \leq \frac{|X|^p}{x^p} I(|X| > x) \leq \frac{|X|^p}{x^p},$$

and we can get the Markov inequality:

$$\mathbb{V}(|X| \geq x) \leq \frac{\hat{\mathbb{E}}|X|^p}{x^p}, \ \ p > 0, \ x > 0.$$

Definition 2.3. [5] The *Choquet* integral/expectation $(\mathrm{C}_{\mathbb{V}}, \ \mathrm{C}_{\mathcal{V}})$ is defined by

$$\mathrm{C}_{\mathrm{V}}(X) = \int X \mathrm{d}\mathrm{V} = \int_{-\infty}^{0} (\mathrm{V}(X \geq t) - 1)\, \mathrm{d}t + \int_{0}^{\infty} \mathrm{V}(X \geq t)\, \mathrm{d}t, \ \ \forall \ X \in \mathcal{H},$$

where V is replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively.

Definition 2.4. [19] (i) A sublinear expectation $\hat{\mathbb{E}}$: $\mathcal{H} \to \mathbb{R}$ is called to be countably sub-additive if it satisfies

$$\hat{\mathbb{E}}(X) \le \sum_{i=1}^{\infty} \hat{\mathbb{E}}(X_i),$$

whenever

$$X \le \sum_{i=1}^{\infty} X_i,$$

$X, X_i \in \mathcal{H}$ and $X \ge 0, X_i \ge 0, i \ge 1$.

(ii) A function V: $\mathcal{F} \to [0,\ 1]$ is called to be countably sub-additive if

$$\mathrm{V}\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mathrm{V}(A_i), \quad \forall A_i \in \mathcal{F}.$$

Definition 2.5. [19] A sequence of random variables $\{X_i,\ i \ge 1\}$ is called to converge to X a.s. V defined by $X_i \to X$ a.s. V as $i \to \infty$, if

$$\mathrm{V}(X_i \nrightarrow X) = 0.$$

Further, by

$$\mathcal{V}(A) + \mathbb{V}(A^c) = 1$$

for any $A \in \mathcal{F}$,

$$X_i \to X \ \ a.s.\ \mathbb{V} \Longleftrightarrow \mathcal{V}(X_i \to X) = 1.$$

Definition 2.6. [20] Suppose $\{X_i,\ i \ge 1\}$ is a sequence of random variables in sublinear expectation space $\left(\Omega,\ \mathcal{H},\ \hat{\mathbb{E}}\right)$. $\{X_i,\ i \ge 1\}$ is called to be WA if there exists a positive sequence $\{g(n),\ n \ge 1\}$ dominating coefficients such that for each $n \ge 1$,

$$\hat{\mathbb{E}}\left[\exp\left(\sum_{i=1}^{n} a_{ni}\varphi_i(X_i)\right)\right] \le g(n) \prod_{i=1}^{n} \hat{\mathbb{E}}\left[\exp\left(a_{ni}\varphi_i(X_i)\right)\right],$$

where $\{a_{ni},\ 1 \le i \le n,\ n \ge 1\}$ is an array of nonnegative constants and

$$\varphi_i(\cdot) \in \mathrm{C}_{b,Lip}(\mathbb{R}), \quad i \ge 1$$

are all nondecreasing (resp., all nonincreasing) real-valued truncation functions.

Definition 2.7. [20] Let $m \ge 1$ be a fixed integer. A sequence of random variables $\{X_i,\ i \ge 1\}$ is said to be m-WA if for any $i \ge 2$ and any $n_1, n_2, n_3, \cdots, n_i$ satisfying

$$\left|n_k - n_j\right| \ge m$$

for all $1 \le k \ne j \le i$, we have that $X_{n_1}, X_{n_2}, \cdots, X_{n_i}$ are WA.

Remark 2.1. It is easily seen that m-WA random variables are a natural extension of WA random variables. It follows by the definition of m-WA random variables that sequences

$$\{X_1,\ X_{1+m},\ X_{1+2m}, \cdots\}, \quad \{X_2,\ X_{2+m},\ X_{2+2m}, \cdots\}, \quad \cdots, \{X_m,\ X_{2m},\ X_{3m}, \cdots\}$$

are WA and m-WA is WA if $m = 1$. m-WA random variables include negatively dependent (ND) random variables, END random variables, widely negative orthant dependent (WOD) random variables, m-END random variables, m-WOD random variables, etc. Thus, it is meaningful to research probability limit theories for m-WA random variables.

Definition 2.8. [24] A function $L(x)$: $(0, \infty) \to (0, \infty)$ is called a slowly varying function, if for any $\lambda > 0$,

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1.$$

In this paper, the symbol c stands for a positive constant which may not be the same in various places. Let C be a concrete positive constant. $I(\mathrm{A})$ is the indicator function of the event A. $a_n = \mathrm{O}(b_n)$ means there exists a constant $c > 0$ such that $a_n \le cb_n$ for all $n \ge 1$. $a_n \ll b_n$ means that there exists a constant $c > 0$ such that $a_n \le cb_n$ for sufficiently large n. The symbol $\#A$ is on behalf of the number of elements in set A.

Lemma 2.1. [20] Let $\{X_i,\ i \ge 1\}$ be a sequence of m-WA random variables with dominating coefficients $\{g(n),\ n \ge 1\}$ in sublinear expectation space $\left(\Omega,\ \mathcal{H},\ \hat{\mathbb{E}}\right)$. If $\{\varphi_i(\cdot),\ i \ge 1\} \in \mathrm{C}_{b,Lip}(\mathbb{R})$ are all non-decreasing (resp., all nonincreasing), then the sequence $\{\varphi_i(X_i),\ i \ge 1\}$ is still m-WA random variables with dominating coefficients $\{g(n),\ n \ge 1\}$.

Lemma 2.2. [25] (Borel-Cantelli's lemma) Let $\{A_n, n \ge 1\}$ be a sequence of events in $\mathcal{F}$. Suppose that V is a countably sub-additive capacity. If

$$\sum_{n=1}^{\infty} \mathrm{V}(A_n) < \infty,$$

then

$$\mathrm{V}(A_n,\ i.o.) = 0,$$

where

$$\{A_n,\ i.o.\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

Lemma 2.3. (1) C_r inequality [3]: Let $X_1, X_2, \cdots, X_n \in \mathcal{H}$ for $n \ge 1$, then

$$\hat{\mathbb{E}}|X_1 + X_2 + \cdots + X_n|^r \le \mathrm{C}_r\left[\hat{\mathbb{E}}|X_1|^r + \hat{\mathbb{E}}|X_2|^r + \cdots + \hat{\mathbb{E}}|X_n|^r\right],$$

where

$$\mathrm{C}_r = \begin{cases} 1, & 0 < r \le 1, \\ n^{r-1}, & r > 1. \end{cases}$$

(2) Jensen inequality [25]: Let $f(\cdot)$ be a convex function on $\mathbb{R}$. Assume that $\hat{\mathbb{E}}(X)$ and $\hat{\mathbb{E}}(f(X))$ exist. Then,

$$\hat{\mathbb{E}}[f(X)] \ge f\left(\hat{\mathbb{E}}(X)\right).$$

(3) [15] For all $X \in \mathcal{H}$ and $0 < r < s$,

$$\left(\hat{\mathbb{E}}|X|^r\right)^{1/r} \le \left(\hat{\mathbb{E}}|X|^s\right)^{1/s}.$$

Lemma 2.4. [14] For any $X, Y \in \mathcal{H}$, it holds that

$$\left|\hat{\mathbb{E}}(X) - \hat{\mathbb{E}}(Y)\right| \le \hat{\mathbb{E}}|X - Y|.$$

Lemma 2.5. [26] (Toeplitz lemma) Let k_n be a positive number and $\{a_{nk},\ 1 \le k \le k_n,\ n \ge 1\}$ be an array of real numbers fulfilling for any $k \ge 1$,

$$\lim_{n\to\infty} a_{nk} = 0$$

and

$$\sup_{n\ge 1} \sum_{k=1}^{k_n} |a_{nk}| < \infty.$$

Let $\{x_n, n \ge 1\}$ be a sequence of real numbers. If

$$\lim_{n\to\infty} x_n = 0,$$

then,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} a_{nk} x_k = 0.$$

Lemma 2.6. [24] A function $L(x)$: $(0,\ \infty) \to (0,\ \infty)$ is a slowly varying function, then for $\eta > 0$,

$$\lim_{x\to\infty} x^{\eta} L(x) = \infty, \quad \lim_{x\to\infty} x^{-\eta} L(x) = 0.$$

Lemma 2.7. [27] Assume that $X \in \mathcal{H}$ and

$$f(x) = x^{1/\beta} L(x), \quad 0 < \beta < \infty,$$

and $L(x)$ is a slowly varying function. Then, for any $c > 0$,

$$C_{\mathbb{V}}\left(f^{-}(|X|)\, c^{-\beta}\right) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} \mathbb{V}\left(|X| > c n^{1/\beta} L(n)\right) < \infty,$$

where $f^{-}(x)$ is the inverse function of $f(x)$.

Lemma 2.8. [20] (Fuk-Nagaev type inequality) Let $\{X_i,\ i \ge 1\}$ be a sequence of m-WA random variables in $\left(\Omega,\ \mathcal{H},\ \hat{\mathbb{E}}\right)$ with

$$\hat{\mathbb{E}}[X_i] \le 0$$

for $i \ge 1$. Then, for all $x > 0$ and $d > 0$,

$$\mathbb{V}\left(\sum_{i=1}^{n} X_i > x\right) \le m\mathbb{V}\left(\max_{1\le i\le n} X_i > \frac{d}{m}\right) + mg(n)\exp\left\{\frac{x}{d} - \frac{x}{d}\ln\left(1 + \frac{xd/m^2}{\sum_{i=1}^{n} \hat{\mathbb{E}}|X_i|^2}\right)\right\}.$$

Lemma 2.9. [28] (Kronecker lemma) Let $\{x_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be sequences of real numbers with $0 < b_n \uparrow \infty$. If the series $\sum_{n=1}^{\infty} \frac{x_n}{b_n}$ converges, then

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} x_i = 0.$$

3. Main results

Theorem 3.1. Assume that
$$1<\beta<\alpha,\quad 1\le s<2,\quad \frac{1}{s}=\frac{1}{\alpha}+\frac{1}{\beta},$$
and $\mathbb{V}$ is countably sub-additive. Let $\{X_i,\ i\ge 1\}$ be a sequence of m-WA random variables dominated by
$$g(n)=\mathrm{O}\left(n^{\theta}\right)\ (0\le\theta<\frac{2}{\alpha}-1)$$
in $\left(\Omega,\ \mathcal{H},\ \hat{\mathbb{E}}\right)$. Suppose that
$$f(x)=x^{1/\beta}L(x),$$
where $L(x)>0\ (x>0)$ is a monotonic nondecreasing slowly varying function, and $f^{-}(x)$ is the inverse function of $f(x)$. Further, there exist a random variable X and a constant C satisfying that
$$\hat{\mathbb{E}}\left[\psi(X_i)\right]\le \mathrm{C}\hat{\mathbb{E}}\left[\psi(X)\right],\ \ \forall\ i\ge 1,\ 0\le\psi\in \mathrm{C}_{l,Lip}(\mathbb{R}) \tag{3.1}$$
and
$$\hat{\mathbb{E}}\left[f^{-}(|X|)\right]\le \mathrm{C}_{\mathbb{V}}\left[f^{-}(|X|)\right]<\infty. \tag{3.2}$$
Let $\{a_i,\ i\ge 1\}$ and $\{b_i,\ i\ge 1\}$ be sequences of positive numbers with $a_i\uparrow$, $b_i\uparrow\infty$, and the following two conditions hold:
$$\sum_{i=1}^{n}a_i=\mathrm{O}(b_n), \tag{3.3}$$
$$\frac{a_n}{b_n}=\mathrm{O}\left(n^{-1/s}\right). \tag{3.4}$$
Then,
$$\limsup_{n\to\infty}\ b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n}a_i\left(X_i-\hat{\mathbb{E}}(X_i)\right)\le 0,\ \ a.s.\ \mathbb{V} \tag{3.5}$$
and
$$\liminf_{n\to\infty}\ b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n}a_i\left(X_i-\hat{\varepsilon}(X_i)\right)\ge 0,\ \ a.s.\ \mathbb{V}. \tag{3.6}$$
Furthermore, if
$$\hat{\mathbb{E}}(X_i)=\hat{\varepsilon}(X_i),$$
we have
$$\lim_{n\to\infty}b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n}a_i\left(X_i-\hat{\mathbb{E}}(X_i)\right)=0,\ \ a.s.\ \mathbb{V}. \tag{3.7}$$

Theorem 3.2. Let $\{X_i,\ i \geq 1\}$ be a sequence of m-WA random variables dominated by

$$g(n) = \mathrm{O}\left(n^{\theta}\right)\ (0 \leq \theta < 1)$$

in $\left(\Omega,\ \mathcal{H},\ \hat{\mathbb{E}}\right)$. Assume that

$$1 < \gamma \leq \gamma + \theta < 2,$$

$\hat{\mathbb{E}}$ and $\mathbb{V}$ are countably sub-additive,

$$h(x) = x^{1/\gamma} L(x),$$

where $L(x) > 0$ $(x > 0)$ is a monotonic nondecreasing slowly varying function, and $h^{-}(x)$ is the inverse function of $h(x)$. There exist a random variable X and a constant C satisfying (3.1). For any $c > 0$,

$$\sum_{n=1}^{\infty} n^{\theta/\gamma} \mathbb{V}\left(|X| > cn^{1/\gamma} L(n)\right) < \infty, \tag{3.8}$$

and $\mathrm{C}_{\mathbb{V}}\left(h^{-}(|X|)\right)$ exists. Suppose that $\{a_i,\ i \geq 1\}$ and $\{b_i,\ i \geq 1\}$ are sequences of positive numbers with $b_i \uparrow \infty$ and

$$c_n = b_n/a_n \uparrow \infty$$

fulfilling that

$$\sum_{n=i}^{\infty} b_n^{-2} n^{\theta} \left(L(c_n)\right)^{-2} \ll\ b_i^{-2} c_i^{\theta} \left(L(c_i)\right)^{-2}$$

for sufficiently large i. Note that

$$\mathrm{T}(n) = \#\{i,\ c_i \leq n\} \ll n^{\gamma},\ \ n \geq 1. \tag{3.9}$$

Then

$$\limsup_{n\to\infty}\ b_n^{-1} \left(L(c_n)\right)^{-1} \sum_{i=1}^{n} a_i \left(X_i - \hat{\mathbb{E}}(X_i)\right) \leq 0,\ \ a.s.\ \mathbb{V} \tag{3.10}$$

and

$$\liminf_{n\to\infty}\ b_n^{-1} \left(L(c_n)\right)^{-1} \sum_{i=1}^{n} a_i \left(X_i - \hat{\varepsilon}(X_i)\right) \geq 0,\ \ a.s.\ \mathbb{V}. \tag{3.11}$$

Furthermore, if

$$\hat{\mathbb{E}}(X_i) = \hat{\varepsilon}(X_i),$$

we have

$$\lim_{n\to\infty} b_n^{-1} \left(L(c_n)\right)^{-1} \sum_{i=1}^{n} a_i \left(X_i - \hat{\mathbb{E}}(X_i)\right) = 0,\ \ a.s.\ \mathbb{V}. \tag{3.12}$$

Taking $L(x) = 1$, $a_n = 1$, $b_n = n^s$ for $s = 1$ in Theorem 3.1, we get the Kolmogorov-type strong law of large numbers.

Corollary 3.1. Assume that the conditions of Theorem 3.1 hold and

$$\hat{\mathbb{E}}(X_i) = \hat{\varepsilon}(X_i),$$

then

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \left(X_i - \hat{\mathbb{E}}(X_i)\right) = 0, \quad a.s.\ \mathbb{V}.$$

Besides, taking $L(x) = 1$, $a_n = 1$, $b_n = n^s$ for $1 < s < 2$ in Theorem 3.1, we obtain the partial Marcinkiewicz-type strong law of large numbers.

Corollary 3.2. Assume that the conditions of Theorem 3.1 hold and

$$\hat{\mathbb{E}}(X_i) = \hat{\varepsilon}(X_i),$$

then

$$\lim_{n\to\infty} n^{-s} \sum_{i=1}^{n} \left(X_i - \hat{\mathbb{E}}(X_i)\right) = 0, \quad a.s.\ \mathbb{V}.$$

Taking $L(x) = 1$, $a_n = 1$, $b_n = e^n$ in Theorem 3.2, we have

$$c_n = e^n \uparrow \infty.$$

By

$$c_i = e^i \leq n,$$

we can get

$$i \leq \ln n \ll n.$$

Thus,

$$T(n) \ll n \ll n^{\gamma}$$

for $1 < \gamma < 2$, which satisfies the condition of (3.9). By $0 \leq \theta < 1$ and $n \ll e^n$, we get

$$\begin{aligned}
\sum_{n=i}^{\infty} b_n^{-2} n^{\theta} &= \sum_{n=i}^{\infty} e^{-2n} n^{\theta} \ll \sum_{n=i}^{\infty} e^{-2n} e^{n\theta} \\
&= \sum_{n=i}^{\infty} e^{-(2-\theta)n} \ll e^{-(2-\theta)i} \\
&= e^{-2i} \cdot e^{i\theta} \\
&= b_i^{-2} c_i^{\theta}.
\end{aligned}$$

Therefore, the above conditions satisfy Theorem 3.2. So, we obtain Corollary 3.3.

Corollary 3.3. Assume that the conditions of Theorem 3.2 hold and

$$\hat{\mathbb{E}}(X_i) = \hat{\varepsilon}(X_i),$$

then

$$\lim_{n\to\infty} e^{-n} \sum_{i=1}^{n} \left(X_i - \hat{\mathbb{E}}(X_i)\right) = 0, \quad a.s.\ \mathbb{V}.$$

Remark 3.1. Theorems 3.1 and 3.2 extend the results of Ma and Wu [21] from END random variables to m-WA random variables under sublinear expectation space. The dominating coefficient of END random variables is a constant $K \geq 1$, but the dominating coefficients of m-WA random variables is a positive sequence $\{g(n), n \geq 1\}$. Thus, $\{g(n), n \geq 1\}$ has brought us the main technical difficulties of the proofs.

Remark 3.2. Besides, in sublinear expectation space, Ma and Wu [21] studied strong law of large numbers for weighted sums of END random variables under sublinear expectations with the condition

$$\hat{\mathbb{E}}\left(|X|^{\beta}\right) \leq C_{\mathbb{V}}\left(|X|^{\beta}\right) < \infty, \quad \beta > 1$$

and

$$C_{\mathbb{V}}\left(|X|^{r}\right) < \infty, \quad 1 < r < 2.$$

Thus, we introduce the slowly varying function, making our results better than the results of Ma and Wu [21] and our conditions weaker than those in [21]. In particular, taking $\theta = 0$ and $L(x) = 1$ in Theorem 3.1 and Theorem 3.2, we conclude that these results are almost identical to the results of Ma and Wu [21].

Remark 3.3. From Corollaries 3.1 and 3.2, Theorem 3.1 consists of the Kolmogorov-type strong law of large numbers and the partial Marcinkiewicz-type strong law of large numbers for m-WA random variables, which is different from the result of Wu et al. [20]. Theorem 3.2 is the result of almost surely convergence and Corollary 3.3 is the application of Theorem 3.2.

Remark 3.4. In Theorems 3.1 and 3.2, we assume that $\mathbb{V}$ is countably sub-additive. If $\mathbb{V}$ isn't countably sub-additive, we can define an outer capacity $\mathbb{V}^*$ as in Zhang [7] by

$$\mathbb{V}^*(A) = \inf\left\{\sum_{n=1}^{\infty}\mathbb{V}(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n\right\}, \ \mathcal{V}^*(A) = 1 - \mathbb{V}^*(A^c), \ A \in \mathcal{F}.$$

Then $\mathcal{V}^*(A)$ is countably sub-additive with

$$\mathcal{V}^*(A) \leq \mathbb{V}(A).$$

Therefore, we can get the corresponding strong law of large numbers with respect to $\mathcal{V}^*$.

4. The proof of theorems

Proof of Theorem 3.1. It is easily seen that

$$C_{\mathbb{V}}\left(f^{-}(|X|)\right) < \infty$$

is equivalent to

$$C_{\mathbb{V}}\left(f^{-}(|X|)\, c^{-\beta}\right) < \infty$$

from Definition 2.3. So, for any $c > 0$, according to Lemma 2.7, we get

$$\sum_{n=1}^{\infty}\mathbb{V}\left(|X| > cn^{1/\beta}L(n)\right) < \infty. \tag{4.1}$$

We notice that

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{V}\left(|X|>cn^{1/\beta}L(n)\right)&=\sum_{j=1}^{\infty}\sum_{2^{j-1}\leq n<2^{j}}\mathbb{V}\left(|X|>cn^{1/\beta}L(n)\right)\\&\geq\sum_{j=1}^{\infty}\sum_{2^{j-1}\leq n<2^{j}}\mathbb{V}\left(|X|>c2^{j/\beta}L\left(2^{j}\right)\right)\\&\geq\sum_{j=1}^{\infty}\left(2^{j}-2^{j-1}\right)\mathbb{V}\left(|X|>c2^{j/\beta}L\left(2^{j}\right)\right)\\&\geq 2^{-1}\sum_{j=1}^{\infty}2^{j}\mathbb{V}\left(|X|>c2^{j/\beta}L\left(2^{j}\right)\right),\end{aligned}$$

which implies that

$$\sum_{j=1}^{\infty}2^{j}\mathbb{V}\left(|X|>c2^{j/\beta}L\left(2^{j}\right)\right)<\infty. \tag{4.2}$$

For a sequence of m-WA random variables, to ensure that the truncated random variables are also a sequence of m-WA random variables, we choose the function as follows:

$$l_a(x)=-aI(x<-a)+xI(|x|\leq a)+aI(x>a)$$

for any $a>0$. This truncated function $l_a(x)$ belongs to $\mathrm{C}_{b,Lip}(\mathbb{R})$ and is nondecreasing. So, by Lemma 2.1, for fixed $n\geq 1$ and each $1\leq i\leq n$,

$$\begin{aligned}Y_{ni}&:=-n^{1/\beta}L(n)I\left(X_i<-n^{1/\beta}L(n)\right)+X_iI\left(|X_i|\leq n^{1/\beta}L(n)\right)+n^{1/\beta}L(n)I\left(X_i>n^{1/\beta}L(n)\right),\\Z_{ni}&:=X_i-Y_{ni}=\left(X_i+n^{1/\beta}L(n)\right)I\left(X_i<-n^{1/\beta}L(n)\right)+\left(X_i-n^{1/\beta}L(n)\right)I\left(X_i>n^{1/\beta}L(n)\right).\end{aligned} \tag{4.3}$$

Then, $\{Y_{ni},\ n\geq 1,\ 1\leq i\leq n\}$ is also a sequence of m-WA random variables. It is easy to obtain that

$$\begin{aligned}b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n}a_i\left(X_i-\hat{\mathbb{E}}(X_i)\right)=&b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n}a_iZ_{ni}+b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n}a_i\left(Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right)\\&+b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n}a_i\left(\hat{\mathbb{E}}(Y_{ni})-\hat{\mathbb{E}}(X_i)\right)\\=&\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3.\end{aligned}$$

In order to prove the Eq (3.5), it suffices to verify that

$$\limsup_{n\to\infty}\ \mathrm{J}_1\leq 0\ a.s.\ \mathbb{V},\quad \limsup_{n\to\infty}\ \mathrm{J}_2=0\ a.s.\ \mathbb{V} \tag{4.4}$$

and

$$\lim_{n\to\infty}\ \mathrm{J}_3=0. \tag{4.5}$$

In classical probability space $(\Omega,\ \mathcal{F},\ \mathrm{P})$, we know that the equation

$$\mathrm{P}(A) = \mathrm{E}(I_A)$$

was established for $A \in \mathcal{F}$. However, in the sublinear expected space $\left(\Omega,\ \mathcal{H},\ \hat{\mathbb{E}}\right)$, to ensure continuity, we need to adjust the indicator function through the function in $\mathrm{C}_{l,Lip}(\mathbb{R})$. So, we define the function as follows. For $0 < \mu < 1$, $\bar{g}(x)$ is an even function and $\bar{g}(x) \in \mathrm{C}_{l,Lip}(\mathbb{R})$ fulfilling

$$0 \le \bar{g}(x) \le 1$$

for all x. $\bar{g}(x) = 1$ if $|x| < \mu$; $\bar{g}(x) = 0$ if $|x| > 1$, and $\bar{g}(x)$ is nonincreasing as $x > 0$. Then,

$$\begin{aligned} &I(|x| \le \mu) \le \bar{g}(|x|) \le I(|x| \le 1), \\ &I(|x| > 1) \le 1 - \bar{g}(|x|) \le I(|x| > \mu). \end{aligned} \tag{4.6}$$

By (3.1), (4.1), (4.3), and (4.6), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{V}(Z_{ni} \ne 0) &\le \sum_{n=1}^{\infty} \mathbb{V}\left(|X_i| > n^{1/\beta} L(n)\right) \\ &\le \sum_{n=1}^{\infty} \hat{\mathbb{E}}\left(1 - \bar{g}\left(\frac{|X_i|}{n^{1/\beta} L(n)}\right)\right) \\ &\le \mathrm{C} \sum_{n=1}^{\infty} \hat{\mathbb{E}}\left(1 - \bar{g}\left(\frac{|X|}{n^{1/\beta} L(n)}\right)\right) \\ &\le \mathrm{C} \sum_{n=1}^{\infty} \mathbb{V}\left(|X| > \mu n^{1/\beta} L(n)\right) \\ &< \infty. \end{aligned} \tag{4.7}$$

Then, by (4.7), Lemma 2.2, and $\mathbb{V}$ being countably sub-additive, we obtain

$$\mathbb{V}(Z_{ni} \ne 0,\ i.o.) = 0.$$

Since (3.4), $a_i \uparrow$, and $L(x)$ is a nondecreasing function, we obtain

$$\begin{aligned} |\mathrm{J}_1| &\le b_n^{-1} (L(n))^{-1} \max_{1 \le i \le n} a_i \sum_{i=1}^{n} |Z_{ni}| \\ &\le b_n^{-1} (L(n))^{-1} a_n \sum_{i=1}^{n} |Z_{ni}| \\ &\le c n^{-1/s} (L(n))^{-1} \sum_{i=1}^{n} |Z_{ni}| \to 0, \quad a.s.\ \mathbb{V}. \end{aligned}$$

Hence,

$$\limsup_{n \to \infty} \mathrm{J}_1 \le 0\ a.s.\ \mathbb{V}$$

has been proved, and we will turn to prove (4.5).

Because $f(x)$ $(x > 0)$ is a regularly varying function with an exponent of $1/\beta$, $f^{-}(x)$ is a regularly varying function with an exponent of β from Bingham et al. [29, Theorem 1.5.12]. Thus, by (3.2) and Lemma 2.3 (3),

$$\hat{\mathbb{E}}\left(f^{-}(|X|)\right) < \infty$$

implies

$$\hat{\mathbb{E}}|X|^{\delta} < \infty, \ \ \forall\, \delta \in (0, \beta)\,. \tag{4.8}$$

We choose

$$\eta = 1/\beta > 0$$

in Lemma 2.6, then we have $n^{1/\beta}L(n) \to \infty$ as $n \to \infty$. We take $\delta \in (1, \beta)$. By (3.1), (4.3), (4.6), (4.8), Lemma 2.4, and $1 - \delta < 0$, and we have

$$\begin{aligned}
\left|\hat{\mathbb{E}}(Y_{ni}) - \hat{\mathbb{E}}(X_i)\right| \leq & \hat{\mathbb{E}}|Y_{ni} - X_i| \\
\leq & \hat{\mathbb{E}}\left[\left|-n^{1/\beta}L(n) - X_i\right| I\left(X_i < -n^{1/\beta}L(n)\right) + \left|n^{1/\beta}L(n) - X_i\right| I\left(X_i > n^{1/\beta}L(n)\right)\right] \\
\ll & \hat{\mathbb{E}}\left[|X_i|\left(1 - \bar{g}\left(\frac{|X_i|}{n^{1/\beta}L(n)}\right)\right)\right] \\
\leq & \mathrm{C}\hat{\mathbb{E}}\left[|X|\left(1 - \bar{g}\left(\frac{|X|}{n^{1/\beta}L(n)}\right)\right)\right] \\
= & \mathrm{C}\hat{\mathbb{E}}\left[|X|^{\delta}|X|^{1-\delta}\left(1 - \bar{g}\left(\frac{|X|}{n^{1/\beta}L(n)}\right)\right)\right] \\
\leq & \mathrm{C}\mu^{1-\delta}n^{(1-\delta)/\beta}(L(n))^{1-\delta}\,\hat{\mathbb{E}}|X|^{\delta} \to 0, \ \ n \to \infty.
\end{aligned} \tag{4.9}$$

According to $b_n \uparrow \infty$ and the fact that $L(n)$ is nondecreasing, for fixed a_i, we get

$$\lim_{n\to\infty} b_n^{-1}(L(n))^{-1}a_i = 0.$$

By (3.3),

$$\sup_{n\geq 1} b_n{}^{-1}(L(n))^{-1}\sum_{i=1}^{n} a_i \leq (L(1))^{-1}\sup_{n\geq 1} b_n{}^{-1}\sum_{i=1}^{n} a_i \leq c < \infty. \tag{4.10}$$

Then, by (4.9), (4.10), and Lemma 2.5, the Eq (4.5) is proved.

Finally, we will turn to prove

$$\limsup_{n\to\infty} \mathrm{J}_2 = 0 \ \ a.s.\ \mathbb{V}.$$

We notice that $\{Y_{ni},\ n \geq 1,\ 1 \leq i \leq n\}$ is a sequence of m-WA random variables, then by Lemma 2.1, $\left\{a_i\left(Y_{ni} - \hat{\mathbb{E}}(Y_{ni})\right),\ n \geq 1,\ 1 \leq i \leq n\right\}$ is still a sequence of m-WA random variables and

$$\hat{\mathbb{E}}\left[a_i\left(Y_{ni} - \hat{\mathbb{E}}(Y_{ni})\right)\right] = 0,$$

which satisfies the requirements of Lemma 2.8. For every $\varepsilon > 0$, we take

$$x = d = b_n L(n)\,\varepsilon$$

in Lemma 2.8. By using the Markov inequality, $\mathbb{V}$ being countably sub-additive, and Lemma 2.3 (1) and (2), we get

$$\begin{aligned}
&\mathbb{V}\left[b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n}a_i\left(Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right)>\varepsilon\right]\\
&\leq m\mathbb{V}\left[\max_{1\leq i\leq n}a_i\left(Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right)>\frac{b_nL(n)\cdot\varepsilon}{m}\right]+mg(n)\exp\left\{1-\ln\left(1+\frac{\varepsilon^2\cdot b_n^2(L(n))^2/m^2}{\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right|^2}\right)\right\}\\
&\leq m\sum_{i=1}^{n}\mathbb{V}\left[\left|a_i\left(Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right)\right|>\frac{b_nL(n)\cdot\varepsilon}{m}\right]+mg(n)\cdot\mathrm{e}\cdot\left(1+\frac{\varepsilon^2\cdot b_n^2(L(n))^2/m^2}{\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right|^2}\right)^{-1}\\
&\leq m\sum_{i=1}^{n}\mathbb{V}\left[\left|a_i\left(Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right)\right|>\frac{b_nL(n)\cdot\varepsilon}{m}\right]+mg(n)\cdot\mathrm{e}\cdot\left(\frac{\varepsilon^2b_n^2(L(n))^2}{m^2}\right)^{-1}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right|^2\\
&\leq m\left(\frac{\varepsilon b_nL(n)}{m}\right)^{-\alpha}\sum_{i=1}^{n}a_i^\alpha\hat{\mathbb{E}}\left|Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right|^\alpha+mg(n)\cdot\mathrm{e}\cdot\left(\frac{\varepsilon^2b_n^2(L(n))^2}{m^2}\right)^{-1}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right|^2\\
&\ll(b_nL(n))^{-\alpha}\sum_{i=1}^{n}a_i^\alpha\hat{\mathbb{E}}\left|Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right|^\alpha+g(n)(b_nL(n))^{-2}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right|^2\\
&\ll(b_nL(n))^{-\alpha}\sum_{i=1}^{n}a_i^\alpha\hat{\mathbb{E}}|Y_{ni}|^\alpha+g(n)(b_nL(n))^{-2}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}|Y_{ni}|^2.
\end{aligned}\tag{4.11}$$

Thus, by (4.11) and $g(n)=\mathrm{O}\left(n^\theta\right)$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{V}\left[b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n}a_i\left(Y_{ni}-\hat{\mathbb{E}}(Y_{ni})\right)>\varepsilon\right]\\
&\ll\sum_{n=1}^{\infty}(b_nL(n))^{-\alpha}\sum_{i=1}^{n}a_i^\alpha\hat{\mathbb{E}}|Y_{ni}|^\alpha+\sum_{n=1}^{\infty}n^\theta(b_nL(n))^{-2}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}|Y_{ni}|^2\\
&=\mathrm{J}_{21}+\mathrm{J}_{22}.
\end{aligned}$$

In order to prove $\mathrm{J}_{21}<\infty$, we need to structure an even function, which is similar to (4.6). Let

$$\bar{g}_j(x)\in\mathrm{C}_{l.Lip}(\mathbb{R}),\quad j\geq 1$$

satisfying

$$0\leq\bar{g}_j(x)\leq 1$$

for all $x\in\mathbb{R}$, and if

$$2^{(j-1)/\beta}L\left(2^{j-1}\right)<|x|\leq 2^{j/\beta}L\left(2^j\right),\quad \bar{g}_j\left(\frac{x}{2^{j/\beta}L(2^j)}\right)=1;$$

if

$$|x| \leq \mu 2^{(j-1)/\beta} L\left(2^{j-1}\right)$$

or

$$|x| > (1+\mu)\, 2^{j/\beta} L\left(2^{j}\right), \quad \bar{g}_j\left(\frac{x}{2^{j/\beta} L\left(2^{j}\right)}\right) = 0.$$

Thus, for every $\rho > 0$,

$$\begin{aligned} \bar{g}_j\left(\frac{|X|}{2^{j/\beta} l\left(2^{j}\right)}\right) &\leq I\left(\mu 2^{(j-1)/\beta} L\left(2^{j-1}\right) < |X| \leq (1+\mu)\, 2^{j/\beta} L\left(2^{j}\right)\right), \\ |X|^{\rho}\, \bar{g}\left(\frac{|X|}{2^{k/\beta} L\left(2^{k}\right)}\right) &\leq 1 + \sum_{j=1}^{k} |X|^{\rho}\, \bar{g}_j\left(\frac{|X|}{2^{j/\beta} L\left(2^{j}\right)}\right). \end{aligned} \tag{4.12}$$

For all $\tau > 0$, by (3.1), (4.3), (4.6),

$$\begin{aligned} \hat{\mathbb{E}}\,|Y_{ni}|^{\tau} &\leq \hat{\mathbb{E}}\left[|X_i|^{\tau}\, I\left(|X_i| \leq n^{1/\beta} L(n)\right) + n^{\tau/\beta} (L(n))^{\tau}\, I\left(|X_i| > n^{1/\beta} L(n)\right)\right] \\ &\leq \hat{\mathbb{E}}\left[|X_i|^{\tau}\, \bar{g}\left(\frac{\mu\,|X_i|}{n^{1/\beta} L(n)}\right)\right] + n^{\tau/\beta} (L(n))^{\tau}\, \hat{\mathbb{E}}\left(1 - \bar{g}\left(\frac{|X_i|}{n^{1/\beta} L(n)}\right)\right) \\ &\leq \mathrm{C}\hat{\mathbb{E}}\left[|X|^{\tau}\, \bar{g}\left(\frac{\mu\,|X|}{n^{1/\beta} L(n)}\right)\right] + \mathrm{C} n^{\tau/\beta} (L(n))^{\tau}\, \hat{\mathbb{E}}\left(1 - \bar{g}\left(\frac{|X|}{n^{1/\beta} L(n)}\right)\right) \\ &\leq \mathrm{C}\hat{\mathbb{E}}\left[|X|^{\tau}\, \bar{g}\left(\frac{\mu\,|X|}{n^{1/\beta} L(n)}\right)\right] + \mathrm{C} n^{\tau/\beta} (L(n))^{\tau}\, \mathbb{V}\left(|X| > \mu n^{1/\beta} L(n)\right). \end{aligned} \tag{4.13}$$

Thus, for all $\alpha > 1$, since (3.4), (4.1), (4.13), $a_i \uparrow$, and

$$\frac{1}{s} = \frac{1}{\alpha} + \frac{1}{\beta},$$

then,

$$\begin{aligned} \mathrm{J}_{21} &\leq \sum_{n=1}^{\infty} b_n^{-\alpha} (L(n))^{-\alpha} \max_{1\leq i\leq n} a_i^{\alpha} \sum_{i=1}^{n} \hat{\mathbb{E}}\,|Y_{ni}|^{\alpha} \\ &\leq \sum_{n=1}^{\infty} b_n^{-\alpha} a_n^{\alpha} (L(n))^{-\alpha} \sum_{i=1}^{n} \hat{\mathbb{E}}\,|Y_{ni}|^{\alpha} \\ &\leq \sum_{n=1}^{\infty} n^{-\alpha/s} (L(n))^{-\alpha} \sum_{i=1}^{n} \left[\mathrm{C}\hat{\mathbb{E}}\left[|X|^{\alpha}\, \bar{g}\left(\frac{\mu\,|X|}{n^{1/\beta} L(n)}\right)\right] + \mathrm{C} n^{\alpha/\beta} (L(n))^{\alpha}\, \mathbb{V}\left(|X| > \mu n^{1/\beta} L(n)\right)\right] \\ &= \mathrm{C} \sum_{n=1}^{\infty} n^{1-\alpha/s} (L(n))^{-\alpha}\, \hat{\mathbb{E}}\left[|X|^{\alpha}\, \bar{g}\left(\frac{\mu\,|X|}{n^{1/\beta} L(n)}\right)\right] + \mathrm{C} \sum_{n=1}^{\infty} n^{1-\alpha/s} n^{\alpha/\beta} \mathbb{V}\left(|X| > \mu n^{1/\beta} L(n)\right) \\ &= \mathrm{C} \sum_{n=1}^{\infty} n^{1-\alpha/s} (L(n))^{-\alpha}\, \hat{\mathbb{E}}\left[|X|^{\alpha}\, \bar{g}\left(\frac{\mu\,|X|}{n^{1/\beta} L(n)}\right)\right] + \mathrm{C} \sum_{n=1}^{\infty} \mathbb{V}\left(|X| > \mu n^{1/\beta} L(n)\right) \\ &= \mathrm{J}_{211} + c. \end{aligned} \tag{4.14}$$

In order to prove $J_{21} < \infty$, we need to show $J_{211} < \infty$. Because $L(x) > 0$ $(x > 0)$ is a monotonic nondecreasing function and $\alpha > \beta$, we have

$$\sum_{k=1}^{\infty} 2^{(1-\alpha/\beta)k} \left(L\left(2^k\right)\right)^{-\alpha} \leq L(2)^{-\alpha} \sum_{k=1}^{\infty} 2^{(1-\alpha/\beta)k} < \infty. \tag{4.15}$$

Otherwise, taking

$$x = 2^{j-1} \quad \text{and} \quad \lambda = 2 > 0$$

in Definition 2.8, we can get

$$L\left(2^j\right) \leq cL\left(2^{j-1}\right)$$

and

$$\left\{|X| > c2^j L\left(2^{j-1}\right)\right\} \subset \left\{|X| > c2^j L\left(2^j\right)\right\}.$$

Thus, by (4.2), (4.12), (4.15), $\alpha > \beta$, $\bar{g}(x) \downarrow$ for all $x > 0$,

$$\frac{1}{s} = \frac{1}{\alpha} + \frac{1}{\beta},$$

we get

$$\begin{aligned}
J_{211} =& C \sum_{n=1}^{\infty} n^{-\alpha/\beta} (L(n))^{-\alpha} \hat{\mathbb{E}}\left[|X|^{\alpha} \bar{g}\left(\frac{\mu |X|}{n^{1/\beta} L(n)}\right)\right] \\
\leq& C \sum_{k=1}^{\infty} \sum_{2^{k-1} \leq n < 2^k} 2^{[-(k-1)\alpha]/\beta} \left(L\left(2^{k-1}\right)\right)^{-\alpha} \hat{\mathbb{E}}\left[|X|^{\alpha} \bar{g}\left(\frac{\mu |X|}{2^{k/\beta} L(2^k)}\right)\right] \\
\ll& \sum_{k=1}^{\infty} 2^{(1-\alpha/\beta)k} \left(L\left(2^{k-1}\right)\right)^{-\alpha} \hat{\mathbb{E}}\left[|X|^{\alpha} \bar{g}\left(\frac{\mu |X|}{2^{k/\beta} L(2^k)}\right)\right] \\
\leq& \sum_{k=1}^{\infty} 2^{(1-\alpha/\beta)k} \left(L\left(2^{k-1}\right)\right)^{-\alpha} \hat{\mathbb{E}}\left[1 + \sum_{j=1}^{k} |X|^{\alpha} \bar{g}_j\left(\frac{\mu |X|}{2^{j/\beta} L(2^j)}\right)\right] \\
\leq& \sum_{k=1}^{\infty} 2^{(1-\alpha/\beta)k} \left(L\left(2^{k-1}\right)\right)^{-\alpha} + \sum_{k=1}^{\infty} 2^{(1-\alpha/\beta)k} \left(L\left(2^{k-1}\right)\right)^{-\alpha} \sum_{j=1}^{k} \hat{\mathbb{E}}\left[|X|^{\alpha} \bar{g}_j\left(\frac{\mu |X|}{2^{j/\beta} L(2^j)}\right)\right] \\
\leq& \sum_{j=1}^{\infty} \hat{\mathbb{E}}\left[|X|^{\alpha} \bar{g}_j\left(\frac{\mu |X|}{2^{j/\beta} L(2^j)}\right)\right] \sum_{k=j}^{\infty} 2^{(1-\alpha/\beta)k} \left(L\left(2^k\right)\right)^{-\alpha} + c \\
\ll& \sum_{j=1}^{\infty} 2^{(1-\alpha/\beta)j} \left(L\left(2^j\right)\right)^{-\alpha} \hat{\mathbb{E}}\left[|X|^{\alpha} \bar{g}_j\left(\frac{\mu |X|}{2^{j/\beta} L(2^j)}\right)\right] + c \\
\ll& \sum_{j=1}^{\infty} 2^{(1-\alpha/\beta)j} \left(L\left(2^j\right)\right)^{-\alpha} \cdot 2^{j\alpha/\beta} \left(L\left(2^j\right)\right)^{\alpha} \mathbb{V}\left(|X| > 2^{(j-1)/\beta} L\left(2^{j-1}\right)\right) + c \\
=& \sum_{j=1}^{\infty} 2^j \mathbb{V}\left(|X| > 2^{-1/\beta} \cdot 2^{j/\beta} L\left(2^{j-1}\right)\right) + c \\
\leq& \sum_{j=1}^{\infty} 2^j \mathbb{V}\left(|X| > c2^{j/\beta} L\left(2^j\right)\right) + c \\
<& \infty.
\end{aligned} \tag{4.16}$$

In the end, we need to prove $J_{22} < \infty$. We take

$$\max\{0,\ \beta(2+\theta-2/\alpha)\} < \delta < \min\{2,\ \beta\}.$$

For all $\eta > 0$, by Lemma 2.6, we can get

$$cx^{\eta} \geq L(x).$$

Since $\delta < \beta$, we have

$$cn^{1-\delta/\beta}(L(n))^{-\delta} \geq 1$$

when n is sufficiently large. According to (4.1) and Lemma 2.2, we obtain

$$n\mathbb{V}\left(|X| > cn^{1/\beta}L(n)\right) \to 0$$

as $n \to \infty$. By (4.6), (4.13), we have

$$\begin{aligned}
\sum_{i=1}^{n}\hat{\mathbb{E}}|Y_{ni}|^2 \leq& C\sum_{i=1}^{n}\hat{\mathbb{E}}\left[|X|^2\bar{g}\left(\frac{\mu|X|}{n^{1/\beta}L(n)}\right)\right] + C\sum_{i=1}^{n}n^{2/\beta}(L(n))^2\mathbb{V}\left(|X| > \mu n^{1/\beta}L(n)\right)\\
=& Cn\hat{\mathbb{E}}\left[|X|^2\bar{g}\left(\frac{\mu|X|}{n^{1/\beta}L(n)}\right)\right] + Cn^{2/\beta+1}(L(n))^2\mathbb{V}\left(|X| > \mu n^{1/\beta}L(n)\right)\\
\ll& n\hat{\mathbb{E}}\left[|X|^2\bar{g}\left(\frac{\mu|X|}{n^{1/\beta}L(n)}\right)\right] + n^{2/\beta}(L(n))^2\\
\leq& (1/\mu)^{2-\delta}\cdot\ n\left[n^{1/\beta}L(n)\right]^{2-\delta}\hat{\mathbb{E}}|X|^{\delta} + n^{2/\beta}(L(n))^2\\
\ll& n^{2/\beta}(L(n))^2\left[n^{1-\delta/\beta}(L(n))^{-\delta} + 1\right]\\
\ll& n^{1+(2-\delta)/\beta}(L(n))^{2-\delta}.
\end{aligned} \tag{4.17}$$

Since

$$\beta(2+\theta-2/\alpha) < \delta,$$

we can get

$$\theta - (2/\alpha - 1 + \delta/\beta) < -1.$$

Thus, by (3.4), (4.17), $a_i \uparrow$,

$$\frac{1}{s} = \frac{1}{\alpha} + \frac{1}{\beta}$$

and $L(x) > 0$ $(x > 0)$ being a monotonic nondecreasing function,

$$\begin{aligned} \mathrm{J}_{22} &\leq \sum_{n=1}^{\infty} n^{\theta} b_n^{-2} (L(n))^{-2} \max_{1\leq i\leq n} a_i^2 \sum_{i=1}^{n} \hat{\mathbb{E}} |Y_{ni}|^2 \\ &\leq \sum_{n=1}^{\infty} n^{\theta} b_n^{-2} a_n^2 (L(n))^{-2} \sum_{i=1}^{n} \hat{\mathbb{E}} |Y_{ni}|^2 \\ &\ll \sum_{n=1}^{\infty} n^{\theta} \cdot n^{-2/s} (L(n))^{-2} \cdot n^{1+(2-\delta)/\beta} (L(n))^{2-\delta} \\ &= \sum_{n=1}^{\infty} n^{\theta-(2/\alpha-1+\delta/\beta)} (L(n))^{-\delta} \\ &\leq (L(1))^{-\delta} \sum_{n=1}^{\infty} n^{\theta-(2/\alpha-1+\delta/\beta)} \\ &< \infty. \end{aligned} \tag{4.18}$$

By (4.16) and (4.18), we get

$$\sum_{n=1}^{\infty} \mathbb{V}\left[b_n^{-1} (L(n))^{-1} \sum_{i=1}^{n} a_i \left(Y_{ni} - \hat{\mathbb{E}}(Y_{ni}) \right) > \varepsilon \right] < \infty.$$

According to Lemma 2.2 and $\mathbb{V}$ being countably sub-additive, we know

$$\mathbb{V}\left[b_n^{-1} (L(n))^{-1} \sum_{i=1}^{n} a_i \left(Y_{ni} - \hat{\mathbb{E}}(Y_{ni}) \right) > \varepsilon,\ i.o. \right] = 0$$

and

$$\mathcal{V}\left[\bigcup_{t=1}^{\infty} \bigcap_{n=t}^{\infty} \left\{ b_n^{-1} (L(n))^{-1} \sum_{i=1}^{n} a_i \left(Y_{ni} - \hat{\mathbb{E}}(Y_{ni}) \right) \leq \varepsilon \right\} \right] = 1.$$

It is obvious that

$$\begin{aligned} \left\{ \bigcup_{t=1}^{\infty} \bigcap_{n=t}^{\infty} \left\{ b_n^{-1} (L(n))^{-1} \sum_{i=1}^{n} a_i \left(Y_{ni} - \hat{\mathbb{E}}(Y_{ni}) \right) \leq \varepsilon \right\} \right\} &\subset \left\{ b_n^{-1} (L(n))^{-1} \sum_{i=1}^{n} a_i \left(Y_{ni} - \hat{\mathbb{E}}(Y_{ni}) \right) \to 0,\ n \to \infty \right\} \\ &= \{ \mathrm{J}_2 \to 0,\ \ n \to \infty \}. \end{aligned}$$

The equation

$$\limsup_{n\to\infty}\ \mathrm{J}_2 = 0\ a.s.\ \mathbb{V}$$

has been proved.

Replacing $\{X_i,\ i \geq 1\}$ by $\{-X_i,\ i \geq 1\}$ for each $1 \leq i \leq n$ in (3.5), by

$$\hat{\varepsilon}(X_i) := -\hat{\mathbb{E}}(-X_i),$$

we have

$$\begin{aligned}0 \geq &\limsup_{n\to\infty}\ b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n} a_i\left(-X_i - \hat{\mathbb{E}}(-X_i)\right)\\ =&\limsup_{n\to\infty}\ b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n} a_i\left(-X_i + \hat{\varepsilon}(X_i)\right)\\ =&\limsup_{n\to\infty}\ b_n^{-1}(L(n))^{-1}\sum_{i=1}^{n} a_i\left(-\left(X_i - \hat{\varepsilon}(X_i)\right)\right),\end{aligned}$$

which implies (3.6). Therefore, by (3.5), (3.6), and

$$\hat{\mathbb{E}}(X_i) = \hat{\varepsilon}(X_i),$$

the Eq (3.7) is obtained.

The proof of Theorem 3.1 is completed. □

Proof of Theorem 3.2. We define for fixed $n \geq 1$ and each $1 \leq i \leq n$,

$$Z'_{ni} := -c_i L(c_i) I(X_i < -c_i L(c_i)) + X_i I(|X_i| \leq c_i L(c_i)) + c_i L(c_i) I(X_i > c_i L(c_i)). \tag{4.19}$$

By Lemma 2.1, it is easy to see that $\left\{Z'_{ni},\ n \geq 1,\ 1 \leq i \leq n\right\}$ is still a sequence of m-WA random variables. Besides, we notice that

$$\begin{aligned}b_n^{-1}(L(c_n))^{-1}\sum_{i=1}^{n} a_i\left(X_i - \hat{\mathbb{E}}(X_i)\right) =& b_n^{-1}(L(c_n))^{-1}\sum_{i=1}^{n} a_i\left(X_i - Z'_{ni}\right) + b_n^{-1}(L(c_n))^{-1}\sum_{i=1}^{n} a_i\left(Z'_{ni} - \hat{\mathbb{E}}(Z'_{ni})\right)\\ &+ b_n^{-1}(L(c_n))^{-1}\sum_{i=1}^{n} a_i\left(\hat{\mathbb{E}}(Z'_{ni}) - \hat{\mathbb{E}}(X_i)\right)\\ =&\mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_3.\end{aligned}$$

In order to prove (3.10), we only need to prove

$$\limsup_{n\to\infty}\ \mathrm{K}_1 \leq 0\ a.s.\ \mathbb{V}, \quad \limsup_{n\to\infty}\ \mathrm{K}_2 = 0\ a.s.\ \mathbb{V} \tag{4.20}$$

and

$$\lim_{n\to\infty}\ \mathrm{K}_3 = 0. \tag{4.21}$$

For any $c > 0$, by (3.8), we easily obtain

$$\sum_{n=1}^{\infty}\mathbb{V}\left(|X| > cn^{1/\gamma}L(n)\right) < \infty.$$

Since Definition 2.3, we know

$$C_{\mathbb{V}}\left(h^{-}(|X|)\right) < \infty$$

is equivalent to

$$\mathrm{C}_{\mathbb{V}}\left(h^{-}(|X|)\,c^{-\gamma}\right)<\infty.$$

According to Lemma 2.7, we obtain

$$\mathrm{C}_{\mathbb{V}}\left(h^{-}(|X|)\right)<\infty.$$

By Definition 2.8, taking

$$x=n^{1/\gamma}$$

and

$$\lambda=n^{1-1/\gamma}>0$$

for $n\geq 1$, we get

$$L(n)\leq cL\left(n^{1/\gamma}\right)$$

and

$$\left\{|X|>cn^{1/r}L(n)\right\}\supset\left\{|X|>cn^{1/r}L\left(n^{1/\gamma}\right)\right\}.$$

By $0\leq\theta<1$, we notice that

$$\begin{aligned}\sum_{n=1}^{\infty}n^{\theta/\gamma}\mathbb{V}\left(|X|>\mathrm{C}n^{1/\gamma}L(n)\right)&\geq\sum_{n=1}^{\infty}n^{\theta/\gamma}\mathbb{V}\left(|X|>cn^{1/\gamma}L\left(n^{1/\gamma}\right)\right)\\&\geq\sum_{k=1}^{\infty}\sum_{2^{k\gamma-1}\leq n<2^{k\gamma}}\left(2^{k\gamma-1}\right)^{\theta/\gamma}\mathbb{V}\left(|X|>c2^{k}L\left(2^{k}\right)\right)\\&\geq\sum_{k=1}^{\infty}\left(2^{k\gamma}-2^{k\gamma-1}\right)\left(2^{k\gamma-1}\right)^{\theta/\gamma}\mathbb{V}\left(|X|>c2^{k}L\left(2^{k}\right)\right)\\&\geq 2^{-1-\theta/\gamma}\sum_{k=1}^{\infty}2^{k(\gamma+\theta)}\mathbb{V}\left(|X|>c2^{k}L\left(2^{k}\right)\right)\\&\geq 2^{-1-\theta/\gamma}\sum_{k=1}^{\infty}2^{k\gamma}\mathbb{V}\left(|X|>c2^{k}L\left(2^{k}\right)\right),\end{aligned}$$

which implies that

$$\sum_{k=1}^{\infty}2^{k(\gamma+\theta)}\mathbb{V}\left(|X|>c2^{k}L\left(2^{k}\right)\right)<\infty \tag{4.22}$$

and

$$\sum_{k=1}^{\infty}2^{k\gamma}\mathbb{V}\left(|X|>c2^{k}L\left(2^{k}\right)\right)<\infty. \tag{4.23}$$

Besides, for every c_i, $i\geq 1$, there exists a k such that

$$2^{k-1}\leq c_i<2^{k}.$$

By (4.22),

$$\left\{|X|>c2^{k}L\left(2^{k-1}\right)\right\}\subset\left\{|X|>c2^{k}L\left(2^{k}\right)\right\}.$$

$L(x) > 0$ $(x > 0)$ is a monotonic nondecreasing function and $0 \le \theta < 1$, and we get

$$\begin{aligned}\sum_{k=1}^{\infty} 2^{k(\gamma+\theta)} \mathbb{V}\left(|X| > c2^k L\left(2^k\right)\right) &= \sum_{k=1}^{\infty} 2^{k(\gamma+\theta)} \mathbb{V}\left(|X| > 2c2^{k-1} L\left(2^k\right)\right) \\ &\ge 2^{\gamma} \sum_{k=1}^{\infty} 2^{k\theta} \mathbb{V}\left(|X| > 2c2^{k-1} L\left(2^k\right)\right) \\ &\ge 2^{\gamma} \sum_{k=1}^{\infty} 2^{k\theta} \mathbb{V}\left(|X| > 2c2^{k-1} L\left(2^{k-1}\right)\right) \\ &\ge 2^{\gamma} \sum_{i=1}^{\infty} c_i^{\theta} \mathbb{V}\left(|X| > 2cc_i L(c_i)\right) \\ &\ge 2^{\gamma} \sum_{i=1}^{\infty} \mathbb{V}\left(|X| > 2cc_i L(c_i)\right),\end{aligned}$$

which implies that

$$\sum_{i=1}^{\infty} c_i^{\theta} \mathbb{V}\left(|X| > cc_i L(c_i)\right) < \infty \tag{4.24}$$

and

$$\sum_{i=1}^{\infty} \mathbb{V}\left(|X| > cc_i L(c_i)\right) < \infty. \tag{4.25}$$

For $0 < \mu < 1$, let $\tilde{g}(x)$ be an even function and

$$\tilde{g}(x) \in \mathrm{C}_{l,Lip}(\mathbb{R})$$

satisfying

$$0 \le \tilde{g}(x) \le 1$$

for all x.

$$\tilde{g}(x) = 1$$

if $|x| < \mu$;

$$\tilde{g}(x) = 0$$

if $|x| > 1$, and $\tilde{g}(x)$ is nonincreasing as $x > 0$. Then,

$$\begin{aligned}I(|x| \le \mu) \le \tilde{g}(|x|) \le I(|x| \le 1), \\ I(|x| > 1) \le 1 - \tilde{g}(|x|) \le I(|x| > \mu).\end{aligned} \tag{4.26}$$

We also define an even function $\tilde{g}_j(x)$ as follows. Let

$$\tilde{g}_j(x) \in \mathrm{C}_{l,Lip}(\mathbb{R}), \quad j \ge 1$$

such that

$$0 \le \tilde{g}_j(x) \le 1$$

for all x and

$$\tilde{g}_j\left(\frac{x}{2^jL\left(2^j\right)}\right)=1$$

if

$$2^{j-1}L\left(2^{j-1}\right)<|X|\leq 2^jL\left(2^j\right);$$
$$\tilde{g}_j\left(\frac{x}{2^jL\left(2^j\right)}\right)=0$$

if

$$|X|<\mu 2^{j-1}L\left(2^{j-1}\right)$$

or

$$|X|>(1+\mu)\,2^jL\left(2^j\right).$$

Then, for all $\rho>0$,

$$\begin{aligned}&\tilde{g}_j\left(\frac{|X|}{2^jL\left(2^j\right)}\right)\leq I\left(\mu 2^{j-1}L\left(2^{j-1}\right)<|X|\leq(1+\mu)\,2^jL\left(2^j\right)\right),\\&|X|^\rho\,\tilde{g}\left(\frac{|X|}{2^kL\left(2^k\right)}\right)\leq 1+\sum_{j=1}^{k}|X|^\rho\,\tilde{g}_j\left(\frac{|X|}{2^jL\left(2^j\right)}\right)\end{aligned}\tag{4.27}$$

and

$$1-\tilde{g}\left(\frac{|X|}{2^kL\left(2^k\right)}\right)\leq\sum_{j=k}^{\infty}\tilde{g}_j\left(\frac{|X|}{2^jL\left(2^j\right)}\right).\tag{4.28}$$

To start, we prove

$$\limsup_{n\to\infty}\ \mathrm{K}_1\leq 0\ a.s.\ \mathbb{V}.$$

Let

$$T(1)=1.$$

By (3.1), (3.9), (4.19), (4.23), (4.26), (4.27), and (4.28), $\tilde{g}(x)\downarrow$ for all $x>0$,

$$\left\{|X|>c2^jL\left(2^{j-1}\right)\right\}\subset\left\{|X|>c2^jL\left(2^j\right)\right\}$$

and $\hat{\mathbb{E}}$ being countably sub-additive, we have

$$
\begin{aligned}
\sum_{i=1}^{\infty} \mathbb{V}\left(X_i \neq Z'_{ni}\right) \leq & \sum_{i=1}^{\infty} \mathbb{V}\left(|X_i| > c_i L\left(c_i\right)\right) \\
\leq & \sum_{i=1}^{\infty} \hat{\mathbb{E}}\left[1 - \tilde{g}\left(\frac{|X_i|}{c_i L\left(c_i\right)}\right)\right] \\
\leq & C \sum_{i=1}^{\infty} \hat{\mathbb{E}}\left[1 - \tilde{g}\left(\frac{|X|}{c_i L\left(c_i\right)}\right)\right] \\
\ll & \sum_{k=1}^{\infty} \sum_{2^{k-1} \leq c_i < 2^k} \hat{\mathbb{E}}\left[1 - \tilde{g}\left(\frac{|X|}{2^{k-1} L\left(2^{k-1}\right)}\right)\right] \\
\leq & \sum_{k=1}^{\infty} \left[T\left(2^k\right) - T\left(2^{k-1}\right)\right] \hat{\mathbb{E}}\left[1 - \tilde{g}\left(\frac{|X|}{2^{k-1} L\left(2^{k-1}\right)}\right)\right] \\
\leq & \sum_{k=1}^{\infty} \left[T\left(2^k\right) - T\left(2^{k-1}\right)\right] \sum_{j=k-1}^{\infty} \hat{\mathbb{E}}\left[\tilde{g}_j\left(\frac{|X|}{2^j L\left(2^j\right)}\right)\right] \\
\leq & \sum_{k=1}^{\infty} \left[T\left(2^k\right) - T\left(2^{k-1}\right)\right] \sum_{j=k-1}^{\infty} \mathbb{V}\left(|X| > \mu 2^{j-1} L\left(2^{j-1}\right)\right) \\
= & \sum_{j=1}^{\infty} \mathbb{V}\left(|X| > \mu 2^{j-1} L\left(2^{j-1}\right)\right) \sum_{k=1}^{j+1} \left[T\left(2^k\right) - T\left(2^{k-1}\right)\right] \\
\leq & \sum_{j=1}^{\infty} T\left(2^{j+1}\right) \mathbb{V}\left(|X| > \mu 2^{j-1} L\left(2^{j-1}\right)\right) \\
\ll & \sum_{j=1}^{\infty} 2^{j\gamma} \mathbb{V}\left(|X| > \mu 2^{j-1} L\left(2^{j-1}\right)\right) \\
\leq & \sum_{j=1}^{\infty} 2^{j\gamma} \mathbb{V}\left(|X| > c 2^{j} L\left(2^{j}\right)\right) < \infty.
\end{aligned}
\tag{4.29}
$$

By (4.29), Lemma 2.2, and $\mathbb{V}$ being countably sub-additive, we have

$$
\mathbb{V}\left(X_i \neq Z'_{ni},\ i.o.\right) = 0.
$$

By $b_n \uparrow \infty$,

$$
c_n = b_n / a_n \uparrow \infty
$$

and $L\left(x\right) > 0$ $\left(x > 0\right)$ being a monotonic nondecreasing function. We have

$$
|K_1| \leq b_n^{-1} \left(L\left(c_n\right)\right)^{-1} \sum_{i=1}^{n} a_i \left|X_i - Z'_{ni}\right| \to 0, \quad a.s.\ \mathbb{V}.
$$

Second, we prove

$$
\limsup_{n\to\infty} \mathrm{K}_2 = 0\ a.s.\ \mathbb{V}.
$$

By Lemma 2.1, $\left\{a_i\left(Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right),\ n\geq 1,\ 1\leq i\leq n\right\}$ is still a sequence of m-WA random variables. We can easily obtain

$$\hat{\mathbb{E}}\left[a_i\left(Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right)\right]=0,$$

which satisfies the condition of Lemma 2.8. Then, for all $\varepsilon>0$, we take

$$x=d=b_nL\left(c_n\right)\varepsilon$$

in Lemma 2.8. By the Markov inequality, Lemma 2.3 (1) (2),

$$g\left(n\right)=\mathrm{O}\left(n^{\theta}\right),$$

and $\mathbb{V}$ being countably sub-additive, we have

$$\begin{aligned}
&\mathbb{V}\left[\sum_{i=1}^{n}a_i\left(Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right)>b_nL\left(c_n\right)\varepsilon\right]\\
&\leq m\mathbb{V}\left[\max_{1\leq i\leq n}a_i\left(Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right)>\frac{b_nL\left(c_n\right)\varepsilon}{m}\right]+mg\left(n\right)\exp\left\{1-\ln\left(1+\frac{\varepsilon^2b_n^2\left(L\left(c_n\right)\right)^2/m^2}{\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right|^2}\right)\right\}\\
&\leq m\sum_{i=1}^{n}\mathbb{V}\left[\left|a_i\left(Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right)\right|>\frac{b_nL\left(c_n\right)\varepsilon}{m}\right]+mg\left(n\right)\cdot\mathrm{e}\cdot\left(1+\frac{\varepsilon^2b_n^2\left(L\left(c_n\right)\right)^2/m^2}{\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right|^2}\right)^{-1}\\
&\leq m\sum_{i=1}^{n}\mathbb{V}\left[\left|a_i\left(Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right)\right|>\frac{b_nL\left(c_n\right)\varepsilon}{m}\right]+mg\left(n\right)\cdot\mathrm{e}\cdot\left(\frac{\varepsilon^2b_n^2\left(L\left(c_n\right)\right)^2}{m^2}\right)^{-1}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right|^2\\
&\leq m\left(\frac{b_nL\left(c_n\right)\varepsilon}{m}\right)^{-2}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right|^2+mg\left(n\right)\cdot\mathrm{e}\cdot\left(\frac{\varepsilon^2b_n^2\left(L\left(c_n\right)\right)^2}{m^2}\right)^{-1}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Z'_{ni}-\hat{\mathbb{E}}\left(Z'_{ni}\right)\right|^2\\
&\leq m^3\varepsilon^{-2}b_n^{-2}\left(L\left(c_n\right)\right)^{-2}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Z'_{ni}\right|^2+\mathrm{e}m^3\varepsilon^{-2}b_n^{-2}\left(L\left(c_n\right)\right)^{-2}g\left(n\right)\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Z'_{ni}\right|^2\\
&\ll n^{\theta}b_n^{-2}\left(L\left(c_n\right)\right)^{-2}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Z'_{ni}\right|^2.
\end{aligned}\tag{4.30}$$

Thus, by (4.30) and

$$\sum_{n=i}^{\infty}b_n^{-2}n^{\theta}\left(L\left(c_n\right)\right)^{-2}\ll b_i^{-2}c_i^{\theta}\left(L\left(c_i\right)\right)^{-2}$$

for sufficiently large i,

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{V}\left[b_n^{-1}(L(c_n))^{-1}\sum_{i=1}^{n}a_i\left(Z'_{ni}-\hat{\mathbb{E}}(Z'_{ni})\right)>\varepsilon\right] &\ll \sum_{n=1}^{\infty}n^{\theta}b_n^{-2}(L(c_n))^{-2}\sum_{i=1}^{n}a_i^2\hat{\mathbb{E}}\left|Z'_{ni}\right|^2\\ &=\sum_{i=1}^{\infty}a_i^2\hat{\mathbb{E}}\left|Z'_{ni}\right|^2\sum_{n=i}^{\infty}n^{\theta}b_n^{-2}(L(c_n))^{-2}\\ &\ll \sum_{i=1}^{\infty}c_i^{\theta-2}(L(c_i))^{-2}\hat{\mathbb{E}}\left|Z'_{ni}\right|^2.\end{aligned}\tag{4.31}$$

Otherwise, by (3.1), (4.19), (4.26), we get

$$\begin{aligned}\hat{\mathbb{E}}\left|Z'_{ni}\right|^2 &\le \hat{\mathbb{E}}\left[|X_i|^2 I(|X_i|\le c_iL(c_i))+c_i^2(L(c_i))^2 I(|X_i|>c_iL(c_i))\right]\\ &\le \hat{\mathbb{E}}\left[|X_i|^2\tilde{g}\left(\frac{\mu|X_i|}{c_iL(c_i)}\right)\right]+c_i^2(L(c_i))^2\hat{\mathbb{E}}\left(1-\tilde{g}\left(\frac{|X_i|}{c_iL(c_i)}\right)\right)\\ &\le \mathrm{C}\hat{\mathbb{E}}\left[|X|^2\tilde{g}\left(\frac{\mu|X|}{c_iL(c_i)}\right)\right]+\mathrm{C}c_i^2(L(c_i))^2\hat{\mathbb{E}}\left[1-\tilde{g}\left(\frac{|X|}{c_iL(c_i)}\right)\right]\\ &\le \mathrm{C}\hat{\mathbb{E}}\left[|X|^2\tilde{g}\left(\frac{\mu|X|}{c_iL(c_i)}\right)\right]+\mathrm{C}c_i^2(L(c_i))^2\,\mathbb{V}\,(|X|>\mu c_iL(c_i)).\end{aligned}\tag{4.32}$$

Since $0\le\theta<1$, we get

$$-2\le\theta-2<-1.$$

Thus, by (4.24), (4.27), (4.31), (4.32), $\tilde{g}(x)\downarrow$ for all $x>0$, $L(x)>0$ $(x>0)$ being a monotonic non-decreasing function,

$$L\left(2^k\right)\le cL\left(2^{k-1}\right)$$

and $\hat{\mathbb{E}}$ being countably sub-additive, we have

$$\begin{aligned}&\sum_{i=1}^{\infty}c_i^{\theta-2}(L(c_i))^{-2}\hat{\mathbb{E}}\left|Z'_{ni}\right|^2\\ &\le \mathrm{C}\sum_{i=1}^{\infty}c_i^{\theta-2}(L(c_i))^{-2}\hat{\mathbb{E}}\left[|X|^2\tilde{g}\left(\frac{\mu|X|}{c_iL(c_i)}\right)\right]+\mathrm{C}\sum_{i=1}^{\infty}c_i^{\theta}\mathbb{V}(|X|>\mu c_iL(c_i))\\ &\ll \sum_{k=1}^{\infty}\sum_{2^{k-1}\le c_i<2^k}c_i^{\theta-2}(L(c_i))^{-2}\hat{\mathbb{E}}\left[|X|^2\tilde{g}\left(\frac{\mu|X|}{c_iL(c_i)}\right)\right]+c\\ &\le \sum_{k=1}^{\infty}\left[T\left(2^k\right)-T\left(2^{k-1}\right)\right]\left(2^{k-1}\right)^{\theta-2}\left(L\left(2^{k-1}\right)\right)^{-2}\hat{\mathbb{E}}\left[|X|^2\tilde{g}\left(\frac{\mu|X|}{2^kL(2^k)}\right)\right]+c\\ &\le \sum_{k=1}^{\infty}\left[T\left(2^k\right)-T\left(2^{k-1}\right)\right]\left(2^{k-1}\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}\hat{\mathbb{E}}\left[1+\sum_{j=1}^{k}|X|^2\tilde{g}_j\left(\frac{\mu|X|}{2^kL(2^k)}\right)\right]+c\\ &\le \sum_{k=1}^{\infty}\left[T\left(2^k\right)-T\left(2^{k-1}\right)\right]\left(2^{k-1}\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}\end{aligned}$$

$$+\sum_{k=1}^{\infty}\left[T\left(2^k\right)-T\left(2^{k-1}\right)\right]\left(2^{k-1}\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}\sum_{j=1}^{k}\hat{\mathbb{E}}\left[|X|^2\tilde{g}_j\left(\frac{\mu|X|}{2^kL\left(2^k\right)}\right)\right]+c$$
$$=\mathrm{K}_{21}+\mathrm{K}_{22}+c.$$

By (3.9), $L(x)>0$ being a monotonic nondecreasing function, and $\gamma+\theta<2$, we get

$$\begin{aligned}\mathrm{K}_{21}&\leq\sum_{k=1}^{\infty}T\left(2^k\right)\left[\left(2^{k-1}\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}-\left(2^k\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}\right]\\&\leq\sum_{k=1}^{\infty}T\left(2^k\right)\left(2^{k-1}\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}\\&\ll\sum_{k=1}^{\infty}2^{k(\gamma+\theta-2)}\left(L\left(2^k\right)\right)^{-2}\\&\leq(L(2))^{-2}\sum_{k=1}^{\infty}2^{k(\gamma+\theta-2)}\\&<\infty.\end{aligned}\tag{4.33}$$

Besides, by (3.9), (4.22), (4.27), $\gamma+\theta<2$,

$$\left\{|X|>c2^jL\left(2^{j-1}\right)\right\}\subset\left\{|X|>c2^jL\left(2^j\right)\right\}$$

and $L(x)>0$ being a monotonic nondecreasing function, we obtain

$$\begin{aligned}\mathrm{K}_{22}&\leq\sum_{k=1}^{\infty}\left[T\left(2^k\right)-T\left(2^{k-1}\right)\right]\left(2^{k-1}\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}\sum_{j=1}^{k}2^{2j}\left(L\left(2^j\right)\right)^2\mathbb{V}\left(|X|>2^{j-1}L\left(2^{j-1}\right)\right)\\&=\sum_{j=1}^{\infty}2^{2j}\left(L\left(2^j\right)\right)^2\mathbb{V}\left(|X|>2^{j-1}L\left(2^{j-1}\right)\right)\sum_{k=j}^{\infty}\left[T\left(2^k\right)-T\left(2^{k-1}\right)\right]\left(2^{k-1}\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}\\&\leq\sum_{j=1}^{\infty}2^{2j}\left(L\left(2^j\right)\right)^2\mathbb{V}\left(|X|>2^{j-1}L\left(2^{j-1}\right)\right)\sum_{k=j}^{\infty}T\left(2^k\right)\left[\left(2^{k-1}\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}-\left(2^k\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}\right]\\&\leq\sum_{j=1}^{\infty}2^{2j}\left(L\left(2^j\right)\right)^2\mathbb{V}\left(|X|>2^{j-1}L\left(2^{j-1}\right)\right)\sum_{k=j}^{\infty}T\left(2^k\right)\left(2^{k-1}\right)^{\theta-2}\left(L\left(2^k\right)\right)^{-2}\\&\ll\sum_{j=1}^{\infty}2^{2j}\left(L\left(2^j\right)\right)^2\mathbb{V}\left(|X|>2^{j-1}L\left(2^{j-1}\right)\right)\sum_{k=j}^{\infty}2^{k(\gamma+\theta-2)}\left(L\left(2^k\right)\right)^{-2}\\&\leq\sum_{j=1}^{\infty}2^{j(\gamma+\theta)}\mathbb{V}\left(|X|>2^{j-1}L\left(2^{j-1}\right)\right)\\&\leq\sum_{j=1}^{\infty}2^{j(\gamma+\theta)}\mathbb{V}\left(|X|>c2^jL\left(2^j\right)\right)\\&<\infty.\end{aligned}\tag{4.34}$$

Thus, by (4.33) and (4.34), we get

$$\sum_{n=1}^{\infty} \mathbb{V}\left[b_n^{-1}(L(c_n))^{-1}\sum_{i=1}^{n} a_i\left(Z'_{ni}-\hat{\mathbb{E}}(Z'_{ni})\right)>\varepsilon\right]<\infty,$$

which implies that

$$\limsup_{n\to\infty}\ \mathrm{K}_2=0\ a.s.\ \mathbb{V}$$

by Lemma 2.2 and $\mathbb{V}$ being countably sub-additive.

Finally, we will turn to prove (4.21). By (3.1), (4.19), (4.26), and Lemma 2.4, we get

$$\begin{aligned}\left|\hat{\mathbb{E}}(Z'_{ni})-\hat{\mathbb{E}}(X_i)\right| &\le\hat{\mathbb{E}}\left|Z'_{ni}-X_i\right|\\ &\le\hat{\mathbb{E}}\left[(|X_i|+c_iL(c_i))\left(1-\tilde{g}\left(\frac{|X_i|}{c_iL(c_i)}\right)\right)\right]\\ &\le\hat{\mathbb{E}}\left[|X_i|\left(1-\tilde{g}\left(\frac{|X_i|}{c_iL(c_i)}\right)\right)\right]+c_iL(c_i)\,\hat{\mathbb{E}}\left[1-\tilde{g}\left(\frac{|X_i|}{c_iL(c_i)}\right)\right]\\ &\le\mathrm{C}\hat{\mathbb{E}}\left[|X|\left(1-\tilde{g}\left(\frac{|X|}{c_iL(c_i)}\right)\right)\right]+\mathrm{C}c_iL(c_i)\,\hat{\mathbb{E}}\left[1-\tilde{g}\left(\frac{|X|}{c_iL(c_i)}\right)\right].\end{aligned}\tag{4.35}$$

Thus, by (4.25), (4.26), (4.35), and $c_n=b_n/a_n$, we have

$$\begin{aligned}&\sum_{i=1}^{\infty}\left|\frac{a_i}{b_iL(c_i)}\left[\hat{\mathbb{E}}(Z'_{ni})-\hat{\mathbb{E}}(X_i)\right]\right|\\ &\le\sum_{i=1}^{\infty}c_i^{-1}(L(c_i))^{-1}\left|\hat{\mathbb{E}}(Z'_{ni})-\hat{\mathbb{E}}(X_i)\right|\\ &\le\mathrm{C}\sum_{i=1}^{\infty}c_i^{-1}(L(c_i))^{-1}\,\hat{\mathbb{E}}\left[|X|\left(1-\tilde{g}\left(\frac{|X|}{c_iL(c_i)}\right)\right)\right]+\mathrm{C}\sum_{i=1}^{\infty}\hat{\mathbb{E}}\left[1-\tilde{g}\left(\frac{|X|}{c_iL(c_i)}\right)\right]\\ &\le\mathrm{C}\sum_{i=1}^{\infty}c_i^{-1}(L(c_i))^{-1}\,\hat{\mathbb{E}}\left[|X|\left(1-\tilde{g}\left(\frac{|X|}{c_iL(c_i)}\right)\right)\right]+\mathrm{C}\sum_{i=1}^{\infty}\mathbb{V}(|X|>\mu c_iL(c_i))\\ &=\mathrm{K}_{31}+c.\end{aligned}\tag{4.36}$$

By (4.23), (4.27), (4.28), $\tilde{g}(x)\downarrow$ for all $x>0$,

$$L\left(2^k\right)\le cL\left(2^{k-1}\right)$$

and $\hat{\mathbb{E}}$ being countably sub-additive, we obtain

$$\begin{aligned}\mathrm{K}_{31}&\ll\sum_{k=1}^{\infty}\sum_{2^{k-1}\le c_i<2^k}\left(2^{k-1}\right)^{-1}\left(L\left(2^{k-1}\right)\right)^{-1}\hat{\mathbb{E}}\left[|X|\left(1-\tilde{g}\left(\frac{|X|}{2^{k-1}L(2^{k-1})}\right)\right)\right]\\ &\le\sum_{k=1}^{\infty}\left[T\left(2^k\right)-T\left(2^{k-1}\right)\right]\left(2^{k-1}\right)^{-1}\left(L\left(2^{k-1}\right)\right)^{-1}\sum_{j=k-1}^{\infty}\hat{\mathbb{E}}\left[|X|\,\tilde{g}_j\left(\frac{|X|}{2^jL(2^j)}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{\infty}\left[T\left(2^{k}\right)-T\left(2^{k-1}\right)\right]\left(2^{k-1}\right)^{-1}\left(L\left(2^{k}\right)\right)^{-1} \sum_{j=k-1}^{\infty} 2^{j} L\left(2^{j}\right) \mathbb{V}\left(|X|>\mu 2^{j-1} L\left(2^{j-1}\right)\right)\\
&\leq \sum_{j=1}^{\infty} 2^{j} L\left(2^{j}\right) \mathbb{V}\left(|X|>\mu 2^{j-1} L\left(2^{j-1}\right)\right) \sum_{k=1}^{j+1}\left[T\left(2^{k}\right)-T\left(2^{k-1}\right)\right]\left(2^{k-1}\right)^{-1}\left(L\left(2^{k}\right)\right)^{-1}\\
&\leq \sum_{j=1}^{\infty} 2^{j} L\left(2^{j}\right) \mathbb{V}\left(|X|>\mu 2^{j-1} L\left(2^{j-1}\right)\right) \sum_{k=1}^{j+1} T\left(2^{k}\right)\left[\left(2^{k-1}\right)^{-1}\left(L\left(2^{k}\right)\right)^{-1}-\left(2^{k}\right)^{-1}\left(L\left(2^{k}\right)\right)^{-1}\right]\\
&=\sum_{j=1}^{\infty} 2^{j} L\left(2^{j}\right) \mathbb{V}\left(|X|>\mu 2^{j-1} L\left(2^{j-1}\right)\right) \sum_{k=1}^{j} T\left(2^{k}\right)\left[\left(2^{k-1}\right)^{-1}\left(L\left(2^{k}\right)\right)^{-1}-\left(2^{k}\right)^{-1}\left(L\left(2^{k}\right)\right)^{-1}\right]\\
&\quad+\sum_{j=1}^{\infty} 2^{j} L\left(2^{j}\right) \mathbb{V}\left(|X|>\mu 2^{j-1} L\left(2^{j-1}\right)\right)\left\{T\left(2^{j+1}\right)\left[\left(2^{j}\right)^{-1}\left(L\left(2^{j+1}\right)\right)^{-1}-\left(2^{j+1}\right)^{-1}\left(L\left(2^{j+1}\right)\right)^{-1}\right]\right\}\\
&=\mathrm{K}_{311}+\mathrm{K}_{312}.
\end{aligned}
$$

By (3.9), (4.23),

$$\left\{|X|>c 2^{j} L\left(2^{j-1}\right)\right\} \subset\left\{|X|>c 2^{j} L\left(2^{j}\right)\right\}$$

and $L(x)>0$ $(x>0)$ being a monotonic nondecreasing function, we have

$$
\begin{aligned}
\mathrm{K}_{312} &\leq \sum_{j=1}^{\infty} 2^{j} L\left(2^{j}\right) \mathbb{V}\left(|X|>\mu 2^{j-1} L\left(2^{j-1}\right)\right) \cdot T\left(2^{j+1}\right)\left(2^{j}\right)^{-1}\left(L\left(2^{j+1}\right)\right)^{-1}\\
&\ll \sum_{j=1}^{\infty} L\left(2^{j}\right) \mathbb{V}\left(|X|>\mu 2^{j-1} L\left(2^{j-1}\right)\right) \cdot 2^{(j+1) \gamma}\left(L\left(2^{j}\right)\right)^{-1}\\
&\leq \sum_{j=1}^{\infty} 2^{j \gamma} \mathbb{V}\left(|X|>\mu 2^{-1} \cdot 2^{j} L\left(2^{j-1}\right)\right)\\
&\leq \sum_{j=1}^{\infty} 2^{j \gamma} \mathbb{V}\left(|X|>c 2^{j} L\left(2^{j}\right)\right)\\
&<\infty.
\end{aligned} \tag{4.37}
$$

Besides, taking

$$\lambda=2^{j-k}>0$$

for $j \geq k$ and $x=2^{k}$ in Definition 2.8, we get

$$L\left(2^{j}\right) \leq c L\left(2^{k}\right).$$

According to (3.9), (4.23), $\gamma>1$, $L(x)>0$ $(x>0)$, and

$$\left\{|X|>c 2^{j} L\left(2^{j-1}\right)\right\} \subset\left\{|X|>c 2^{j} L\left(2^{j}\right)\right\},$$

we obtain

$$\begin{aligned}
\mathrm{K}_{311} &\leq \sum_{j=1}^{\infty} 2^j L\left(2^j\right) \mathbb{V}\left(|X| > \mu 2^{j-1} L\left(2^{j-1}\right)\right) \sum_{k=1}^{j} T\left(2^k\right)\left(2^{k-1}\right)^{-1}\left(L\left(2^k\right)\right)^{-1} \\
&\ll \sum_{j=1}^{\infty} 2^j L\left(2^j\right) \mathbb{V}\left(|X| > \mu 2^{j-1} L\left(2^{j-1}\right)\right) \sum_{k=1}^{j} 2^{k(\gamma-1)}\left(L\left(2^k\right)\right)^{-1} \\
&\leq \sum_{j=1}^{\infty} 2^j L\left(2^j\right) \mathbb{V}\left(|X| > \mu 2^{j-1} L\left(2^{j-1}\right)\right) \sum_{k=1}^{j} 2^{k(\gamma-1)}\left(L\left(2^j\right)\right)^{-1} \\
&\leq \sum_{j=1}^{\infty} 2^{j\gamma} \mathbb{V}\left(|X| > \mu 2^{-1} \cdot 2^j L\left(2^{j-1}\right)\right) \\
&\leq \sum_{j=1}^{\infty} 2^{j\gamma} \mathbb{V}\left(|X| > c 2^j L\left(2^j\right)\right) \\
&< \infty.
\end{aligned} \tag{4.38}$$

By (4.36)–(4.38), we get

$$\sum_{i=1}^{\infty} \left| \frac{a_i}{b_i L(c_i)} \left[\hat{\mathbb{E}}(Z'_{ni}) - \hat{\mathbb{E}}(X_i)\right] \right| < \infty.$$

Using Lemma 2.9, we obtain (4.22). Thus, (3.10) has been proved.

Replacing $\{X_i,\ i \geq 1\}$ by $\{-X_i,\ i \geq 1\}$ for each $1 \leq i \leq n$ in (3.10), by

$$\hat{\varepsilon}(X_i) := -\hat{\mathbb{E}}(-X_i),$$

we have

$$\begin{aligned}
0 &\geq \limsup_{n\to\infty}\ b_n^{-1}(L(c_n))^{-1} \sum_{i=1}^{n} a_i\left(-X_i - \hat{\mathbb{E}}(-X_i)\right) \\
&= \limsup_{n\to\infty}\ b_n^{-1}(L(c_n))^{-1} \sum_{i=1}^{n} a_i(-X_i + \hat{\varepsilon}(X_i)) \\
&= \limsup_{n\to\infty}\ b_n^{-1}(L(c_n))^{-1} \sum_{i=1}^{n} a_i(-(X_i - \hat{\varepsilon}(X_i))),
\end{aligned}$$

which implies (3.11). Furthermore, by (3.10), (3.11), and

$$\hat{\mathbb{E}}(X_i) = \hat{\varepsilon}(X_i),$$

we can get (3.12) immediately.

The proof of Theorem 3.2 is completed. □

5. Conclusions

In this article, by using the Fuk-Nagaev type inequality, C_r inequality, Jensen inequality, and so on under the sublinear expectation space, we obtain general strong law of large numbers of m-WA random variables on different conditions under sublinear expectation space. The key of solving this problem makes full use of the Fuk-Nagaev type inequality. One of the results includes the Kolmogorov-type strong law of large numbers and the partial Marcinkiewicz-type strong law of large numbers for m-WA random variables under sublinear expectation space. Additionally, we obtain almost surely convergence for weighted sums of m-WA random variables under sublinear expectation space. However, the Kronecker Lemma is not applied for arrays of row-wise random variables. Thus, we will try our best to choose other ways to prove almost surely convergence for arrays of row-wise m-WA random variables under sublinear expectation space in the future.

Author contributions

Qingfeng Wu: conceptualization, formal analysis, investigation, methodology, writing-original draft, writing-review and editing; Xili Tan: funding acquisition, project administration, supervision, methodology, formal analysis, writing-review and editing; Shuang Guo: formal analysis, writing-review and editing; Peiyu Sun: writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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