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Research article

Hopf bifurcation and hybrid control of a delayed diffusive semi-ratio-dependent predator-prey model

Hairong Li[1], Yanling Tian[2,*], Ting Huang[1] and Pinghua Yang[1]

[1] College of Computer Engineering, Guangzhou City University of Technology, Guangzhou, Guangdong 510800, China

[2] School of Mathematics, South China Normal University, Guangzhou, Guangdong 510631, China

* **Correspondence:** Email: tianyl@scnu.edu.cn.

Abstract: A delayed diffusive predator-prey system with nonmonotonic functional response subject to Neumann boundary conditions is introduced in this paper. First, we analyze the associated characteristic equation to research the conditions for local stability of the positive equilibrium point and the occurrence of Turing instability induced by diffusion in the absence of delay. Second, we provide conditions for the existence of Hopf bifurcation driven by time delay. By utilizing the normal theory and center manifold theorem, we derive explicit formulas for Hopf bifurcation properties such as direction and stability from the positive equilibrium. Third, a hybrid controller is added to the system. By judiciously adjusting the control parameters, we effectively enhance the stability domain of the system, resulting in a modification of the position of the Hopf bifurcation periodic solutions. Numerical simulations demonstrate the presence of rich dynamical phenomena within the system. Moreover, sensitivity analysis was conducted using Latin hypercube sampling (LHS)/partial rank correlation coefficient (PRCC) to explore the impact of parameter variations on the output of prey and predator populations.

Keywords: time-delayed; diffusive; Turing instability; Hopf bifurcation; hybrid control
Mathematics Subject Classification: 92D25, 35Q92, 35B32

1. Introduction

The rich dynamics of nonlinear systems have significant implications for modern technology, advancing developments in fields such as natural sciences and engineering applications. Non-linear dynamical systems typically serve as mathematical formalizations of conventional scientific concepts and complex entities, with widespread applications in biology, ecology, and chemistry [1–3]. Zhang and Lu [4] studied a type of predator-prey system with a Holling-IV type functional response function

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1 - b_1(t)x_1(t) - \dfrac{a_1(t)x_2(t)}{m^2 + nx_1(t) + x_1(t)^2}], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2 - \dfrac{a_2(t)x_2(t)}{x_1(t)}], \end{cases} \tag{1.1}$$

where $x_1(t), x_2(t)$ represent the densities of the prey and the predator at time t, respectively, $m \neq 0, n \geq 0$ are all constant, and r_1, r_2 represent the intrinsic growth rates of the prey and the predator respectively, $b_1(t)$ is the intra-specific competition rate of the prey. $a_1(t)$ is the capturing rate of the predator, $a_2(t)$ is a measure of the food quality that the prey provided for conversion into predator birth [4]. In [4], Zhang and Lu demonstrated the global stability of the positive equilibrium point using the Lyapunov function. In [5–7], many scholars have studied the properties of periodic solutions for the semi-ratio-dependent prey-predator model. Zhao [8] investigated the stability and bifurcation of model (1.1) with two delays.

With the continuous deepening and expansion of research, scholars have found that in the natural world, the emergence of many phenomena is not only influenced by the current state but also closely related to the state of a certain moment or period in the past. This phenomenon is called time delay. Many scholars have investigated predator-prey models with various types of time delays, such as infinite delay, time-varying delays, and multiple delays [9–11]. Under specific conditions, delays induce variations in the stability of equilibrium states, leading to the emergence of bifurcations or spiral wave patterns in the considered systems [12–17]. Therefore, the application of time delay in predator-prey models is important for studying the dynamic behavior of ecological systems and predicting their future development trends.

Reaction-diffusion models are commonly employed to characterize the movement and evolution. In both macroscopic and microscopic worlds, every particle, such as bacteria, cells, and organisms, moves in an apparently random manner, commonly referred to as diffusion processes [18]. The diffusion process may lead to environmental changes. Turing's initial studies demonstrated that stable homogeneous states can become unstable under certain conditions in reaction-diffusion systems, leading to the formation of patterns [19]. The corresponding instability is referred to as diffusion-driven instability [20]. Li et al. [21, 22] studied the relevant issues of Turing patterns. Dynamics studies on the stability and bifurcation issues of reaction-diffusion systems have also been quite extensive [23–27]. Hence, the factor of diffusion is considered.

Stability and bifurcation are crucial considerations in the study of predator-prey models. Hopf bifurcation is a significant mathematical tool for describing periodic behavior, and studying Hopf bifurcation contributes to the understanding and modeling of important phenomena in biological systems such as periodic behavior, stability transitions, and dynamic oscillations [28–31]. Considering the system instability caused by Hopf bifurcation, there has been an increasing depth of research on bifurcation control by scholars in recent years [32–34]. Control strategies can effectively enhance dynamic behavior in model research. Jiang et al. [35] proposed a new control scheme that effectively controls bifurcations. Control strategies such as PD control, state feedback control, time-delay feedback control, and hybrid control have been continuously evolving [36–39]. The concept of a hybrid control strategy was initially introduced by Luo et al. [40], research has shown that compared to pure feedback control, the control parameters in hybrid control can be adjusted over a wider range, making it more convenient and feasible for practical implementation. Peng [41] explored the role of hybrid controllers in predator-prey models within the field of biological systems. From a biological perspective, adjusting the stable range of bifurcation period solutions enables the coexistence of two species under oscillatory patterns. These control strategies have predominantly been formulated and examined based on established principles of ordinary differential equations (ODEs). However, there has been limited exploration into control in predator-prey systems described by PDEs. Ghosh [42] investigated a practical approach of utilizing time-delayed feedback for bifurcation control in reaction-diffusion systems, thus highlighting the substantial theoretical research significance of the work presented in our paper.

A delayed diffusive system established in this paper is as follows:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = d_1\Delta u(t,x) + u(x,t)[r_1 - b_1u(x,t) - \frac{a_1v(x,t)}{m^2+nu(x,t)+u^2(x,t)}], & \\ & x \in (0,\Omega), t > 0, \\ \frac{\partial}{\partial t}v(x,t) = d_2\Delta v(t,x) + v(x,t)[r_2 - \frac{a_2v(x,t-\tau)}{u(x,t-\tau)}], & x \in (0,\Omega), t > 0, \\ u_x(0,t) = v_x(0,t) = 0, u_x(\Omega,t) = v_x(\Omega,t) = 0, & t > 0, \\ u(x,t) = u_0(x,t) \geq 0, v(x,t) = v_0(x,t) \geq 0, & (x,t) \in [0,\Omega] \times [-\tau, 0], \end{cases} \tag{1.2}$$

here $u(x,t)$ and $v(x,t)$ represent the population densities of the prey and predator, respectively. Parameters of the system (1.2) are regarded as constants. d_1 and d_2 represent the constant diffusion coefficient. τ represents that the consumption of prey by predators needs some time to be converted into effective energy. Δ represents the Laplacian operator, the system subject to the Neumann boundary condition, and we assume that area $[0,\Omega]$ is closed, where $\Omega = l\pi (l > 0)$; the population cannot go through the boundary of the region.

The main contributions of this paper include: (1) To better approximate real ecological systems, the semi-ratio-dependent prey-predator model with Holling-IV functional response in ordinary differential equations is being extended. Time delays and diffusion terms are being introduced into the model. By analyzing the system's corresponding characteristic equations, the conditions for Turing instability induced by diffusion in the absence of time delays are being deduced. (2) The impact of time delays on the stability of positive equilibrium points in reaction-diffusion systems is being investigated. Sufficient conditions for the existence of spatially homogeneous and inhomogeneous Hopf bifurcation are being provided. Utilizing partial differential equation normalization theory and the center manifold theorem, explicit methodologies for determining the bifurcation direction and the stability of periodic solutions at positive equilibrium points are being derived. (3) The application of control strategies for Hopf bifurcations in reaction-diffusion systems formed by partial differential equations is currently limited. A hybrid controller is being integrated to regulate the bifurcation behavior in reaction-diffusion systems with a Holling-IV functional response. Results demonstrate that by adjusting control parameters, the spatial stability range is expanded, leading to modifications in the positions of periodic solutions of the Hopf bifurcation and enhancing the system's performance and controllability. (4) Currently, there are few researchers conducting sensitivity analysis on the reaction-diffusion prey-predator system, at least from our current perspective. Thus sensitivity analysis of the system is being carried out using Latin hypercube sampling/partial rank correlation coefficient (LHS/PRCC), exploring the parameter space of the model. This analysis offers valuable insights for comprehending the uncertainty and intricacy of the system.

The paper is arranged as follows: In Section 2, we analyze the stability of the model with the inclusion of reaction-diffusion terms and the conditions that lead to Turing instability; We also study the existence of spatially homogeneous and inhomogeneous Hopf bifurcations. In Section 3, We present the properties of the Hopf bifurcation. In Section 4, the dynamical behavior and control strategy of a controlled diffusion system are investigated by incorporating a hybrid controller into the prey-predator model. In Section 5, numerical simulations are accomplished to substantiate conclusions. At last, it is a conclusion of this paper.

2. Stability of equilibrium points and Hopf bifurcation analysis

It is calculated that the system (1.2) has two possible equilibrium points.
(1) The boundary equilibrium $E_1 = (\frac{r_1}{b_1}, 0)$.
(2) The coexisting equilibrium point $E_* = (u_*, v_*)$, which $v_* = \dfrac{r_2 u_*}{a_2}$ and the following equation have at least one positive root u_* .

$$b_1 u^3 + (nb_1 - r_1)u^2 + (\frac{a_1 r_2}{a_2} - nr_1 + m^2 b_1)u - m^2 r_1 = 0. \tag{2.1}$$

Let $u(t) = u(x, t), v(t) = v(x, t), u(t - \tau) = u(x, t - \tau)$, and $v(t - \tau) = v(x, t - \tau)$. The system (1.2) is linearized at (u_*, v_*):

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = D\Delta \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + L_1 \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + L_2 \begin{pmatrix} u(t-\tau) \\ v(t-\tau) \end{pmatrix}, \tag{2.2}$$

where

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, L_1 = \begin{pmatrix} a_{11} & -a_{12} \\ 0 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 \\ a_{21} & -a_{22} \end{pmatrix},$$

and

$$a_{11} = r_1 - 2b_1 u_* - \frac{a_1 v_* m^2 - a_1 u_*^2 v_*}{(m^2 + nu_* + u_*^2)^2}, \quad a_{12} = \frac{a_1 u_*}{m^2 + nu_* + u_*^2},$$
$$a_{21} = \frac{a_2 v_*^2}{u_*^2}, \quad a_{22} = \frac{a_2 v_*}{u_*}.$$

The characteristic equation of (3.1) is

$$det(\lambda I - M_k - L_1 - L_2 e^{-\lambda\tau}) = 0, \tag{2.3}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $M_k = -\frac{k^2}{l^2} D^2$ for $k \in \{0, 1, 2, ...\} := N_0$. Then we have

$$\lambda^2 + A_k \lambda + B_k + \lambda a_{22} e^{-\lambda\tau} + C_k e^{-\lambda\tau} = 0, \tag{2.4}$$

where

$$A_k = \frac{k^2}{l^2}(d_1 + d_2) - a_{11}, \quad B_k = \frac{k^4}{l^4} d_1 d_2 - \frac{k^2}{l^2} d_2 a_{11}, \quad C_k = \frac{k^2}{l^2} d_1 a_{22} - a_{11} a_{22} + a_{12} a_{21}.$$

2.1. Stability of the equilibria as $\tau = 0$

(1) The characteristic equation to $E_1 = (\frac{r_1}{b_1}, 0)$ is

$$(\lambda + \frac{k^2}{l^2}d_1 + r_1)(\lambda + \frac{k^2}{l^2}d_2 - r_2) = 0, k \in N_0. \tag{2.5}$$

Obviously,

$$\lambda_1 = -\frac{k^2}{l^2}d_1 - r_1 < 0, \qquad \lambda_2 = -\frac{k^2}{l^2}d_2 + r_2 \leq r_2, k \in N_0.$$

There is at least one positive characteristic root of (2.5), hence $E_1 = (\frac{r_1}{b_1}, 0)$ is unstable.

(2) When $\tau = 0$, the characteristic equation to $E_* = (u_*, v_*)$ is

$$\lambda^2 + [\frac{k^2}{l^2}(d_1 + d_2) - a_{11} + a_{22}]\lambda + \frac{k^4}{l^4}d_1 d_2 - \frac{k^2}{l^2}(d_2 a_{11} - d_1 a_{22}) + a_{12}a_{21} - a_{11}a_{22} = 0. \tag{2.6}$$

Define

$$\begin{aligned} \sigma_k &= -(d_1 + d_2)\frac{k^2}{l^2} + a_{11} - a_{22}, \\ \varphi_k &= \frac{k^4}{l^4}d_1 d_2 - \frac{k^2}{l^2}(d_2 a_{11} - d_1 a_{22}) + a_{12}a_{21} - a_{11}a_{22}. \end{aligned} \tag{2.7}$$

Assume that

$$(\boldsymbol{H1}) \quad a_{11} - a_{22} < 0, \qquad a_{12}a_{21} - a_{11}a_{22} > 0.$$

When $d_1 = d_2 = 0$ and $\tau = 0$, all roots of (2.6) have negative real parts under hypothesis $(\boldsymbol{H1})$, $E_* = (u_*, v_*)$ is locally asymptotically stable.

Here are the three cases for dividing the parameters,

$$\begin{aligned} &\boldsymbol{Case1}: d_2 a_{11} - d_1 a_{22} \leq 0; \\ &\boldsymbol{Case2}: d_2 a_{11} - d_1 a_{22} > 0, \ and \ (d_2 a_{11} - d_1 a_{22})^2 - 4d_1 d_2(a_{12}a_{21} - a_{11}a_{22}) < 0; \\ &\boldsymbol{Case3}: d_2 a_{11} - d_1 a_{22} > 0, \ and \ (d_2 a_{11} - d_1 a_{22})^2 - 4d_1 d_2(a_{12}a_{21} - a_{11}a_{22}) > 0. \end{aligned}$$

Denote

$$\mathbb{S} = \{k | \varphi_k < 0, k \in N_+\}.$$

Theorem 1. *Suppose* $(\boldsymbol{H1})$ *holds and* $\tau = 0$.

(1) In $\boldsymbol{Case1}$*(or* $\boldsymbol{Case2}$*),* $E_* = (u_*, v_*)$ *of system (1.2) is locally asymptotically stable.*

(2) In $\boldsymbol{Case3}$*, when* $k \in \mathbb{S}$*, then* $E_* = (u_*, v_*)$ *of system (1.2) is Turing unstable.*

Proof. If $\boldsymbol{Case1}$(or $\boldsymbol{Case2}$) hold, we have $\sigma_k < 0$ and $\varphi_k > 0$ for $k \in N_0$, the above (1) holds. If $\boldsymbol{Case3}$ hold, we have $\varphi_k < 0$ for $k \in \mathbb{S}$. This ensures Eq (2.6) has a positive real part root, so $E_* = (u_*, v_*)$ of system (1.2) undergoes Turing bifurcation. □

Remark 1. *The system (1.2) exhibits an unstable predator-free equilibrium point, which may not occur in natural systems. At the positive equilibrium point* $E_* = (u_*, v_*)$*, there is a richer dynamic behavior. According to the conditions of Turing, spatial diffusion can lead to the destabilization of the stable equilibrium of the ordinary differential equations corresponding to reaction-diffusion systems, forming regular pattern structures in space, known as diffusion-induced instability. Example 1 of numerical simulations can illustrate the occurrence of Turing instability.*

2.2. Hopf bifurcation ($\tau > 0$)

We will derive the conditions for the Hopf bifurcation. Suppose $i\omega(\omega > 0)$ be a solution of Eq (2.4), then

$$-\omega^2 + i\omega A_k + B_k + (i\omega a_{22} + C_k)(cos(\omega\tau) - isin(\omega\tau)) = 0. \tag{2.8}$$

Then we have

$$\begin{cases} C_k cos(\omega\tau) + \omega a_{22} sin(\omega\tau) = \omega^2 - B_k, \\ \omega a_{22} cos(\omega\tau) - C_k sin(\omega\tau) = -A_k\omega, \end{cases} \tag{2.9}$$

which lead to

$$\omega^4 + \omega^2(A_k^2 - 2B_k - a_{22}^2) + B_k^2 - C_k^2 = 0. \tag{2.10}$$

Let $z = \omega^2$, then (2.10) becomes

$$z^2 + z(A_k^2 - 2B_k - a_{22}^2) + B_k^2 - C_k^2 = 0, \tag{2.11}$$

where

$$\begin{aligned} A_k^2 - 2B_k - a_{22}^2 &= \frac{k^4}{l^4}(d_1^2 + d_2^2) - \frac{2k^2}{l^2}a_{11}d_1 + a_{11}^2 - a_{22}^2, \\ B_k^2 - C_k^2 &= \left[\frac{k^4}{l^4}d_1d_2 - \frac{k^2}{l^2}(d_2a_{11} - d_1a_{22}) + (a_{12}a_{21} - a_{11}a_{22})\right]\left[\frac{k^4}{l^4}d_1d_2 - \frac{k^2}{l^2}(d_2a_{11}\right. \\ &\quad \left. + d_1a_{22}) - (a_{12}a_{21} - a_{11}a_{22})\right]. \end{aligned} \tag{2.12}$$

We have $B_0^2 - C_0^2 < 0$, there must exist some $k_0 \in \{1, 2, \cdots\}$ satisfied $B_k^2 - C_k^2 < 0$ for $k \in \{0, 1, 2, \cdots, k_0 - 1\}$, and $B_k^2 - C_k^2 \geq 0$ for $k \in \{k_0, k_0 + 1, \cdots\}$.

When $a_{11} \in (-\infty, -a_{22})$, then we have $-A_k^2 + 2B_k + a_{22}^2 \leq a_{22}^2 - a_{11}^2 < 0$ for any $k \in N_0$.

When $a_{11} \in (-a_{22}, a_{22})$, we make the following assumption:

$$\textbf{(}\boldsymbol{H2}\textbf{)} \quad -\frac{k_0^4}{l^4}(d_1^2 + d_2^2) + \frac{2k_0^2}{l^2}a_{11}d_1 + a_{22}^2 - a_{11}^2 < 0.$$

If **(*H*2)** holds, then we have

$$-A_k^2 + 2B_k + a_{22}^2 \leq -\frac{k_0^4}{l^4}(d_1^2 + d_2^2) + \frac{2k_0^2}{l^2}a_{11}d_1 + a_{22}^2 - a_{11}^2 < 0,$$

for any $k \geq k_0$. These imply that there are no purely imaginary roots in Eq (2.4).

Since $B_k^2 - C_k^2 < 0$, when $k \in \{0, 1, 2, \cdots, k_0 - 1\}$, the equation (2.11) has one positive root z_k, namely

$$z_k = \frac{-(A_k^2 - 2B_k - a_{22}^2) + \sqrt{(A_k^2 - 2B_k - a_{22}^2)^2 - 4(B_k^2 - C_k^2)}}{2}.$$

Equation (2.4) has a pair of roots with purely imaginary $\pm i\omega_k$ at $\tau_k^j(k \in \{0, 1, 2, \cdots, k_0 - 1\}, j \in N_0)$, where

$$\omega_k = \sqrt{\frac{-(A_k^2 - 2B_k - a_{22}^2) + \sqrt{(A_k^2 - 2B_k - a_{22}^2)^2 - 4(B_k^2 - C_k^2)}}{2}},$$

and

$$\tau_k^j = \frac{1}{\omega_k} arccos \frac{\omega_k^2(C_k - a_{22}A_k) - B_kC_k}{C_k^2 + \omega_k^2 a_{22}^2} + \frac{2j\pi}{w_k}, j \in N_0. \tag{2.13}$$

Obviously $\tau_k^0 = \min_{j \in N_0}\{\tau_k^j\}$. Denote

$$\widetilde{\tau} = \min_{k \in \{0,1,2,\cdots,k_0-1\}} \{\tau_k^0\}. \tag{2.14}$$

Let $\lambda_k(\tau) = \alpha_k(\tau) \pm i\omega_k(\tau)$ be the root of (2.4) near $\tau = \tau_k^j, k \in \{0, 1, 2, \cdots, k_0 - 1\}$ satisfying $\alpha_k(\tau_k^j) = 0$ for $\omega_k(\tau_k^j) = \omega_k$.

Lemma 1. *The following traversal condition holds:* $Re(\frac{d\lambda}{d\tau})|_{\tau=\tau_k^j} > 0.$

Proof. Take the derivative of both sides of (2.4) with respect to τ, and we have

$$2\lambda\frac{d\lambda}{d\tau} + A_k\frac{d\lambda}{d\tau} - e^{-\lambda\tau}\tau(\lambda a_{22} + C_k)\frac{d\lambda}{d\tau} - \lambda e^{-\lambda\tau}(\lambda a_{22} + C_k) + e^{-\lambda\tau}a_{22}\frac{d\lambda}{d\tau} = 0,$$

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda + A_k + a_{22}e^{-\lambda\tau}}{\lambda e^{-\lambda\tau}(\lambda a_{22} + C_k)} - \frac{\tau}{\lambda},$$

then

$$\begin{aligned} Re(\frac{d\lambda}{d\tau})^{-1}_{\tau=\tau_k^j} &= Re[\frac{(2i\omega_k + A_k)(cos\omega_k\tau + isin\omega_k\tau) + a_{22}}{i\omega_k(i\omega_k a_{22} + C_k)}]_{\tau=\tau_k^j}, \\ &= [\frac{2\omega_k^2 - 2B_k + A_k^2 - a_{22}^2}{C_k^2 + \omega_k^2 a_{22}^2}]_{\tau=\tau_k^j}, \\ &= \frac{\sqrt{(A_k^2 - 2B_k - a_{22}^2)^2 - 4(B_k^2 - C_k^2)}}{C_k^2 + \omega_k^2 a_{22}^2} > 0. \end{aligned}$$

□

These critical values τ_k^j for $j \in \{0, 1, 2, \cdots\}; k \in \{0, 1, 2, \cdots, k_0 - 1\}$ are possible Hopf bifurcations. Suppose $\tau_{k_1}^i \neq \tau_{k_2}^j$ for any $i, j \in \{0, 1, 2, \cdots\}; k_1, k_2 \in \{0, 1, 2, \cdots, k_0 - 1\}$. Based on the above, we derive the main result.

Theorem 2. *When* $a_{11} \in (-\infty, -a_{22})$ *, if* **(*H*1)** *holds; When* $a_{11} \in (-a_{22}, a_{22})$ *, if* **(*H*1)(*H*2)** *hold, there are the following conclusions.*

(1) When $\tau \in [0, \widetilde{\tau})$, the coexisting equilibrium $E_* = (u_*, v_*)$ of system (1.2) is locally asymptotically stable, where $\widetilde{\tau}$ is defined in (2.14).
(2) System (1.2) undergoes a Hopf bifurcation at the equilibrium $E_* = (u_*, v_*)$ when $\tau = \tau_k^j, j = 0, 1, 2, \cdots; k = 0, 1, 2, \cdots, k_0 - 1$. Furthermore, when $k = 0$, bifurcating periodic solutions are spatially homogeneous; otherwise, they are spatially inhomogeneous.

3. Stability and direction of Hopf bifurcation

We will apply Wu [23] and Hassard's [28] method to compute properties of periodic solutions for system (1.2). For fixed $k \in \{0, 1, 2, \cdots, k_0 - 1\}$, $j \in N_0$, we denote $\tau^* = \tau_k^j$. We perform a transformation $\overline{u}(x,t) = u(x,\tau t) - u_*$ and $\overline{v}(x,t) = v(x,\tau t) - v_*$, but we still use $u(x,t), v(x,t)$ instead of $\overline{u}(x,t), \overline{v}(x,t)$. Then (1.2) can be transformed into the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \tau[d_1 \Delta u + (u+u_*)\left(r_1 - b_1(u+u_*) - \dfrac{a_1(v+v_*)}{m^2 + n(u+u_*) + (u+u_*)^2}\right)], \\ \dfrac{\partial v}{\partial t} = \tau[d_2 \Delta v + (v+v_*)\left(r_2 - \dfrac{a_2 v(t-1) + v_*}{u(t-1)+u_*}\right)], \end{cases} \tag{3.1}$$

for $x \in (0, l\pi)$, and $t > 0$. Let

$$\tau = \tau^* + \mu, u_1(t) = u(\cdot, t), u_2(t) = v(\cdot, t) \text{ and } U = (u_1, u_2)^T.$$

We use $C := C([-1, 0], X)$ to represent the phase space, then (3.1) can be written as an abstract differential equation.

$$\frac{dU(t)}{dt} = \tau^* D \Delta U(t) + L(\tau^*)(U_t) + F(U_t, \mu), \tag{3.2}$$

where $L(\tau)(\phi) : C \to X$ and $F(\phi, \mu) : C \times R \to X$ are defined by

$$L(\mu)(\phi) = \mu \begin{pmatrix} a_{11}\phi_1(0) - a_{12}\phi_2(0) \\ a_{21}\phi_1(-1) - a_{22}\phi_2(-1) \end{pmatrix}, \tag{3.3}$$

$$F(\phi, \mu) = \mu D \Delta \phi + L(\mu)(\phi) + f(\phi, \mu), \tag{3.4}$$

with

$$f(\phi, \mu) = (\tau^* + \mu)\left(F_1(\phi, \mu), F_2(\phi, \mu)\right)^T, \tag{3.5}$$

$$\begin{aligned} F_1(\phi,\mu) &= a_{13}\phi_1^2(0) + a_{14}\phi_1(0)\phi_2(0) + a_{15}\phi_1^3(0) + a_{16}\phi_1^2(0)\phi_2(0) + o(4), \\ F_2(\phi,\mu) &= a_{23}\phi_1^2(-1) + a_{24}\phi_1(-1)\phi_2(0) + a_{25}\phi_2(-1)\phi_2(0) + a_{27}\phi_1^3(-1) \\ &\quad + a_{28}\phi_1^2(-1)\phi_2(0) + a_{29}\phi_1^2(-1)\phi_2(-1) \\ &\quad + a_{30}\phi_1(-1)\phi_2(0)\phi_2(-1) + o(4), \end{aligned} \tag{3.6}$$

where

$$a_{13} = -b_1 - \frac{a_1 u_*^3 v_* - a_1 m^2 n v_* - 3a_1 m^2 u_* v_*}{(m^2 + nu_* + u_*^2)^3}, \quad a_{14} = \frac{a_1 u_*^2 - a_1 m^2}{(m^2 + nu_* + u_*^2)^2},$$

$$a_{15} = \frac{a_1 u_* v_* (u_*^3 - 4m^2 n - 6u_* v_* m^2) + a_1 v_* m (m^2 - n^2)}{(m^2 + nu_* + u_*^2)^4},$$

$$a_{16} = \frac{3a_1 u_* m^2 - a_1 u_*^3 + a_1 m^2 n}{(m^2 + nu_* + u_*^2)^3}, \quad a_{23} = -\frac{a_2 v_*^2}{u_*^3}, \quad a_{24} = \frac{a_2 v_*}{u_*^2},$$

$$a_{25} = -\frac{a_2}{u_*}, \quad a_{26} = \frac{a_2 v_*}{u_*^2}, \quad a_{27} = \frac{a_2 v_*^2}{u_*^4}, \quad a_{28} = -\frac{a_2 v_*}{u_*^3},$$

$$a_{29}=-\frac{a_2 v_*}{u_*^3},\quad a_{30}=\frac{a_2}{u_*^2},$$

with $\phi(\theta)=(\phi_1(\theta),\phi_2(\theta))^T\in C$.

Consider the linear equation

$$\frac{dU(t)}{dt}=\tau^* D\Delta U(t)+L(\tau^*)\,(U_t)\,. \tag{3.7}$$

Clearly, $(0,0)$ is an equilibrium point of the system (3.1). Consider the functional differential equation

$$\dot{z}=-\tau^* D\frac{k^2}{l^2}z(t)+L(\tau^*)(z_t). \tag{3.8}$$

By the Riesz representation, there exists a bounded variation function $\eta(\theta,\tau^*)(-1\le\theta\le 0)$, such that

$$-\tau^* D\frac{k^2}{l^2}\phi(0)+L\,(\tau^*)\,(\phi)=\int_{-1}^{0}\left[d\eta\,(\theta,\tau^*)\right]\phi(\theta),\qquad \text{for } \phi\in C\left([-1,0],\mathbb{R}^2\right). \tag{3.9}$$

We choose

$$\eta(\theta,\tau^*)=\begin{cases}\tau^*\begin{pmatrix} a_{11}-d_1\frac{k^2}{l^2} & -a_{12}\\ 0 & -d_2\frac{k^2}{l^2}\end{pmatrix}, & \theta=0,\\ 0, & \theta\in(-1,0),\\ -\tau^*\begin{pmatrix}0 & 0\\ a_{21} & -a_{22}\end{pmatrix}, & \theta=-1.\end{cases} \tag{3.10}$$

For $\phi\in C^1\left([-1,0],\mathbb{R}^2\right),\psi\in C^1\left([0,1],\mathbb{R}^2\right)$, we define

$$A(\mu)\phi=\begin{cases}\dfrac{d\phi(\theta)}{d\theta},-1\le\theta<0,\\[2ex] \int_{-1}^{0}d\eta(\theta,\mu)\phi(\theta),\theta=0,\end{cases}$$

$$A^*(\mu)\psi=\begin{cases}-\dfrac{d\psi(s)}{ds},0<s\le 1,\\[2ex] \int_{-1}^{0}d\eta^T(s,\mu)\psi(-s),s=0.\end{cases}$$

Define the following bilinear pairing

$$\begin{aligned}(\psi,\phi)&=\psi(0)\phi(0)-\int_{-1}^{0}\int_{\epsilon=0}^{\theta}\psi(\epsilon-\theta)d\eta(\theta,0)\phi(\epsilon)d\epsilon\\ &=\psi(0)\phi(0)+\tau^*\int_{-1}^{0}\psi(\epsilon+1)\begin{pmatrix}0 & 0\\ a_{21} & -a_{22}\end{pmatrix}\phi(\epsilon)d\epsilon.\end{aligned} \tag{3.11}$$

Through the above analysis, $\pm i\omega_k\tau^*$ are the eigenvalues of $A\,(\tau^*)$ and A^*. Let P and P^* be the two-dimensional center spaces of $A\,(\tau^*)$ and A^* associated with $\pm i\omega_k\tau^*$, then P^* is the adjoint space of P.

Let $p_1(\theta) = (1, \xi)^T e^{i\omega_k \tau^* \theta}$ and $p_2(\theta) = \overline{p}_1(\theta)(\theta \in [-1, 0])$ be the bases of $A(\tau^*)$ and A^* corresponding to $i\omega_k\tau^*, -i\omega_k\tau^*$ respectively. By calculations, we have

$$\xi = \frac{1}{a_{12}}\left(a_{11} - i\omega_k - d_1\frac{k^2}{l^2}\right), \quad \eta = \frac{a_{12}}{i\omega_k - d_1 k^2/l^2}.$$

Let $\Phi = (\Phi_1, \Phi_2)$, and $\Psi^* = \left(\Psi_1^*, \Psi_2^*\right)^T$ with

$$\Phi_1(\theta) = \frac{p_1(\theta) + p_2(\theta)}{2} = \begin{pmatrix} \mathrm{Re}\left(e^{i\omega_k\tau^*\theta}\right) \\ \mathrm{Re}\left(\xi \mathrm{e}^{i\omega_k\tau^*\theta}\right) \end{pmatrix}, \Phi_2(\theta) = \frac{p_1(\theta) - p_2(\theta)}{2i} = \begin{pmatrix} \mathrm{Im}\left(e^{i\omega_k\tau^*\theta}\right) \\ \mathrm{Im}\left(\xi \mathrm{e}^{i\omega_k\tau^*\theta}\right) \end{pmatrix},$$

for $\theta \in [-1, 0]$, and

$$\Psi_1^*(t) = \frac{q_1(t) + q_2(t)}{2} = \begin{pmatrix} \mathrm{Re}\left(e^{-i\omega_k\tau^* t}\right) \\ \mathrm{Re}\left(\eta \mathrm{e}^{-i\omega_k\tau^* t}\right) \end{pmatrix}, \Psi_2^*(t) = \frac{q_1(t) - q_2(t)}{2i} = \begin{pmatrix} \mathrm{Im}\left(e^{-i\omega_k\tau^* t}\right) \\ \mathrm{Im}\left(\eta \mathrm{e}^{-i\omega_k\tau^* t}\right) \end{pmatrix},$$

for $t \in [0, 1]$.

Now we define $(\Psi^*, \Phi) = \begin{pmatrix} \left(\Psi_1^*, \Phi_1\right) & \left(\Psi_1^*, \Phi_2\right) \\ \left(\Psi_2^*, \Phi_1\right) & \left(\Psi_2^*, \Phi_2\right) \end{pmatrix}$, and it can be computed by (3.11), then we construct a new basis P^* by

$$\Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)^{-1}\, \Psi^*.$$

Then $(\Psi, \Phi) = I_2$. In addition, define $f_k := \left(\alpha_k^1, \alpha_k^2\right)$, where

$$\alpha_k^1 = \begin{pmatrix} \cos\frac{k}{l}x \\ 0 \end{pmatrix}, \quad \alpha_k^2 = \begin{pmatrix} 0 \\ \cos\frac{k}{l}x \end{pmatrix}.$$

Define

$$c \cdot f_k = c_1\alpha_k^1 + c_2\alpha_k^2, \text{ for } c = (c_1, c_2)^T \in C.$$

We have $P_{CN}C$ to represent the center space of (3.7), where

$$P_{CN}C(\phi) = \Phi\left(\Psi, \langle\phi, f_k\rangle\right) \cdot f_k, \quad \phi \in C. \tag{3.12}$$

And $C = P_{CN}C \oplus P_S C$, here $P_S C$ and $P_{CN}C$ are complementary in C, where

$$\langle u, v\rangle := \frac{1}{l\pi}\int_0^{l\pi} u_1\overline{v_1}dx + \frac{1}{l\pi}\int_0^{l\pi} u_2\overline{v_2}dx,$$

for $u = (u_1, u_2)^T, v = (v_1, v_2)^T, u, v \in X$ and $\langle\phi, f_k\rangle = \left(\left\langle\phi, \alpha_k^1\right\rangle, \left\langle\phi, \alpha_k^2\right\rangle\right)^T$.

Let A_{τ^*} be the infinitesimal generator generated by the solution of the linear Eq (3.7), and rewrite (3.1) as

$$\frac{dU(t)}{dt} = A_{\tau^*}U_t + R\left(U_t, \mu\right), \tag{3.13}$$

where

$$R\left(U_t, \mu\right) = \begin{cases} 0, & \theta \in [-1, 0), \\ F\left(U_t, \mu\right), & \theta = 0. \end{cases} \tag{3.14}$$

Induced by $C = P_{CN}C \oplus P_S C$, the solution can be obtained as

$$U_t = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_k + h\left(x_1, x_2, \mu\right), \tag{3.15}$$

where $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\Psi, \langle U_t, f_k \rangle)$, $h\left(x_1, x_2, \mu\right) \in P_S C$, $h(0,0,0) = 0$, $Dh(0,0,0) = 0$. The solution of (3.2) can be obtained as

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} f_k + h\left(x_1, x_2, 0\right). \tag{3.16}$$

Let $z = x_1 - ix_2$, and notice that $p_1 = \Phi_1 + i\Phi_2$, then

$$\Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_k = (\Phi_1, \Phi_2) \begin{pmatrix} \frac{z+\bar{z}}{2} \\ \frac{i(z-\bar{z})}{2} \end{pmatrix} f_k = \frac{1}{2}\left(p_1 z + \overline{p_1 z}\right) f_k, \tag{3.17}$$

and (3.16) can be transformed into

$$U_t = \frac{1}{2}\left(p_1 z + \overline{p_1 z}\right) f_k + W(z, \bar{z}), \tag{3.18}$$

where

$$W(z, \bar{z}) = h\left(\frac{z+\bar{z}}{2}, \frac{i(z-\bar{z})}{2}, 0\right) \triangleq W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{3.19}$$

From [23], z satisfies

$$\dot{z} = i\omega_k \tau^* z + g(z, \bar{z}), \tag{3.20}$$

where

$$g(z, \bar{z}) = (\Psi_1(0) - i\Psi_2(0)) \langle F\left(U_t, 0\right), f_k \rangle \triangleq g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots, \tag{3.21}$$

from Eqs (3.18) and (3.19), we have

$$\begin{aligned}
u_t(0) &= \frac{1}{2}(z + \bar{z}) \cos\left(\frac{kx}{l}\right) + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots, \\
v_t(0) &= \frac{1}{2}(\xi + \bar{\xi}\bar{z}) \cos\left(\frac{kx}{l}\right) + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots, \\
u_t(-1) &= \frac{1}{2}\left(ze^{-i\omega_k\tau*} + \bar{z}e^{i\omega_k\tau*}\right) \cos\left(\frac{kx}{l}\right) + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots, \\
v_t(-1) &= \frac{1}{2}\left(\xi ze^{-i\omega_k\tau*} + \bar{\xi}\bar{z}e^{i\omega_k\tau*}\right) \cos\left(\frac{kx}{l}\right) + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + \cdots.
\end{aligned}$$

Hence

$$\begin{aligned}
F_1\left(U_t, 0\right) &= \cos^2\left(\frac{kx}{l}\right)\left(\frac{z^2}{2}c_{11} + z\bar{z}c_{12} + \frac{\bar{z}^2}{2}\bar{c}_{11}\right) + \frac{z^2\bar{z}}{2}\left(c_{13}\cos\frac{kx}{l} + c_{14}\cos^3\frac{kx}{l}\right) + \cdots, \\
F_2\left(U_t, 0\right) &= \cos^2\left(\frac{kx}{l}\right)\left(\frac{z^2}{2}c_{21} + z\bar{z}c_{22} + \frac{\bar{z}^2}{2}\bar{c}_{21}\right) + \frac{z^2\bar{z}}{2}\left(c_{23}\cos\frac{kx}{l} + c_{24}\cos^3\frac{kx}{l}\right) + \cdots,
\end{aligned} \tag{3.22}$$

$$
\begin{aligned}
\langle F\left(U_{t},0\right), f_{k}\rangle &= \tau^{*}\left(F_{1}\left(U_{t},0\right)\alpha_{k}^{1}+F_{2}\left(U_{t},0\right)\alpha_{k}^{2}\right) \\
&= \frac{z^{2}}{2}\tau^{*}\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}\chi + z\bar{z}\tau^{*}\begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix}\chi + \frac{\bar{z}^{2}}{2}\tau^{*}\begin{pmatrix} \bar{c}_{11} \\ \bar{c}_{21} \end{pmatrix}\chi + \frac{z^{2}\bar{z}}{2}\tau^{*}\begin{pmatrix} \varsigma_{1} \\ \varsigma_{2} \end{pmatrix} + \cdots,
\end{aligned}
\tag{3.23}
$$

with

$$
\begin{aligned}
\chi &= \frac{1}{l\pi}\int_{0}^{l\pi}\cos^{3}\left(\frac{kx}{l}\right)dx, \\
\varsigma_{1} &= \frac{c_{13}}{l\pi}\int_{0}^{l\pi}\cos^{2}\left(\frac{kx}{l}\right)dx + \frac{c_{14}}{l\pi}\int_{0}^{l\pi}\cos^{4}\left(\frac{kx}{l}\right)dx, \\
\varsigma_{2} &= \frac{c_{23}}{l\pi}\int_{0}^{l\pi}\cos^{2}\left(\frac{kx}{l}\right)dx + \frac{c_{24}}{l\pi}\int_{0}^{l\pi}\cos^{4}\left(\frac{kx}{l}\right)dx, \\
c_{11} &= \frac{1}{2}\left(a_{13}+\xi a_{14}\right), \quad c_{12} = \frac{1}{4}\left(2a_{13}+(\bar{\xi}+\xi)a_{14}\right), \\
c_{13} &= W_{11}^{(1)}(0)\left(2a_{13}+\xi a_{14}\right) + W_{20}^{(1)}(0)\left(a_{13}+\frac{1}{2}\bar{\xi}a_{14}\right) + W_{11}^{(2)}(0)a_{14} + \frac{1}{2}W_{20}^{(2)}(0)a_{14}, \\
c_{14} &= \frac{1}{4}\left(3a_{15}+(\bar{\xi}+2\xi)a_{16}\right), \\
c_{21} &= \frac{1}{2}(a_{23}+a_{26}\xi)e^{-2i\omega_{k}\tau^{*}} + \frac{1}{2}(a_{24}\xi+a_{25}\xi^{2})e^{-i\omega_{k}\tau^{*}}, \\
c_{22} &= \frac{1}{2}a_{23} + \frac{1}{4}(a_{24}\xi+a_{25}\xi\bar{\xi})e^{i\omega_{k}\tau^{*}} + \frac{1}{4}(a_{24}\bar{\xi}+a_{25}\xi\bar{\xi})e^{-i\omega_{k}\tau^{*}} + \frac{1}{4}a_{26}(\xi+\bar{\xi}), \\
c_{23} &= W_{20}^{(1)}(-1)(a_{23}e^{i\omega_{k}\tau^{*}} + \frac{1}{2}a_{26}\bar{\xi} + \frac{1}{2}a_{24}\bar{\xi}) + W_{11}^{(1)}(-1)(2a_{23}e^{-i\omega_{k}\tau^{*}} + a_{24} + a_{26}\xi e^{-i\omega_{k}\tau^{*}}) \\
&\quad + \frac{1}{2}W_{20}^{(2)}(0)(a_{24}e^{i\omega_{k}\tau^{*}} + a_{25}\bar{\xi} + a_{25}\bar{\xi}e^{i\omega_{k}\tau^{*}}) + W_{11}^{(2)}(0)e^{-i\omega_{k}\tau^{*}}(a_{24}+a_{25}\xi) \\
&\quad + W_{11}^{(2)}(-1)(a_{25}\xi + a_{26}e^{-i\omega_{k}\tau^{*}}) + \frac{1}{2}a_{26}W_{20}^{(2)}(-1)e^{i\omega_{k}\tau^{*}}, \\
c_{24} &= \frac{1}{2}a_{27}(e^{-i\omega_{k}\tau^{*}} + \frac{1}{2}e^{i\omega_{k}\tau^{*}}) + \frac{1}{2}a_{28}(\xi + \frac{1}{2}\bar{\xi}e^{-2i\omega_{k}\tau^{*}}) + \frac{1}{2}a_{29}e^{-i\omega_{k}\tau^{*}}(\xi + \frac{1}{2}\bar{\xi}) \\
&\quad + \frac{1}{4}a_{30}(\xi^{2} + \bar{\xi}\xi e^{-2i\omega_{k}\tau^{*}} + \xi\bar{\xi}).
\end{aligned}
$$

Let $(\nu_{1},\nu_{2}) = \Psi_{1}(0) - i\Psi_{2}(0)$, and notice that

$$
\frac{1}{l\pi}\int_{0}^{l\pi}\cos^{3}\frac{kx}{l}dx = 0, \frac{1}{l\pi}\int_{0}^{l\pi}\cos^{4}\frac{kx}{l}dx = \frac{3}{8}, k = 1,2,3,\ldots
$$

Then, by the (3.19) and (3.21), we can obtain the following quantities:

$$
\begin{aligned}
g_{20} &= \begin{cases} 0, & k \in N, \\ \nu_{1}c_{11}\tau^{*} + \nu_{2}c_{21}\tau^{*}, & k = 0, \end{cases} \\
g_{11} &= \begin{cases} 0, & k \in N, \\ \nu_{1}c_{12}\tau^{*} + \nu_{2}c_{22}\tau^{*}, & k = 0, \end{cases} \\
g_{02} &= \begin{cases} 0, & k \in N, \\ \nu_{1}\overline{c_{11}}\tau^{*} + \nu_{2}\overline{c_{21}}\tau^{*}, & k = 0, \end{cases}
\end{aligned}
$$

$$g_{21} = \nu_1\varsigma_1\tau^* + \nu_2\varsigma_2\tau^*, \qquad k \in N_0.$$

To calculate g_{21}, we need to find $W_{20}(\theta), W_{11}(\theta)$ for $\theta \in [-1, 0]$. From (3.19) we have

$$\begin{aligned} \dot{W}(z,\bar{z}) &= W_{20}z\dot{z} + W_{11}\dot{z} + W_{11}z\dot{\bar{z}} + W_{02}\bar{z}\,\dot{\bar{z}} + \cdots, \\ A_{\tau^*}W(z,\bar{z}) &= A_{\tau^*}W_{20}\tfrac{z^2}{2} + A_{\tau^*}W_{11}z\bar{z} + A_{\tau^*}W_{02}\tfrac{\bar{z}^2}{2} + \cdots, \end{aligned} \tag{3.24}$$

and by [23], $\dot{W}(z,\bar{z})$ satisfy

$$\dot{W} = A_{\tau^*}W + H(z,\bar{z}), \tag{3.25}$$

where

$$\begin{aligned} H(z,\bar{z}) &= H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots \\ &= X_0F(U_t,0) - \Phi\left(\Psi, \langle X_0F(U_t,0), f_k\rangle \cdot f_k\right), \end{aligned} \tag{3.26}$$

and $X_0 : [-1, 0] \to B(X, X)$ is given by $X_0(\theta) = \begin{cases} 0, & -1 \le \theta < 0, \\ I, & \theta = 0. \end{cases}$

Hence, we have

$$(2i\omega_k\tau^* - A_{\tau^*})W_{20} = H_{20}, \quad -A_{\tau^*}W_{11} = H_{11}, \quad (-2i\omega_k\tau^* - A_{\tau^*})W_{02} = H_{02}, \tag{3.27}$$

that is

$$W_{20} = (2i\omega_k\tau^* - A_{\tau^*})^{-1}H_{20}, \quad W_{11} = -A_{\tau^*}^{-1}H_{11}, \quad W_{02} = (-2i\omega_k\tau^* - A_{\tau^*})^{-1}H_{02}. \tag{3.28}$$

From (3.26), we have that for $\theta \in [-1, 0]$,

$$\begin{aligned} H(z,\bar{z}) &= -\Phi(0)\Psi(0)\langle F(U_t,0), f_k\rangle \cdot f_k \\ &= -\left(\frac{p_1(\theta)+p_2(\theta)}{2}, \frac{p_1(\theta)-p_2(\theta)}{2i}\right)\begin{pmatrix} \Psi_1(0) \\ \Psi_2(0) \end{pmatrix}\langle F(U_t,0), f_k\rangle \cdot f_k \\ &= -\frac{1}{2}\left[p_1(\theta)(\Psi_1(0) - i\Psi_2(0)) + p_2(\theta)(\Psi_1(0) + i\Psi_2(0))\right]\langle F(U_t,0), f_k\rangle \cdot f_k \\ &= -\frac{1}{2}\left[\left(p_1(\theta)g_{20} + p_2(\theta)\bar{g}_{02}\right)\frac{z^2}{2} + \left(p_1(\theta)g_{11} + p_2(\theta)\bar{g}_{11}\right)z\bar{z} + \left(p_1(\theta)g_{02} + p_2(\theta)\bar{g}_{20}\right)\frac{\bar{z}^2}{2}\right] + \cdots \end{aligned} \tag{3.29}$$

Therefore, by (3.26), for $\theta \in [-1, 0]$

$$H_{20}(\theta) = \begin{cases} 0, & k \in N, \\ -\frac{1}{2}\left(p_1(\theta)g_{20} + p_2(\theta)\bar{g}_{02}\right) \cdot f_0, & k = 0, \end{cases}$$

$$H_{11}(\theta) = \begin{cases} 0, & k \in N, \\ -\frac{1}{2}\left(p_1(\theta)g_{11} + p_2(\theta)\bar{g}_{11}\right) \cdot f_0, & k = 0, \end{cases}$$

$$H_{02}(\theta) = \begin{cases} 0, & k \in N, \\ -\frac{1}{2}\left(p_1(\theta)g_{02} + p_2(\theta)\bar{g}_{20}\right) \cdot f_0, & k = 0, \end{cases}$$

and

$$H(z,\overline{z})(0) = F\left(U_t, 0\right) - \Phi\left(\Psi, \left\langle F\left(U_t, 0\right), f_k\right\rangle\right) \cdot f_k,$$

where

$$H_{20}(0) = \begin{cases} \tau^* \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} \cos^2\left(\frac{kx}{l}\right), & k \in N. \\ \tau^* \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} - \frac{1}{2}\left(p_1(0)g_{20} + p_2(0)\overline{g}_{02}\right) \cdot f_0, & k = 0, \end{cases} \tag{3.30}$$

$$H_{11}(0) = \begin{cases} \tau^* \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} \cos^2\left(\frac{kx}{l}\right), & k \in N, \\ \tau^* \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} - \frac{1}{2}\left(p_1(0)g_{11} + p_2(0)\overline{g}_{11}\right) \cdot f_0, & k = 0. \end{cases} \tag{3.31}$$

By the definition of A_{τ^*} and (3.27), we have

$$\dot{W}_{20} = A_{\tau^*} W_{20} = 2i\omega_k\tau^* W_{20} + \frac{1}{2}\left(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02}\right) \cdot f_k, \quad -1 \leq \theta < 0.$$

That is

$$W_{20}(\theta) = \frac{i}{2\omega_k\tau^*}\left(g_{20}p_1(\theta) + \frac{\overline{g}_{02}}{3}p_2(\theta)\right) \cdot f_k + E_1 e^{2i\omega_k\tau^*\theta},$$

where

$$E_1 = \begin{cases} W_{20}(0), & k \in N, \\ W_{20}(0) - \frac{i}{2\omega_k\tau^*}\left(g_{20}p_1(\theta) + \frac{\overline{g}_{02}}{3}p_2(\theta)\right) \cdot f_0, & k = 0. \end{cases} \tag{3.32}$$

Using the definition of A_{τ^*} and (3.27), we have that for $-1 \leq \theta < 0$,

$$\begin{aligned} & 2i\omega_k\tau^*\left[\frac{ig_{20}}{2\omega_k\tau^*}p_1(0) \cdot f_0 + \frac{i\overline{g}_{02}}{6\omega_k\tau^*}p_2(0) \cdot f_0 + E\right] \\ & -\tau^* D\Delta\left[\frac{ig_{20}}{2\omega_k\tau^*}p_1(0) \cdot f_0 + \frac{i\overline{g}_{02}}{6\omega_k\tau^*}p_2(0) \cdot f_0 + E\right] \\ & -L(\tau^*)\left[\frac{ig_{20}}{2\omega_k\tau^*}p_1(\theta) \cdot f_0 + \frac{i\overline{g}_{02}}{6\omega_k\tau^*}p_2(\theta) \cdot f_0 + Ee^{2i\omega_k\tau^*\theta}\right] \\ & = \tau^*\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} - \frac{1}{2}\left(p_1(0)g_{20} + p_2(0)\overline{g}_{02}\right) \cdot f_0. \end{aligned} \tag{3.33}$$

Notice that

$$\begin{cases} \tau^* D\Delta\left[p_1(0) \cdot f_0\right] + L(\tau^*)\left[p_1(\theta) \cdot f_0\right] = i\omega_0\tau^* p_1(0) \cdot f_0, \\ \tau^* D\Delta\left[p_2(0) \cdot f_0\right] + L(\tau^*)\left[p_2(\theta) \cdot f_0\right] = -i\omega_0\tau^* p_2(0) \cdot f_0. \end{cases}$$

We have

$$2i\omega_k\tau^* E_1 - \tau^* D\Delta E_1 - L(\tau^*)\left(E_1 e^{2i\omega_k\tau^*\theta}\right) = \tau^*\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}\cos^2\left(\frac{kx}{l}\right), \quad k \in N_0.$$

Therefore

$$E_1 = \tau^*\begin{pmatrix} 2i\omega_k\tau^* + d_1\frac{k^2}{l^2} - a_{11} & a_{12} \\ -a_{21}e^{-2i\omega_k\tau^*} & 2i\omega_k\tau^* + d_2\frac{k^2}{l^2} + a_{22}e^{-2i\omega_k\tau^*} \end{pmatrix}^{-1}\begin{pmatrix} c_{11} \\ c_{12} \end{pmatrix}\cos^2\left(\frac{kx}{l}\right).$$

Similarly, we have

$$W_{11}(\theta) = \frac{i}{2\omega_k\tau^*}\left(p_1(\theta)\overline{g}_{11} - p_1(\theta)g_{11}\right) + E_2.$$

Calculate W_{20} using the same method, we have

$$E_2 = \tau^*\begin{pmatrix} d_1\frac{k^2}{l^2} - a_{11} & a_{12} \\ -a_{21} & d_2\frac{k^2}{l^2} + a_{22} \end{pmatrix}^{-1}\begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix}\cos^2\left(\frac{kx}{l}\right).$$

Thus, we can evaluate the following values:

$$c_1(0) = \frac{i}{2\omega_k\tau^*}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{\left|g_{02}\right|^2}{3}\right) + \frac{1}{2}g_{21},$$

$$\mu_2 = -\frac{\mathrm{Re}\left(c_1(0)\right)}{\mathrm{Re}\left(\lambda'\left(\tau^*\right)\right)}, \quad \beta_2 = 2\,\mathrm{Re}\left(c_1(0)\right),$$

$$T_2 = -\frac{1}{\omega_{\mathrm{k}}\tau^*}\left[\mathrm{Im}\left(c_1(0)\right) + \mu_2\,\mathrm{lm}\left(\lambda'\left(\tau^*\right)\right)\right].$$

Theorem 3. *For any critical value* τ_k^j*, we have the Hopf bifurcation is forward* ($\mu_2 > 0$) *or backward* ($\mu_2 < 0$)*. The bifurcating periodic solutions are orbitally asymptotically stable* ($\beta_2 < 0$) *or unstable* ($\beta_2 > 0$)*. The period increases* ($T_2 > 0$) *or decreases* ($T_2 < 0$)*.*

4. Hybrid control of bifurcation

In this section, we design a hybrid controller to control Hopf bifurcations and expand the stability range of equilibrium points. The system (1.2) with hybrid controller control can be described as follows:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = d_1\Delta u(t,x) + u(x,t)[r_1 - b_1u(x,t) - \frac{a_1v(x,t)}{m^2+nu(x,t)+u^2(x,t)}], \\ \frac{\partial}{\partial t}v(x,t) = d_2\Delta v(t,x) + Kv(x,t)[r_2 - \frac{a_2v(x,t-\tau)}{u(x,t-\tau)}] + (1-K)(v(x,t-\tau) - v_*), \\ x \in (0,\Omega), t > 0, \end{cases} \tag{4.1}$$

where the parameter K is treated as a feedback parameter, and v_* represents the value of $v(t,x)$ at the equilibrium point.

Linearize the system affected by the hybrid controller at the equilibrium point, and obtain the linearized controlled model as follows:

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = D\Delta\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + L_1\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + L_3\begin{pmatrix} u(t-\tau) \\ v(t-\tau) \end{pmatrix}, \tag{4.2}$$

where

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, L_1 = \begin{pmatrix} a_{11} & -a_{12} \\ 0 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & 0 \\ b_{21} & b_{22} \end{pmatrix},$$

and

$$a_{11} = r_1 - 2b_1u_* - \frac{a_1v_*m^2 - a_1u_*^2v_*}{(m^2+nu_*+u_*^2)^2}, \quad a_{12} = \frac{a_1u_*}{m^2+nu_*+u_*^2},$$
$$b_{21} = \frac{Ka_2v_*^2}{u_*^2}, \quad b_{22} = \frac{-Ka_2v_*}{u_*} + (1-K).$$

The characteristic equation of (4.2) is

$$det(\lambda I + Dk^2 - L_1 - L_3e^{-\lambda\tau}) = 0, \tag{4.3}$$

then we have

$$\lambda^2 + \lambda(\Lambda - b_{22}e^{-\lambda\tau}) + \Theta + \Upsilon e^{-\lambda\tau} = 0, \tag{4.4}$$

where

$$\begin{aligned}\Lambda(k^2) &= (d_1+d_2)k^2 - a_{11},\\ \Theta(k^2) &= k^2(d_1d_2k^2 - a_{11}d_2),\\ \Upsilon(k^2) &= -d_1b_{22}k^2 + a_{11}b_{22} + a_{12}b_{21}.\end{aligned}$$

When $\tau > 0$, let $i\widetilde{\omega}(\widetilde{\omega} > 0)$ be a solution of Eq (4.4), then decompose into the real and imaginary parts, we have

$$\begin{cases} \Upsilon cos(\widetilde{\omega}\tau) - \widetilde{\omega}b_{22}sin(\widetilde{\omega}\tau) = \widetilde{\omega}^2 - \Theta,\\ \widetilde{\omega}b_{22}cos(\widetilde{\omega}\tau) - \Upsilon sin(\widetilde{\omega}\tau) = \Lambda\widetilde{\omega}, \end{cases} \tag{4.5}$$

which lead to

$$\widetilde{\omega}^4 + \widetilde{\omega}^2(\Lambda^2 - 2\Theta - b_{22}^2) + \Theta^2 - \Upsilon^2 = 0, \tag{4.6}$$

where

$$P_k = 2\Theta + b_{22}^2 - \Lambda^2 = -(d_1^2+d_2^2)k^4 + 2a_{11}d_1k^2 + b_{22}^2 - a_{11}^2,$$
$$Q_k = \Theta^2 - \Upsilon^2 = (d_1d_2k^4 - a_{11}b_{22}k^2)^2 - (a_{11}d_{22} + a_{12}d_{21} - d_2b_{22}k^2)^2.$$

We propose the following hypotheses:

(*H*3) $(d_1d_2 - a_{11}b_{22})^2 - (a_{11}b_{22} + a_{12}d_{21} - d_2b_{22})^2 > 0;$
(*H*4) $a_{11}d_1 \le d_1^2 + d_2^2,$ *and* $P_1 < 0;$ *or*
(*H*5) $a_{11}d_1 > d_1^2 + d_2^2,$ *and* $a_{11}^2d_1^2 < (d_1^2+d_2^2)(a_{11}^2 - b_{22}^2).$

Suppose **(*H*3)** holds, $Q_k > 0, \forall k \ge 1$; Suppose **(*H*4)** or **(*H*5)** holds, $P_k < 0, \forall k \ge 1$. These imply that there are no purely imaginary roots in Eq (4.6) for $k \ge 1$. On the other hand, when $k = 0$, we have $Q_0 < 0$, then the Eq (4.6) has one positive root :

$$\omega_0 = [\frac{1}{2}(b_{22}^2 - a_{11}^2 + \sqrt{(b_{22}^2 - a_{11}^2)^2 + 4(a_{11}b_{22} + a_{12}b_{21})^2})]^{\frac{1}{2}}.$$

On account of (4.5), it is obvious that

$$cos\omega_0\tau = \frac{\omega_0^2 a_{12}b_{21}}{a_{11}b_{22} + a_{12}b_{21}{}^2 + \omega_0^2 b_{22}^2}.$$

Therefore

$$\tau_0^j = \frac{1}{\omega_0} arccos\frac{\omega_0^2 a_{12}b_{21}}{a_{11}b_{22} + a_{12}b_{21}{}^2 + \omega_0^2 b_{22}^2} + \frac{2j\pi}{w_0}, j \in N_0. \tag{4.7}$$

Denote $\tau_0^0 = \min_{j\in N_0}\{\tau_0^j\}$.

Let $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ be the root of (4.4) near $\tau = \tau_0^j, j = 0, 1, 2, \cdots$, satisfying $\alpha(\tau_0^j) = 0$ for $\omega(\tau_0^j) = \omega_0$. Take the derivative of both sides of (4.4) with respect to τ, and we have

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda - (b_{22} + \Upsilon\tau - b_{22}\tau\lambda)e^{-\lambda\tau}}{(\Upsilon - b_{22}\lambda)\lambda e^{-\lambda\tau}}.$$

Then

$$Re(\frac{d\lambda}{d\tau})^{-1}_{\tau=\tau_0^j} = \frac{\Lambda\Upsilon\omega_0 - 2b_{22}\omega_0^3 sin\omega_0\tau_0^j + (\Lambda b_{22}\omega_0^2 + 2\Upsilon\omega_0^2)cos\omega_0\tau_0^j - b_{22}\omega_0^2}{\Upsilon^2\omega_0^2 + \omega_0^4 b_{22}^2}.$$

If $Re(\frac{d\lambda}{d\tau})^{-1}_{\tau=\tau_0^j} \neq 0$, the traversal condition will hold. Based on the above, we derive the main result.

Theorem 4. *Assume that* **(*H*1)(*H*3)(*H*4)** *or* **(*H*1)(*H*3)(*H*5)** *hold. There are the following conclusions.*

(1) The coexisting equilibrium $E_* = (u_*, v_*)$ is locally asymptotically stable for $\tau \in [0, \tau_0^0)$.
(2) The coexisting equilibrium $E_* = (u_*, v_*)$ is unstable for $\tau \in (\tau_0^0, +\infty)$. And system (4.1) undergoes a Hopf bifurcation at the equilibrium $E_* = (u_*, v_*)$ when $\tau = \tau_0^j, j = 0, 1, 2, \cdots$.

Remark 2. *The above analysis demonstrates that by adjusting the feedback control gains, it is possible to change the values of the Hopf bifurcation without altering the original equilibrium point. This allows the system to transition from its original unstable state back to a stable state, effectively expanding the stability region and maintaining the predator-prey dynamic equilibrium (see Section 5).*

5. Numerical simulations

Example 1. *Consider system (1.2) with the following parameters: $d_1 = 0, d_2 = 0, r_1 = 0.35, r_2 = 0.22, b_1 = 0.27, a_1 = 0.57, a_2 = 0.11, m = 0.36, n = 0.30$, and $\tau = 0$. Hypothesis* **(*H*1)** *is satisfied, then $(u_*, v_*) = (0.043, 0.086)$ is locally asymptotically stable (Figure 1). We take $d_1 = 5, d_2 = 12$, by Theorem 1, $(u_*, v_*) = (0.043, 0.086)$ is still locally asymptotically stable (Figure 2). We choose $d_1 = 0.01, d_2 = 5$, and we have that $(u_*, v_*) = (0.043, 0.086)$ is Turing unstable, and Turing patterns appear (Figures 3 and 4).*

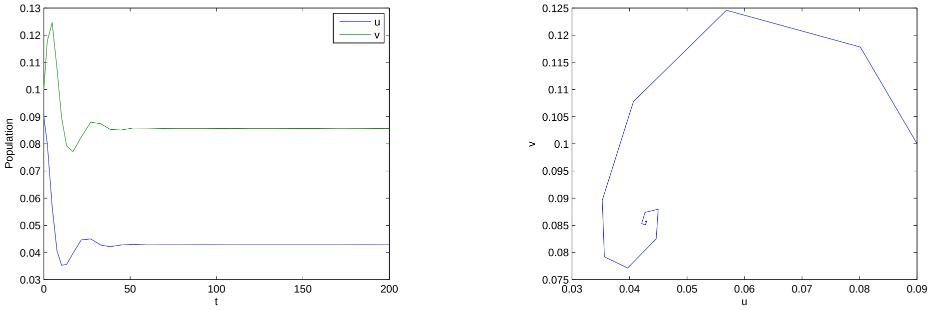


Figure 1. The numerical results are acquired with $\tau = 0, d_1 = d_2 = 0$.

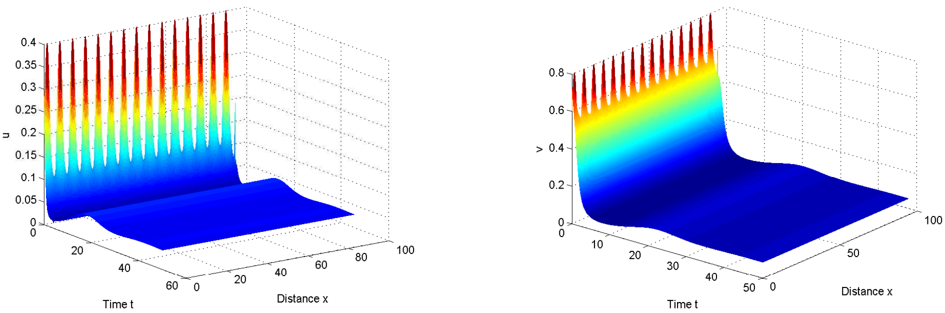


Figure 2. Behaviors of appearance for Turing stable conditions with $\tau = 0, d_1 = 5, d_2 = 12$.

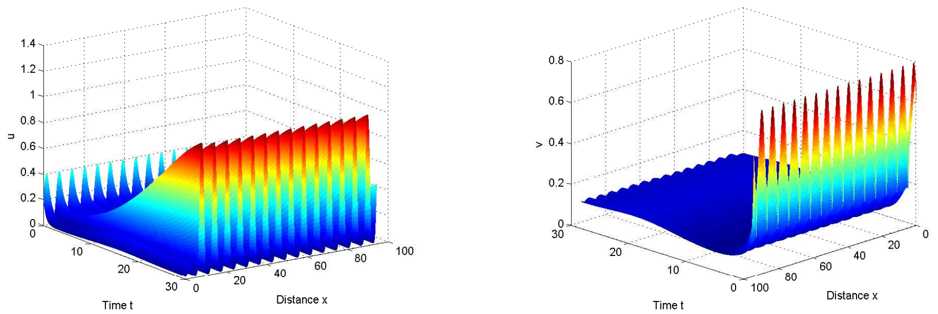
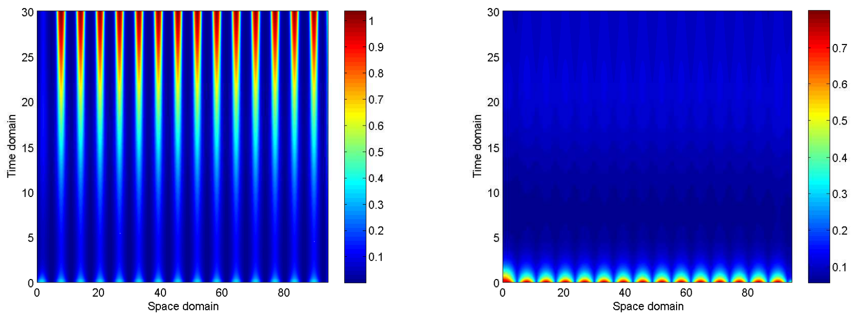


Figure 3. Behaviors of appearance for Turing unstable conditions with $\tau = 0, d_1 = 0.01, d_2 = 5$.



(a) (b)

Figure 4. Pattern appearance of **(a)** prey and **(b)** predator for Turing unstable conditions with $\tau = 0, d_1 = 0.01, d_2 = 5$.

Example 2. *Consider system* (1.2) *with the following parameters:* $d_1 = 0.49$, $d_2 = 0.8$, $r_1 = 0.97$, $r_2 = 0.96, b_1 = 0.14, a_1 = 0.42, a_2 = 0.92, m = 0.79$ *and* $n = 0.96$ *. The system* (1.2) *has a unique coexisting equilibrium* $(u_*, v_*) = (6.5, 6.5)$ *, we compute that the critical value is* $\widetilde{\tau} \approx 1.5551$ *. By Theorem 2, system* (1.2) *is locally asymptotically stable for* $\tau = 1.4 \in [0, \widetilde{\tau}]$ *(Figure 5). As* $\tau = 1.8 > \widetilde{\tau}$ *, the system (1.2) undergoes oscillations (Figure 6). In addition, we calculate* $Re(C_1(0)) = -12.687$*, then we have* $\mu_2 > 0, \beta_2 < 0$*.*

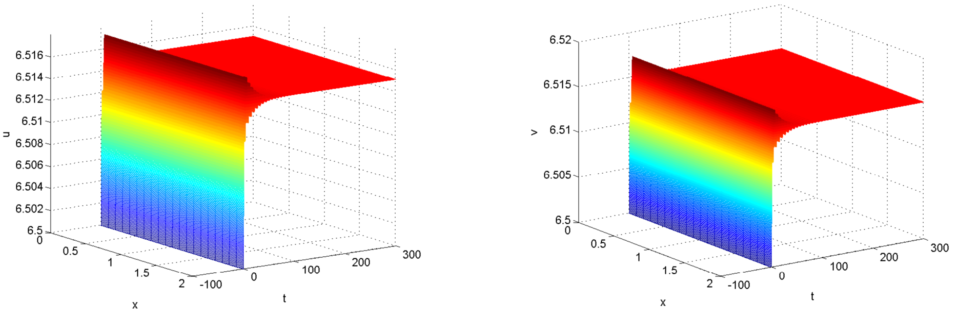


Figure 5. The coexisting equilibrium E_* is locally stable where $\tau = 1.4 < \widetilde{\tau}$.

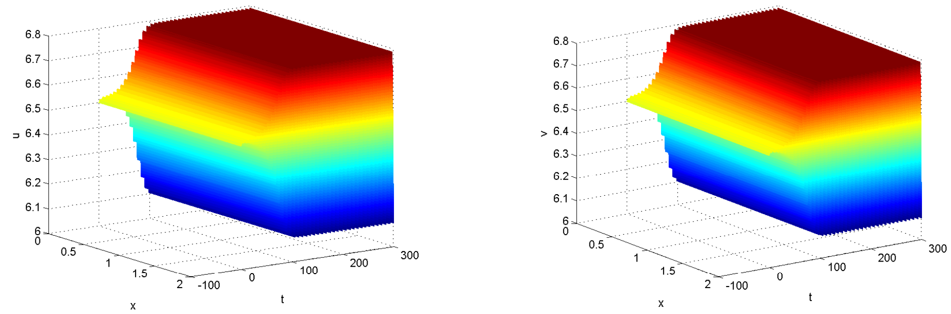
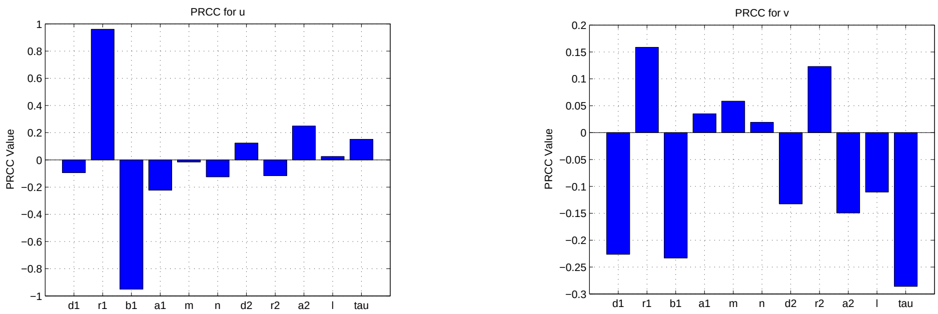


Figure 6. The periodic solutions bifurcating from the coexisting equilibrium E_* where $\tau = 1.8 > \widetilde{\tau}$.

In the process of selecting parameter values for numerical simulations of the model, there exists a certain degree of uncertainty in parameter selection. In order to identify parameters that significantly impact the densities of prey and predator populations, those parameters that have a substantial effect on model outputs, precise values should be assigned, while parameters with minor impacts on model outputs can be assigned rough estimates [43, 44]. This allocation of values is crucial for assessing the sensitivity of the model relative to parameter manipulations. We employed Latin hypercube sampling/partial rank correlation coefficient (LHS/PRCC) sensitivity analysis [45] to explore the entire parameter space of the model. In this study, parameter values were obtained from Example 2 with a reference deviation of ±25%, and a uniform spread was assigned to each model parameter. Each LHS run consisted of 200 simulations, and the sampling was conducted autonomously. The PRCC values range between -1 and 1, where positive and negative PRCC values respectively reveal the positive or negative correlation between model parameters and model outputs, while the magnitude indicates the strength of the linear relationship.

From Figure 7, it can be observed that the growth rate of the prey, food conversion rate, and digestion delay have a positive impact on the prey population density. That is, an increase in these parameters will lead to an increase in the output of the prey population density. On the other hand, the internal competition rate and predation rate have a negative impact on the prey population density. A decrease in these parameters will result in a decrease in the output of the prey population density. Additionally, we find that the intrinsic growth rate of both the prey and predator has a significant positive effect on the predator population density. The competition rate within the prey population, digestion delay, and the diffusion coefficient of the prey have a significant negative impact on the predator population density, and the predator population is more sensitive to theses parameters.



(a) (b)

Figure 7. Impact of uncertainty of system (1.2) on **(a)** prey population and **(b)** predator population.

Example 3. *We consider the influence of the hybrid bifurcation control strategy of system (4.1). We choose* $K = 0.83$*, other parameters remain the same as in Example 2; the bifurcation critical point of the controlled system (4.1) is* $\tau_0^0 \approx 2.3$*. By Theorem 4 that when* $\tau = 1.8 < \tau_0^0$*, the controlled system (4.1) returns to locally stability at the equilibrium point (Figure 8), when* $\tau = 2.5 > \tau_0^0$*, the system (4.1) undergoes oscillations (Figure 9). Therefore, adjusting the controller coefficient can effectively expand the stable region and change the position of the bifurcation point. It is evident that reducing the feedback gain leads to a faster convergence of the system towards a stable state; in other words, a smaller feedback gain results in better control of the controller's impact on Hopf bifurcation (Figure 10).*

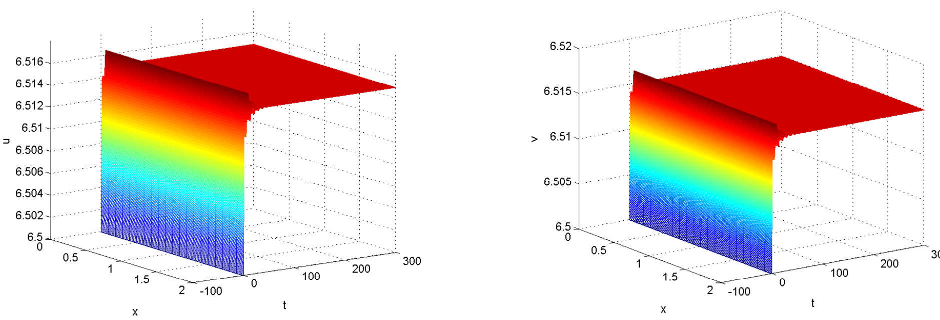


Figure 8. Waveform plots of the controlled system (4.1) with $\tau = 1.8 < \tau_0^0 = 2.3$; the feedback gain is $K = 0.83$. The controlled model (4.1) returns to stability at equilibrium point E_* .

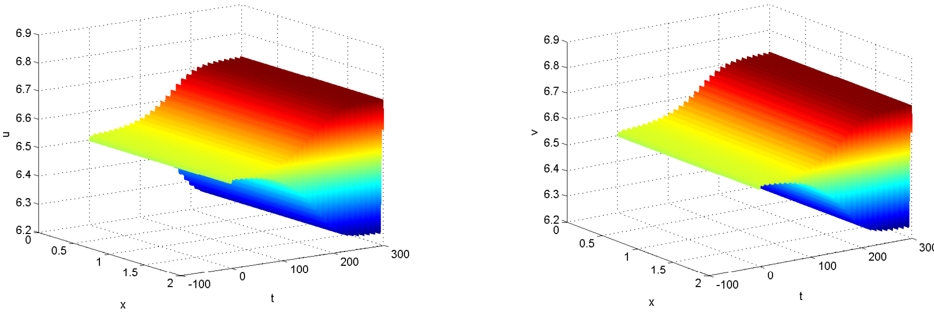


Figure 9. Waveform plots of controlled system (4.1) with $\tau = 2.5 > \tau_0^0 = 2.3$; the feedback gain is $K = 0.83$.

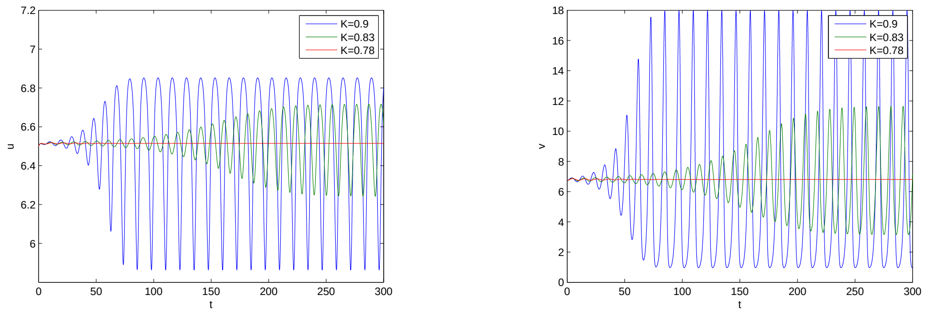


Figure 10. Waveform plots of controlled system (4.1), the feedback gain is $K = 0.9, K = 0.83, K = 0.78$ for $\tau = 2.5$. The control effect increases as the feedback gain decreases.

6. Conclusions

This paper delves into a delayed diffusive semi-ratio-dependent predator-prey model. We first analyzed how diffusion can lead to Turing instability for the system without time delay. Second, we considered the time delay τ as a parameter for bifurcation and provided conditions for the occurrence of Hopf bifurcation. Our results indicated that time delay can induce complex dynamical phenomena; the model can bifurcate from the normal equilibrium solution to spatially homogeneous and inhomogeneous periodic solutions. The time delay effect causes the system to transition from a stable state to periodic oscillations, reflecting a dynamic imbalance between predator and prey populations. Over time, the populations of predators and prey alternate between increase and decrease, forming a periodic fluctuation pattern. Furthermore, through calculations, it was determined that the Hopf bifurcation is forward and the system possesses stable branch periodic solutions. This implies that the interaction between prey and predators is regulated through stable periodic oscillations, maintaining ecological balance and species diversity. Third, a hybrid controller has been incorporated into the system (4.1) to optimize the dynamic characteristics of the predator-prey model. With the addition of a hybrid controller and reasonable parameter adjustment, the previously oscillating waveforms regain stability. This signifies that the introduction of the controller has effectively controlled the range of model stability. Loading the controller onto the predator and adjusting its control parameters can achieve objectives such as population control, population dynamics regulation, and ecosystem management, playing a significant role in the fields of biology and ecology. Furthermore, through PRCC analysis, we have obtained the sensitivity relationships between the two population densities and the parameters. It is evident that the inclusion of diffusion and time delay has a more significant impact on the predator population density. Finally, numerical examples are introduced to validate the theoretical results.

The bifurcation studied in this paper is limited to one-dimensional space. To be more realistic, future research will continue to explore higher codimension branching problems, such as Turing-Hopf bifurcations of codimension two or even three, and their more complex dynamical phenomena. We also will explore the analysis and comparison of alternative control strategies and their corresponding simulation scenarios, thereby enriching the development of the current stage of research.

Author contributions

Hairong Li: Conceptualization, Methodology, Software, Validation, Writing-original and editing; Yanling Tian: Conceptualization, Methodology, Validation, Writing-review, Supervision; Ting Huang: Software; Pinghua Yang: Supervision, Resources. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interest.

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