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Research article

Results for fractional bilinear Hardy operators in central varying exponent Morrey space

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Abstract: This paper intends to demonstrate the boundedness of the fractional bilinear Hardy operator and its adjoint on the λ-central Morrey space with variable exponents. Analogous outcomes for their commutators are derived when the symbol functions are elements of the λ-central bounded mean oscillation (λ-central BMO) space.

Keywords: bilinear Hardy operators; fractional integral; boundedness; function space
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1. Introduction

In 1920, Hardy [1] presented a mathematical operator tailored for functions h that are locally integrable within the space R^n, commonly denoted as the Hardy operator:

$$\mathcal{H}h(t) = t^{-1}\int_0^t h(\xi)d\xi, \quad t > 0. \tag{1.1}$$

He derived the subsequent inequality

$$|\mathcal{H}h|_{L^{\mathrm{p}}} \leq \mathrm{p}'|h|_{L^{\mathrm{p}}}, \tag{1.2}$$

where it was demonstrated that $\mathrm{p}' = \mathrm{p}/(\mathrm{p}-1)$ serves as the optimal parameter. Subsequently, Faris [2] extended (1.1) to n dimensions, with the equivalent form

$$Hh(t) = |S(0,|t|)|^{-1}\int_{S(0,|t|)} h(\xi)d\xi.$$

In a recent study [3], it was shown that H meets the condition where $|S(0,|t|)|$ represents the Lebesgue measure of the ball $S(0,|t|)$ within the n-dimensional Euclidean space $\mathbb{R}^n$

$$|Hh|L^{\mathrm{p}} \leq \mathrm{p}'|h|L^{\mathrm{p}}, \qquad 1 < \mathrm{p} \leq \infty, \tag{1.3}$$

with p' identified as the precise parameter. The inequalities represented by Eqs (1.2) and (1.3) have been expanded to encompass power-weighted Lebesgue spaces as discussed in [4, 5], with the precise constants determined by the indices of the weights involved. The inequalities denoted by Eqs (1.2) and (1.3) are termed as Hardy strong-type (p,p) inequalities, as the Hardy operator in these inequalities maps functions from L^{p} to L^{p}. The writers in [6] confirmed Hardy weak-type (p,p) inequalities, where the Hardy operator transforms functions from the space L^{p} to $L^{\mathrm{p},\infty}$. Yet, it has been shown that the best constant for Hardy weak-type inequalities is 1. Subsequently, the precise parameter for Hardy weak-type inequalities on Morrey spaces was determined in [7]. Similarly, the determination of the precise constant for the fractional high-dimensional Hardy operator [8] remained unresolved up to 2015. Zhao and Lu [9] tackled this problem by expanding Bliss's findings regarding the fractional one-dimensional Hardy operator. In [9], the bounded nature of the Hardy operator H_γ was confirmed, resulting in the subsequent inequality:

$$||H_\gamma h||_{L^{\mathrm{q}}} \leq k||h||_{L^{\mathrm{p}}}, \tag{1.4}$$

where k is a constant defined in [9]. Grafakos introduced an m-linear Hardy operator in [4] for functions $h_1, h_2, \ldots, h_m$ belonging to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where m is a natural number. The operator is expressed as follows:

$$H(h_1, ..., h_m) = \frac{1}{|t|^{nm}} \int_{|(\xi_1,...,\xi_m)|<|t|} \prod_{i=1}^{m} h_i(\xi_i) d\xi_1, ..., d\xi_m.$$

The bilinear operator, alternatively referred to as the 2-linear operator, has been studied extensively. In work by [10, 11], the authors investigated the commutator associated with the bilinear Hardy operator, defined as follows:

$$[b_i, H^i](h_1, ..., h_m)(\xi) = b_i(\xi)H(h_1, ..., h_m)(\xi) - H(h_1, ..., h_{i-1}, h_i b_i, h_{i+1}, ..., h_m)(\xi).$$

They established the boundedness of commutators produced by the bilinear operator. In the present manuscript, we present the concept of fractional m-linear Hardy operators as follows:

$$H_\gamma(h_1, ..., h_m) = \frac{1}{|t|^{nm-\gamma}} \int_{|(\xi_1,...,\xi_m)|<|t|} \prod_{i=1}^{m} h_i(\xi_i) d\xi_1, ..., d\xi_m.$$

$$H^*_\gamma(h_1, ..., h_m) = \int_{|(\xi_1,...,\xi_m)|>|t|} \frac{1}{|\xi|^{nm-\gamma}} \prod_{i=1}^{m} h_i(\xi_i) d\xi_1, ..., d\xi_m.$$

We also provide a definition for the commutator of fractional m-linear Hardy operators.

$$[b, H_\gamma](h_1, ..., h_m)(\xi) = \sum_{i=1}^{m} [b_i, H^i_\gamma](h_1, ..., h_m)(\xi)$$

$$[b, H^*_\gamma](h_1, ..., h_m)(\xi) = \sum_{i=1}^{m} [b_i, H^{*i}_\gamma](h_1, ..., h_m)(\xi)$$

$$[b_i, H_\gamma^i](h_1, ..., h_m)(\xi) = b_i(\xi)H_\gamma(h_1, ..., h_m)(\xi) - H_\gamma(h_1, ..., h_{i-1}, h_i b_i, h_{i+1}, ..., h_m)(\xi).$$

$$[b_i, H_\gamma^{*i}](h_1, ..., h_m)(\xi) = b_i(\xi)H_\gamma^*(h_1, ..., h_m)(\xi) - H_\gamma^*(h_1, ..., h_{i-1}, h_i b_i, h_{i+1}, ..., h_m)(\xi),$$

where $\xi = (\xi_1, \xi_2, ..., \xi_m)$. The primary focus of attention in various monographs, such as [12, 13], has been on Hardy inequalities. Optimal bounds for Hardy-type inequalities have been established in only a few instances, and ongoing research in this field continues to be a vibrant area within contemporary analysis. Current works addressing this topic include [14, 15]. Additionally, investigations into the sharp constants associated with Hardy-type inequalities on the product of certain function spaces have been presented in the published work [16]. We highlight several key pieces of literature in the review of Hardy operators across diverse function spaces, including references such as [4, 5, 17–20].

The research showcased in [21] sparked the notion of expanding function spaces. The notion of varying Lebesgue spaces, symbolized as $L^{p(\cdot)}$, was first proposed by Rákosník in [22]. Following this, the advancement of Lebesgue spaces with varying exponents began, along with inquiries into the boundedness of different operators. One significant instance is the investigation into the boundedness of the maximum operator on the Lebesgue space characterized by varying exponents $L^{p(\cdot)}$ [23, 24]. Lately, the theory of generalized function spaces has garnered interest across various realms of mathematical analysis, such as image processing [25], electrorheological fluid modeling [26], and the examination of differential equations [27].

At the same time, central Morrey space and associated function spaces have been applied in intriguing ways to investigate estimates for operators [28, 29]. Simultaneously, researchers in [30] introduced the concept of variable exponent central Morrey and presented important findings related to estimating certain operators. Recent works [31,32] that discuss the continuity of multilinear integral operators on these function spaces have made significant contributions to the existing literature in this area. In [33], the researchers established the boundedness of the Hardy-Littlewood maximal operator M given by

$$Mh(\eta) = \sup_{o:\text{ball},\eta\in o} \frac{1}{|o|} \int_o |h(\xi)| d\xi,$$

on the Lebesgue space $L^{p(\cdot)}$.

In this paper, we will investigate the boundedness of the m-linear fractional Hardy operator on central Morrey space with a varying exponent. Furthermore, this manuscript encompasses novel findings that explore the boundedness properties of commutators engendered by H_γ (or H_γ^*) and the λ-central BMO function b within the context of the variable central Morrey space. To regulate the continuity conditions of the m-linear fractional Hardy operator, we will leverage the boundedness of the fractional integral defined as

$$I_\gamma(h)(\eta) = \int_{\mathbb{R}^n} \frac{h(\xi)}{|\eta - \xi|^{n-\gamma}} d\xi.$$

The bounded nature of the Riesz potential on Lebesgue spaces with varying exponents is documented in [34].

This article is organized into four sections. The second section comprises several definitions, while the third section presents key lemmas that will be utilized in the fourth section to derive our principal findings.

2. Symbols and descriptions

In this document, the symbol C is employed to denote a constant, and it is important to note that these constants may assume different values in different contexts. Consider a collection S that is both not empty and can be measured within $\mathbb{R}^n$. Here, χ_S stands for the characteristic function of S, and $|S|$ represents the Lebesgue measure. To initiate, we will define Lebesgue spaces with a variable exponent by referring to fundamental literature, such as articles and books [22, 24, 35, 36].

Definition 2.1. Define $P(\mathbb{R}^n)$ as the set comprising all measurable functions $\mathrm{q}(\cdot) : \mathbb{R}^n \to (1, \infty)$ that meet the conditions

$$\infty > \mathrm{q}_+ \geq \mathrm{q}(x) \geq \mathrm{q}_- > 1,$$

where $\mathrm{q}_+ := \operatorname{esssup}_{x\in\mathbb{R}^n} \mathrm{q}(x)$, $\mathrm{q}_- := \operatorname{essinf}_{x\in\mathbb{R}^n} \mathrm{q}(x)$.

Definition 2.2. Consider a measurable function $q(\cdot) : \mathbb{R}^n \to [1, \infty]$. The Lebesgue space with a variable exponent, denoted as $L^{q(\cdot)}(\mathbb{R}^n)$, consists of all measurable functions h such that the following integral, denoted as $\mathbb{F}_q(h)$, is finite:

$$\mathbb{F}_q(h) = \int_{\mathbb{R}^n} \Big(|h(x)|\Big)^{q(x)} dx < \infty.$$

The space $L^{q(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm defined as

$$\|h\|_{L^{q(\cdot)}} = \inf\Big\{\nu > 0 : \mathbb{F}_q(\frac{h}{\nu}) = \int_{\mathbb{R}^n} \Big(\frac{|h(x)|}{\nu}\Big)^{q(x)} dx \leq 1\Big\}.$$

Definition 2.3. Consider a function $\mathrm{q}(\cdot)$ defined on $\mathbb{R}^n$. We present the following:
(i) $C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ comprises all locally logarithmic Holder continuous functions $\mathrm{q}(\cdot)$ satisfying

$$|\mathrm{q}(\xi) - \mathrm{q}(\eta)| \lesssim \frac{-C}{\log(|\xi - \eta|)}, \quad |\xi - \eta| < \frac{1}{2}, \quad \xi \neq \eta, \quad \xi, \eta \in \mathbb{R}^n.$$

(ii) If $\mathrm{q}(\cdot) \in C^{\log}_0(\mathbb{R}^n)$, then in this case, it satisfies the following condition at the origin:

$$|\mathrm{q}(\xi) - \mathrm{q}(0)| \lesssim \frac{C}{\log(|\frac{1}{|\xi|} + e|)}, \qquad \xi \in \mathbb{R}^n.$$

(iii) If $\mathrm{q}(\cdot)$ belongs to the class $C^{\log}_\infty(\mathbb{R}^n)$, then in this case, it satisfies the following condition at infinity:

$$|\mathrm{q}(\xi) - \mathrm{q}_\infty| \leq \frac{C_\infty}{\log(|\xi| + e)}, \quad \xi \in \mathbb{R}^n$$

for some real number q_∞.
(iv) $C^{\log}$ is the set of all globally logarithmic Holder continuous functions $\mathrm{q}(\cdot)$, given by the intersection of $C^{\log}_{\mathrm{loc}}$ and $C^{\log}_\infty$.

We represent by $\mathbb{B}(\mathbb{R}^n)$ a collection of $\mathrm{q}(\cdot)$ belonging to $P(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, meeting the requirement that the Hardy-Littlewood maximal operator is bounded on $L^{\mathrm{q}(\cdot)}(\mathbb{R}^n)$ is satisfied.

In [37], it was demonstrated that if $\mathrm{q}(\cdot) \in P(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, then M is bounded on $L^{\mathrm{q}(\cdot)}(\mathbb{R}^n)$.

Definition 2.4. [38] Let $f \in L^1(\mathbb{R}^n)$. Define

$$\|b\|_{BMO(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |b(x) - b_B|dx,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $b_B = \frac{1}{|B|}\int_B f(y)dy$. The function b is said to have bounded mean oscillation if $\|b\|_{BMO(\mathbb{R}^n)} < \infty$. The bounded mean oscillation space $BMO(\mathbb{R}^n)$ comprises all $f \in L^1(\mathbb{R}^n)$ for which $BMO(\mathbb{R}^n) < \infty$.

Definition 2.5. [39] For any p$(\cdot)$ from $P(\mathbb{R}^n)$ and any λ in $\mathbb{R}$, the variable exponent central Morrey space $\dot{B}^{\mathrm{p}(\cdot),\lambda}(\mathbb{R}^n)$ is characterized by

$$\dot{B}^{\mathrm{p}(\cdot),\lambda}(\mathbb{R}^n) = \left\{ f \in L^{\mathrm{p}(\cdot)}_{Loc}(\mathbb{R}^n) : \|f\|_{\dot{B}^{\mathrm{p}(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{B}^{\mathrm{p}(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)}}.$$

Replacing the variable exponent with a constant exponent results in the classical λ-central Morrey space.

Definition 2.6. [39] Let $p(\cdot) \in P(\mathbb{R}^n)$ and $\lambda < \frac{1}{n}$. The variable exponent λ-central BMO spaces, denoted as $CBMO^{p(\cdot),\lambda}(\mathbb{R}^n)$, are delineated as

$$CBMO^{p(\cdot),\lambda}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}(\mathbb{R}^n) : \|f\|_{CBMO^{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{CBMO^{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|(f - f_{B(0,R)}) \cdot \chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$$

3. Primary propositions and lemmas

This section starts by introducing a number of pertinent lemmas that will assist in demonstrating the boundedness of our main results.

Lemma 3.1. *[40] Let $q(\cdot)$, $q_1(\cdot)$, and $q_2(\cdot)$ belong to the set $P(E)$.*
(a) For $h \in L^{q(\cdot)}(E)$ and $f \in L^{q'(\cdot)}(E)$, the following inequality holds:

$$\int_E |f(x)h(x)| \le r_q\|h\|_{L^{q(\cdot)}(E)}\|f\|_{L^{q'(\cdot)}(E)}$$

where $r_q = 1 + \frac{1}{q_-} - \frac{1}{q_+}$ and $\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1$.
(b) If $h \in L^{q_1(\cdot)}(E)$, $f \in L^{q_2(\cdot)}(E)$, and $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$, then the following inequality holds:

$$\|fh\|_{L^{q(\cdot)}(E)} \le r_q, r_{q_1}\|h\|_{L^{q_1(\cdot)}(E)}\|f\|_{L^{q_2(\cdot)}(E)}$$

where $r_q, r_{q_1} = (1 + \frac{1}{(q_1)_-} - \frac{1}{(q_1)_+})^{\frac{1}{q_-}}$.

Proposition 3.2. *[33] Let E be an open set and let $p(\cdot) \in P(E)$ satisfy the following conditions:*

$$|p(t) - p(z)| \leq \frac{-c}{log(|t-z|)}, \frac{1}{2} \geq |t-z| \tag{3.1}$$

$$|p(t) - p(z)| \leq \frac{-c}{log(|t|+e)}, |t| \leq |z|. \tag{3.2}$$

Then, $p(\cdot) \in \mathbb{B}(\mathbb{R}^n)$, where C is a positive constant independent of t and z.

Lemma 3.3. *[41] If $q(\cdot) \in \mathbb{B}(\mathbb{R}^n)$, there exists a constant $0 < \delta < 1$ and a positive constant C such that for all B in $\mathbb{R}^n$ and all measurable subsets $O \subset B$*

$$\frac{||\chi_B||_{L^{q(\cdot)}(\mathbb{R}^n)}}{||\chi_O||_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq \frac{|B|}{|O|}$$

$$\frac{||\chi_O||_{L^{q(\cdot)}(\mathbb{R}^n)}}{||\chi_B||_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|O|}{|B|}\right)^{\delta}.$$

Remark. *Suppose $q(\cdot) \in P(\mathbb{R}^n)$ and satisfies conditions (3.1) and (3.2) in Proposition 3.2. Then, $q^{'}(\cdot)$ also meets these conditions, implying that both $q(\cdot)$ and $q^{'}(\cdot)$ belong to $\mathbb{B}(\mathbb{R}^n)$. Utilizing Lemma 3.3, we obtain constants $\delta_{11} \in (0, \frac{1}{(q_1)+})$ and $\delta_{22} \in (0, \frac{1}{(q_2)+})$ such that the inequalities*

$$\frac{||\chi_O||_{L^{q_1(\cdot)}(\mathbb{R}^n)}}{||\chi_B||_{L^{q_1(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|O|}{|B|}\right)^{\delta_{11}} \tag{3.3}$$

$$\frac{||\chi_O||_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{||\chi_B||_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|O|}{|B|}\right)^{\delta_{22}} \tag{3.4}$$

hold for all balls $B \subset \mathbb{R}^n$ and subsets $O \subset B$. Similarly, if $q_1^{'}(\cdot), q_2^{'}(\cdot) \in \mathbb{B}(\mathbb{R}^n)$, then by Lemma 3.3 we have constants $\delta_{33} \in (0, \frac{1}{(q_1)+})$ and $\delta_{44} \in (0, \frac{1}{(q_2)_+})$ such that

$$\frac{||\chi_O||_{L^{q_1^{'}(\cdot)}(\mathbb{R}^n)}}{||\chi_B||_{L^{q_1^{'}(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|O|}{|B|}\right)^{\delta_{33}} \tag{3.5}$$

$$\frac{||\chi_O||_{L^{q_2^{'}(\cdot)}(\mathbb{R}^n)}}{||\chi_B||_{L^{q_2^{'}(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|O|}{|B|}\right)^{\delta_{44}} \tag{3.6}$$

for all balls $B \subset \mathbb{R}^n$ and $O \subset B$.

Lemma 3.4. *[41] If the function $q(\cdot)$ is a member of the set $P(\mathbb{R}^n)$ for all balls O in $\mathbb{R}^n$ and there exists a positive constant C, then the following inequality holds:*

$$C^{-1} < |O|^{-1}||\chi_O||_{L^{q'(\cdot)}}||\chi_O||_{L^{q(\cdot)}} < C.$$

Proposition 3.5. *[42] Let $q(\cdot)$ be an element of the set $P(\mathbb{R}^n)$, where $0 < \gamma < \frac{n}{(p_1)_+}$. Define $q^{'}(\cdot)$ as follows:*

$$\frac{1}{q^{'}(\cdot)} = \frac{1}{q(\cdot)} - \frac{\gamma}{n}.$$

Then, the inequality

$$\|I_\gamma f\|_{L^{q^{'}(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

holds. Proposition (3.5) is instrumental in establishing the subsequent lemma (refer to [43]).

Lemma 3.6. *Assume that γ, $q(\cdot)$, and $q^{'}(\cdot)$ are defined as in Proposition 3.5. The following inequality holds for all balls $B_k = x \in \mathbb{R}^n : |x| \leq 2^k$ with $k \in \mathbb{Z}$:*

$$\|\chi_{B_k}\|_{L^{q^{'}(\cdot)}(\mathbb{R}^n)} \leq C2^{-k\gamma}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Lemma 3.7. *[43] Assume $p(\cdot) \in P(\mathbb{R}^n)$. Then, for every $b \in BMO$ and any j, k such that $j > k$, the following holds:*

$$C^{-1}\|b\|_{BMO} \leq \sup_B \frac{1}{\|\chi_B\|}\|(b - b_B)\chi_B\|_{L^{p(\cdot)}} \leq C\|b\|_{BMO}.$$

$$\|(b - b_{B_k})\chi_{B_j}\| \leq (j - k)\|b\|_{BMO}\|\chi_{B_j}\|_{L^{p(\cdot)}}.$$

4. Principal findings

Lemma 4.1. *If $q(\cdot)$ belongs to both $P(\mathbb{R}^n)$and $C^{log}(\mathbb{R}^n)$, and $p(\cdot)$ satisfies $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\gamma}{n}$ with $\frac{1}{p(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$, then the inequality can be expressed as follows:*

$$\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C2^{j(2n-\gamma)}\|\chi_{B_j}\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|^{-1}_{L^{q_2'(\cdot)}(\mathbb{R}^n)}.$$

Proof. We suppose $f = \chi_{B_j}$ and employ the definition of I_γ:

$$I_\gamma(\chi_{B_j})(x) \geq C2^{j\gamma}\chi_{B_j}(x),$$

$$\chi_{B_j}(x) \leq C2^{-j\gamma}I_\gamma(\chi_{B_j})(x).$$

On both sides, we apply the norm and utilize the outcomes of Proposition 3.5 and Lemma 3.6, correspondingly. This leads to

$$\begin{aligned}
\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq C2^{-j\gamma}\|I_\gamma\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C2^{-j\gamma}\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq C2^{-j\gamma}\|\chi_{B_j}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\leq C2^{j(2n-\gamma)}\|\chi_{B_j}\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|^{-1}_{L^{q_2'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{4.1}$$

Theorem 4.2. *Suppose $p(\cdot) \in P(\mathbb{R}^n)$ satisfies conditions (3.1) and (3.2) as stated in Proposition 3.2. We define the variable exponent $q(\cdot)$ as $\frac{1}{q(\cdot)} + \frac{\gamma}{n} = \frac{1}{p(\cdot)}$, where $\frac{1}{p(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$. If $\lambda = \lambda_1 + \lambda_2 + \frac{\gamma}{n}$ and $\lambda > (\delta_{33} + \delta_{44} + \delta)$, where δ_{33}, δ_{44}, and δ are the same constants as those appearing in equalities (3.5) and (3.6), then we have*

$$\|H_\gamma(f_1, f_2)\|_{\dot{B}^{q(\cdot),\lambda}} \leq C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}.$$

Proof. If we represent $f_{1i} = f_1 \cdot \chi_i = f_1 \cdot \chi_{A_i}$ and $f_{2i} = f_2 \cdot \chi_i = f_2 \cdot \chi_{A_i}$ for any $i \in \mathbb{Z}$, then we can express

$$f_1(x) = \sum_{i=-\infty}^{\infty} f_1(x) \cdot \chi_i(x) = \sum_{i=-\infty}^{\infty} f_{1i}(x).$$

$$f_2(x) = \sum_{i=-\infty}^{\infty} f_2(x) \cdot \chi_i(x) = \sum_{i=-\infty}^{\infty} f_{2i}(x).$$

Utilizing the generalized Hölder inequality, we obtain

$$\begin{aligned}
|H_\gamma(f_1, f_2)(x) \cdot \chi_j(x)| &\leq \frac{1}{|x|^{2n-\gamma}} \int_{B_j} \int_{B_j} |f_1(y_1)||f_2(y_2)| dy_1 dy_2 \cdot \chi_j(x) \\
&= \frac{1}{|x|^{2n-\gamma}} \int_{B_j} |f_1(y_1)| dy_1 \int_{B_j} |f_2(y_2)| dy_2 \cdot \chi_j(x) \\
&\leq C2^{-2jn} \sum_{i=-\infty}^{j} \|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\times \|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)} 2^{j\gamma} \chi_j(x).
\end{aligned}$$

Using Lemmas 4.1 and 3.3, we have derived the following inequality:

$$\begin{aligned}
&\|H_\gamma(f_1, f_2) \cdot \chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C2^{j\gamma} \sum_{i=-\infty}^{j} \|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)} 2^{-2jn} \|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leq C \sum_{i=-\infty}^{j} \|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)} 2^{-j(2n-\gamma)} \|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{4.2}$$

To proceed, inserting (4.1) into (4.2) yields

$$\begin{aligned}
\|H_\gamma(f_1, f_2) \cdot \chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq C \sum_{i=-\infty}^{j} \|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)} \\
&\times \|\chi_j\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_j\|^{-1}_{L^{q_2'(\cdot)}(\mathbb{R}^n)} \\
&\leq C \sum_{i=-\infty}^{j} \|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}} \frac{\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \\
&\leq C \sum_{i=-\infty}^{j} 2^{n\delta_{33}(i-j)} 2^{n\delta_{44}(i-j)} \|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\leq C \sum_{i=-\infty}^{j} 2^{(n\delta_{33}+n\delta_{44})(i-j)} \|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

$$\|H_\gamma(f_1, f_2) \cdot \chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}} \sum_{i=-\infty}^{j} 2^{(n\delta_{33}+n\delta_{44})(i-j)} |B_i|^{\lambda_1} |B_i|^{\lambda_2} \|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.$$

Given the conditions $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)} - \frac{\gamma}{n}$ and $\lambda = \lambda_1 + \lambda_2 + \frac{\gamma}{n}$, we can express the equation as follows:

$$\|\chi_i\|_{L^{q(\cdot)}(\mathbb{R}^n)} = |B_i|^{\frac{1}{q(\cdot)}} = |B_i|^{\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}-\frac{\gamma}{n}} = \|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}|B_i|^{-\frac{\gamma}{n}}.$$

The inequality can be rewritten as

$$\begin{aligned}\|H_\gamma(f_1,f_2)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=-\infty}^{j}2^{(n\delta_{33}+n\delta_{44})(i-j)}|B_i|^{\lambda_1+\lambda_2+\frac{\gamma}{n}}\|\chi_i\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\le C\|f_1\|_{\dot{B}^{p_1,\lambda_1}}\|f_2\|_{\dot{B}^{p_2,\lambda_2}}\sum_{i=-\infty}^{j}2^{(n\delta_{33}+n\delta_{44})(i-j)}|B_j|^{\lambda}\frac{|B_i|^{\lambda}}{|B_j|^{\lambda}}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_i\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.\\ &\le C\|f_1\|_{\dot{B}^{p_1,\lambda_1}}\|f_2\|_{\dot{B}^{p_2,\lambda_2}}\sum_{i=-\infty}^{j}2^{(\delta_{33}+\delta_{44}+\delta+\lambda)n(i-j)}|B_j|^{\lambda}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

The inequality can be simplified further to

$$\|H_\gamma(f_1,f_2)\|_{\dot{B}^{q(\cdot),\lambda}} \le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=-\infty}^{j}2^{(\delta_{33}+\delta_{44}+\delta+\lambda)n(i-j)}.$$

Utilizing the condition $\lambda > -(\delta_{33}+\delta_{44}+\delta)$, we obtain the desired condition:

$$\|H_\gamma(f_1,f_2)\|_{\dot{B}^{q(\cdot),\lambda}} \le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}.$$

Theorem 4.3. *Let $q_1(\cdot)$ and $q_2(\cdot)$ belong to $P(\mathbb{R}^n)$ and satisfy the conditions (3.1) and (3.2) as stated in Proposition 3.2. Define the variable exponent $q(\cdot)$ as follows:*

$$\frac{1}{q(\cdot)} = \frac{1}{q_2(\cdot)} + \frac{1}{q_1(\cdot)} - \frac{\gamma}{n}.$$

If $\lambda = \lambda_1 + \lambda_2 + \frac{\gamma}{n}$ and $\gamma < n(\delta_{11}+\delta_{22}-\delta+\lambda)$, where δ_{11}, δ_{22}, and δ are constants as they appear in inequality (3.3) and (3.4), then we have

$$\|H^*_\gamma(f_1,f_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}} \le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}.$$

Proof. Using Holder's inequality, we derive

$$\begin{aligned}|H^*_\gamma(f_1,f_2)(x)\cdot\chi_j(x)| &\le \int_{R^n\setminus B_j}\int_{R^n\setminus B_j}\frac{1}{|y|^{2n-\gamma}}|f_1(y)||f_2(y)|dy_1dy_2\cdot\chi_j(x)\\ &\le C\sum_{i=j+1}^{\infty}2^{-i(2n-\gamma)}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\ &\times\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\chi_j(x).\end{aligned}$$

$$\begin{aligned}\|H^*_\gamma(f_1,f_2)(x)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\sum_{i=j+1}^{\infty}2^{-i(2n-\gamma)}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\ &\times\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n))}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Based on inequality (4.1), we obtain

$$\begin{aligned}\|H_\gamma^*(f_1,f_2)(x)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\sum_{i=j+1}^{\infty}\|f_{1i}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\ &\times\|\chi_i\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{-1}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\le C\sum_{i=j+1}^{\infty}2^{n\delta(j-i)}\|f_{1i}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\ &\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=j+1}^{\infty}2^{n\delta(j-i)}\frac{|B_i|^{\lambda_1}}{|B_j|^{\lambda_1}}|B_j|^{\lambda_1}\frac{|B_i|^{\lambda_2}}{|B_j|^{\lambda_2}}|B_j|^{\lambda_2}\\ &\times\frac{\|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\ &\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=j+1}^{\infty}2^{(n\delta-n\delta_{11}-n\delta_{22})(j-i)}\left|\frac{B_i}{B_j}\right|^{\lambda_1+\lambda_2}|B_j|^{\lambda_1+\lambda_2}\\ &\times\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Here we employ $\frac{1}{q(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}-\frac{\gamma}{n}$ and $\lambda=\lambda_1+\lambda_2+\frac{\gamma}{n}$:

$$\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}=\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}|B_j|^{-\frac{\gamma}{n}}$$

$$\begin{aligned}\|H_\gamma^*(f_1,f_2)(x)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=j+1}^{\infty}2^{(n\delta-n\delta_{11}-n\delta_{22})(j-i)}\\ &\times\left|\frac{B_i}{B_j}\right|^{\lambda_1+\lambda_2}|B_j|^{\lambda_1+\lambda_2+\frac{\gamma}{n}}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=j+1}^{\infty}2^{(n\delta-n\delta_{11}-n\delta_{22}-n\lambda+\gamma)(j-i)}|B_j|^{\lambda}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

$$\|H_\gamma^*(f_1,f_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}}\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=j+1}^{\infty}2^{(n\delta-n\delta_{11}-n\delta_{22}-n\lambda+\gamma)(j-i)}.$$

By employing the condition $\gamma<n(\delta_{11}+\delta_{22}-\delta+\lambda)$, we achieve the desired result:

$$\|H_\gamma^*(f_1,f_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}}\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}.$$

Theorem 4.4. *Suppose $p(\cdot)\in P(\mathbb{R}^n)$ satisfies conditions (3.1) and (3.2) as stated in Proposition 3.2. We define the variable exponent $q(\cdot)$ as $\frac{1}{q(\cdot)}+\frac{\gamma}{n}=\frac{1}{p(\cdot)}$, where $\frac{1}{p(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}$. If $\lambda=\nu+\lambda_1+\lambda_2+\frac{\gamma}{n}$ and $\lambda>(\delta_{33}+\delta_{44}+\delta)$, where δ_{33}, δ_{44}, and δ are the same constants as those appearing in equalities (3.5) and (3.6), then we have*

$$\|[b,H_\gamma](f_1,f_2)\|_{\dot{B}^{q(\cdot),\lambda}}\le C\|b\|_{CBMO^{q(\cdot),\nu}}\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}},$$

where $b=(b_1,b_2)$ and $b\in\|b\|_{CBMO^{q(\cdot),\nu}}$.

Proof. Utilizing the generalized Hölder inequality and Lemma 3.7, we obtain

$$
\begin{aligned}
|[b_1, H_\gamma](f_1, f_2)(x)\cdot\chi_j(x)| &\le \frac{1}{|x|^{2n-\gamma}}\int_{B_j}\int_{B_j}|f_1(y_1)f_2(y_2)(b_1(x)-b_1(y_1))|dy_1dy_2\cdot\chi_j(x)\\
&\le \frac{1}{|x|^{2n-\gamma}}\sum_{i=-\infty}^{j}\int_{B_j}\int_{B_j}|f_1(y_1)f_2(y_2)(b_1(x)-(b_1)_{B_i}+(b_1)_{B_i}-b_1(y_1))|dy_1dy_2\cdot\chi_j(x)\\
&\le \frac{1}{|x|^{2n-\gamma}}\sum_{i=-\infty}^{j}\int_{B_j}\int_{B_j}|f_1(y_1)f_2(y_2)(b_1(x)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
&+ \frac{1}{|x|^{2n-\gamma}}\sum_{i=-\infty}^{j}\int_{B_j}\int_{B_j}|f_1(y_1)f_2(y_2)(b_1(y_1)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
&= I+II
\end{aligned}
$$

$$
\begin{aligned}
I =&\frac{1}{|x|^{2n-\gamma}}\sum_{i=-\infty}^{j}\int_{B_j}\int_{B_j}|f_1(y_1)f_2(y_2)(b_1(x)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
&\le C2^{-j(2n-\gamma)}\sum_{i=-\infty}^{j}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\times\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}|(b_1(x)-(b_1)_{B_i})\chi_j(x)|.
\end{aligned}
$$

$$
\begin{aligned}
\|I\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C2^{-j(2n-\gamma)}\sum_{i=-\infty}^{j}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\times\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|(b_1(x)-(b_1)_{B_i})\chi_j(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-j(2n-\gamma)}\sum_{i=-\infty}^{j}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\times\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}(j-i)\|b_1\|_{BMO}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \qquad (4.3)
\end{aligned}
$$

$$
\begin{aligned}
II =& \frac{1}{|x|^{2n-\gamma}}\sum_{i=-\infty}^{j}\int_{B_j}\int_{B_j}|f_1(y_1)f_2(y_2)(b_1(y_1)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
&\le C2^{-j(2n-\gamma)}\sum_{i=-\infty}^{j}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|b_1\|_{BMO}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\times\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}|\chi_j(x)|
\end{aligned}
$$

$$
\begin{aligned}
\|II\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C2^{-j(2n-\gamma)}\sum_{i=-\infty}^{j}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|b_1\|_{BMO}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\times\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \qquad (4.4)
\end{aligned}
$$

From inequalities (4.3) and (4.4), we have

$$\|[b_1,H_\gamma](f_1,f_2)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C2^{-j(2n-\gamma)}\sum_{i=-\infty}^{j}(j-i)\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$
$$\times\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|b_1\|_{BMO}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Using Lemmas 4.1 and 3.3, we have derived the following inequality:

$$\begin{aligned}
&\|[b_1,H_\gamma](f_1,f_2)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{j\gamma}\sum_{i=-\infty}^{j}(j-i)\|b_1\|_{BMO}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}2^{-2jn}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{i=-\infty}^{j}(j-i)\|b_1\|_{BMO}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}2^{-j(2n-\gamma)}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{4.5}$$

To proceed, inserting (4.1) into (4.5) yields

$$\begin{aligned}
\|[b_1,H_\gamma](f_1,f_2)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}&\le C\sum_{i=-\infty}^{j}(j-i)\|b_1\|_{BMO}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\|\chi_j\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_j\|^{-1}_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{i=-\infty}^{j}(j-i)\|b_1\|_{BMO}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}}\frac{\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}\\
&\le C\sum_{i=-\infty}^{j}2^{n\delta_{33}(i-j)}2^{n\delta_{44}(i-j)}(j-i)\|b_1\|_{BMO}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{i=-\infty}^{j}2^{(n\delta_{33}+n\delta_{44})(i-j)}(j-i)\|b_1\|_{BMO}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

$$\begin{aligned}
\|[b_1,H_\gamma](f_1,f_2)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}&\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=-\infty}^{j}2^{(n\delta_{33}+n\delta_{44})(i-j)}(j-i)\|b_1\|_{BMO}|B_i|^{\lambda_1}|B_i|^{\lambda_2}\\
&\times\|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=-\infty}^{j}2^{(n\delta_{33}+n\delta_{44})(i-j)}(j-i)\frac{\|(b_1-(b_1)_{b_i})\chi_{B_i}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_i}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}|B_i|^{\lambda_1}|B_i|^{\lambda_2}\\
&\times\|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=-\infty}^{j}2^{(n\delta_{33}+n\delta_{44})(i-j)}(j-i)\frac{\|(b_1-(b_1)_{b_i})\chi_{B_i}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B_i|^{\nu}\|\chi_{B_i}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}|B_i|^{\nu}|B_i|^{\lambda_1}|B_i|^{\lambda_2}\\
&\times\|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=-\infty}^{j}2^{(n\delta_{33}+n\delta_{44})(i-j)}(j-i)\|b_1\|_{CBMO^{q(\cdot),\nu}}|B_i|^{\nu+\lambda_1+\lambda_2}\\
&\times\|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Given the conditions $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)} - \frac{\gamma}{n}$ and $\lambda = \nu + \lambda_1 + \lambda_2 + \frac{\gamma}{n}$, we can express the equation as follows:

$$\|\chi_i\|_{L^{q(\cdot)}(\mathbb{R}^n)} = |B_i|^{\frac{1}{q(\cdot)}} = |B_i|^{\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}-\frac{\gamma}{n}} = \|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}|B_i|^{-\frac{\gamma}{n}}.$$

The inequality can be rewritten as

$$\begin{aligned}\|[b_1,H_\gamma](f_1,f_2)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq C\|b_1\|_{CBMO^{q(\cdot),\nu}}\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=-\infty}^{j}2^{(n\delta_{33}+n\delta_{44})(i-j)}|B_i|^{\nu+\lambda_1+\lambda_2+\frac{\gamma}{n}}\|\chi_i\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\leq C\|b_1\|_{CBMO^{q(\cdot),\nu}}\|f_1\|_{\dot{B}^{p_1,\lambda_1}}\|f_2\|_{\dot{B}^{p_2,\lambda_2}}\sum_{i=-\infty}^{j}2^{(n\delta_{33}+n\delta_{44})(i-j)}|B_j|^{\lambda}\frac{|B_i|^{\lambda}}{|B_j|^{\lambda}}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_i\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.\\ &\leq C\|b_1\|_{CBMO^{q(\cdot),\nu}}\|f_1\|_{\dot{B}^{p_1,\lambda_1}}\|f_2\|_{\dot{B}^{p_2,\lambda_2}}\sum_{i=-\infty}^{j}2^{(\delta_{33}+\delta_{44}+\delta+\lambda)n(i-j)}|B_j|^{\lambda}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

The inequality can be simplified further to

$$\|[b_1,H_\gamma](f_1,f_2)\|_{\dot{B}^{q(\cdot),\lambda}} \leq C\|b_1\|_{CBMO^{q(\cdot),\nu}}\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=-\infty}^{j}2^{(\delta_{33}+\delta_{44}+\delta+\lambda)n(i-j)}.$$

Utilizing the condition $\lambda > -(\delta_{33}+\delta_{44}+\delta)$, we obtain the desired condition:

$$\|[b_1,H_\gamma](f_1,f_2)\|_{\dot{B}^{q(\cdot),\lambda}} \leq C\|b_1\|_{CBMO^{q(\cdot),\nu}}\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}.$$

Similarly, we can easily estimate the following result:

$$\|[b_2,H_\gamma](f_1,f_2)\|_{\dot{B}^{q(\cdot),\lambda}} \leq C\|b_2\|_{CBMO^{q(\cdot),\nu}}\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}.$$

Theorem 4.5. *Let $q_1(\cdot)$ and $q_2(\cdot)$ belong to $P(\mathbb{R}^n)$ and satisfy the conditions (3.1) and (3.2) as stated in Proposition 3.2. Define the variable exponent $q(\cdot)$ as follows:*

$$\frac{1}{q(\cdot)} = \frac{1}{q_2(\cdot)} + \frac{1}{q_1(\cdot)} - \frac{\gamma}{n}.$$

If $\lambda = \nu+\lambda_1+\lambda_2+\frac{\gamma}{n}$ and $\lambda < n\delta_{11}+n\delta_{22}-n\delta-\frac{\gamma}{n}$, where δ_{11}, δ_{22}, and δ are constants as they appear in (3.3) and (3.4), then we have

$$\|[b,H_\gamma^*](f_1,f_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}} \leq C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\|b_1\|_{CBMO^{q(\cdot),\nu}},$$

where $b = (b_1,b_2)$ and $b \in \|b\|_{CBMO^{q(\cdot),\nu}}$.

Proof. Using Hölder's inequality, we derive

$$\begin{aligned}|[b_1,H_\gamma^*](f_1,f_2)(x)\cdot\chi_j(x)| &\leq \int_{R^n\backslash B_j}\int_{R^n\backslash B_j}\frac{1}{|y|^{2n-\gamma}}|f_1(y)f_2(y)(b_1(x)-b_1(y_1))|dy_1dy_2\cdot\chi_j(x)\\ &= \int_{R^n\backslash B_j}\int_{R^n\backslash B_j}\frac{1}{|y|^{2n-\gamma}}|f_1(y)f_2(y)(b_1(x)-(b_1)_{B_i}+(b_1)_{B_i}-b_1(y_1))|dy_1dy_2\cdot\chi_j(x)\\ &\leq \int_{R^n\backslash B_j}\int_{R^n\backslash B_j}\frac{1}{|y|^{2n-\gamma}}|f_1(y)f_2(y)(b_1(x)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\ &+ \int_{R^n\backslash B_j}\int_{R^n\backslash B_j}\frac{1}{|y|^{2n-\gamma}}|f_1(y)f_2(y)(b_1(y_1)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\ &= I+II.\end{aligned}$$

$$I=\int_{R^n\backslash B_j}\int_{R^n\backslash B_j}\frac{1}{|y|^{2n-\gamma}}|f_1(y)f_2(y)(b_1(x)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)$$

$$II=\int_{R^n\backslash B_j}\int_{R^n\backslash B_j}\frac{1}{|y|^{2n-\gamma}}|f_1(y)f_2(y)(b_1(y_1)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)$$

$$\begin{aligned}|I|\leq\ &C\sum_{i=j+1}^{\infty}2^{-i(2n-\gamma)}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\&\times\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}(b_1(x_1)-(b_1)_{B_i})\cdot\chi_j(x).\end{aligned}$$

$$\begin{aligned}\|I\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leq\ &C\sum_{i=j+1}^{\infty}2^{-i(2n-\gamma)}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\&\times\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|(b_1(x_1)-(b_1)_{B_i})\cdot\chi_j(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\\leq\ &C\sum_{i=j+1}^{\infty}2^{-i(2n-\gamma)}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\&\times\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|b_1\|_{BMO}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{4.6}$$

Now,

$$\begin{aligned}II=&\int_{R^n\backslash B_j}\int_{R^n\backslash B_j}\frac{1}{|y|^{2n-\gamma}}|f_1(y)f_2(y)(b_1(y_1)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\\leq&\ C\sum_{i=j+1}^{\infty}2^{-i(2n-\gamma)}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|(b_1(y_1)-(b_1)_{B_i})\cdot\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\cdot\chi_j(x).\end{aligned}$$

Using Lemma 3.7, we derive

$$\begin{aligned}\|II\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leq\ &C\sum_{i=j+1}^{\infty}2^{-i(2n-\gamma)}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|(b_1(y_1)-(b_1)_{B_i})\cdot\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\\leq\ &C\sum_{i=j+1}^{\infty}2^{-i(2n-\gamma)}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|b_1\|_{BMO}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{4.7}$$

From the combination of Eqs (4.6) and (4.7), we have

$$\|[b_1,H_\gamma^*](f_1,f_2)(x)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leq C\sum_{i=j+1}^{\infty}2^{-i(2n-\gamma)}\|f_{1i}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|b_1\|_{BMO}\|\chi_i\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_i\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Based on inequality (4.1) and Lemma 3.3, we obtain

$$
\begin{aligned}
\|[b_1, H_\gamma^*](f_1, f_2)(x)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\sum_{i=j+1}^{\infty}\|b_1\|_{BMO}\|f_{1i}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_{2i}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\times\|\chi_i\|^{-1}_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=j+1}^{\infty}2^{n\delta(j-i)}\frac{\|(b_1-(b_1)_{b_i})\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_i}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\frac{|B_i|^{\lambda_1}}{|B_j|^{\lambda_1}}|B_j|^{\lambda_1}\frac{|B_i|^{\lambda_2}}{|B_j|^{\lambda_2}}|B_j|^{\lambda_2}\\
&\times\frac{\|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=j+1}^{\infty}2^{n\delta(j-i)}\frac{\|(b_1-(b_1)_{b_i})\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B_i|^{\nu}\|\chi_{B_i}\|_{L^{q(\cdot)}(;2\mathbb{R}^n)}}|B_i|^{\nu}\frac{|B_i|^{\lambda_1}}{|B_j|^{\lambda_1}}|B_j|^{\lambda_1}\frac{|B_i|^{\lambda_2}}{|B_j|^{\lambda_2}}|B_j|^{\lambda_2}\\
&\times\frac{\|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\sum_{i=j+1}^{\infty}2^{n\delta(j-i)}\frac{\|(b_1-(b_1)_{b_i})\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B_i|^{\nu}\|\chi_{B_i}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\frac{|B_i|^{\nu}}{|B_j|^{\nu}}|B_j|^{\nu}\frac{|B_i|^{\lambda_1}}{|B_j|^{\lambda_1}}|B_j|^{\lambda_1}\\
&\times\frac{|B_i|^{\lambda_2}}{|B_j|^{\lambda_2}}|B_j|^{\lambda_2}\frac{\|\chi_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_i\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\|b_1\|_{CBMO^{q(\cdot),\nu}}\sum_{i=j+1}^{\infty}2^{(n\delta-n\delta_{11}-n\delta_{22})(j-i)}\left|\frac{B_i}{B_j}\right|^{\lambda_1+\lambda_2+\nu}|B_j|^{\lambda_1+\lambda_2+\nu}\\
&\times\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Here, we employ $\frac{1}{q(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}-\frac{\gamma}{n}$ and $\lambda=\nu+\lambda_1+\lambda_2+\frac{\gamma}{n}$:

$$
\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}=\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}|B_j|^{-\frac{\gamma}{n}}
$$

$$
\begin{aligned}
\|[b_1, H_\gamma^*](f_1, f_2)(x)\cdot\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\|b_1\|_{CBMO^{q(\cdot),\nu}}\sum_{i=j+1}^{\infty}2^{(n\delta-n\delta_{11}-n\delta_{22})(j-i)}\\
&\times\left|\frac{B_i}{B_j}\right|^{\nu+\lambda_1+\lambda_2}|B_j|^{\nu+\lambda_1+\lambda_2+\frac{\gamma}{n}}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\|b_1\|_{CBMO^{q(\cdot),\nu}}\sum_{i=j+1}^{\infty}2^{(n\delta-n\delta_{11}-n\delta_{22}-\lambda+\frac{\gamma}{n})(j-i)}|B_j|^{\lambda}\|\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}
\end{aligned}
$$

$$
\|[b_1, H_\gamma^*](f_1, f_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}}\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\|b_1\|_{CBMO^{q(\cdot),\nu}}\sum_{i=j+1}^{\infty}2^{(n\delta-n\delta_{11}-n\delta_{22}-\lambda+\frac{\gamma}{n})(j-i)}.
$$

By employing the condition $\lambda<n\delta_{11}+n\delta_{22}-n\delta-\frac{\gamma}{n}$, we achieve the desired result:

$$
\|[b_1, H_\gamma^*](f_1, f_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}}\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\|b_1\|_{CBMO^{q(\cdot),\nu}}.
$$

Analogously, we can expeditiously approximate the subsequent outcome:

$$
\|[b_2, H_\gamma^*](f_1, f_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}}\le C\|f_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|f_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}}\|b_2\|_{CBMO^{q(\cdot),\nu}}.
$$

5. Conclusions

This manuscript makes significant contributions to the study of m-linear fractional Hardy operators in central Morrey spaces with varying exponents. The results presented provide a deeper understanding of the boundedness properties of commutators associated with H_γ (or H_γ^*) and the λ-central BMO function.

Author contributions

The authors contributed equally to this work. All authors read and approved the final copy of this paper.

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Conflict of interest

The authors declare that they have no competing interests.

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