AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 29633–29644.
DOI:10.3934/math.20241435
Received: 19 August 2024
Revised: 19 September 2024
Accepted: 09 October 2024
Published: 18 October 2024

Research article

A topological approach for rough semigroups

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Abstract: This study presents a novel approach to defining topological rough semigroups on an approximation space. The concepts of topological space and rough semigroup are naturally combined to achieve this goal. Also, some basic results and examples are presented. Furthermore, some compactness properties are also studied. In addition, their rough subsemigroups and rough ideals are analysed.

Keywords: rough sets; rough semigroup; rough ideal; topological semigroup; topological rough semigroup
Mathematics Subject Classification: 03E99, 22A99

1. Introduction

The theory of rough sets, presented by Z. Pawlak in 1982, is a modern mathematical approach to modelling uncertainty and imprecision in data mining [1]. Since its introduction, this theory has attracted a great deal of attention from researchers and has developed into a broad field with a variety of applications in a wide range of scientific fields. Its applications extend to economics, finance, machine learning, data mining, and medicine [2–5].

Rough set theory precisely defines lower and upper approximations of a subset within a universal set, based on an equivalence relation defined on the universal set. The lower approximation set contains elements definitely belonging to the target set, while the upper approximation set contains elements probably belonging to the target set. The difference between these two approximation sets forms the boundary region, which represents the regions of uncertainty within the target set. In this context, some researchers have taken some algebraic structures (semigroup, group, ring, module, etc.) instead of universal sets and congruence relations (normal groups, ideals, etc.) instead of equivalence relations and studied rough approximations within these algebraic structures [8–12, 14–17]. On the other hand, some researchers have redefined and analysed rough algebraic structures based on the rough upper approximation, similar to the definitions of classical algebraic structures. In this direction, the concept of rough semigroup on approximation spaces was defined by Bagirmaz and Özcan [6]. Rough groups and rough subgroups were defined, and their basic properties were analysed by Biswas and Nanda [7]. In the work of Miao et al. [13], the definitions of rough groups and rough subgroups have been improved and new properties have been proved. Zhang et al. proposed the idea of rough modules and studied its characteristics [18]. In [19], Agusfrianto et al. researched the definitions of the rough ring and the rough subring. Almohammadi and Özel [20] introduced rough vector spaces and proved several algebraic results.

The intersection of topology and rough sets has important implications in both applied and theoretical work. In this context, Skowron [21] and Wiweger [22] studied rough set theory in the framework of topological concepts. Lashin et al. focused on generalising the basic concepts of rough sets with topology induced by binary relations [23]. In this direction, many researchers have analysed the relationship between rough sets and topologies by considering reflexive, pre-order, or tolerance relations instead of the equivalence relation of rough set theory [24–27]. Later, Kandi et al. [28] presented an innovative method for constructing approximation spaces based on the structure of ideals, with the aim of improving accuracy measures. Hosny [29, 30] has worked on generating different topologies based on ideals. Mustafa et al. [31] have developed new types of approximation spaces, directly inspired by containment neighbourhoods and ideals. Al-Shami [32] developed a new type of rough set model by applying a topological approach called "slightly open and slightly closed sets". There is a significant amount of research in this context; see [33–37]. On the other hand, there are a lot of studies on the theoretical side of the matter. On this part, Bagirmaz et al. [38] defined the concept of a topological rough group and examined its basic topological properties. Altassan et al. [39] introduced rough action and topological rough group homeomorphisms. Li et al. studied the separation axioms of topological rough groups in [40]. Also, see [41–45].

The principal objective of this paper is to present the first study of the notion of a rough semigroup in conjunction with the notion of topology. It also aims to lay the groundwork for potential areas of theoretical and practical applications by examining the relationship between these two mathematical topics. In this framework, the text of the paper is organised as follows: In Section 2, the reader will find an introduction to the concepts of rough set, topological semigroup, and rough semigroup and some basic properties of these concepts related to each other. We also define the notion of compactness and give some elementary notations. In Section 3, the concept of a topological rough semigroup is introduced and its significant properties are delineated, with the concept itself being demonstrated through an exemplification. In Section 4, the definitions of a topological rough subsemigroup and a rough ideal are presented, and some properties of these structures are established. Finally, the study results were summarised in the conclusion.

2. Preliminaries

Here, we present the definitions and results used in the paper. We also present some propositions concerning the topological properties of the upper approximation of a set on an approximation space and the notion of a rough semigroup.

Let us consider a non-empty set $\boldsymbol{U}$ and an equivalence relation $\boldsymbol{\sigma}$ on $\boldsymbol{U}$. The pair $(\boldsymbol{U}, \boldsymbol{\sigma})$ is referred to as the approximation space. The equivalence class of an element $s \in \boldsymbol{U}$ is denoted by $\boldsymbol{\sigma}(s)$. For a subset $S \subseteq \boldsymbol{U}$, the lower and upper approximations of S with respect to $(\boldsymbol{U}, \boldsymbol{\sigma})$ are denoted by

$\underline{S} = \underset{s\in U}{\cup} \{\sigma(s) : \sigma(s) \subseteq S\}$, and $\overline{S} = \underset{s\in U}{\cup} \{\sigma(s) : \sigma(s) \cap S \neq \emptyset\}$, respectively [1].

To illustrate, consider an approximation space (U, σ) , where $U = \{r, s, t, m, n, k, l\}$ and σ is an equivalence relation with the following classes:

$$\sigma_1 = \{r, s, k\}, \quad \sigma_2 = \{m, n\}, \quad \sigma_3 = \{t\}.$$

Let $S = \{r, m, n\}$. Then, $\underline{S} = \{m, n\}$ and $\overline{S} = (r, s, m, n, k)$.

From this point forward, we will take the triple $(U, \sigma, *)$ as an approximation space, where " $*$ " is a binary operation defined on U. For simplicity, we will use mn instead of $m * n$, where m and n are elements of U. If M and N are subsets of U, then MN denotes the subset of all elements of the form mn, where $m \in M$, $n \in N$. If (U, τ) is a topological space and $m \in U$, $\mathcal{N}(m)$ denotes the set of neighbourhoods of m. Additionally, we will denote the closure of M by clM.

Definition 1. *[46] A topological semigroup is a semigroup S together with the product topology on $S \times S$ such that the maping $f : S \times S \to S$ defined by $f(m, n) \to mn$ is continuous.*

This statement implies that whenever $W \subseteq S$ is open and $W \in \mathcal{N}(mn)$, then there exist open sets M and N such that $M \in \mathcal{N}(m)$, $N \in \mathcal{N}(n)$, and $MN = \{mn : m \in M; n \in N\} \subseteq W$.

Definition 2. *[6] Let $(U, \sigma, *)$ be an approximation space. A subset S of U is called a rough semigroup on approximation space, provided the following properties are satisfied:*

(i) For all $m, n \in S$, $mn \in \overline{S}$,

(ii) For all $m, n, r \in \overline{S}$, the $(mn)r = m(nr)$ property holds in $\overline{S}$.

Let $(U, \sigma, *)$ be an approximation space and $S \subseteq U$ be a rough semigroup. A non-void subset K of S is said to be a rough subsemigroup of S, if $KK \subseteq \overline{K}$. A non-void subset I of S is said to be a rough left (resp. right) ideal of S if $SI \subseteq \overline{I}$ (resp. $IS \subseteq \overline{I}$). If I is both the rough left and rough right ideal of S, then I is called the rough ideal of S.

Example 3. *Let $U = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ be a set of surplus classes with respect to module 4 and $(*)$ be the plus of surplus class. Consider that σ is an equivalence relation forming the classes $C_1 = \{\overline{0}, \overline{1}\}$, $C_2 = \{\overline{2}\}$, $C_3 = \{\overline{3}\}$.*

Let $S_1 = \{\overline{0}, \overline{2}, \overline{3}\}$, then $\overline{S_1} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. From Definition 2, $S_1 \subseteq U$ is a rough semigroup. Besides, S_1 is also not a semigroup.

Let $S_2 = \{\overline{0}, \overline{2}\}$, then $\overline{S_2} = \{\overline{0}, \overline{1}, \overline{2}\}$. Because $\overline{1} * \overline{2} = \overline{3} \notin \overline{S_2}$, we have S_2 is not a rough semigroup. On the other hand, S_2 is a semigroup.

In order to clarify the situations in the above example, let us give the following proposition.

Proposition 4. *Let $(U, \sigma, *)$ be an approximation space and $S \subseteq U$. Let S be a rough semigroup. If $S = \overline{S}$, then S is a semigroup.*

Proof. Let S be a topological rough semigroup and $S = \overline{S}$.

1) $\forall m, n \in S$, $mn \in \overline{S} = S$,

2) $\forall m, n, r \in \overline{S}$, $(mn)r = m(nr)$, association property holds in $\overline{S} = S$.

Thus, S is a rough semigroup. □

The following proposition constitutes a modified version of Proposition 12 in [6], which can be found on page 3448 in [43] for further details.

Proposition 5. *Let $(\boldsymbol{U},\boldsymbol{\sigma},*)$ be an approximation space, $S\subseteq\boldsymbol{U}$. Let M, N be two rough subsemigroups of S and $M\cap N\neq\emptyset$. If $\overline{M}\cap\overline{N}=\overline{M\cap N}$, then the intersection of M and N is a rough subsemigroup.*

Proof. Assuming $\overline{M}\cap\overline{N}=\overline{M\cap N}$. Since $M\cap N\neq\emptyset$, let us take any $m,n\in M\cap N$. Since M and N are rough subsemigroups, $mn\in\overline{M}$, $mn\in\overline{N}$, i.e., $mn\in\overline{M}\cap\overline{N}$. It follows that $mn\in\overline{M\cap N}$. Thus, $M\cap N$ is a rough subsemigroup of S. □

Proposition 6. *Let $(\boldsymbol{U},\boldsymbol{\sigma},*)$ be an approximation space, $S\subseteq\boldsymbol{U}$. Let I_1,I_2 be two rough lefts (resp. right) ideal of S and $I_1\cap I_2\neq\emptyset$. If $\overline{I_1}\cap\overline{I_1}=\overline{I_1\cap I_1}$, then the intersection of I_1 and I_1 is a rough left (resp. right) ideal.*

Proof. The proof is straightforward to execute in a manner comparable to that demonstrated in Proposition 5. □

Proposition 7. *Let $(\boldsymbol{U},\boldsymbol{\sigma},*)$ be an approximation space, and $S\subseteq\boldsymbol{U}$. Let (S,τ_1) and $\left(\overline{S},\tau_2\right)$ be topological spaces and let $f:S\to\overline{S}$ be a function. Then, the following are equivalent:*

(i) $f:S\to\overline{S}$ is continuous on S,

(ii) $f^{-1}(M)$ is open in S for all open sets M of $\overline{S}$,

(iii) $f^{-1}(N)$ is closed in S for all closed sets M of $\overline{S}$.

Proof. It is clear. □

Proposition 8. *Let $(\boldsymbol{U},\boldsymbol{\sigma},*)$ be an approximation space, and $S\subseteq\boldsymbol{U}$. Let (S,τ_1), and $\left(\overline{S},\tau_2\right)$ be topological spaces and let $f:S\to\overline{S}$ be a continuous function. If S is a compact set, then $f(S)$ is a compact set.*

Proof. Let $\mathcal{M}$ be an open cover of $f(S)$ in $\left(\overline{S},\tau_2\right)$. This implies that $\{f^{-1}(M):M\in\mathcal{M}\}$ is an open cover for S. Since S is compact, a finite subcover exists. Therefore, there exist $M_1,M_2,...,M_n\in\mathcal{M}$ such that $S\subset f^{-1}(M_1)\cup f^{-1}(M_2)\cup...\cup f^{-1}(M_n)$. Hence, it follows that $f(S)\subset M_1\cup M_2\cup...\cup M_n$. As a consequence, an arbitrary open cover of $f(S)$ is found to have a finite subcover. This completes the proof. □

Proposition 9. *Let $(\boldsymbol{U},\boldsymbol{\sigma},*)$ be an approximation space, and $S\subseteq\boldsymbol{U}$. Let $\left(\overline{S},\tau\right)$ be a compact topological spaces, and let S be a closed set. Then, S is a compact set.*

Proof. Let $\mathcal{M}$ be an open cover of S. Since S is closed in $\overline{S}$, $\overline{S}\backslash S$ is open in $\overline{S}$. Consequently, $\mathcal{M}\cup\overline{S}\backslash S$ constitute an open cover for $\overline{S}$. Since S is compact, it must have a finite subcover. Thus, there exist $M_1,M_2,...,M_n\in\mathcal{M}$ such that $S\subset M_1\cup M_2\cup...\cup M_n\cup\left(\overline{S}\backslash S\right)$. Hence, $S\subset M_1\cup M_2\cup...\cup M_n$. Since there is a finite subcover of an arbitrary open cover of S, S is compact. □

3. Topology on a rough semigroup

This section is devoted to the study of topological rough semigroups in an approximation space equipped with relative topology. It also presents some basic properties and examples of great importance. In addition, we will define the notion of compactness and give a few basic notations for it.

Definition 10. *Let $(\boldsymbol{U}, \boldsymbol{\sigma}, *)$ be an approximation space and $S \subseteq \boldsymbol{U}$. A topological rough semigroup is a rough semigroup S together with a topology τ on $\overline{S}$ such that the mapping $f : S \times S \to \overline{S}$ defined by $f(m,n) = mn$ is continuous with respect to product topology on $S \times S$, and the topology τ_S on S induced by τ.*

Definition 10 implies that whenever $W \subseteq \overline{S}$ is open and $W \in \mathcal{N}(mn)$, then there exist open sets $M \subseteq S$ and $N \subseteq S$ such that $M \in \mathcal{N}(m)$; $N \in \mathcal{N}(n)$ and $MN = \{mn : m \in M; n \in N\} \subseteq W$.

Example 11. *Let $\boldsymbol{U} = \left\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\right\}$ be a set of surplus classes with respect to module 5 and $(*)$ be the plus of surplus class. A classification of $\boldsymbol{U}$ is $\boldsymbol{U}/\boldsymbol{\sigma} = \{\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3\}$, where $\boldsymbol{C}_1 = \left\{\overline{0}, \overline{1}\right\}$, $\boldsymbol{C}_2 = \left\{\overline{2}, \overline{4}\right\}$, $\boldsymbol{C}_3 = \left\{\overline{3}\right\}$.*

Let $S = \left\{\overline{1}, \overline{2}, \overline{3}\right\}$, then $\overline{S} = \boldsymbol{U}$.
It is easy to see that S is a rough semigroup from Definition 2.
Let $\tau = \left\{\emptyset, \overline{S}, \left\{\overline{2}\right\}, \left\{\overline{3}\right\}, \left\{\overline{2}, \overline{3}\right\}, \left\{\overline{1}, \overline{2}, \overline{3}\right\}\right\}$.
It is obvious that τ is a topology on $\overline{S}$.
Then, $\tau_S = \left\{\emptyset, S, \left\{\overline{2}\right\}, \left\{\overline{3}\right\}, \left\{\overline{2}, \overline{3}\right\}\right\}$ is a relative topology on G.

The product mapping $f : S \times S \to \overline{S}$ is continuous with respect to product topology on $S \times S$ and the topology τ_S on S induced by the topology τon $\overline{S}$. For example, the open set $\{\overline{2}, \overline{3}\}$ in τ_S has an inverse $\{\{\overline{1}\} \times \{\overline{1}\} \cup \{\overline{1}\} \times \{\overline{2}\}\}$ which is open in the product topology.

It is evident from Definition 10 that S is a topological rough semigroup.

Example 12. *Let $\boldsymbol{U} = \mathbb{R}$ and $(*)$ be the usual addition. Let a partition of $\boldsymbol{U}$ as $\boldsymbol{U}/\boldsymbol{\sigma} = \{\boldsymbol{C}_1, \boldsymbol{C}_2\}$, where*

$$\boldsymbol{C}_1 = \{x : 0 \leq x\}, \quad \boldsymbol{C}_2 = \{x : x < 0\}.$$

Let $S = \mathbb{R} \setminus \{0\}$. Then, $\overline{S} = \mathbb{R}$. Let us define a topology τ on $\overline{S}$ with base $\mathfrak{B} = \{[a,b) : a \leq x < b\}$ in $\overline{S}$. It is easy to see that S is a rough semigroup.

It is well known that the map

$$f : (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$$

defined by $f(a,b) = a + b$ is continuous with respect to product topology on $S \times S$ and the topology τ_S on S induced by τ.

Example 13. *Let $\boldsymbol{U} = S^1$ be the unit circle. Consider a partition of $\boldsymbol{U}$ of the form $\boldsymbol{U}/\boldsymbol{\sigma} = \{\boldsymbol{C}_1, \boldsymbol{C}_2\}$, where*

$$\boldsymbol{C}_1 = \{(a,b) \in \mathbb{R} : a^2 + b^2 = 1,\ b \leq 0\},$$
$$\boldsymbol{C}_2 = \{(a,b) \in \mathbb{R} : a^2 + b^2 = 1\ , b > 0\}.$$

The unit circle S^1 is defined as

$$S^1 = \{(a,b) \in \mathbb{R} : a^2 + b^2 = 1\}.$$

Let

$$H = \{(a,b) \in \mathbb{R} : a^2 + b^2 = 1,\ -\frac{1}{2} \leq b \leq \frac{1}{2}\}.$$

Then, $\overline{H} = S^1$. The topology on S^1 is considered to be the relative topology induced by $\mathbb{R}^2$. So, it is also obvious that H is a topological rough semigroup according to multiplying by

$$* : H \times H \to \overline{H} = S^1,$$
$$((a,b),(c,d)) \mapsto (a,b) * (c,d) = (ac - bd, ac + bd).$$

It is known that the product of two topological semigroups is also a topological semigroup. Let us give the following proposition with the restriction that the product of two topological rough semigroups is also a topological rough semigroup.

Proposition 14. *Let $(S,*)$ and $(H,\circ)$ be two topological rough semigroups on approximation space, $(\boldsymbol{U},\boldsymbol{\sigma})$ with the binary operations "$*$" and "$\circ$". If $\overline{S} \times \overline{H} = \overline{S \times H}$, then $S \times H$ is a topological rough semigroup.*

Proof. With respect to Definition 10, we have the maps

$$f : S \times S \to \overline{S},\quad f(m,n) = m * n,$$

$$g : H \times H \to \overline{H},\quad g(h,k) = h \circ k.$$

Since S and H are topological rough semigroups and $\overline{S} \times \overline{H} = \overline{S \times H}$, we have that $S \times H$ is a topological rough semigroups with the following product operation:

$$f \times g : (S \times H) \times (S \times H) \to \overline{S} \times \overline{H} = \overline{S \times H},$$
$$((m,h),(n,k)) \mapsto (f \times g)\,((a,h),(b,k)) = (f(m,n), g(h,k)) = (m * n, h \circ k)\,.$$

□

Example 15. *Let us consider the same knowledge as in Example 13 and let $\overline{H} \times \overline{H} = \overline{H \times H}$. Then, $\overline{H \times H} = S^1 \times S^1$. Thus, from Proposition 14, we have the product $H \times H$ is a topologically rough semigroup. That is, the torus can be associated with a topologically rough semigroup.*

Proposition 16. *Let $(\boldsymbol{U},\boldsymbol{\sigma},*)$ be an approximation space and $S \subseteq \boldsymbol{U}$. Let S be a topological rough semigroup and fix $s \in S$. Then,*

(i) The map $L_s : S \to \overline{S}$ defined by $L_s(m) = sm$ is continuous for every $m \in S$.

(ii) The map $R_s : S \to \overline{S}$ defined by $R_s(m) = ms$ is continuous for every $m \in S$.

Proof. (i) For every $m \in S$ then $L_s(m) = sm$. Let $W \in \mathcal{N}(sm)$. Then, by Definition 10, $\exists M \in \mathcal{N}(s) \in \tau_S$ and $\exists N \in \mathcal{N}(m) \in \tau_S$ such that $MN \subseteq W$. Since, $sN \subseteq MN \subseteq W$, then $L_s(N) = sN \subseteq W$. Therefore, L_s is continuous on m. Since m be any element of S, then L_s is continuous in S.

(ii) Similar to evidence (i). □

Proposition 17. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space and $S \subseteq \boldsymbol{U}$. Let S be a topological rough semigroup. If $S = \overline{S}$, then S is a topological semigroup.*

Proof. From Proposition 4, S is a rough semigroup. Since S is a topological rough semigroup and $S = \overline{S}$, we have the maps

$$f : S \times S \to S, \quad f(m, n) \to mn$$

is continuous. Hence, S is a topological semigroup. □

Definition 18. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space and let $S \subseteq \boldsymbol{U}$. If S is a topological rough semigroup and S is a compact subset of $\overline{S}$, then S is called a compact topological rough semigroup.*

Remark 19. *It is clear that when we combine Definition 10 and Definition 18, the space (S, τ_S) becomes compact.*

Proposition 20. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space and let $S \subseteq \boldsymbol{U}$. Let S be a compact topological rough semigroup and $s \in S$. Then, sS and sS are compact in $\overline{S}$.*

Proof. From Proposition 16, the maps L_s and R_s *are continuous. Because* S is compact and $L_s(S) = sS$, $R_s(S) = S\,s$, then, by Proposition 8, we have that sS and sS are compact in $\overline{S}$. □

The following proposition can be derived from Proposition 9 and Definition 18.

Proposition 21. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space and let $S \subseteq \boldsymbol{U}$. Let $\left(\overline{S}, \tau\right)$ be a compact topological spaces, and let S be a topological rough semigroup. If S is closed, then S is a compact topological rough semigroup.*

Proposition 22. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space and $S \subseteq \boldsymbol{U}$. Let M and N be subsets of a compact topological rough semigroup S. If M and N are compact, then MN is compact.*

Proof. Since M and N are compact, $M \times N$ is also compact with respect to the product topology on $S \times S$. It follows that since S is a topological rough semigroup and Proposition 8, $f(M \times N) = MN$ is compact. □

Proposition 23. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space and $S \subseteq \boldsymbol{U}$. Let A, B, and C be subsets of a topological rough semigroup S.*

(i) If B is compact in $\overline{S}$ and $A \subset CB$, then C is closed in $\overline{S}$.

(ii) If B is closed in $\overline{S}$ and $CA \subset B$, then C is closed in $\overline{S}$.

Proof. (i) Let $y \in \overline{S}\backslash C$. Then, $a \notin yB$ for some element a of A. Since yB is compact, there exist disjoint open sets M and N such that $a \in M$ and $yB \subset N$. Hence, there are open sets U and V such that $y \in U$, $B \subset V$ and $UV \subset N$. Hence, we obtain $UB \subset N$, $UB \cap M = \emptyset$, $a \notin UB$, $U \subset \overline{S}\backslash C$. So, $\overline{S}\backslash C$ is open, and therefore, C is closed.

(ii) Let $y \in \overline{S}\backslash C$. Then, any element $a \in A$ implies that $ya \in \overline{S}\backslash B$. Since $\overline{S}\backslash B$ is open, there exist open sets M and N such that $y \in M$, $a \in N$, and $MN \subset \overline{S}\backslash B$. Therefore, $Ma \subset \overline{S}\backslash B$ and $y \in M \subset \overline{S}\backslash C$. Consequently, $\overline{S}\backslash C$ is open and C is closed. □

3.1. Topological rough subsemigroups and ideals

This section is devoted to the study of topological rough subgroups and topological rough ideals in an approximation space endowed with the relative topology. It also presents some elementary properties.

Definition 24. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space, $S \subseteq \boldsymbol{U}$ be a topological rough semigroup and let K be a subsemigroup of S. Then, K is called a topological rough subsemigroup of S if the map $f_K : K \times K \to \overline{K}$ defined by $f_K(k_1, k_2) = k_1k_2$ is continuous where $\overline{K}$ carries the topology induced by $\overline{S}$.*

Proposition 25. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space, and $S \subseteq \boldsymbol{U}$ be a topological rough semigroup. Then, every rough subsemigroup K of S with relative topology is a topological rough subsemigroup.*

Proof. Since $f : S \times S \to \overline{S}$ is continuous, its restriction, $f_K : K \times K \to \overline{K}$, is also continuous. Hence, K is a rough subsemigroup. □

Proposition 26. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space, and $S \subseteq \boldsymbol{U}$ be a topological rough semigroup. If K is a topological rough subsemigroup of S, then clK is a topological rough subsemigroup of S.*

Proof. Assuming K is a topological subsemigroup of S, let $k_1, k_2 \in clK$. It is required to prove that $k_1k_2 \in \overline{clK}$. Let W be a neighbourhood of k_1, k_2. Then, given a neighbourhood W of k_1k_2, there are M and N neighbourhoods of k_1 and k_1, respectively, such that $MN \subseteq W$. Since $k_1 \in clK$ and $k_2 \in clK$ there exist elements m and n of K such that $m \in M \cap K$, $m \in M \cap \overline{K}$ and $n \in N \cap K$, $n \in N \cap \overline{K}$. In this case, $mn \in MN$ and $mn \in \overline{K}$. This means that $mn \in MN \cap \overline{K}$. Therefore, $MN \cap \overline{K} \neq \varnothing$ and hence, $W \cap \overline{K} \neq \varnothing$. Consequently, $k_1k_2 \in \overline{clK}$.

This means that clK is a topologically rough subsemigroup of S. □

Proposition 27. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space, $S \subseteq \boldsymbol{U}$. Let M, N be two topological rough subsemigroups of S and $M \cap N \neq \emptyset$. If $\overline{M} \cap \overline{N} = \overline{M \cap N}$, then the intersection of M and N is a topologically rough subsemigroup.*

Proof. The proof is complete when Proposition 5 is applied first and then Definition 24. □

Definition 28. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space, $S \subseteq \boldsymbol{U}$ be a topological rough semigroup, and let I be a rough left (resp. right) ideal of S. Then, I is called a topological rough left (resp. right) ideal of S if the map $f_I : S \times I \to \overline{I}$ defined by $f_I(s, n) = sn$ is continuous where $\overline{I}$ carries the topology induced by $\overline{S}$, $s \in S$, $n \in I$.*

Proposition 29. *Let $(\boldsymbol{U}, \sigma, *)$ be an approximation space, and $S \subseteq \boldsymbol{U}$ be a topological rough semigroup. Then, every rough left (resp. right) ideal I of S with relative topology is a topological rough left (resp. right) ideal of S.*

Proof. Since $f : S \times S \rightarrow \overline{S}$ is continuous, its restriction, $f_I : S \times I \rightarrow \overline{I}$, is also continuous. Consequently, I is a rough left ideal. Similarly, I is a rough right ideal. □

Proposition 30. *Let $(\boldsymbol{U}, \boldsymbol{\sigma}, *)$ be an approximation space, and $S \subseteq \boldsymbol{U}$ be a topological rough semigroup. If I is a topological rough left (resp. right) ideal of S, then clI is a topological rough left (resp. right) ideal of S.*

Proof. Assuming I is a topological left ideal of S, let $s, k \in clI$. We must prove that $sk \in \overline{clI}$. Let W be a neighbourhood of sk. Then, there exist M and N neighbourhoods of s and k, in S and I, respectively, such that $MN \subseteq W$. Since $s, k \in clI$, there are elements m and n of I such that $m \in M \cap I$, $m \in M \cap \overline{I}$, and $n \in N \cap I$, $n \in N \cap \overline{I}$. So we have $mn \in MN$ and $mn \in \overline{I}$. This follows that $mn \in MN \cap \overline{I}$. Therefore, $MN \cap \overline{I} \neq \emptyset$, and thus, $W \cap \overline{I} \neq \emptyset$. Consequently, $sk \in \overline{clI}$.

This proves that clI is a topological rough-left ideal of S. Similarly, I is a rough-right ideal of S. □

Proposition 31. *Let $(\boldsymbol{U}, \boldsymbol{\sigma}, *)$ be an approximation space, $S \subseteq \boldsymbol{U}$. Let I_1, I_2 be two topological rough left (resp. right) ideal of S and $I_1 \cap I_2 \neq \emptyset$. If $\overline{I_1} \cap \overline{I_1} = \overline{I_1 \cap I_1}$, then the intersection of I_1 and I_1 is a topological rough left (resp. right) ideal.*

Proof. The proof is complete when Proposition 6 is applied first and then Definition 28. □

4. Conclusions

Topology and rough sets are two very important fields that contribute to the understanding of current problems and relationships, since topological spaces model continuity, while rough sets deal with imprecision and uncertainty. This paper focusses on one of the new arguments, the notion of a topological rough semigroup. First, the notion of a topological rough semigroup is defined and analysed with the help of a number of illustrative examples. In addition, the notion of compactness is defined, and some basic properties are given. Finally, the concepts of topological rough subgroup and rough ideal are presented.

Our theory provides a bridge between rough sets, topological spaces, and semigroups. We believe that this new topological structure will evolve in the future and has the potential to be applied in different fields.

Acknowledgments

The author would like to thank the academic editor and referees for their appreciative comments and additions, which greatly improved the quality of this article.

Conflict of interest

The author declares that he has no conflict of interest.

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