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Research article

Rough topological structures by various types of maximal neighborhoods

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Abstract: This manuscript centers on creating various topologies utilizing different sorts of maximal neighborhoods. The comparison of these topologies with the previous ones reveal that the earlier topology is weaker than the current ones. The core properties of the proposed topologies are examined, and the necessary conditions for achieving certain equivalences among them are outlined. Additionally, this study provides a distinctive characterization of these topologies by pinpointing the coarsest and largest one among all types, whereas previous methods were limited to characterizing only disjoint pairs of sets. Thereafter, these topologies are utilized to evolve new approximations. One of the major benefits of the current extension is that it adheres to all the properties of the original approximations without the constraints or limitations imposed by earlier versions. The significance of this paper lies not only in introducing new types of approximations based primarily on different kinds of topologies, but also in the fact that these approximations maintain the monotonic property for any given relation, enabling effective evaluation of uncertainty in the data. The monotonic property is crucial for various applications, as it guarantees that the approximation process is logically coherent and robust in the face of evolving information. The proposed models distinguish from their predecessors by their ability to compare all types of the suggested approximations. Moreover, comparisons reveal that the optimal approximations and accuracy are achieved with a specific type of generating topologies. The results demonstrate that topological notions can be a potent technique for studying rough set models. Furthermore, advanced topological features of approximate sets aid in finding rough measures, which assists in identifying missing feature values. Afterward, a numerical example is presented to highlight and emphasize the importance of the present results. Ultimately, the benefits of the followed manner are scrutinized and also some of their limitations are pointed out.

Keywords: rough sets; lower approximations; upper approximations; neighborhood; topology
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1. Introduction

A rough set [35,36] is fundamentally based on the observation that objects cannot be distinguished with the available information. In essence, imperfect data leads to the indiscernibility of things. This indiscernibility forms an approximation space composed of equivalence classes of indistinguishable objects. Initially, a rough set was characterized by two approximation operators (lower and upper) represented in equivalence classes. However, in practical situations, obtaining an equivalence relation can be challenging due to the uncertainty and shortcomings of human knowledge. This challenge served as motivation and encouragement to generalize and interpret rough sets (see [31, 33]).

Topology is inherently present in almost all branches of mathematics (see [26, 38]), establishing it as a crucial unifying concept. Currently, ordinary topology is utilized in several areas of artificial intelligence. Pawlak's operators (lower and upper) are equivalent to topology's operators (interior and closure). So, numerous studies have explored topological space and rough sets. Among the most prominent studies were the works undertaken by Skowron [40] and Wiweger [42]. The main benefit of integrating topological spaces into this theory lies in their ability to diminish conceptual ambiguity. This improvement enhances the certainty of knowledge and boosts the reliability of decision-making methods. Consequently, employing topologies is a powerful method for clarifying and accurately defining concepts. Some of the topological concepts that were applied to develop this theory were nearly open sets [15,16,21] and ideals (see [17–20,22]). Therefore, the investigation of rough set theory with topologies has emerged as a prominent and compelling research area, garnering considerable interest from scholars (see [30, 37, 39, 46]).

Neighborhoods are essential to topological spaces and for addressing topological issues. Consequently, they are employed to broaden topological rough sets by substituting equivalence classes by neighborhoods in Pawlak's operators (see [9, 19, 23–25]). It is the most influential tools for examining the extension of rough sets. This trend emerged from the paper of Yao [43, 44]. He established approximation spaces derived from four types of neighborhoods relative to an arbitrary relation. Afterward, various types of neighborhoods have been applied to evolve lower and upper approximations [4, 5, 28]. Later on, Abo-Tabl [1] developed approximations based on specific types of neighborhoods which formed the basis of topology under limited conditions on relations. In [2, 3, 29], approximations utilizing several topologies formed by eight kinds of neighborhoods were proposed. Meanwhile, Dai et al. [12] exhibited a new neighborhood inspired by similarity relations. They employed them to suggest three innovative types of approximations. In two types of approximations presented by Dai et al. [12], they satisfied monotonicity, which was one difference from the work of Abo-Tabl [1]. One of the limitations of these approximations is that some of Pawlak's properties are lost. Furthermore, similarity relations are not always applicable in various real-life situations which hinders the broader application of this manner. Following this, the remaining seven kinds of maximal neighborhoods were proposed in [7] and he utilized this system to propose new approximations. These approximations achieved better accuracy measures compared to those of Dai et al. [12] in certain kinds of maximal neighborhoods, but also some of Pawlak's properties were not satisfied. More recently, Taher et al. [41] utilized the maximal right neighborhoods and formed a topology.

1.1. Motivation for the work

A major motivation for the proposed paper is the significant relationships between topological techniques and rough sets. Recent progress in rough set has resulted in the emergence of topological rough sets methods. This work analyzes rough sets from a topological perspective. The topological notions can be a potent technique for studying rough set models. The advanced topological features of approximate sets aid in finding rough measures, which assists in identifying missing feature values. Hence, the topological concepts can serve as an effective approach for exploring rough set models. This motivates us to employ different sorts of maximal neighborhoods as essential topological tools to generate different topologies and then propose new approximations.

1.2. Contributions of the work

The contributions of this work are detailed as follows: The principal concepts and essential properties are presented in Section 2. Section 3 concentrates on creating various topologies $\mathfrak{T}^{\mathfrak{y}}_{\mathfrak{x}}$ by employing $\mathfrak{y}_{\mathfrak{x}}$-neighborhoods. Relationships among these topologies are given and we determine the smallest and the largest one among all possible maximal neighborhoods under any relation (see Theorem 3.2, and Corollary 3.2). Although, the foregoing manner [2, 3, 29, 34, 45] and their generalization by ideals [14, 17, 34] can only compare topologies confined to the distinct sets $\{r, \mathfrak{l}, i, u\}$ and $\{\langle r\rangle, \langle \mathfrak{l}\rangle, \langle i\rangle, \langle u\rangle\}$. It is explained that Taher et al.'s topology [41] ($\mathfrak{T}^{\mathfrak{y}}_{r}$) in Theorem 2.3 is weaker than the suggested topologies $\forall \mathfrak{x} \in \{\langle r\rangle, i, \langle i\rangle\}$ (see Proposition 3.1, Theorem 3.2, and Corollary 3.2). Thereafter, conditions are established on the relation to derive some equivalences among the proposed topologies. The findings show that the current manners and the prior ones in [2, 3, 29], G*ü*ler et al. [14], and Yildirim [45] are independent when the relation is a binary relation. In Section 4, the suggested topologies are employed to inspect new approximations. The fundamental properties of these approximations are outlined and shown that they achieve all Pawlak's properties without any limitations as in [1, 4, 7, 12, 32]. Moreover, these approximations have the monotonic property under any binary relation (see Proposition 4.3) which is in contrast to the previous ones [1, 4, 8, 12, 19]. The monotonic property specifically addresses how these approximations behave as the information available about the system changes. This means that as more data or knowledge is added to the system, the lower approximation should remain the same or become more precise, and the upper approximation should remain the same or become less inclusive, which guarantees that the approximations remain stable. It is showed that the proposed approximations and the previous ones [2, 3, 29, 34, 45] are independent, in the case of any binary relation. The purpose of Section 5, is to compare $\mathfrak{y}_{\mathfrak{x}}$-approximations, $\mathfrak{y}_{\mathfrak{x}}$-accuracy values, $\mathfrak{y}_{\langle\mathfrak{x}\rangle}$-approximations, and $\mathfrak{y}_{\langle\mathfrak{x}\rangle}$-accuracy values (see Theorem 5.1, and Corollary 5.3). These comparisons in particular are the most important thing that distinguishes these approximations which were not possible in earlier methods [6, 34, 45]. Section 6 presents a numerical example to demonstrate the significance of employing $\mathfrak{y}_{\mathfrak{x}}$-neighborhoods in the current manner. Finally, Section 7 elaborates on the discussions, while Section 8 summarizes the conclusions.

2. Preliminaries

Definition 2.1. [2, 4, 5, 11, 28, 29] Take $\mathfrak{d}$ as an arbitrary binary relation on a finite set $\mathfrak{W} \neq \emptyset$ and $\mathfrak{r} \in \mathfrak{W}$. Then,

(i) $\mathfrak{h}_r(\mathfrak{r}) = \{\mathfrak{s} \in \mathfrak{W} : (\mathfrak{r}, \mathfrak{s}) \in \mathfrak{d}\}$,

(ii) $\mathfrak{h}_l(\mathfrak{r}) = \{\mathfrak{s} \in \mathfrak{W} : (\mathfrak{s}, \mathfrak{r}) \in \mathfrak{d}\}$,

(iii) $\mathfrak{h}_i(\mathfrak{r}) = \mathfrak{h}_r(\mathfrak{r}) \cap \mathfrak{h}_l(\mathfrak{r})$,

(iv) $\mathfrak{h}_u(\mathfrak{r}) = \mathfrak{h}_r(\mathfrak{r}) \cup \mathfrak{h}_l(\mathfrak{r})$,

(v) $\mathfrak{h}_{\langle r\rangle}(\mathfrak{r}) = \bigcap_{\mathfrak{r}\in\mathfrak{h}_r(\mathfrak{s})} \mathfrak{h}_r(\mathfrak{s})$,

(vi) $\mathfrak{h}_{\langle \mathfrak{l}\rangle}(\mathfrak{r}) = \bigcap_{\mathfrak{r}\in\mathfrak{h}_l(\mathfrak{s})} \mathfrak{h}_l(\mathfrak{s})$,

(vii) $\mathfrak{h}_{\langle i\rangle}(\mathfrak{r}) = \mathfrak{h}_{\langle r\rangle}(\mathfrak{r}) \cap \mathfrak{h}_{\langle l\rangle}(\mathfrak{r})$,

(viii) $\mathfrak{h}_{\langle u\rangle}(\mathfrak{r}) = \mathfrak{h}_{\langle r\rangle}(\mathfrak{r}) \cup \mathfrak{h}_{\langle l\rangle}(\mathfrak{r})$,

(ix) the triple $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ is known as an $\mathfrak{x}$-neighborhood space (for short, $\mathfrak{x}$-NS), where $\mathfrak{x} \in \{r, \mathfrak{l}, i, u, \langle r\rangle, \langle \mathfrak{l}\rangle, \langle i\rangle, \langle u\rangle\}$, and $\mathfrak{H}_{\mathfrak{x}}$ is a mapping from $\mathfrak{W}$ to $P(\mathfrak{W})$ which associates each $\mathfrak{r} \in \mathfrak{W}$ with an $\mathfrak{x}$-neighborhood.

Theorem 2.1. *[2, 3, 29] Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS. Then,* $\mathfrak{T}^{\mathfrak{h}}_{\mathfrak{x}} = \{\mathfrak{U} \subseteq \mathfrak{W} : \mathfrak{h}_{\mathfrak{x}}(\mathfrak{r}) \subseteq \mathfrak{U}, \forall \mathfrak{r} \in \mathfrak{U}\}$ *represents a* $\mathfrak{h}_{\mathfrak{x}}$*-topology on* $\mathfrak{W}$.

Definition 2.2. [2, 3, 29] The $\mathfrak{h}_{\mathfrak{x}}$-lower and $\mathfrak{h}_{\mathfrak{x}}$-upper approximations of a set $\mathfrak{U}$ are

$$\underline{\mathfrak{d}}^{\mathfrak{h}}_{\mathfrak{x}}(\mathfrak{U}) = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{h}}_{\mathfrak{x}} : \mathfrak{O} \subseteq \mathfrak{U}\},$$

$$\overline{\mathfrak{d}}^{\mathfrak{h}}_{\mathfrak{x}}(\mathfrak{U}) = \cap\{\mathfrak{V} : \mathfrak{V}' \in \mathfrak{T}^{\mathfrak{h}}_{\mathfrak{x}} : \mathfrak{U} \subseteq \mathfrak{V}\}.$$

Definition 2.3. [7, 12] Take $\mathfrak{d}$ as an arbitrary binary relation on a finite set $\mathfrak{W} \neq \emptyset$. Then the maximal neighborhoods of $\mathfrak{r} \in \mathfrak{W}$ are

(i) $\mathfrak{y}_r(\mathfrak{r}) = \bigcup_{\mathfrak{r}\in\mathfrak{h}_r(\mathfrak{s})} \mathfrak{h}_r(\mathfrak{s})$,

(ii) $\mathfrak{y}_{\mathfrak{l}}(\mathfrak{r}) = \bigcup_{\mathfrak{r}\in\mathfrak{h}_{\mathfrak{l}}(\mathfrak{s})} \mathfrak{h}_{\mathfrak{l}}(\mathfrak{s})$,

(iii) $\mathfrak{y}_i(\mathfrak{r}) = \mathfrak{y}_r(\mathfrak{r}) \cap \mathfrak{y}_l(\mathfrak{r})$,

(iv) $\mathfrak{y}_u(\mathfrak{r}) = \mathfrak{y}_r(\mathfrak{r}) \cup \mathfrak{y}_l(\mathfrak{r})$,

(v) $\mathfrak{y}_{\langle r\rangle}(\mathfrak{r}) = \bigcap_{\mathfrak{r}\in\mathfrak{y}_r(\mathfrak{s})} \mathfrak{y}_r(\mathfrak{s})$,

(vi) $\mathfrak{y}_{\langle \mathfrak{l}\rangle}(\mathfrak{r}) = \bigcap_{\mathfrak{r}\in\mathfrak{y}_{\mathfrak{l}}(\mathfrak{s})} \mathfrak{y}_{\mathfrak{l}}(\mathfrak{s})$,

(vii) $\mathfrak{y}_{\langle i\rangle}(\mathfrak{r}) = \mathfrak{y}_{\langle r\rangle}(\mathfrak{r}) \cap \mathfrak{y}_{\langle l\rangle}(\mathfrak{r})$,

(viii) $\mathfrak{y}_{\langle u\rangle}(\mathfrak{r}) = \mathfrak{y}_{\langle r\rangle}(\mathfrak{r}) \cup \mathfrak{y}_{\langle l\rangle}(\mathfrak{r})$.

Theorem 2.2. *[7] Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS and* $\mathfrak{r} \in \mathfrak{W}$*. Then*

(i) $\mathfrak{y}_{\langle\mathfrak{x}\rangle}(\mathfrak{r}) \subseteq \mathfrak{y}_{\mathfrak{x}}(\mathfrak{r}), \mathfrak{x} \in \{r, \mathfrak{l}, i, u\}$,

(ii) $\mathfrak{y}_r(\mathfrak{r}) = \mathfrak{y}_l(\mathfrak{r}) = \mathfrak{y}_i(\mathfrak{r}) = \mathfrak{y}_u(\mathfrak{r})$ *and* $\mathfrak{y}_{\langle r\rangle}(\mathfrak{r}) = \mathfrak{y}_{\langle l\rangle}(\mathfrak{r}) = \mathfrak{y}_{\langle i\rangle}(\mathfrak{r}) = \mathfrak{y}_{\langle u\rangle}(\mathfrak{r})$, *when* $\mathfrak{d}$ *is symmetric,*

(iii) $\mathfrak{y}_{\mathfrak{x}}(\mathfrak{r}) = \mathfrak{y}_{\langle\mathfrak{x}\rangle}(\mathfrak{r}), \forall \mathfrak{x} \in \{r, l, i, u\}$, *when* $\mathfrak{d}$ *is symmetric and transitive,*

(iv) *all types of* $\mathfrak{y}_{\mathfrak{x}}(\mathfrak{r})$ *are equal, when* $\mathfrak{d}$ *is equivalence.*

Proposition 2.1. *[7] Let* $(\mathfrak{W}, \mathfrak{d}_1, \mathfrak{H}_{1\mathfrak{x}})$ *and* $(\mathfrak{W}, \mathfrak{d}_2, \mathfrak{H}_{2\mathfrak{x}})$ *be two* $\mathfrak{x}$*-NSs. If* $\mathfrak{d}_1 \subseteq \mathfrak{d}_2$, *then* $\mathfrak{y}_{1\mathfrak{x}}(\mathfrak{r}) \subseteq \mathfrak{y}_{2\mathfrak{x}}(\mathfrak{r}), \forall \mathfrak{r} \in \mathfrak{W}$ *and* $\mathfrak{x} \in \{r, \mathfrak{l}, i, u\}$.

Definition 2.4. [7, 12] Let $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ be an $\mathfrak{x}$-NS. Then $\mathfrak{y}_{\mathfrak{x}*}$-lower and $\mathfrak{y}_{\mathfrak{x}*}$-upper approximations of a set $\mathfrak{U}$ are

$$\underline{\mathfrak{d}}_{*}^{\mathfrak{y}_{\mathfrak{x}}}(\mathfrak{U}) = \{\mathfrak{r} \in \mathfrak{W} : \mathfrak{y}_{\mathfrak{x}}(\mathfrak{r}) \subseteq \mathfrak{U}\},$$

$$\overline{\mathfrak{d}}_{*}^{\mathfrak{y}_{\mathfrak{x}}}(\mathfrak{U}) = \{\mathfrak{r} \in \mathfrak{W} : \mathfrak{y}_{\mathfrak{x}}(\mathfrak{r}) \cap \mathfrak{U} \neq \emptyset\}.$$

Definition 2.5. [10, 12] Let $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ be an $\mathfrak{x}$-NS. Then $\mathfrak{y}_{\mathfrak{x}**}$-lower and $\mathfrak{y}_{\mathfrak{x}**}$-upper approximations of a set $\mathfrak{U}$ are

$$\underline{\mathfrak{d}}_{**}^{\mathfrak{y}_{\mathfrak{x}}}(\mathfrak{U}) = \cup\{\mathfrak{y}_{\mathfrak{x}}(\mathfrak{r}) : \mathfrak{y}_{\mathfrak{x}}(\mathfrak{r}) \subseteq \mathfrak{U}\},$$

$$\overline{\mathfrak{d}}_{**}^{\mathfrak{y}_{\mathfrak{x}}}(\mathfrak{U}) = [\underline{\mathfrak{d}}_{**}^{\mathfrak{y}_{\mathfrak{x}}}(\mathfrak{U}')]'.$$

Definition 2.6. [10, 12] Let $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ be an $\mathfrak{x}$-NS. Then $\mathfrak{y}_{\mathfrak{x}***}$-lower and $\mathfrak{y}_{\mathfrak{x}***}$-upper approximations of a set $\mathfrak{U}$ are

$$\overline{\mathfrak{d}}_{***}^{\mathfrak{y}_{\mathfrak{x}}}(\mathfrak{U}) = \cup\{\mathfrak{y}_{\mathfrak{x}}(\mathfrak{r}) : \mathfrak{y}_{\mathfrak{x}}(\mathfrak{r}) \cap \mathfrak{U} \neq \emptyset\},$$

$$\underline{\mathfrak{d}}_{***}^{\mathfrak{y}_{\mathfrak{x}}}(\mathfrak{U}) = [\overline{\mathfrak{d}}_{***}^{\mathfrak{y}_{\mathfrak{x}}}(\mathfrak{U}')]'.$$

Theorem 2.3. *[41] Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS. Then,* $\mathfrak{T}_r^{\mathfrak{y}} = \{\mathfrak{U} \subseteq \mathfrak{W} : \mathfrak{y}_r(\mathfrak{r}) \subseteq \mathfrak{U}, \forall \mathfrak{r} \in \mathfrak{U}\}$ *represents a* $\mathfrak{y}_r$*-topology on* $\mathfrak{W}$.

Definition 2.7. Take $\mathfrak{d}$ as an arbitrary binary relation on a finite set $\mathfrak{W} \neq \emptyset$ and $\mathfrak{r} \in \mathfrak{W}$. Then the subset neighborhoods of $\mathfrak{r} \in \mathfrak{W}$ are

(i) $\mathbb{I}_r(\mathfrak{r}) = \{\mathfrak{s} \in \mathfrak{W} : \mathfrak{h}_r(\mathfrak{r}) \subseteq \mathfrak{h}_r(\mathfrak{s})\}$ [13],

(ii) $\mathbb{I}_{\mathfrak{l}}(\mathfrak{r}) = \{\mathfrak{s} \in \mathfrak{W} : \mathfrak{h}_{\mathfrak{l}}(\mathfrak{r}) \subseteq \mathfrak{h}_{\mathfrak{l}}(\mathfrak{s})\}$ [8],

(iii) $\mathbb{I}_i(\mathfrak{r}) = \mathbb{I}_r(\mathfrak{r}) \cap \mathbb{I}_{\mathfrak{l}}(\mathfrak{r})$ [8],

(iv) $\mathbb{I}_u(\mathfrak{r}) = \mathbb{I}_r(\mathfrak{r}) \cup \mathbb{I}_{\mathfrak{l}}(\mathfrak{r})$ [8],

(v) $\mathbb{I}_{\langle r\rangle}(\mathfrak{r}) = \{\mathfrak{s} \in \mathfrak{W} : \mathfrak{h}_{\langle r\rangle}(\mathfrak{r}) \subseteq \mathfrak{h}_{\langle r\rangle}(\mathfrak{s})\}$ [8],

(vi) $\mathbb{I}_{\langle\mathfrak{l}\rangle}(\mathfrak{r}) = \{\mathfrak{s} \in \mathfrak{W} : \mathfrak{h}_{\langle\mathfrak{l}\rangle}(\mathfrak{r}) \subseteq \mathfrak{h}_{\langle\mathfrak{l}\rangle}(\mathfrak{s})\}$ [8],

(vii) $\mathbb{I}_{\langle i\rangle}(\mathfrak{r}) = \mathbb{I}_{\langle r\rangle}(\mathfrak{r}) \cap \mathbb{I}_{\langle\mathfrak{l}\rangle}(\mathfrak{r})$ [8],

(viii) $\mathbb{I}_{\langle u\rangle}(\mathfrak{r}) = \mathbb{I}_{\langle r\rangle}(\mathfrak{r}) \cup \mathbb{I}_{\langle\mathfrak{l}\rangle}(\mathfrak{r})$ [8].

Lemma 2.1. *[13] Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS and* $\mathfrak{d}$ *be a reflexive relation. Then* $\mathbb{I}_r(\mathfrak{r}) \subseteq \mathfrak{y}_r(\mathfrak{r}), \forall \mathfrak{r} \in \mathfrak{W}$.

It is easy to prove the following lemma.

Lemma 2.2. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS and* $\mathfrak{d}$ *be a reflexive relation. Then* $\mathbb{I}_{\mathfrak{x}}(\mathfrak{r}) \subseteq \mathfrak{y}_{\mathfrak{x}}(\mathfrak{r}), \forall\ \mathfrak{x} \in \{\mathfrak{l}, i, u, \langle r\rangle, \langle \mathfrak{l}\rangle, \langle i\rangle, \langle u\rangle\}, \forall\ \mathfrak{r} \in \mathfrak{W}$.

Theorem 2.4. *[45] Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS. Then* $\mathfrak{T}^{\mathbb{I}_{\mathfrak{x}}} = \{\mathfrak{U} \subseteq \mathfrak{W} : \mathbb{I}_{\mathfrak{x}}(\mathfrak{r}) \subseteq \mathfrak{U}, \forall \mathfrak{r} \in \mathfrak{U}\}$ *constitutes a topology on* $\mathfrak{W}$.

Definition 2.8. [45] Let $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ be an $\mathfrak{x}$-NS. Then the $\mathbb{I}_{\mathfrak{x}}$-lower approximation, $\mathbb{I}_{\mathfrak{x}}$-upper approximation, $\mathbb{I}_{\mathfrak{x}}$-boundary, $\mathbb{I}_{\mathfrak{x}}$-accuracy, and $\mathbb{I}_{\mathfrak{x}}$-roughness of a set $\mathfrak{U} \subseteq \mathfrak{W}$ are

$$\underline{\mathfrak{d}}^{\mathbb{I}_{\mathfrak{x}}}(\mathfrak{U}) = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathbb{I}_{\mathfrak{x}}} : \mathfrak{O} \subseteq \mathfrak{U}\},$$

$$\overline{\mathfrak{d}}^{\mathbb{I}_{\mathfrak{x}}}(\mathfrak{U}) = \cap\{\mathfrak{V} : \mathfrak{V}' \in \mathfrak{T}^{\mathbb{I}_{\mathfrak{x}}} : \mathfrak{U} \subseteq \mathfrak{V}\},$$

$$\mathfrak{P}^{\mathbb{I}_{\mathfrak{x}}}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathbb{I}_{\mathfrak{x}}}(\mathfrak{U}) \setminus \underline{\mathfrak{d}}^{\mathbb{I}_{\mathfrak{x}}}(\mathfrak{U}),$$

$$\mathfrak{C}^{\mathbb{I}_{\mathfrak{x}}}(\mathfrak{U}) = \frac{|\underline{\mathfrak{d}}^{\mathbb{I}_{\mathfrak{x}}}(\mathfrak{U})|}{|\overline{\mathfrak{d}}^{\mathbb{I}_{\mathfrak{x}}}(\mathfrak{U})|}, \text{ where } \mathfrak{U} \neq \emptyset,$$

$$\mathfrak{R}^{\mathbb{I}_{\mathfrak{x}}}(\mathfrak{U}) = 1 - \mathfrak{C}^{\mathbb{I}_{\mathfrak{x}}}(\mathfrak{U}).$$

Definition 2.9. [6] Take $\mathfrak{d}$ as an arbitrary binary relation on a finite set $\mathfrak{W} \neq \emptyset$ and $\mathfrak{r} \in \mathfrak{W}$. Then the containment neighborhoods of $\mathfrak{r} \in \mathfrak{W}$ are

(i) $\mathbb{B}_r(\mathfrak{r}) = \{\mathfrak{s} \in \mathfrak{W} : \mathfrak{h}_r(\mathfrak{s}) \subseteq \mathfrak{h}_r(\mathfrak{r})\}$,

(ii) $\mathbb{B}_{\mathfrak{l}}(\mathfrak{r}) = \{\mathfrak{s} \in \mathfrak{W} : \mathfrak{h}_{\mathfrak{l}}(\mathfrak{s}) \subseteq \mathfrak{h}_{\mathfrak{l}}(\mathfrak{r})\}$,

(iii) $\mathbb{B}_i(\mathfrak{r}) = \mathbb{B}_r(\mathfrak{r}) \cap \mathbb{B}_{\mathfrak{l}}(\mathfrak{r})$,

(iv) $\mathbb{B}_u(\mathfrak{r}) = \mathbb{B}_r(\mathfrak{r}) \cup \mathbb{B}_{\mathfrak{l}}(\mathfrak{r})$,

(v) $\mathbb{B}_{\langle r\rangle}(\mathfrak{r}) = \{\mathfrak{s} \in \mathfrak{W} : \mathfrak{h}_{\langle r\rangle}(\mathfrak{s}) \subseteq \mathfrak{h}_{\langle r\rangle}(\mathfrak{r})\}$,

(vi) $\mathbb{B}_{\langle \mathfrak{l}\rangle}(\mathfrak{r}) = \{\mathfrak{s} \in \mathfrak{W} : \mathfrak{h}_{\langle \mathfrak{l}\rangle}(\mathfrak{s}) \subseteq \mathfrak{h}_{\langle \mathfrak{l}\rangle}(\mathfrak{r})\}$,

(vii) $\mathbb{B}_{\langle i\rangle}(\mathfrak{r}) = \mathbb{B}_{\langle r\rangle}(\mathfrak{r}) \cap \mathbb{B}_{\langle \mathfrak{l}\rangle}(\mathfrak{r})$,

(viii) $\mathbb{B}_{\langle u\rangle}(\mathfrak{r}) = \mathbb{B}_{\langle r\rangle}(\mathfrak{r}) \cup \mathbb{B}_{\langle \mathfrak{l}\rangle}(\mathfrak{r})$.

Lemma 2.3. *[7] Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS and* $\mathfrak{d}$ *be a reflexive relation. Then,* $\forall \mathfrak{r} \in \mathfrak{W}$

(i) $\mathbb{B}_{\mathfrak{x}}(\mathfrak{r}) \subseteq \mathfrak{h}_{\mathfrak{x}}(\mathfrak{r})$,

(ii) $\mathbb{B}_{\mathfrak{x}}(\mathfrak{r}) \subseteq \mathfrak{y}_{\mathfrak{x}}(\mathfrak{r})$.

Theorem 2.5. *[14] Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS. Then* $\mathfrak{T}^{\mathbb{B}_{\mathfrak{x}}} = \{\mathfrak{U} \subseteq \mathfrak{W} : \mathbb{B}_{\mathfrak{x}}(\mathfrak{r}) \subseteq \mathfrak{U}, \forall \mathfrak{r} \in \mathfrak{U}\}$ *constitutes a topology on* $\mathfrak{W}$.

Definition 2.10. [34] Let $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ be an $\mathfrak{x}$-NS. Then the $\mathbb{B}_{\mathfrak{x}}$-lower and $\mathbb{B}_{\mathfrak{x}}$-upper approximations of a set $\mathfrak{U}$ are

$$\underline{\mathfrak{d}}^{\mathbb{B}_{\mathfrak{x}}}(\mathfrak{U}) = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathbb{B}_{\mathfrak{x}}} : \mathfrak{O} \subseteq \mathfrak{U}\},$$

$$\overline{\mathfrak{d}}^{\mathbb{B}_{\mathfrak{x}}}(\mathfrak{U}) = \cap\{\mathfrak{V} : \mathfrak{V}' \in \mathfrak{T}^{\mathbb{B}_{\mathfrak{x}}} : \mathfrak{U} \subseteq \mathfrak{V}\}.$$

3. Topologies inspired by various sorts of maximal neighborhoods

This section is devoted to suggesting new manners for generating various topologies by $\mathfrak{y}_\mathfrak{x}$-neighborhoods. Additionally, the essential relationships among them are outlined. Moreover, the fundamental differences between current and foregoing topologies are outlined.

The maximal right neighborhoods were proposed by Dai et al. [12], while the remaining seven types of maximal neighborhoods were presented in [7]. Based on the fact that $\mathfrak{y}_\mathfrak{x}$-neighborhoods have properties that differ from other neighborhoods, for example, the property in the Theorem 2.2 [7] distinguishes these neighborhoods while it is not verified in $\mathfrak{y}_\mathfrak{x}$-neighborhoods, so it is necessary to verify that they can create topology as in the following findings.

Theorem 3.1. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_\mathfrak{x})$ *be an* $\mathfrak{x}$*-NS. Then,*

$$\mathfrak{T}^{\mathfrak{y}}_\mathfrak{x} = \{\mathfrak{U} \subseteq \mathfrak{W} : \mathfrak{y}_\mathfrak{x}(\mathfrak{r}) \subseteq \mathfrak{U}, \forall \mathfrak{r} \in \mathfrak{U}\}, \forall \mathfrak{x} \in \{\mathfrak{l}, i, u, \langle r\rangle, \langle \mathfrak{l}\rangle, \langle i\rangle, \langle u\rangle\}$$

represents an $\mathfrak{y}_\mathfrak{x}$*-topology on* $\mathfrak{W}$.

Proof. **(i)** Clearly, $\mathfrak{W}, \emptyset \in \mathfrak{T}^{\mathfrak{y}}_\mathfrak{x}$.

(ii) Let $\mathfrak{U}_i \in \mathfrak{T}^{\mathfrak{y}}_\mathfrak{x}(\forall i \in I)$ and $\mathfrak{r} \in \bigcup_{i\in I} \mathfrak{U}_i$. Then, $\exists i_0 \in I$ such that $\mathfrak{r} \in \mathfrak{U}_{i_0}$.
$\Rightarrow \mathfrak{y}_\mathfrak{x}(\mathfrak{r}) \subseteq \mathfrak{U}_{i_0}$.
$\Rightarrow \mathfrak{y}_\mathfrak{x}(\mathfrak{r}) \subseteq \bigcup_{i\in I} \mathfrak{U}_i$, so $\bigcup_{i\in I} \mathfrak{U}_i \in \mathfrak{T}^{\mathfrak{y}}_\mathfrak{x}$.

(iii) Let $\mathfrak{U}_1, \mathfrak{U}_2 \in \mathfrak{T}^{\mathfrak{y}}_\mathfrak{x}, \mathfrak{r} \in \mathfrak{U}_1 \cap \mathfrak{U}_2$.
$\Rightarrow \mathfrak{r} \in \mathfrak{U}_1$ and $\mathfrak{r} \in \mathfrak{U}_2$.
$\Rightarrow \mathfrak{y}_\mathfrak{x}(\mathfrak{r}) \subseteq \mathfrak{U}_1$ and $\mathfrak{y}_\mathfrak{x}(\mathfrak{r}) \subseteq \mathfrak{U}_2$.
$\Rightarrow \mathfrak{y}_\mathfrak{x}(\mathfrak{r}) \subseteq \mathfrak{U}_1 \cap \mathfrak{U}_2$.
$\Rightarrow \mathfrak{U}_1 \cap \mathfrak{U}_2 \in \mathfrak{T}^{\mathfrak{y}}_\mathfrak{x}$.
So, $\mathfrak{T}^{\mathfrak{y}}_\mathfrak{x}$ is an $\mathfrak{y}_\mathfrak{x}$-topology on $\mathfrak{W}$. □

Definition 3.1. The triple system $(\mathfrak{W}, \mathfrak{d}, \mathfrak{T}^{\mathfrak{y}}_\mathfrak{x})$ is said to be an $\mathfrak{y}_\mathfrak{x}$-topological space (for short, $\mathfrak{h}_\mathfrak{x}TS$), and $\mathfrak{T}^{\mathfrak{y}}_\mathfrak{x}$ is an $\mathfrak{y}_\mathfrak{x}$-topology on $\mathfrak{W}$ induced by Theorem 3.1.

$\mathfrak{U} \subseteq \mathfrak{W}$ is known as an $\mathfrak{y}_\mathfrak{x}$-open set if $\mathfrak{U} \in \mathfrak{T}^{\mathfrak{y}}_\mathfrak{x}$, and an $\mathfrak{y}_\mathfrak{x}$-closed set if $\mathfrak{U}' \in \mathfrak{T}^{\mathfrak{y}}_\mathfrak{x}$. All $\mathfrak{y}_\mathfrak{x}$-closed subsets of $\mathfrak{W}$ are symbolized by $\mathfrak{S}^{\mathfrak{y}}_\mathfrak{x}$.

Definition 3.2. The $\mathfrak{y}_\mathfrak{x}$-interior and $\mathfrak{y}_\mathfrak{x}$-closure of a subset $\mathfrak{U}$ of an $\mathfrak{y}_\mathfrak{x}TS$ $(\mathfrak{W}, \mathfrak{d}, \mathfrak{T}^{\mathfrak{y}}_\mathfrak{x})$ are defined as follows.

$$Int^{\mathfrak{y}}_\mathfrak{x}(\mathfrak{U}) = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_\mathfrak{x} : \mathfrak{O} \subseteq \mathfrak{U}\}, \text{ and}$$

$$Cl^{\mathfrak{y}}_\mathfrak{x}(\mathfrak{U}) = \cap\{\mathfrak{V} \in \mathfrak{S}^{\mathfrak{y}}_\mathfrak{x} : \mathfrak{U} \subseteq \mathfrak{V}\}.$$

Proposition 3.1. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_\mathfrak{x})$ *be an* $\mathfrak{x}$*-NS. Then*
(1) $\mathfrak{T}^{\mathfrak{y}}_u \subseteq \mathfrak{T}^{\mathfrak{y}}_r$ *and* $\mathfrak{T}^{\mathfrak{y}}_u \subseteq \mathfrak{T}^{\mathfrak{y}}_l$,
(2) $\mathfrak{T}^{\mathfrak{y}}_r \subseteq \mathfrak{T}^{\mathfrak{y}}_i$ *and* $\mathfrak{T}^{\mathfrak{y}}_l \subseteq \mathfrak{T}^{\mathfrak{y}}_i$,
(3) $\mathfrak{T}^{\mathfrak{y}}_{\langle u\rangle} \subseteq \mathfrak{T}^{\mathfrak{y}}_{\langle r\rangle}$ *and* $\mathfrak{T}^{\mathfrak{y}}_{\langle u\rangle} \subseteq \mathfrak{T}^{\mathfrak{y}}_{\langle l\rangle}$,
(4) $\mathfrak{T}^{\mathfrak{y}}_{\langle r\rangle} \subseteq \mathfrak{T}^{\mathfrak{y}}_{\langle i\rangle}$ *and* $\mathfrak{T}^{\mathfrak{y}}_{\langle l\rangle} \subseteq \mathfrak{T}^{\mathfrak{y}}_{\langle i\rangle}$,

Proof.

(1) Let $\mathfrak{U} \in \mathfrak{T}_u^{\mathfrak{y}}$. Then, $\mathfrak{y}_u(\mathfrak{r}) \subseteq \mathfrak{U}\ \forall \mathfrak{r} \in \mathfrak{U}$. Thus, $(\mathfrak{y}_r(\mathfrak{r}) \cup \mathfrak{y}_l(\mathfrak{r})) \subseteq \mathfrak{U}\ \forall \mathfrak{r} \in \mathfrak{U}$. Hence, $\mathfrak{y}_r(\mathfrak{r}) \subseteq \mathfrak{U}\ \forall \mathfrak{r} \in \mathfrak{U}$ and $\mathfrak{y}_\mathfrak{l}(\mathfrak{r}) \subseteq \mathfrak{U}\ \forall \mathfrak{l} \in \mathfrak{U}$. Therefore, $\mathfrak{U} \in \mathfrak{T}_r^{\mathfrak{y}}$ and $\mathfrak{U} \in \mathfrak{T}_l^{\mathfrak{y}}$. Hence, $\mathfrak{T}_u^{\mathfrak{y}} \subseteq \mathfrak{T}_r^{\mathfrak{y}}$ and $\mathfrak{T}_u^{\mathfrak{y}} \subseteq \mathfrak{T}_l^{\mathfrak{y}}$. Similarly, (3) can be proved.

(2) Let $\mathfrak{U} \in \mathfrak{T}_r^{\mathfrak{y}}$. Then, $\mathfrak{y}_r(\mathfrak{r}) \subseteq \mathfrak{U}\ \forall \mathfrak{r} \in \mathfrak{U}$. Thus, $(\mathfrak{y}_r(\mathfrak{r}) \cap \mathfrak{y}_l(\mathfrak{r})) \subseteq \mathfrak{U}\ \forall \mathfrak{r} \in \mathfrak{U}$. Hence, $\mathfrak{y}_r(\mathfrak{i}) \subseteq \mathfrak{U}\ \forall \mathfrak{r} \in \mathfrak{U}$. Therefore, $\mathfrak{U} \in \mathfrak{T}_i^{\mathfrak{y}}$. Hence, $\mathfrak{T}_r^{\mathfrak{y}} \subseteq \mathfrak{T}_i^{\mathfrak{y}}$. Similarly, (4) can be proved.

Corollary 3.1. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_\mathfrak{x})$ *be a* $\mathfrak{x}$*-NS. Then*

(1) $\mathfrak{T}_u^{\mathfrak{y}} \subseteq \mathfrak{T}_r^{\mathfrak{y}} \subseteq \mathfrak{T}_i^{\mathfrak{y}}$,
(2) $\mathfrak{T}_u^{\mathfrak{y}} \subseteq \mathfrak{T}_l^{\mathfrak{y}} \subseteq \mathfrak{T}_i^{\mathfrak{y}}$,
(3) $\mathfrak{T}_{\langle u\rangle}^{\mathfrak{y}} \subseteq \mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}} \subseteq \mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}}$,
(4) $\mathfrak{T}_{\langle u\rangle}^{\mathfrak{y}} \subseteq \mathfrak{T}_{\langle l\rangle}^{\mathfrak{y}} \subseteq \mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}}$.

Theorem 3.2. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_\mathfrak{x})$ *be an* $\mathfrak{x}$*-NS. Then* $\mathfrak{T}_\mathfrak{x}^{\mathfrak{y}} \subseteq \mathfrak{T}_{\langle \mathfrak{x}\rangle}^{\mathfrak{y}}$, $\mathfrak{x} \in \{r, \mathfrak{l}, i, u\}$.

Proof. Let $\mathfrak{U} \in \mathfrak{T}_r^{\mathfrak{y}}$. Then $\mathfrak{y}_r(\mathfrak{r}) \subseteq \mathfrak{U}, \forall \mathfrak{r} \in \mathfrak{U}$. Thus $\mathfrak{y}_{\langle r\rangle}(\mathfrak{r}) \subseteq \mathfrak{U}, \forall \mathfrak{r} \in \mathfrak{U}$ by Theorem 2.2. Therefore, $\mathfrak{U} \in \mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}}$. Hence, $\mathfrak{T}_r^{\mathfrak{y}} \subseteq \mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}}$ and the other cases exhibit a similar pattern. □

Remark 3.1. The prior topology in Theorem 2.3 [41] is weaker than the present ones as manifested by Proposition 3.1, Corollary 3.1, and Theorem 3.2, which show that $\mathfrak{T}_r^{\mathfrak{y}} \subseteq \mathfrak{T}_i^{\mathfrak{y}}$, $\mathfrak{T}_r^{\mathfrak{y}} \subseteq \mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}}$, and $\mathfrak{T}_r^{\mathfrak{y}} \subseteq \mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}}$. Additionally, Example 3.1 interprets that $\mathfrak{T}_r^{\mathfrak{y}} \subsetneq \mathfrak{T}_i^{\mathfrak{y}}$, $\mathfrak{T}_r^{\mathfrak{y}} \subsetneq \mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}}$, and $\mathfrak{T}_r^{\mathfrak{y}} \subsetneq \mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}}$. So, the current topologies can be seen as a broader and an extension of the prior work [41].

The following corollary provides a distinctive characterization of $\mathfrak{y}_\mathfrak{x}$-topology by pinpointing the smallest and largest $\mathfrak{y}_\mathfrak{x}$-topology among all types. The previous manner [2, 3, 29, 34, 45] lacks this characterization and only discusses by disjoint pairs of sets $\{r, \mathfrak{l}, i, u\}$ and $\{\langle r\rangle, \langle \mathfrak{l}\rangle, \langle i\rangle, \langle u\rangle\}$.

Corollary 3.2. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_\mathfrak{x})$ *be an* $\mathfrak{x}$*-NS. Then* $\mathfrak{T}_u^{\mathfrak{y}} \subseteq \mathfrak{T}_\mathfrak{x}^{\mathfrak{y}} \subseteq \mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}}$, $\mathfrak{x} \in \{r, \mathfrak{l}, i, \langle r\rangle, \langle \mathfrak{l}\rangle, \langle u\rangle\}$.

Example 3.1. *Consider* $\mathfrak{d} = \{(\mathfrak{r},\mathfrak{r}), (\mathfrak{s},\mathfrak{f}), (\mathfrak{s},\mathfrak{k}), (\mathfrak{f},\mathfrak{k}), (\mathfrak{f},\mathfrak{v}), (\mathfrak{k},\mathfrak{r}), (\mathfrak{k},\mathfrak{v}), (\mathfrak{v},\mathfrak{s}), (\mathfrak{v},\mathfrak{v})\}$ *as a relation on* $\mathfrak{W} = \{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{k}, \mathfrak{v}\}$. *Then,* $\mathfrak{y}_\mathfrak{x}$*-neighborhoods* $\forall \mathfrak{r} \in \mathfrak{W}$ *are introduced in Table 1.*

Table 1. $\mathfrak{y}_\mathfrak{x}$-neighborhoods.

	$\mathfrak{r}$	$\mathfrak{s}$	$\mathfrak{f}$	$\mathfrak{k}$	$\mathfrak{v}$
$\mathfrak{h}_\mathfrak{r}$	$\{\mathfrak{r}\}$	$\{\mathfrak{f},\mathfrak{k}\}$	$\{\mathfrak{k},\mathfrak{v}\}$	$\{\mathfrak{r},\mathfrak{v}\}$	$\{\mathfrak{s},\mathfrak{v}\}$
$\mathfrak{h}_\mathfrak{l}$	$\{\mathfrak{r},\mathfrak{k}\}$	$\{\mathfrak{v}\}$	$\{\mathfrak{s}\}$	$\{\mathfrak{s},\mathfrak{f}\}$	$\{\mathfrak{f},\mathfrak{k},\mathfrak{v}\}$
$\mathfrak{y}_\mathfrak{r}$	$\{\mathfrak{r},\mathfrak{v}\}$	$\{\mathfrak{s},\mathfrak{v}\}$	$\{\mathfrak{f},\mathfrak{k}\}$	$\{\mathfrak{f},\mathfrak{k},\mathfrak{v}\}$	$\{\mathfrak{r},\mathfrak{s},\mathfrak{k},\mathfrak{v}\}$
$\mathfrak{y}_\mathfrak{l}$	$\{\mathfrak{r},\mathfrak{k}\}$	$\{\mathfrak{s},\mathfrak{f}\}$	$\{\mathfrak{s},\mathfrak{f},\mathfrak{k},\mathfrak{v}\}$	$\{\mathfrak{r},\mathfrak{f},\mathfrak{k},\mathfrak{v}\}$	$\{\mathfrak{f},\mathfrak{k},\mathfrak{v}\}$
$\mathfrak{y}_\mathfrak{i}$	$\{\mathfrak{r}\}$	$\{\mathfrak{s}\}$	$\{\mathfrak{f},\mathfrak{k}\}$	$\{\mathfrak{f},\mathfrak{k},\mathfrak{v}\}$	$\{\mathfrak{k},\mathfrak{v}\}$
$\mathfrak{y}_\mathfrak{u}$	$\{\mathfrak{r},\mathfrak{k},\mathfrak{v}\}$	$\{\mathfrak{s},\mathfrak{f},\mathfrak{v}\}$	$\{\mathfrak{s},\mathfrak{f},\mathfrak{k},\mathfrak{v}\}$	$\{\mathfrak{r},\mathfrak{f},\mathfrak{k},\mathfrak{v}\}$	$\mathfrak{W}$
$\mathfrak{y}_{\langle r\rangle}$	$\{\mathfrak{r},\mathfrak{v}\}$	$\{\mathfrak{s},\mathfrak{v}\}$	$\{\mathfrak{f},\mathfrak{k}\}$	$\{\mathfrak{k}\}$	$\{\mathfrak{v}\}$
$\mathfrak{y}_{\langle l\rangle}$	$\{\mathfrak{r},\mathfrak{k}\}$	$\{\mathfrak{s},\mathfrak{f}\}$	$\{\mathfrak{f}\}$	$\{\mathfrak{k}\}$	$\{\mathfrak{f},\mathfrak{k},\mathfrak{v}\}$
$\mathfrak{y}_{\langle i\rangle}$	$\{\mathfrak{r}\}$	$\{\mathfrak{s}\}$	$\{\mathfrak{f}\}$	$\{\mathfrak{k}\}$	$\{\mathfrak{v}\}$
$\mathfrak{y}_{\langle u\rangle}$	$\{\mathfrak{r},\mathfrak{k},\mathfrak{v}\}$	$\{\mathfrak{s},\mathfrak{f},\mathfrak{v}\}$	$\{\mathfrak{f},\mathfrak{k}\}$	$\{\mathfrak{k}\}$	$\{\mathfrak{f},\mathfrak{k},\mathfrak{v}\}$

Consequently,
(1) $\mathfrak{T}_r^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset\}$,

(2) $\mathfrak{T}_l^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset\}$,

(3) $\mathfrak{T}_i^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset, \{\mathfrak{r}\}, \{\mathfrak{s}\}, \{\mathfrak{r}, \mathfrak{s}\}, \{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}\}$,

(4) $\mathfrak{T}_u^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset\}$,

(5) $\mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset, \{\mathfrak{t}\}, \{\mathfrak{v}\}, \{\mathfrak{r}, \mathfrak{v}\}, \{\mathfrak{s}, \mathfrak{v}\}, \{\mathfrak{f}, \mathfrak{t}\}, \{\mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}, \{\mathfrak{r}, \mathfrak{v}, \mathfrak{t}\}, \{\mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}\}$,

(6) $\mathfrak{T}_{\langle l\rangle}^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset, \{\mathfrak{f}\}, \{\mathfrak{t}\}, \{\mathfrak{r}, \mathfrak{t}\}, \{\mathfrak{s}, \mathfrak{f}\}, \{\mathfrak{f}, \mathfrak{t}\}, \{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}, \{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}, \{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}, \{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}\}$,

(7) $\mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}} = P(\mathfrak{W})$,

(8) $\mathfrak{T}_{\langle u\rangle}^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset, \{\mathfrak{t}\}, \{\mathfrak{f}, \mathfrak{t}\}, \{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}, \{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}\}$.

So,

(1) $\mathfrak{T}_i^{\mathfrak{y}} \nsubseteq \mathfrak{T}_r^{\mathfrak{y}}, \mathfrak{T}_l^{\mathfrak{y}}, \mathfrak{T}_u^{\mathfrak{y}}$,

(2) $\mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}} \nsubseteq \mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}}, \mathfrak{T}_{\langle l\rangle}^{\mathfrak{y}}, \mathfrak{T}_{\langle u\rangle}^{\mathfrak{y}}$,

(3) $\mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}} \nsubseteq \mathfrak{T}_{\langle u\rangle}^{\mathfrak{y}}$ *and* $\mathfrak{T}_{\langle l\rangle}^{\mathfrak{y}} \nsubseteq \mathfrak{T}_{\langle u\rangle}^{\mathfrak{y}}$,

(4) $\mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}} \nsubseteq \mathfrak{T}_r^{\mathfrak{y}}$,

(5) $\mathfrak{T}_{\langle \mathfrak{l}\rangle}^{\mathfrak{y}} \nsubseteq \mathfrak{T}_{\mathfrak{l}}^{\mathfrak{y}}$,

(6) $\mathfrak{T}_{\langle u\rangle}^{\mathfrak{y}} \nsubseteq \mathfrak{T}_u^{\mathfrak{y}}$,

(7) $\mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}} \nsubseteq \mathfrak{T}_{\mathfrak{x}}^{\mathfrak{y}}, \mathfrak{x} \in \{r, \mathfrak{l}, i, \langle r\rangle, \langle \mathfrak{l}\rangle, \langle u\rangle\}$.

Example 3.2. *Let* $\mathfrak{d} = \{(\mathfrak{r}, \mathfrak{r}), (\mathfrak{r}, \mathfrak{t}), (\mathfrak{s}, \mathfrak{r}), (\mathfrak{s}, \mathfrak{f}), (\mathfrak{f}, \mathfrak{f}), (\mathfrak{f}, \mathfrak{t}), (\mathfrak{t}, \mathfrak{r})\}$ *be a relation on* $\mathfrak{W} = \{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$. *Then,*

(1) $\mathfrak{T}_r^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset, \{\mathfrak{s}\}, \{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}\}$,

(2) $\mathfrak{T}_l^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset\}$,

(3) $\mathfrak{T}_u^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset\}$.

Thus,

(1) $\mathfrak{T}_r^{\mathfrak{y}} \nsubseteq \mathfrak{T}_l^{\mathfrak{y}}$,

(2) $\mathfrak{T}_r^{\mathfrak{y}} \nsubseteq \mathfrak{T}_u^{\mathfrak{y}}$.

Example 3.3. *Take* $\mathfrak{d} = \{(\mathfrak{r}, \mathfrak{r}), (\mathfrak{r}, \mathfrak{s}), (\mathfrak{r}, \mathfrak{t}), (\mathfrak{f}, \mathfrak{s}), (\mathfrak{f}, \mathfrak{f}), (\mathfrak{t}, \mathfrak{r}), (\mathfrak{t}, \mathfrak{f})\}$ *as a binary relation on* $\mathfrak{W} = \{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$. *Then,*

(1) $\mathfrak{T}_r^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset\}$,

(2) $\mathfrak{T}_l^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset, \{\mathfrak{s}\}, \{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}\}$,

(3) $\mathfrak{T}_u^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset\}$.

Therefore,

(1) $\mathfrak{T}_l^{\mathfrak{y}} \nsubseteq \mathfrak{T}_r^{\mathfrak{y}}$,

(2) $\mathfrak{T}_l^{\mathfrak{y}} \nsubseteq \mathfrak{T}_u^{\mathfrak{y}}$.

Remark 3.2. The current topology distinguishes that $\mathfrak{T}_r^{\mathfrak{y}}$ is not dual to $\mathfrak{T}_l^{\mathfrak{y}}$ (see Examples 3.2 and 3.3), but $\mathfrak{T}_r^{\mathfrak{h}}$ is dual to $\mathfrak{T}_l^{\mathfrak{h}}$ as in [2, 3, 29].

Theorem 3.3. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS. Then if* $\mathfrak{d}$ *is*

(i) *symmetric, then* $\mathfrak{T}_r^{\mathfrak{y}} = \mathfrak{T}_l^{\mathfrak{y}} = \mathfrak{T}_i^{\mathfrak{y}} = \mathfrak{T}_u^{\mathfrak{y}}$ *and* $\mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}} = \mathfrak{T}_{\langle \mathfrak{l}\rangle}^{\mathfrak{y}} = \mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}} = \mathfrak{T}_{\langle u\rangle}^{\mathfrak{y}}$,

(ii) *symmetric and transitive, then* $\mathfrak{T}_{\mathfrak{x}}^{\mathfrak{y}} = \mathfrak{T}_{\langle \mathfrak{x}\rangle}^{\mathfrak{h}}, \forall \mathfrak{x} \in \{r, \mathfrak{l}, i, u\}$,

(iii) *an equivalence, then* $\mathfrak{T}_r^{\mathfrak{y}} = \mathfrak{T}_l^{\mathfrak{h}} = \mathfrak{T}_i^{\mathfrak{y}} = \mathfrak{T}_u^{\mathfrak{y}} = \mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}} = \mathfrak{T}_{\langle \mathfrak{l}\rangle}^{\mathfrak{y}} = \mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}} = \mathfrak{T}_{\langle u\rangle}^{\mathfrak{y}}$.

Proof. **(i)** Let $\mathfrak{U} \in \mathfrak{T}_r^{\mathfrak{y}}$. Then $\mathfrak{y}_r(\mathfrak{r}) \subseteq \mathfrak{U}, \forall\ \mathfrak{r} \in \mathfrak{U}$.

$\Leftrightarrow \mathfrak{y}_l(\mathfrak{r}) \subseteq \mathfrak{U}, \mathfrak{y}_i(\mathfrak{r}) \subseteq \mathfrak{U}, \mathfrak{y}_u(\mathfrak{r}) \subseteq \mathfrak{U}, \forall\ \mathfrak{r} \in \mathfrak{U}$ (by Theorem 2.2).

Hence, $\mathfrak{T}_r^{\mathfrak{y}} = \mathfrak{T}_l^{\mathfrak{y}} = \mathfrak{T}_i^{\mathfrak{y}} = \mathfrak{T}_u^{\mathfrak{h}}$. By the same method, $\mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}} = \mathfrak{T}_{\langle l\rangle}^{\mathfrak{y}} = \mathfrak{T}_{\langle i\rangle}^{\mathfrak{y}} = \mathfrak{T}_{\langle u\rangle}^{\mathfrak{y}}$.

(ii) Let $\mathfrak{U} \in \mathfrak{T}_r^{\mathfrak{y}}$. Then $\mathfrak{y}_r(\mathfrak{r}) \subseteq \mathfrak{U}, \forall\ \mathfrak{r} \in \mathfrak{U}$.
$\Leftrightarrow \mathfrak{y}_{\langle r\rangle}(\mathfrak{r}), \forall\ \mathfrak{r} \in \mathfrak{U}$ (by Theorem 2.2).
Hence, $\mathfrak{T}_r^{\mathfrak{y}} = \mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}}$, and the remaining cases are similar.

(iii) The proof is straightforward, so we will omit it here.

□

Remark 3.3. Example 3.1 is interpreted that in Theorem 3.3 the
(1) Symmetric (i) is indispensable as $\mathfrak{T}_r^{\mathfrak{y}} \neq \mathfrak{T}_i^{\mathfrak{y}}$.
(2) Symmetric and transitive (ii) are indispensable as $\mathfrak{T}_r^{\mathfrak{y}} \neq \mathfrak{T}_{\langle r\rangle}^{\mathfrak{y}}$.
(3) Equivalence in (iii) is indispensable as the types of topologies are not equal.

Proposition 3.2. *Let* $(\mathfrak{W}, \mathfrak{d}_1, \mathfrak{H}_{1\mathfrak{x}})$ *and* $(\mathfrak{W}, \mathfrak{d}_2, \mathfrak{H}_{2\mathfrak{x}})$ *be two* $\mathfrak{x}$*-NSs and* $\mathfrak{d}_1 \subseteq \mathfrak{d}_2$. *Then, for* $\mathfrak{x} \in \{r, \mathfrak{l}, i, u\}$,
(1) $\mathfrak{T}_{2\mathfrak{x}}^{\mathfrak{y}} \subseteq \mathfrak{T}_{1\mathfrak{x}}^{\mathfrak{y}}$,
(2) $int_{2\mathfrak{x}}^{\mathfrak{y}} \subseteq int_{1\mathfrak{x}}^{\mathfrak{y}}$,
(3) $Cl_{1\mathfrak{x}}^{\mathfrak{y}} \subseteq Cl_{2\mathfrak{x}}^{\mathfrak{y}}$.

Proof. To prove (1), let $\mathfrak{U} \in \mathfrak{T}_{2r}^{\mathfrak{y}}$. Then $\mathfrak{y}_{2r}(\mathfrak{r}) \subseteq \mathfrak{U}, \forall \mathfrak{r} \in \mathfrak{U}$. Thus, $\mathfrak{y}_{1r}(\mathfrak{r}) \subseteq \mathfrak{U}, \forall \mathfrak{r} \in \mathfrak{U}$ (by Proposition 2.1). So, $\mathfrak{U} \in \mathfrak{T}_{1r}^{\mathfrak{y}}$ and hence $\mathfrak{T}_{2r}^{\mathfrak{y}} \subseteq \mathfrak{T}_{1r}^{\mathfrak{y}}$. The remaining cases follow a similar pattern. □

Remark 3.4. Example 3.3 is elucidated that
(1) The topologies created by $\mathfrak{x}$-neighborhoods [2, 3, 29], $\mathbb{B}_{\mathfrak{x}}$-neighborhoods [14], and $\mathbb{I}_{\mathfrak{x}}$-neighborhoods [45] are distinct from the topologies exhibited in this section as $\mathfrak{T}_l^{\mathfrak{h}} = \{\mathfrak{W}, \emptyset, \{\mathfrak{r}, \mathfrak{k}\}, \{\mathfrak{f}, \mathfrak{k}\}, \{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}\}\}, \mathfrak{T}^{\mathbb{I}_l} = \{\mathfrak{W}, \emptyset, \{\mathfrak{r}\}, \{\mathfrak{s}\}, \{\mathfrak{f}\}, \{\mathfrak{r}, \mathfrak{s}\}, \{\mathfrak{r}, \mathfrak{f}\}, \{\mathfrak{s}, \mathfrak{f}\}, \{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}\}, \{\mathfrak{r}, \mathfrak{s}, \mathfrak{k}\}\}, \mathfrak{T}^{\mathbb{B}_l} = \{\mathfrak{W}, \emptyset, \{\mathfrak{f}\}, \{\mathfrak{k}\}, \{\mathfrak{r}, \mathfrak{k}\}, \{\mathfrak{f}, \mathfrak{k}\}, \{\mathfrak{r}, \mathfrak{s}, \mathfrak{k}\}, \{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}\}\} \neq \mathfrak{T}_l^{\mathfrak{y}} = \{\mathfrak{W}, \emptyset, \{\mathfrak{s}\}, \{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}\}\}$.
(2) The primary distinctions between the current manner and the previous ones [2, 3, 14, 29, 45] are that $\mathfrak{T}_{\mathfrak{x}}^{\mathfrak{y}} \subseteq \mathfrak{T}_{\langle\mathfrak{x}\rangle}^{\mathfrak{y}}, \mathfrak{x} \in \{r, \mathfrak{l}, i, u\}$, even though $\mathfrak{T}_{\mathfrak{x}}^{\mathfrak{h}}$, $\mathfrak{T}_{\langle\mathfrak{x}\rangle}^{\mathfrak{h}}$ in [2, 3, 29], $\mathfrak{T}^{\mathbb{B}_{\mathfrak{x}}}$, $\mathfrak{T}^{\mathbb{B}\langle\mathfrak{x}\rangle}$ [14], and $\mathfrak{T}^{\mathbb{I}_{\mathfrak{x}}}$, $\mathfrak{T}^{\mathbb{I}\langle\mathfrak{x}\rangle}$ [45] are incomparable.

Proposition 3.3. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS and* $\mathfrak{d}$ *be a reflexive relation. Then,*
(1) $\mathfrak{T}_{\mathfrak{x}}^{\mathfrak{y}} \subseteq \mathfrak{T}_{\mathfrak{x}}^{\mathfrak{h}}$,
(2) $\mathfrak{T}_{\mathfrak{x}}^{\mathfrak{y}} \subseteq \mathfrak{T}^{\mathbb{I}_{\mathfrak{x}}}$,
(3) $\mathfrak{T}_{\mathfrak{x}}^{\mathfrak{y}} \subseteq \mathfrak{T}^{\mathbb{B}_{\mathfrak{x}}}$.

Proof. It is directly obtained from Lemmas 2.1–2.3. □

It is easy to add an example to show that the opposite of Proposition 3.3 does not always hold.

Remark 3.5. According to Example 3.3 and Remark 3.4, the proposed methods and the previous ones in [2, 3, 29], G*ü*ler et al. [14], and Yildirim [45] are not dependent in the general case of a binary relation. Furthermore, Proposition 3.3 shows that the topologies generated by the proposed method are weaker than those presented in the previous works.

4. Approximate models from a topological perspective inspired by various sorts of maximal neighborhoods

This section employs new rough models using $\mathfrak{T}_{\mathfrak{x}}^{\mathfrak{y}}$-topologies obtained from $\mathfrak{y}_{\mathfrak{x}}$-neighborhood systems and outlines their fundamental properties.

Definition 4.1. The $\mathfrak{y}_x$-lower approximation $\underline{\mathfrak{d}}_x^{\mathfrak{y}}$ and $\mathfrak{y}_x$-upper approximation $\overline{\mathfrak{d}}_x^{\mathfrak{y}}$ of a subset $\mathfrak{U}$ of an $\mathfrak{y}_xTS$ $(\mathfrak{W}, \mathfrak{d}, \mathfrak{T}_x^{\mathfrak{y}})$ are defined as follows.

$$\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) = \cup\{\mathfrak{O} \in \mathfrak{T}_x^{\mathfrak{y}} : \mathfrak{O} \subseteq \mathfrak{U}\} = Int_x^{\mathfrak{y}}(\mathfrak{U}), \text{ and}$$

$$\overline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) = \cap\{\mathfrak{V} \in \mathfrak{S}_x^{\mathfrak{y}} : \mathfrak{U} \subseteq \mathfrak{V}\} = Cl_x^{\mathfrak{y}}(\mathfrak{U}).$$

Proposition 4.1. *Let $\mathfrak{U}$ and $\mathfrak{Q}$ be subsets of an $\mathfrak{y}_xTS$ $(\mathfrak{W}, \mathfrak{d}, \mathfrak{T}_x^{\mathfrak{y}})$. Then*

(i) $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \subseteq \mathfrak{U}$,

(ii) $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\emptyset) = \emptyset$,

(iii) $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{W}) = \mathfrak{W}$,

(iv) *If* $\mathfrak{U} \subseteq \mathfrak{Q}$, *then* $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{Q})$,

(v) $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U} \cap \mathfrak{Q}) = \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \cap \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{Q})$,

(vi) $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}') = (\overline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}))'$,

(vii) $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U})) = \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U})$.

Proof. (i)–(iii) are evident from Definition 4.1.

(iv) Let $\mathfrak{U} \subseteq \mathfrak{Q}$. Then $\cup\{\mathfrak{O} \in \mathfrak{T}_x^{\mathfrak{y}} : \mathfrak{O} \subseteq \mathfrak{U}\} \subseteq \cup\{\mathfrak{O} \in \mathfrak{T}_x^{\mathfrak{y}} : \mathfrak{O} \subseteq \mathfrak{Q}\}$ and so $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{Q})$.

(v) $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U} \cap \mathfrak{Q}) \subseteq \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \cap \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{Q})$ by (iv), $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \subseteq \mathfrak{U}$, and $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{Q}) \subseteq \mathfrak{Q}$ by (i). Therefore, $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \cap \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{Q}) \subseteq \mathfrak{U} \cap \mathfrak{Q}$. So, $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \cap \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{Q})) \subseteq \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U} \cap \mathfrak{Q})$. Then $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \cap \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U})) \subseteq \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U} \cap \mathfrak{Q})$. Thus $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U} \cap \mathfrak{Q}) = \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \cap \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{Q})$.

(vi) Let $\mathfrak{r} \in \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}')$. Then, $\exists \mathfrak{O} \in \mathfrak{T}_x^{\mathfrak{y}}$ such that $\mathfrak{r} \in \mathfrak{O} \subseteq \mathfrak{U}'$, so $\mathfrak{O} \cap \mathfrak{U} = \emptyset$. Therefore, $\mathfrak{r} \notin \overline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U})$. Thus, $\mathfrak{r} \in (\overline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}))'$. Let $\mathfrak{r} \in (\overline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}))'$. Then, $\mathfrak{r} \notin \overline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U})$, and so, $\exists U \in \mathfrak{T}_x^{\mathfrak{y}}$ such that $\mathfrak{r} \in \mathfrak{O}$ and $\mathfrak{O} \cap \mathfrak{U} = \emptyset$. So, $\mathfrak{r} \in \mathfrak{O} \subseteq \mathfrak{U}'$. Hence, $\mathfrak{r} \in \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}')$.

(vii) From (i), we get $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U})) \subseteq \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U})$. Conversely, let $\mathfrak{r} \in \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U})$. Then, $\exists \mathfrak{O} \in \mathfrak{T}_x^{\mathfrak{y}}$ such that $\mathfrak{r} \in \mathfrak{O} \subseteq \mathfrak{U}$. Therefore, $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{O}) \subseteq \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U})$ (by (iv)). According to Definition 4.1, we have $\mathfrak{O} = \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{O})$, so $\{\mathfrak{r}\} \in \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{O}) \subseteq \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}))$. Thus, $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}))$.

□

Corollary 4.1. *Let $\mathfrak{U}$ and $\mathfrak{Q}$ be subsets of an $\mathfrak{y}_xTS$ $(\mathfrak{W}, \mathfrak{d}, \mathfrak{T}_x^{\mathfrak{y}})$. Then,* $\underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U}) \cup \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{Q}) \subseteq \underline{\mathfrak{d}}_x^{\mathfrak{y}}(\mathfrak{U} \cup \mathfrak{Q})$.

Proof. It is directly obtained from (iv) of Proposition 4.1. □

Remark 4.1. In Example 3.1,

(i) $\underline{\mathfrak{d}}_i^{\mathfrak{y}}(\{\mathfrak{f}\}) = \emptyset \subset \{\mathfrak{f}\}$,

(ii) $\underline{\mathfrak{d}}_i^{\mathfrak{y}}(\{\mathfrak{f}\}) = \emptyset \subset \{\mathfrak{s}\} = \underline{\mathfrak{d}}_i^{\mathfrak{y}}(\{\mathfrak{s}\})$ but $\{\mathfrak{f}\} \not\subseteq \{\mathfrak{s}\}$,

(iii) $\underline{\mathfrak{d}}_i^{\mathfrak{y}}(\{\mathfrak{v}\}) \cup \underline{\mathfrak{d}}_i^{\mathfrak{y}}(\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{k}\}) = \{\mathfrak{r}, \mathfrak{s}\} \subset \mathfrak{W} = \underline{\mathfrak{d}}_i^{\mathfrak{y}}(\{\mathfrak{v}\} \cup \{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{k}\})$.

Proposition 4.2. *Let $\mathfrak{U}$ and $\mathfrak{Q}$ be subsets of an $\mathfrak{y}_xTS$ $(\mathfrak{W}, \mathfrak{d}, \mathfrak{T}_x^{\mathfrak{y}})$. Then*

(i) $\mathfrak{U} \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})$,

(ii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\emptyset) = \emptyset$,

(iii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \mathfrak{U}$,

(iv) *if* $\mathfrak{U} \subseteq \mathfrak{Q}$, *then* $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{Q})$,

(v) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \cup \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{Q}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U} \cup \mathfrak{Q})$,

(vi) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}') = (\underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}))'$,

(vii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})$.

Proof. This is just as in the proof of Proposition 4.1. □

Corollary 4.2. *Let* $\mathfrak{U}$ *and* $\mathfrak{Q}$ *be subsets of an* $\mathfrak{y}_{\mathfrak{x}}TS$ $(\mathfrak{W}, \mathfrak{d}, \mathfrak{T}^{\mathfrak{y}}_{\mathfrak{x}})$. *Then,* $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U} \cap \mathfrak{Q}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \cap \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{Q})$.

Proof. It is directly derived from (iv) of Proposition 4.2. □

Remark 4.2. In Example 3.1,

(i) $\{\mathfrak{r}, \mathfrak{k}\} \subset \overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\{\mathfrak{r}, \mathfrak{k}\}) = \{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}, \mathfrak{v}\}$,

(ii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\{\mathfrak{f}\}) = \{\mathfrak{f}, \mathfrak{k}, \mathfrak{v}\} \subset \{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}, \mathfrak{v}\} = \overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\{\mathfrak{r}, \mathfrak{k}\})$, but $\{\mathfrak{f}\} \not\subseteq \{\mathfrak{r}, \mathfrak{k}\}$,

(iii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\{\mathfrak{f}\} \cap \{\mathfrak{k}\}) = \emptyset \subset \{\mathfrak{f}, \mathfrak{k}, \mathfrak{v}\} = \overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\{\mathfrak{f}\}) \cap \overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\{\mathfrak{k}\})$.

It is evident from Propositions 4.1 and 4.2, and Corollaries 4.1 and 4.2 that the suggested approximations adhere to all of Pawlak's properties [35] without any constraints.

Definition 4.2. The $\mathfrak{y}_{\mathfrak{x}}$-accuracy and $\mathfrak{y}_{\mathfrak{x}}$-roughness of a set $\mathfrak{U}$ in a $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ are

$$\mathfrak{C}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \frac{|\underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})|}{|\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})|}, \text{ where } \mathfrak{U} \neq \emptyset,$$

$$\mathfrak{R}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = 1 - \mathfrak{C}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}).$$

Definition 4.3. Let $\mathfrak{d}_1 \subseteq \mathfrak{d}_2$. Then, $(\mathfrak{W}, \mathfrak{d}_1, \mathfrak{H}_{1\mathfrak{x}})$, and $(\mathfrak{W}, \mathfrak{d}_2, \mathfrak{H}_{2\mathfrak{x}})$ have the monotonicity property of accuracy (roughness) if $\mathfrak{C}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U}) \geq \mathfrak{C}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U})$ $(\mathfrak{R}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U}) \leq \mathfrak{R}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U}))$.

The following proposition ensures that the lower approximation, which represents the definitively known elements, does not decrease as more data becomes available. Similarly, the upper approximation, which includes all elements that could possibly be in the set, does not decrease either. In essence, the monotonic property guarantees that the rough set approximations become more precise or remain stable as additional information is incorporated, never becoming less accurate. This characteristic is crucial for maintaining the reliability and consistency of the rough set analysis, ensuring that the approximations provide a robust framework for understanding and interpreting data.

Proposition 4.3. *Let* $(\mathfrak{W}, \mathfrak{d}_1, \mathfrak{H}_{1\mathfrak{x}})$ *and* $(\mathfrak{W}, \mathfrak{d}_2, \mathfrak{H}_{2\mathfrak{x}})$ *be two* $\mathfrak{x}$*-NSs and* $\mathfrak{d}_1 \subseteq \mathfrak{d}_2$. *Then,* $\forall \mathfrak{x} \in \{r, l, i, u\}$ *and* $\mathfrak{U} \subseteq \mathfrak{W}$,

(1) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U})$,

(2) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U})$,

(3) $\mathfrak{C}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U})$,

(4) $\mathfrak{R}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U}) \leq \mathfrak{R}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U})$.

Proof.

(1) Let $\overline{\mathfrak{d}}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U}) = Cl^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U}) \subseteq Cl^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U})$ (by Proposition 3.2). Hence, $\overline{\mathfrak{d}}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U})$,

(2) Let $\underline{\mathfrak{d}}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U}) = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{2\mathfrak{x}} : \mathfrak{O} \subseteq \mathfrak{U}\} \subseteq= \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{1\mathfrak{x}} : \mathfrak{O} \subseteq \mathfrak{U}\} = \underline{\mathfrak{d}}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U})$ (by Proposition 3.2). Hence, $\underline{\mathfrak{d}}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U})$,

(3)

$$\begin{aligned}\mathfrak{C}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U}) &= |\frac{\underline{\mathfrak{d}_2}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})|}{\overline{\mathfrak{d}_2}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})}|,\\ &\leq \frac{|\underline{\mathfrak{d}_1}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})|}{|\overline{\mathfrak{d}_1}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})|},\\ &= \mathfrak{C}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U}).\end{aligned}$$

(4) This proof is straightforward, so we will omit it here.

Definition 4.4. The $\mathfrak{y}_{\mathfrak{x}}$-positive, $\mathfrak{y}_{\mathfrak{x}}$-boundary, and $\mathfrak{y}_{\mathfrak{x}}$-negative regions of a set $\mathfrak{U}$ in an $\mathfrak{y}_{\mathfrak{x}}TS$ $(\mathfrak{W}, \mathfrak{d}, \mathfrak{T}^{\mathfrak{x}})$ are

$$\mathfrak{d}^{\mathfrak{y}+}_{\mathfrak{x}}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}),$$

$$\mathfrak{P}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \setminus \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}), \text{ and}$$

$$\mathfrak{d}^{\mathfrak{y}-}_{\mathfrak{x}}(\mathfrak{U}) = \mathfrak{W} \setminus \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}).$$

Proposition 4.4. *Let* $(\mathfrak{W}, \mathfrak{d}_1, \mathfrak{H}_{1\mathfrak{x}})$ *and* $(\mathfrak{W}, \mathfrak{d}_2, \mathfrak{H}_{2\mathfrak{x}})$ *be two* $\mathfrak{x}$*-NSs and* $\mathfrak{d}_1 \subseteq \mathfrak{d}_2$*. Then,* $\forall \mathfrak{x} \in \{r, l, i, u\}$ *and* $\mathfrak{U} \subseteq \mathfrak{W}$,

(i) $\mathfrak{P}^{\mathfrak{y}}_{\mathfrak{x}_1}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{\mathfrak{x}_2}(\mathfrak{U})$,

(ii) $\mathfrak{d}^{\mathfrak{y}}_{\mathfrak{x}_2}{}^{-}(\mathfrak{U}) \subseteq \mathfrak{d}^{\mathfrak{y}}_{\mathfrak{x}_1}{}^{-}(\mathfrak{U})$.

Proof. **(i)** Let $\mathfrak{r} \in \mathfrak{P}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U})$. Then, $\mathfrak{r} \in \overline{\mathfrak{d}_1}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \setminus \underline{\mathfrak{d}_1}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})$. So, $\mathfrak{r} \in \overline{\mathfrak{d}_1}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})$ and $\mathfrak{r} \in (\underline{\mathfrak{d}_1}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}))'$. Thus, $\mathfrak{r} \in \overline{\mathfrak{d}_2}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})$ and $\mathfrak{r} \in (\underline{\mathfrak{d}_2}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}))'$. Hence, $\mathfrak{r} \in \mathfrak{P}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U})$. Therefore, $\mathfrak{P}^{\mathfrak{y}}_{1\mathfrak{x}}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{2\mathfrak{x}}(\mathfrak{U})$.

(ii) This is derived from Proposition 4.3.

□

Definition 4.5. A subset $\mathfrak{U}$ of a $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ is called $\mathfrak{y}_{\mathfrak{x}}$-exact if $\underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \mathfrak{U}$. Otherwise, it is known as an $\mathfrak{y}_{\mathfrak{x}}$-rough set.

Proposition 4.5. *A subset* $\mathfrak{U}$ *of a* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *is* $\mathfrak{y}_{\mathfrak{x}}$*-exact iff* $\mathfrak{P}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \emptyset$.

Proof. Let $\mathfrak{U}$ be an $\mathfrak{x}$-exact set. Then, $\mathfrak{P}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \setminus \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \emptyset$. Conversely, $\mathfrak{P}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \emptyset$ implies that $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \setminus \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \emptyset$, so, $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})$. But $\underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})$. Thus, $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})$. Hence, $\mathfrak{U}$ is $\mathfrak{y}_{\mathfrak{x}}$-exact. □

Remark 4.3. These approximations count on a topology constructed from several types of maximal neighborhoods, differing significantly from the manners that relied on topology formed by another kind of neighborhood to create approximations [2, 3, 29, 34, 45]. Example 3.3 indicates that

(1) if $\mathfrak{U} =$

- $\{\mathfrak{r}, \mathfrak{k}\}$, then $\underline{\mathfrak{d}}_l^{\mathfrak{h}}(\mathfrak{U}) = \{\mathfrak{r}, \mathfrak{k}\} \nsubseteq \emptyset = \underline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U})$,
- $\{\mathfrak{s}, \mathfrak{v}\}$, then $\overline{\mathfrak{d}}_l^{\mathfrak{h}}(\mathfrak{U}) = \{\mathfrak{s}, \mathfrak{v}\} \nsupseteq \mathfrak{W} = \overline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U})$.

(2) if $\mathfrak{U} = \{\mathfrak{s}\}$, then

- $\underline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U}) = \{\mathfrak{s}\} \nsubseteq \emptyset = \underline{\mathfrak{d}}_l^{\mathfrak{h}}(\mathfrak{U})$,
- $\overline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U}) = \{\mathfrak{s}\} \nsupseteq \mathfrak{W} = \overline{\mathfrak{d}}_l^{\mathfrak{h}}(\mathfrak{U})$.

(3) if $\mathfrak{U} = \{\mathfrak{r}\}$, then

- $\underline{\mathfrak{d}}^{\mathbb{I}_l}(\mathfrak{U}) = \{\mathfrak{r}\} \nsubseteq \emptyset = \underline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U})$,
- $\overline{\mathfrak{d}}^{\mathbb{I}_l}(\mathfrak{U}) = \{\mathfrak{r}\} \nsupseteq \{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}\} = \overline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U})$.

(4) if $\mathfrak{U} =$

- $\{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}\}$, then $\underline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U}) = \{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}\} \nsubseteq \{\mathfrak{r}, \mathfrak{f}\} = \underline{\mathfrak{d}}^{\mathbb{I}_l}(\mathfrak{U})$,
- $\{\mathfrak{s}\}$, then $\overline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U}) = \{\mathfrak{s}\} \nsupseteq \{\mathfrak{s}, \mathfrak{k}\} = \overline{\mathfrak{d}}^{\mathbb{I}_l}(\mathfrak{U})$.

(5) if $\mathfrak{U} = \{\mathfrak{f}\}$, then

- $\underline{\mathfrak{d}}^{\mathbb{B}_l}(\mathfrak{U}) = \{\mathfrak{f}\} \nsubseteq \emptyset = \underline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U})$,
- $\overline{\mathfrak{d}}^{\mathbb{B}_l}(\mathfrak{U}) = \{\mathfrak{f}\} \nsupseteq \{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}\} = \overline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U})$.

(6) if $\mathfrak{U} =$

- $\{\mathfrak{s}\}$, then $\underline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U}) = \{\mathfrak{s}\} \nsubseteq \emptyset = \underline{\mathfrak{d}}^{\mathbb{B}_l}(\mathfrak{U})$,
- $\{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}\}$, then $\overline{\mathfrak{d}}_l^{\mathfrak{y}}(\mathfrak{U}) = \{\mathfrak{r}, \mathfrak{f}, \mathfrak{k}\} \nsupseteq \mathfrak{W} = \overline{\mathfrak{d}}^{\mathbb{B}_l}(\mathfrak{U})$.

Proposition 4.6. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_\mathfrak{x})$ *be an* $\mathfrak{x}$*-NS,* $\mathfrak{U} \subseteq \mathfrak{W}$*, and* $\mathfrak{d}$ *be a reflexive relation. Then,*

(1) $\underline{\mathfrak{d}}_\mathfrak{x}^{\mathfrak{y}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}_\mathfrak{x}^{\mathfrak{h}}(\mathfrak{U})$,
(2) $\overline{\mathfrak{d}}_\mathfrak{x}^{\mathfrak{h}}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}_\mathfrak{x}^{\mathfrak{y}}(\mathfrak{U})$,
(3) $\underline{\mathfrak{d}}_\mathfrak{x}^{\mathfrak{y}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathbb{I}_\mathfrak{x}}(\mathfrak{U})$,
(4) $\overline{\mathfrak{d}}^{\mathbb{I}_\mathfrak{x}}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}_\mathfrak{x}^{\mathfrak{y}}(\mathfrak{U})$,
(5) $\underline{\mathfrak{d}}_\mathfrak{x}^{\mathfrak{y}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathbb{B}_\mathfrak{x}}(\mathfrak{U})$,
(6) $\overline{\mathfrak{d}}^{\mathbb{B}_\mathfrak{x}}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}_\mathfrak{x}^{\mathfrak{y}}(\mathfrak{U})$.

Proof. It is directly obtained from Proposition 3.3. □

It is simple to present an example showing that the Proposition's reverse 4.6 is not always true.

Remark 4.4. According to

(i) Example 3.3, and Remark 4.3, in the general case of the relation, the suggested approximations and the prior ones [2, 3, 29, 34, 45] are independent.

(ii) Proposition 4.6, in the instance of a reflexive relation, the previous rough set models [2,3,29,34,45] are more accurate than the proposed methods.

The comparisons between the approximations using several types of maximal neighborhoods directly in Definition 2.4 [7, 12], Definitions 2.5, and 2.6 [10, 12], and those using the topology generated by these neighborhoods, as in the methods in this section, are studied in the following findings.

Proposition 4.7. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_\mathfrak{x})$ *be an* $\mathfrak{x}$*-NS and* $\mathfrak{U} \subseteq \mathfrak{W}$*. Then,*

(1) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}_\mathfrak{x}}_{*}(\mathfrak{U})$,

(2) $\overline{\mathfrak{d}}^{\mathfrak{y}_\mathfrak{x}}_{*}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})$.

Proof. We prove (1) only, and (2) is in the same manner. Let $\mathfrak{r} \in \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U})$. Then, $\exists\ \mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{\mathfrak{x}}$ such that $\mathfrak{r} \in \mathfrak{O} \subseteq \mathfrak{U}$. Thus, $\mathfrak{y}_\mathfrak{x}(\mathfrak{r}) \subseteq \mathfrak{U}$. Hence, $\mathfrak{r} \in \underline{\mathfrak{d}}^{\mathfrak{y}_\mathfrak{x}}_{*}(\mathfrak{U})$. □

Remark 4.5. It should be emphasized that although the prior approximations in Definition 2.4 [7, 12] are better than the current ones, the suggested ones are characterized by fulfilling all of Pawlak's properties, whereas the previous manner lacked this capability and required imposing constraints on the relations to achieve them which hinders applications. Additionally,

(i) Examples 3.2 and 3.3 clarify that Proposition 4.7's converse is not always true.

(ii) Examples 3.1–3.3 confirm that the prior approximations in Definitions 2.5 and 2.6 [10, 12], and the present ones, are incomparable.

5. Relationships among various types of the proposed approximations

The intent of this section is to highlight some distinctive features of $\mathfrak{y}_\mathfrak{x}$-approximations and $\mathfrak{y}_\mathfrak{x}$-accuracy measures. We explain and present the most important features that distinguish this method from its predecessors [2, 3, 29, 34, 45].

Proposition 5.1. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_\mathfrak{x})$ *be an* $\mathfrak{x}$*-NS and* $\mathfrak{U} \subseteq \mathfrak{W}$*. Then*

(i) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$,

(ii) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{l}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$,

(iii) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$,

(iv) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle l\rangle}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$,

(v) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U})$,

(vi) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{l}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U})$,

(vii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$,

(viii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle l\rangle}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$.

Proof. To demonstrate (i) and (ii), let $\mathfrak{r} \in \underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U})$. Then there is $\mathfrak{Q} \in \mathfrak{T}^{\mathfrak{h}}_{u}$ such that $\mathfrak{r} \in \mathfrak{Q} \subseteq \mathfrak{U}$. By Proposition 3.1, we have $\mathfrak{Q} \in \mathfrak{T}^{\mathfrak{h}}_{r}$ and $\mathfrak{Q} \in \mathfrak{T}^{\mathfrak{h}}_{\mathfrak{l}}$. Thus, $\mathfrak{r} \in Int^{\mathfrak{y}}_{r}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U})$ and $\mathfrak{r} \in Int^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U})$. Hence, $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U})$ and $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U})$. Similarly, the relations $\underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$ and $\underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$ are proved. By applying analogous manners, the other cases can be demonstrated. □

Corollary 5.1. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_\mathfrak{x})$ *be an* $\mathfrak{x}$*-NS and* $\mathfrak{U} \subseteq \mathfrak{W}$*. Then*

(i) $\mathfrak{P}^{\mathfrak{y}}_{i}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{r}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{u}(\mathfrak{U})$,

(ii) $\mathfrak{P}^{\mathfrak{y}}_{i}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{u}(\mathfrak{U})$,

(iii) $\mathfrak{P}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$,

(iv) $\mathfrak{P}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$,

(v) $\mathfrak{C}^{\mathfrak{y}}_{u}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{r}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{i}(\mathfrak{U})$,

(vi) $\mathfrak{C}^{\mathfrak{y}}_{u}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{i}(\mathfrak{U})$,

(vii) $\mathfrak{C}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$,

(viii) $\mathfrak{C}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$.

Proof. (v): $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$ and $\overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U})$ (by Proposition 5.1), so we get

$$| \underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) | \leq | \underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) | \leq | \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U}) |, \tag{5.1}$$

and

$$\frac{1}{| \overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) |} \leq \frac{1}{| \overline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) |} \leq \frac{1}{| \overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U}) |}. \tag{5.2}$$

By (5.1) and (5.2), we get

$\frac{|\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U})|}{|\overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U})|} \leq \frac{|\underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U})|}{|\overline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U})|} \leq \frac{|\underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})|}{|\overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})|}$ which is equivalent to $\mathfrak{C}^{\mathfrak{y}}_{u}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{r}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{i}(\mathfrak{U})$.

In a comparable manner, we establish the other cases. □

The current work stands out with its ability to compare $\mathfrak{y}_{\mathfrak{x}}$-approximations and $\mathfrak{y}_{\mathfrak{x}}$-accuracy values with $\mathfrak{y}_{\langle\mathfrak{x}\rangle}$-approximations and $\mathfrak{y}_{\langle\mathfrak{x}\rangle}$-accuracy values, as shown in the following important findings. This comparison is not present in previous methods [2, 3, 29, 34, 45].

Theorem 5.1. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS and* $\mathfrak{U} \subseteq \mathfrak{W}$. *Then*

(i) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U})$,

(ii) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U})$,

(iii) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$,

(iv) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$,

(v) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U})$,

(vi) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U})$,

(vii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$,

(viii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U})$.

Proof. $\underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{r} : \mathfrak{O} \subseteq \mathfrak{U}\} \subseteq \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{\langle r\rangle} : \mathfrak{O} \subseteq \mathfrak{U}\} = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U})$ by Theorem 3.2. Hence, $\underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U})$. In a comparable manner, we establish the other cases. □

Corollary 5.2. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be a* $\mathfrak{x}$*-NS and* $\mathfrak{U} \subseteq \mathfrak{W}$. *Then*

(i) $\mathfrak{P}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{r}(\mathfrak{U})$,

(ii) $\mathfrak{P}^{\mathfrak{y}}_{\langle \mathfrak{l} \rangle}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{l}(\mathfrak{U})$,

(iii) $\mathfrak{P}^{\mathfrak{y}}_{\langle i \rangle}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{i}(\mathfrak{U})$,

(iv) $\mathfrak{P}^{\mathfrak{y}}_{\langle u \rangle}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{u}(\mathfrak{U})$,

(v) $\mathfrak{C}^{\mathfrak{y}}_{r}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\langle r \rangle}(\mathfrak{U})$,

(vi) $\mathfrak{C}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U})$,

(vii) $\mathfrak{C}^{\mathfrak{y}}_{i}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\langle i \rangle}(\mathfrak{U})$,

(viii) $\mathfrak{C}^{\mathfrak{y}}_{u}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\langle u \rangle}(\mathfrak{U})$.

Corollary 5.3. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS. Then,* $\forall \mathfrak{x} \in \{r, \mathfrak{l}, i, \langle r \rangle, \langle \mathfrak{l} \rangle, \langle u \rangle\}$,

(i) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \subseteq \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i \rangle}(\mathfrak{U})$,

(ii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i \rangle}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \subseteq \overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U})$,

(iii) $\mathfrak{P}^{\mathfrak{y}}_{\langle i \rangle}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \subseteq \mathfrak{P}^{\mathfrak{y}}_{u}(\mathfrak{U})$,

(iv) $\mathfrak{C}^{\mathfrak{y}}_{u}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\mathfrak{x}}(\mathfrak{U}) \leq \mathfrak{C}^{\mathfrak{y}}_{\langle i \rangle}(\mathfrak{U})$.

The computations presented in Tables 2 and 3 are computed by Example 3.1. These substantiate the results established in Proposition 5.1, Theorem 5.1, and Corollaries 5.1–5.3.

Table 2. The proposed Definitions 4.1 and 4.2, $\mathfrak{x} \in \{\langle i\rangle, \langle r\rangle, \langle \mathfrak{l}\rangle, \langle u\rangle\}$.

$\mathfrak{U}$	The proposed Definitions 4.1, 4.2 for $\mathfrak{x} = \langle i\rangle$			The proposed Definitions 4.1, 4.2 for $\mathfrak{x} = \langle r\rangle$			The proposed Definitions 4.1, 4.2 for $\mathfrak{x} = \langle \mathfrak{l}\rangle$			The proposed Definitions 4.1, 4.2 for $\mathfrak{x} = \langle u\rangle$		
	$\underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$	$\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$	$\mathfrak{C}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$	$\underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U})$	$\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U})$	$\mathfrak{C}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U})$	$\underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U})$	$\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U})$	$\mathfrak{C}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U})$	$\underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$	$\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$	$\mathfrak{C}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$
$\{\mathfrak{r}\}$	$\{\mathfrak{r}\}$	$\{\mathfrak{r}\}$	1	$\emptyset$	$\{\mathfrak{r}\}$	0	$\emptyset$	$\{\mathfrak{r}\}$	0	$\emptyset$	$\{\mathfrak{r}\}$	0
$\{\mathfrak{s}\}$	$\{\mathfrak{s}\}$	$\{\mathfrak{s}\}$	1	$\emptyset$	$\{\mathfrak{s}\}$	0	$\emptyset$	$\{\mathfrak{s}\}$	0	$\emptyset$	$\{\mathfrak{s}\}$	0
$\{\mathfrak{f}\}$	$\{\mathfrak{f}\}$	$\{\mathfrak{f}\}$	1	$\emptyset$	$\{\mathfrak{f}\}$	0	$\{\mathfrak{f}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{1}{3}$	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	0
$\{\mathfrak{t}\}$	$\{\mathfrak{t}\}$	$\{\mathfrak{t}\}$	1	$\{\mathfrak{t}\}$	$\{\mathfrak{f}, \mathfrak{t}\}$	$\frac{1}{2}$	$\{\mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{3}$	$\{\mathfrak{t}\}$	$\mathfrak{W}$	$\frac{1}{5}$
$\{\mathfrak{v}\}$	$\{\mathfrak{v}\}$	$\{\mathfrak{v}\}$	1	$\{\mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	$\frac{1}{3}$	$\emptyset$	$\{\mathfrak{v}\}$	0	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	0
$\{\mathfrak{r}, \mathfrak{s}\}$	$\{\mathfrak{r}, \mathfrak{s}\}$	$\{\mathfrak{r}, \mathfrak{s}\}$	1	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}\}$	0	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}\}$	0	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}\}$	0
$\{\mathfrak{r}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{f}\}$	1	$\emptyset$	$\{\mathfrak{r}, \mathfrak{f}\}$	0	$\{\mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{1}{4}$	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	0
$\{\mathfrak{r}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{t}\}$	1	$\{\mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}$	$\frac{1}{3}$	$\{\mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{3}$	$\{\mathfrak{t}\}$	$\mathfrak{W}$	$\frac{1}{5}$
$\{\mathfrak{r}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{v}\}$	1	$\{\mathfrak{r}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	$\frac{2}{3}$	$\emptyset$	$\{\mathfrak{r}, \mathfrak{v}\}$	0	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	0
$\{\mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{s}, \mathfrak{f}\}$	1	$\emptyset$	$\{\mathfrak{s}, \mathfrak{f}\}$	0	$\{\mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{2}{3}$	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	0
$\{\mathfrak{s}, \mathfrak{t}\}$	$\{\mathfrak{s}, \mathfrak{t}\}$	$\{\mathfrak{s}, \mathfrak{t}\}$	1	$\{\mathfrak{t}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\frac{1}{3}$	$\{\mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$	$\{\mathfrak{t}\}$	$\mathfrak{W}$	$\frac{1}{5}$
$\{\mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{v}\}$	1	$\{\mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	$\frac{2}{3}$	$\emptyset$	$\{\mathfrak{s}, \mathfrak{v}\}$	0	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	0
$\{\mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{f}, \mathfrak{t}\}$	1	$\{\mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{f}, \mathfrak{t}\}$	1	$\{\mathfrak{f}, \mathfrak{t}\}$	$\mathfrak{W}$	$\frac{2}{5}$	$\{\mathfrak{f}, \mathfrak{t}\}$	$\mathfrak{W}$	$\frac{2}{5}$
$\{\mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{f}, \mathfrak{v}\}$	1	$\{\mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{1}{4}$	$\{\mathfrak{f}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{1}{3}$	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	0
$\{\mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{t}, \mathfrak{v}\}$	1	$\{\mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{2}{5}$	$\{\mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{1}{4}$	$\{\mathfrak{t}\}$	$\mathfrak{W}$	$\frac{1}{5}$
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}\}$	1	$\{\mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}\}$	$\frac{1}{3}$	$\{\mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{2}$	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	0
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}\}$	1	$\{\mathfrak{r}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\frac{1}{2}$	$\{\mathfrak{r}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{2}$	$\{\mathfrak{t}\}$	$\mathfrak{W}$	$\frac{1}{5}$
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	1	$\{\mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	$\frac{2}{3}$	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	0	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	0
$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}$	1	$\{\mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}$	$\frac{2}{3}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}$	$\mathfrak{W}$	$\frac{3}{5}$	$\{\mathfrak{f}, \mathfrak{t}\}$	$\mathfrak{W}$	$\frac{2}{5}$
$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{v}\}$	1	$\{\mathfrak{r}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{1}{2}$	$\{\mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{1}{4}$	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	0
$\{\mathfrak{r}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{t}, \mathfrak{v}\}$	1	$\{\mathfrak{r}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{3}{5}$	$\{\mathfrak{r}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{2}{3}$	$\{\mathfrak{t}\}$	$\mathfrak{W}$	$\frac{1}{5}$
$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	1	$\{\mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\frac{2}{3}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\mathfrak{W}$	$\frac{3}{5}$	$\{\mathfrak{f}, \mathfrak{t}\}$	$\mathfrak{W}$	$\frac{2}{5}$
$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	1	$\{\mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{1}{2}$	$\{\mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{2}{3}$	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	0
$\{\mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	1	$\{\mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{3}{5}$	$\{\mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$	$\{\mathfrak{t}\}$	$\mathfrak{W}$	$\frac{1}{5}$
$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	1	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{3}{5}$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{3}{5}$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{3}{5}$
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	1	$\{\mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\frac{1}{2}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\mathfrak{W}$	$\frac{4}{5}$	$\{\mathfrak{f}, \mathfrak{t}\}$	$\mathfrak{W}$	$\frac{2}{5}$
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	1	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{3}{4}$	$\{\mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\frac{1}{2}$	$\emptyset$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	0
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	1	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{4}{5}$	$\{\mathfrak{r}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{2}$	$\{\mathfrak{t}\}$	$\mathfrak{W}$	$\frac{1}{5}$
$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	1	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{4}{5}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{4}{5}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{4}{5}$
$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	1	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{4}{5}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{4}{5}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\mathfrak{W}$	$\frac{4}{5}$

Table 3. The proposed Definitions 4.1 and 4.2, $\mathfrak{x} \in \{\langle i\rangle, u, i\}$.

$\mathfrak{U}$	The proposed Definitions 4.1, 4.2 for $\mathfrak{x} = \langle i\rangle$			The proposed Definitions 4.1, 4.2 for $\mathfrak{x} = u$			The proposed Definitions 4.1, 4.2 for $\mathfrak{x} = i$		
	$\underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$	$\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$	$\mathfrak{C}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$	$\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U})$	$\overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U})$	$\mathfrak{C}^{\mathfrak{y}}_{u}(\mathfrak{U})$	$\underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$	$\overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$	$\mathfrak{C}^{\mathfrak{y}}_{i}(\mathfrak{U})$
$\{\mathfrak{r}\}$	$\{\mathfrak{r}\}$	$\{\mathfrak{r}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}\}$	$\{\mathfrak{r}\}$	1
$\{\mathfrak{s}\}$	$\{\mathfrak{s}\}$	$\{\mathfrak{s}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{s}\}$	$\{\mathfrak{s}\}$	1
$\{\mathfrak{f}\}$	$\{\mathfrak{f}\}$	$\{\mathfrak{f}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\emptyset$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	0
$\{\mathfrak{t}\}$	$\{\mathfrak{t}\}$	$\{\mathfrak{t}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\emptyset$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	0
$\{\mathfrak{v}\}$	$\{\mathfrak{v}\}$	$\{\mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\emptyset$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	0
$\{\mathfrak{r}, \mathfrak{s}\}$	$\{\mathfrak{r}, \mathfrak{s}\}$	$\{\mathfrak{r}, \mathfrak{s}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}, \mathfrak{s}\}$	$\{\mathfrak{r}, \mathfrak{s}\}$	1
$\{\mathfrak{r}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{f}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{r}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{t}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{o}\}$	$\frac{1}{4}$
$\{\mathfrak{r}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{s}, \mathfrak{f}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{s}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{s}, \mathfrak{t}\}$	$\{\mathfrak{s}, \mathfrak{t}\}$	$\{\mathfrak{s}, \mathfrak{t}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{s}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{s}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{f}, \mathfrak{t}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\emptyset$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	0
$\{\mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{f}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\emptyset$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	0
$\{\mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{t}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\emptyset$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	0
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}, \mathfrak{s}\}$	$\mathfrak{W}$	$\frac{2}{5}$
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}, \mathfrak{s}\}$	$\mathfrak{W}$	$\frac{2}{5}$
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}, \mathfrak{s}\}$	$\mathfrak{W}$	$\frac{2}{5}$
$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{r}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{t}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{s}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{s}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{s}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\frac{1}{4}$
$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	1
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{t}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}, \mathfrak{s}\}$	$\mathfrak{W}$	$\frac{2}{5}$
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{f}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}, \mathfrak{s}\}$	$\mathfrak{W}$	$\frac{2}{5}$
$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}, \mathfrak{s}\}$	$\mathfrak{W}$	$\frac{2}{5}$
$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{r}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	1
$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	1	$\emptyset$	$\mathfrak{W}$	0	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	$\{\mathfrak{s}, \mathfrak{f}, \mathfrak{t}, \mathfrak{v}\}$	1

Proposition 5.2. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS,* $\mathfrak{U} \subseteq \mathfrak{W}$*, and* $\mathfrak{d}$ *be symmetric. Then,*

(i) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$ *and* $\overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$,

(ii) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$ *and* $\overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$.

Proof. $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{u} : \mathfrak{O} \subseteq \mathfrak{U}\} = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{r} : \mathfrak{O} \subseteq \mathfrak{U}\} = \underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U})$, $\underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{r} : \mathfrak{O} \subseteq \mathfrak{U}\} = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{\mathfrak{l}} : \mathfrak{O} \subseteq \mathfrak{U}\} = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U})$, and $\underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{\mathfrak{l}} : \mathfrak{O} \subseteq \mathfrak{U}\} = \cup\{\mathfrak{O} \in \mathfrak{T}^{\mathfrak{y}}_{i} : \mathfrak{O} \subseteq \mathfrak{U}\} = \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$ (by Theorem 3.3). Hence, $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U})$. Similarly, the remaining cases are derived. □

Corollary 5.4. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS,* $\mathfrak{U} \subseteq \mathfrak{W}$*, and* $\mathfrak{d}$ *be symmetric. Then,*

(i) $\mathfrak{P}^{\mathfrak{y}}_{i}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{l}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{u}(\mathfrak{U})$,

(ii) $\mathfrak{P}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle l\rangle}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$,

(iii) $\mathfrak{C}^{\mathfrak{y}}_{i}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{l}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{u}(\mathfrak{U})$,

(iv) $\mathfrak{C}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle l\rangle}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$.

Proposition 5.3. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS,* $\mathfrak{U} \subseteq \mathfrak{W}$*, and* $\mathfrak{d}$ *be symmetric and transitive. Then,*

(i) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$,

(ii) $\overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$.

Proof. This is analogous to the proof of Proposition 5.2. □

Corollary 5.5. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS,* $\mathfrak{U} \subseteq \mathfrak{W}$*, and* $\mathfrak{d}$ *be symmetric and transitive. Then,*

(i) $\mathfrak{P}^{\mathfrak{y}}_{i}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{l}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle l\rangle}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$,

(ii) $\mathfrak{C}^{\mathfrak{y}}_{i}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{l}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle l\rangle}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$.

Proposition 5.4. *Let* $(\mathfrak{W}, \mathfrak{d}, \mathfrak{H}_{\mathfrak{x}})$ *be an* $\mathfrak{x}$*-NS,* $\mathfrak{U} \subseteq \mathfrak{W}$*, and* $\mathfrak{d}$ *be equivalence. Then,*

(i) $\underline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{i}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) = \underline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) = \overline{\mathfrak{d}}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U})$,

(ii) $\mathfrak{P}^{\mathfrak{y}}_{i}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) = \mathfrak{P}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$,

(iii) $\mathfrak{C}^{\mathfrak{y}}_{i}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{r}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\mathfrak{l}}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{u}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle i\rangle}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle r\rangle}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle \mathfrak{l}\rangle}(\mathfrak{U}) = \mathfrak{C}^{\mathfrak{y}}_{\langle u\rangle}(\mathfrak{U})$.

Proof. This is comparable to the proof of Proposition 5.2. □

6. Numerical example

This section presents the data derived from six people $\mathfrak{W} = \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\}$ who go to the gym four days a week with the aim of taking care of their health and enjoying physical fitness and strength. Each of them performs differently from the other. Individuals' performance is evaluated through four levels: excellent, very good, good, and bad, as shown in Table 4. The levels are arranged in the following order: excellent $\gneqq$ very good $\gneqq$ good $\gneqq$ bad, where $\gneqq$ signifies "greater than and not equal".

Table 4. Information system of people's level for four days.

$\mathfrak{W}$	Day 1	Day 2	Day 3	Day 4
$\mathfrak{r}_1$	good	bad	excellent	excellent
$\mathfrak{r}_2$	very good	good	excellent	bad
$\mathfrak{r}_3$	very good	good	failed	good
$\mathfrak{r}_4$	excellent	bad	very good	excellent
$\mathfrak{r}_5$	very good	bad	very good	excellent
$\mathfrak{r}_6$	good	good	excellent	bad

Two people are related by $\mathfrak{d}$, where: $x\mathfrak{d}y$ iff person x has two days or more with levels exceeding those of the corresponding days for person y. For example, $\mathfrak{r}_4\mathfrak{d}\mathfrak{r}_3$ because the person achieves a level $\mathfrak{r}_4$ in Day 1, Day 2, and Day 3, which surpasses the person's level of $\mathfrak{r}_3$ in these days. But, $(\mathfrak{r}_3, \mathfrak{r}_3) \notin \mathfrak{d}$ because the person $\mathfrak{r}_3$ has only one day's level greater than person $\mathfrak{r}_3$. Hence, $\mathfrak{d} = \{(\mathfrak{r}_1, \mathfrak{r}_3), (\mathfrak{r}_2, \mathfrak{r}_1), (\mathfrak{r}_2, \mathfrak{r}_4), (\mathfrak{r}_2, \mathfrak{r}_5), (\mathfrak{r}_4, \mathfrak{r}_2), (\mathfrak{r}_4, \mathfrak{r}_3), (\mathfrak{r}_4, \mathfrak{r}_6), (\mathfrak{r}_5, \mathfrak{r}_3), (\mathfrak{r}_5, \mathfrak{r}_6), (\mathfrak{r}_6, \mathfrak{r}_4)\}$.

So, $\mathfrak{h}_\mathfrak{l}$-neighborhoods and $\mathfrak{y}_\mathfrak{l}$-neighborhoods are obtained $\forall \mathfrak{r} \in \mathfrak{W}$:

$$\mathfrak{h}_\mathfrak{l}(\mathfrak{r}_1) = \mathfrak{h}_\mathfrak{l}(\mathfrak{r}_5) = \{\mathfrak{r}_2\}, \qquad \mathfrak{h}_\mathfrak{l}(\mathfrak{r}_4) = \{\mathfrak{r}_2, \mathfrak{r}_6\},$$
$$\mathfrak{h}_\mathfrak{l}(\mathfrak{r}_2) = \{\mathfrak{r}_4\}, \qquad \mathfrak{h}_\mathfrak{l}(\mathfrak{r}_6) = \{\mathfrak{r}_4, \mathfrak{r}_5\},$$
$$\mathfrak{h}_\mathfrak{l}(\mathfrak{r}_3) = \{\mathfrak{r}_1, \mathfrak{r}_4, \mathfrak{r}_5\},$$

$$\mathfrak{y}_\mathfrak{l}(\mathfrak{r}_1) = \mathfrak{y}_\mathfrak{l}(\mathfrak{r}_4) = \mathfrak{y}_l(\mathfrak{r}_5) = \{\mathfrak{r}_1, \mathfrak{r}_4, \mathfrak{r}_5\},$$
$$\mathfrak{y}_\mathfrak{l}(\mathfrak{r}_2) = \mathfrak{y}_\mathfrak{l}(\mathfrak{r}_6) = \{\mathfrak{r}_2, \mathfrak{r}_6\},$$
$$\mathfrak{y}_\mathfrak{l}(\mathfrak{r}_3) = \emptyset.$$

Additionally, the $\mathfrak{h}_r$-neighborhood and $\mathfrak{y}_r$-neighborhood are given $\forall \mathfrak{r} \in \mathfrak{W}$:

$$\mathfrak{h}_r(\mathfrak{r}_1) = \{\mathfrak{r}_3\}, \qquad \mathfrak{h}_r(\mathfrak{r}_4) = \{\mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_6\},$$
$$\mathfrak{h}_r(\mathfrak{r}_2) = \{\mathfrak{r}_1, \mathfrak{r}_4, \mathfrak{r}_5\}, \qquad \mathfrak{h}_r(\mathfrak{r}_5) = \{\mathfrak{r}_3, \mathfrak{r}_6\},$$
$$\mathfrak{h}_r(\mathfrak{r}_3) = \emptyset, \qquad \mathfrak{h}_r(\mathfrak{r}_6) = \{\mathfrak{r}_4\},$$

$$\mathfrak{y}_r(\mathfrak{r}_1) = \mathfrak{y}_r(\mathfrak{r}_4) = \mathfrak{y}_r(\mathfrak{r}_5) = \{\mathfrak{r}_1, \mathfrak{r}_4, \mathfrak{r}_5\},$$
$$\mathfrak{y}_r(\mathfrak{r}_2) = \mathfrak{y}_r(\mathfrak{r}_3) = \mathfrak{y}_r(\mathfrak{r}_6) = \{\mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_6\}.$$

Therefore,

(i) $\mathfrak{T}_\mathfrak{l}^\mathfrak{h} = \{\emptyset, \mathfrak{W}, \{\mathfrak{r}_2, \mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\}, \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\}\}$,

(ii) $\mathfrak{T}_r^\mathfrak{h} = \{\emptyset, \mathfrak{W}, \{\mathfrak{r}_3\}, \{\mathfrak{r}_1, \mathfrak{r}_3\}\}$,

(iii) $\mathfrak{T}_{\langle r\rangle}^\mathfrak{h} = \{\emptyset, \mathfrak{W}, \{\mathfrak{r}_3\}, \{\mathfrak{r}_4\}, \{\mathfrak{r}_3, \mathfrak{r}_4\}, \{\mathfrak{r}_3, \mathfrak{r}_6\}, \{\mathfrak{r}_1, \mathfrak{r}_4, \mathfrak{r}_5\}, \{\mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_6\}, \{\mathfrak{r}_3, \mathfrak{r}_4, \mathfrak{r}_6\}, \{\mathfrak{r}_1, \mathfrak{r}_3, \mathfrak{r}_4, \mathfrak{r}_5\}, \{\mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_4, \mathfrak{r}_6\}, \{\mathfrak{r}_1, \mathfrak{r}_3, \mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\}\}$,

(iv) $\mathfrak{T}^{\mathbb{I}_r} = \{\emptyset, \mathfrak{W}, \{\mathfrak{r}_2\}, \{\mathfrak{r}_4\}, \{\mathfrak{r}_2, \mathfrak{r}_4\}, \{\mathfrak{r}_4, \mathfrak{r}_5\}, \{\mathfrak{r}_1, \mathfrak{r}_4, \mathfrak{r}_5\}, \{\mathfrak{r}_2, \mathfrak{r}_4, \mathfrak{r}_5\}, \{\mathfrak{r}_2, \mathfrak{r}_4, \mathfrak{r}_6\}, \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_4, \mathfrak{r}_5\}, \{\mathfrak{r}_2, \mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\}, \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\}\}$,

(v) $\mathfrak{T}^{\mathbb{B}_r} = \{\emptyset, \mathfrak{W}, \{\mathfrak{r}_3\}, \{\mathfrak{r}_1, \mathfrak{r}_3\}, \{\mathfrak{r}_1, \mathfrak{r}_3, \mathfrak{r}_6\}, \{\mathfrak{r}_1, \mathfrak{r}_3, \mathfrak{r}_4, \mathfrak{r}_6\}, \{\mathfrak{r}_1, \mathfrak{r}_3, \mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\}\}$,

(vi) $\mathfrak{T}^{\mathfrak{y}}_{\mathfrak{l}} = \mathfrak{T}^{\mathfrak{y}}_{\langle \mathfrak{l} \rangle} = \mathfrak{T}^{\mathfrak{y}}_{i} = \mathfrak{T}^{\mathfrak{y}}_{\langle i \rangle} = \{\emptyset, \mathfrak{W}, \{\mathfrak{r}_3\}, \{\mathfrak{r}_2, \mathfrak{r}_6\}, \{\mathfrak{r}_1, \mathfrak{r}_4, \mathfrak{r}_5\}, \{\mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_6\}, \{\mathfrak{r}_1, \mathfrak{r}_3, \mathfrak{r}_4, \mathfrak{r}_5\}, \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_4, \mathfrak{r}_5, \mathfrak{r}_6\}\}$,

(vii) $\mathfrak{T}^{\mathfrak{y}}_{r} = \mathfrak{T}^{\mathfrak{y}}_{\langle r \rangle} = \mathfrak{T}^{\mathfrak{y}}_{u} = \mathfrak{T}^{\mathfrak{y}}_{\langle u \rangle} = \{\emptyset, \mathfrak{W}, \{\mathfrak{r}_1, \mathfrak{r}_4, \mathfrak{r}_5\}, \{\mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_6\}\}$.

Consequently,

(i) clearly, $\mathfrak{T}^{\mathfrak{y}}_{r} \subsetneq \mathfrak{T}^{\mathfrak{y}}_{i}, \mathfrak{T}^{\mathfrak{y}}_{\langle i \rangle}$, so the suggested topologies are superior than the prior topology in Theorem 2.3 [41],

(ii) evidently, $\mathfrak{T}^{\mathfrak{y}}_{\mathfrak{x}} \subseteq \mathfrak{T}^{\mathfrak{y}}_{\langle \mathfrak{x} \rangle}$, $\mathfrak{x} \in \{r, \mathfrak{l}, i, u\}$, and $\mathfrak{T}^{\mathfrak{y}}_{u} \subseteq \mathfrak{T}^{\mathfrak{y}}_{\mathfrak{x}} \subseteq \mathfrak{T}^{\mathfrak{y}}_{\langle i \rangle}$, $\mathfrak{x} \in \{r, \mathfrak{l}, i, \langle r \rangle, \langle \mathfrak{l} \rangle, \langle u \rangle\}$. These results confirm the merit of the proposed topologies formed by $\mathfrak{y}_{\mathfrak{x}}$-neighborhoods and this leads us to compare $\mathfrak{y}_{\mathfrak{x}}$-approximations, $\mathfrak{y}_{\mathfrak{x}}$-accuracy values, $\mathfrak{y}_{\langle \mathfrak{x} \rangle}$-approximations, and $\mathfrak{y}_{\langle \mathfrak{x} \rangle}$-accuracy values (see Theorem 5.1 and Corollary 5.3). These two types have not been compared in prior manners [2, 3, 29, 34, 45], for instance, $\mathfrak{T}^{\mathfrak{h}}_{r} \neq \mathfrak{T}^{\mathfrak{h}}_{\langle r \rangle}$,

(iii) the foregoing topologies created by $\mathfrak{x}$-neighborhoods [2, 3, 29], $\mathbb{B}_{\mathfrak{x}}$-neighborhoods [14], and $\mathbb{I}_{\mathfrak{x}}$-neighborhoods [45] differ from the proposed ones, as $\mathfrak{T}^{\mathfrak{h}}_{r}$, $\mathfrak{T}^{\mathbb{I}_r}$, $\mathfrak{T}^{\mathbb{B}_r}$, and $\mathfrak{T}^{\mathfrak{y}}_{r}$ are not comparable,

(iv) $\mathfrak{T}^{\mathfrak{h}}_{\mathfrak{l}}$ is the dual of $\mathfrak{T}^{\mathfrak{h}}_{r}$, in spite of the fact that $\mathfrak{T}^{\mathfrak{y}}_{\mathfrak{l}}$ is not the dual of $\mathfrak{T}^{\mathfrak{y}}_{r}$,

(v) the relation is irreflexive as $(\mathfrak{r}, \mathfrak{r}) \notin \mathfrak{d}, \mathfrak{r} \in \mathfrak{W}$, not symmetry as $(\mathfrak{r}_3, \mathfrak{r}_1) \notin \mathfrak{d}$, but $(\mathfrak{r}_1, \mathfrak{r}_3) \in \mathfrak{d}$, and also, not transitive as $(\mathfrak{r}_5, \mathfrak{r}_4) \notin \mathfrak{d}$. However, $(\mathfrak{r}_5, \mathfrak{r}_6) \in \mathfrak{d}$ and $(\mathfrak{r}_6, \mathfrak{r}_4) \in \mathfrak{d}$. So, some of the prior methods [1, 12, 32] cannot deal with these exampleS as they put restrictions on relations,

(vi) let $\mathfrak{d} = \mathfrak{d}_2$ and $\mathfrak{d}_1$ be another relation. $x\mathfrak{d}_1 y$ iff person x has three days or more with levels exceeding those of the corresponding days for person y. Thus, $\mathfrak{d}_1 = \{(\mathfrak{r}_2, \mathfrak{r}_1)\} \subseteq \mathfrak{d}_2$. Hence, $\mathfrak{T}^{\mathfrak{y}}_{1r} = P(\mathfrak{W})$, if $\mathfrak{U} = \{\mathfrak{r}_y\}$, then

(1) in the prior manner in [45],

- $\overline{\mathfrak{d}_1}^{\mathbb{I}_r}(\mathfrak{U}) = \mathfrak{W}$, $\overline{\mathfrak{d}_2}^{\mathbb{I}_r}(\mathfrak{U}) = \{\mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_6\}$,
- $\underline{\mathfrak{d}_1}^{\mathbb{I}_r}(\mathfrak{U}) = \underline{\mathfrak{d}_2}^{\mathbb{I}_r}(\mathfrak{U}) = \{\mathfrak{r}_2\}$,
- $\mathfrak{P}^{\mathbb{I}_r}_1(\mathfrak{U}) = \mathfrak{W} \setminus \{\mathfrak{r}_2\} \nsubseteq \{\mathfrak{r}_3, \mathfrak{r}_6\} = \mathfrak{P}^{\mathbb{I}_r}_2(\mathfrak{U})$,
- $\mathfrak{C}^{\mathbb{I}_r}_2(\mathfrak{U}) = \frac{1}{3} > \frac{1}{6} = \mathfrak{C}^{\mathbb{I}_r}_1(\mathfrak{U})$,
- $\mathfrak{K}^{\mathbb{I}_r}_1(\mathfrak{U}) = \frac{5}{6} > \frac{2}{3} = \mathfrak{K}^{\mathbb{I}_r}_2(\mathfrak{U})$.

(2)in the present manner,

- $\overline{\mathfrak{d}_{1_r}}^{\mathfrak{y}}(\mathfrak{U}) = \{\mathfrak{r}_2\} \subseteq \{\mathfrak{r}_2, \mathfrak{r}_3, \mathfrak{r}_6\} = \overline{\mathfrak{d}_{2_r}}^{\mathfrak{y}}(\mathfrak{U})$,
- $\underline{\mathfrak{d}_2}_r^{\mathfrak{y}}(\mathfrak{U}) = \emptyset \subseteq \{\mathfrak{r}_2\} = \underline{\mathfrak{d}_1}_r^{\mathfrak{y}}(\mathfrak{U})$,
- $\mathfrak{P}^{\mathfrak{y}}_{1_r}(\mathfrak{U}) = \emptyset \subseteq \{\mathfrak{r}_1, \mathfrak{r}_4, \mathfrak{r}_5\} = \mathfrak{P}^{\mathfrak{y}}_{2_r}(\mathfrak{U})$,
- $\mathfrak{C}^{\mathfrak{y}}_{2_r}(\mathfrak{U}) = 0 < 1 = \mathfrak{C}^{\mathfrak{y}}_{1_r}(\mathfrak{U})$,
- $\mathfrak{K}^{\mathfrak{y}}_{1_r}(\mathfrak{U}) = 0 < 1 = \mathfrak{K}^{\mathfrak{y}}_{2_r}(\mathfrak{U})$.

Accordingly, the computations show that the present technique is monotonic as it relies on $\mathfrak{y}_{\mathfrak{x}}$-neighborhoods. Whereas, the prior manner in [45] is not monotonic as it depended on $\mathbb{I}_{\mathfrak{x}}$-neighborhoods. Additionally, it is easy to show that Abo Tabl's technique [1], Allam et al.'s method [4], Al-shami's approach [6], Dai et al.'s manner [12] (the second type), Hosny's proposal [19] (third type), and Kandil et al.'s methodology [27] are also not monotonic. The monotonic property is a key concept that ensures the consistency and predictability of the approximation process as more information about a system becomes available. It dictates that the operators of approximations should not decrease when additional information is added. Specifically, the lower approximation should remain the same or become more accurate, while the upper approximation should stay the same or become more restrictive. This property guarantees that rough set approximations are stable and reliable, enhancing the analysis and interpretation of data. For example, introducing more attributes or data into a rough set model should improve or at least maintain the accuracy of set approximations, never diminish it.

7. Discussions: Benefits and limitations

The strengths and weaknesses of the current models relative to the last ones are assessed in this section.

- Benefits
  - **(i)** The topology discussed in Theorem 2.3 [41] is coarser than the suggested topologies, as evidenced by Proposition 3.1, Corollary 3.1, and Theorem 3.2, which indicate that $\mathfrak{T}^{\mathfrak{y}}_r \subseteq \mathfrak{T}^{\mathfrak{y}}_i$, $\mathfrak{T}^{\mathfrak{y}}_r \subseteq \mathfrak{T}^{\mathfrak{y}}_{\langle r\rangle}$, and $\mathfrak{T}^{\mathfrak{y}}_r \subseteq \mathfrak{T}^{\mathfrak{y}}_{\langle i\rangle}$. Furthermore, Example 3.1 demonstrates that $\mathfrak{T}^{\mathfrak{y}}_r \subsetneq \mathfrak{T}^{\mathfrak{y}}_i$, $\mathfrak{T}^{\mathfrak{y}}_r \subsetneq \mathfrak{T}^{\mathfrak{y}}_{\langle r\rangle}$, and $\mathfrak{T}^{\mathfrak{y}}_r \subsetneq \mathfrak{T}^{\mathfrak{y}}_{\langle i\rangle}$. Therefore, the proposed topologies represent a more comprehensive extension of the previous work [41].
  - **(ii)** The proposed topologies are compared, leading to the identification of the largest one $\mathfrak{T}^{\mathfrak{y}}_u$ and the weakest one $\mathfrak{T}^{\mathfrak{y}}_{\langle i\rangle}$ among all of them under any relation. This provides a distinctive depiction of these topologies, even though $\mathfrak{T}^{\mathfrak{h}}_{\mathfrak{x}}$, $\mathfrak{T}^{\mathfrak{h}}_{\langle\mathfrak{x}\rangle}$ in [2, 3, 29], $\mathfrak{T}^{\mathbb{B}_{\mathfrak{x}}}$, $\mathfrak{T}^{\mathbb{B}}_{\langle\mathfrak{x}\rangle}$ in [14], and $\mathfrak{T}^{\mathbb{I}_{\mathfrak{x}}}$, $\mathfrak{T}^{\mathbb{I}}_{\langle\mathfrak{x}\rangle}$, $\mathfrak{x} \in \{r, \mathfrak{l}, i, u\}$, in [14] are incomparable under an arbitrary relation. This distinction leads us to compare $\mathfrak{y}_{\mathfrak{x}}$-approximations, $\mathfrak{y}_{\mathfrak{x}}$-accuracy values, $\mathfrak{y}_{\langle\mathfrak{x}\rangle}$-approximations, and $\mathfrak{y}_{\langle\mathfrak{x}\rangle}$-accuracy values. Meanwhile, this capability is lacking in earlier methods [19, 27]. For instance, two kinds of neighborhoods (approximations and accuracy) created from $\mathfrak{x}$-NS $(\mathfrak{h}_{\mathfrak{x}}, \mathfrak{h}_{\langle\mathfrak{x}\rangle})$ and basic-neighborhoods $(\mathbb{B}_{\mathfrak{x}}, \mathbb{B}_{\langle\mathfrak{x}\rangle})$ are not comparable under an arbitrary relation. Example 3.3 confirms this matter, $\mathfrak{h}_{\mathfrak{l}}(\mathfrak{k}) = \{\mathfrak{r}\} \nsubseteq, \nsupseteq \{\mathfrak{k}\} = \mathfrak{h}_{\langle\mathfrak{l}\rangle}(\mathfrak{k})$ and $\mathbb{B}_{\mathfrak{l}}(\mathfrak{r}) = \{\mathfrak{r}, \mathfrak{k}\} \nsubseteq, \nsupseteq \{\mathfrak{r}, \mathfrak{s}\} = \mathbb{B}_{\langle\mathfrak{l}\rangle}(\mathfrak{r})$.
  - **(iii)** The current results do not necessitate that $\mathfrak{T}^{\mathfrak{y}}_r$ represent the dual of $\mathfrak{T}^{\mathfrak{y}}_l$ (see Examples 3.2 and 3.3), while it was proven that $\mathfrak{T}^{\mathfrak{h}}_r$ is the dual of $\mathfrak{T}^{\mathfrak{h}}_l$ as in [2, 3, 29].
  - **(iv)** The current technique achieves all of Pawlak's properties without any restrictions. It permits us to tackle a range of practical problems using any relation, while Pawlak's manner [35, 36] necessitates an equivalence relation, Abo Tabl's technique [1], Allam et al.'s manner [4], Al-shami's approach [7], and Marei's method [32] need a reflexive relation, and Dai et al.'s methodology [12] requires a similarity relation which hinders applications.
  - **(v)** The proposed technique has the monotonic property. This property specifically examines how these approximations respond to changes in the information available about the system. In

other words, as additional data or knowledge is incorporated, the lower approximation should either remain unchanged or become more precise, while the upper approximation should either stay the same or become more restrictive. This ensures that the approximations remain stable. Meanwhile, this property may be either lost or retained under stringent conditions in some earlier methods [1, 4, 6, 8, 12].

(vi) The current approach is more effective for handling large samples because it relies exclusively on the union of $\mathfrak{x}$-neighborhoods. This is particularly important as it allows us to make more accurate decisions for problems where these cases provide the appropriate framework, such as infectious diseases such as COVID-19, where the spread of infection is relative to the size of the sample.

- Limitations

(i) The present proposal is generally not comparable with the prior approaches [2, 3, 29, 34, 45] if the relation is not reflexive. Additionally, the foregoing approximations [2, 3, 29, 34, 45] are better than the present technique under a reflexive relation (refer to Propositions 3.3 and 4.6).

(ii) Taher et al.'s topology [41] ($\mathfrak{T}_r^{\mathfrak{y}}$) in Theorem 2.3 is better than the proposed topologies $\forall \mathfrak{x} = u$.

(iii) Some of the previous methods did not require as many calculations as the current one, as our method relies solely on the union of $\mathfrak{x}$-neighborhoods.

8. Conclusions

A rough set is a significant approach for tackling problems related to vagueness and uncertainty in knowledge. The recent advancement of rough sets has given rise to topological rough set approaches. Topology has numerous real-life applications as it is vital for knowledge extraction. One of the significant contributions of this study was its effort to connect rough sets with topology, revealing the topological structures embedded within these approximations. This interdisciplinary approach paves the way for more in-depth investigations into topology within the context of rough sets, highlighting the crucial role these approximations play in defining topological structures. Since neighborhoods are fundamental to topological spaces and crucial for solving topological problems, we were motivated to incorporate them into rough sets. Therefore, this paper presented new methods for articulating the fundamental concepts of rough sets in terms of topologies inspired by the union of $\mathfrak{x}$-neighborhoods. In light of utilizing the neighborhoods in rough sets, we had extended rough set concepts by incorporating neighborhoods, as demonstrated in this study. This paper focused on creating various topologies using different types of maximal neighborhoods. Their properties were scrutinized and we used illustrative counterexamples to clarify the results. One of the most important features of this article was that it enabled us to know the smallest and largest topologies among all types, which was not possible in the prior manners [2, 3, 29, 34, 45]. Comparing these topologies with the previous one in Theorem 2.3 [41] demonstrated that the earlier topology was not as robust as the current ones when $\mathfrak{x} \in \{\langle r\rangle, i, \langle i\rangle\}$. These topologies were used to propose new approximations. Comparisons among them were studied and in this context, we confirmed that the optimal results were achieved with $\langle i\rangle$. Additionally, all types of the proposed approximations were compared. Meanwhile, this comparison was absent in earlier methods [2, 3, 29, 34, 45]. These approximations preserved the main characteristics of Pawlak without restriction as in [1, 4, 12, 32]. Moreover, they had the monotonic property, where in contrast, this merit may either be missed or kept with limitation as in [19, 27]. Furthermore, to enhance the robustness and clarity of the current work, a numerical example had been proposed to elucidate the core concepts of the results. This study wrapped up with an overview of the main strengths and weaknesses of the current method.

An exciting avenue for future research will involve

- Using ideals to develop the present manner.
- Introducing near open sets based on the current results.
- Broadening the current study to rough multisets.

Conflict of interest

The author declares no conflict of interest.

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