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Research article

The invariance of the peak point(s) in a non-symmetrical graph via CETD matrix under varying α-levels

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Abstract: Events or attributes occur at different ages or times but, in some circumstances, for effective planning and policy formulation, the peak point, where the events or attributes has its peak value, is of interest. Usually, the graphs depicting peak values are not symmetrical. In determining the peak point(s) of events that occur over time, a set of α_s of α-levels, chosen from an antisymmetric interval $(0, 1]$, was used on an ATD matrix. This was done to obtain an RTD matrix which was then aggregated to obtain a CETD matrix. Most authors chose α without any condition. The problem associated with this was that two different sets of α may not necessarily produce the same peak point for the same data set. In this study, the condition to guarantee that the row which had the highest sum (the peak value) in a CETD matrix was invariant, regardless of the set of α-levels, was established. To establish the authenticity of this method, there were experiments conducted and numerical examples were given in this paper.

Keywords: peak value; average time dependent matrix; revised time dependent matrix; combined effect time dependent matrix; fuzzy matrix
Mathematics Subject Classification: 15B15, 90C70, 28E10

1. Introduction

The first introduction of the word matrix into the mathematics literature was by James Joseph Sylvester, but Cayley was the first author of an expository article on the subject [1]. Since then, a lot has been done on the study and use of matrices. It is also important to note that the notion of the fuzzy set was introduced by Zadeh [2]. Thomason [3] was the first to introduce fuzziness into matrix theory. Since then, there have been diverse uses of fuzzy matrices (Ref. [3–7]). However, in this study, average time-dependent (ATD), revised time-dependent (RTD), and combined effect time-dependent (CETD) matrices will be used to determine the peak age at which some attributes occur, and it can even be extended to analyzing transportation problems such as the peak time (peak point) of traffic and some other problems where peak is of interest. While we provide some basic information, readers can please refer to [8–10] for more information on ATD, RTD, and CETD matrices.

When an attribute (of multi-dimension) is observed over a period of time and the various frequencies of occurrences of each dimension are recorded, these observations are grouped into intervals of times according to the researcher's interest. The matrix (or table) obtained by this procedure is called the raw data (RD) matrix. The RD matrix is then transformed into an ATD matrix upon dividing each entry in its row by the corresponding number of years in the time interval. An ATD matrix is a special matrix which represents data that is uniform. In each column of the ATD matrix, the respective mean and standard deviation of the data is found and is used to transform the ATD matrix into special fuzzy matrices (RTD matrices) at different α-levels, using the formula

$$e_{ij} = \begin{cases} -1, if\ a_{ij} \leq \mu_j - \alpha\delta_j, \\ 0, if\ a_{ij} \in (\mu_j - \alpha\delta_j, \mu_j + \alpha\delta_j), \\ 1, if\ a_{ij} \geq \mu_j + \alpha\delta_j, \end{cases} \tag{1.1}$$

where μ_j is the mean of the entries in j-th column of the ATD matrix and δ_j is their standard deviation. The aggregation of RTD matrices at different α-levels produces a CETD matrix.

The CETD matrix is used, most times, to determine the peak value of the occurrence of events or attributes over time. Many researchers have used these fuzzy matrices in the field of health, sociology, and transportation, to mention just a few. In particular, [11–13] used the CETD matrix approach to analyze the personality of individuals. Also, it was used in [14] to determine the maximum age group affected by cardiovascular disease in some men. In [15], the CETD matrix was used to determine the maximum age at which the problem of housemaid occurs. It has been used in [16] to study the peak age when issues of divorce come up, and it was used in [17] to study the peak age when self-actualization occurs. The CETD matrix has also been used to study traffic flow [18]. The CETD matrix is used in a number of ways to analyze health related issues [19–22] and it was also used to analyze the effect of computer use on the vision of women [23].

Since it is not certain for one α-level (just one RTD matrix) to give a precise result, all the authors (Ref. [14–23]) choose three to four α-levels, from an antisymmetric interval (0, 1], and then aggregate the results. This means three or more values of α leads to three or more RTD matrices. The sum of all the RTD matrices form the CETD matrix. The highest row sum of the CETD matrix actually gives the required peak value. As noted above, the determination of this peak value is done with the choice of some values of α without any explanation on the criteria for the selection. No researcher, to the best of our knowledge, has ever shown if choosing and applying another set of values for α will guarantee the same peak point. It was opined that the values of the entries in the RTD matrix is determined in some undisclosed special way [10]. Most authors choose α without any condition. The problem associated with this is that two different sets of α may not necessarily produce the same peak point for the same dataset. Our research indeed shows that some conditions should be applied in the choice of α levels to preserve the point of the CETD matrix possessing the peak value. So, we have established the condition which guarantees that the peak point (the class at which peak value occurs for an attribute) obtained in a CETD matrix is invariant.

The worry that one could have whether the row with the peak point for a set of chosen values for α will be different for another set is eliminated as this paper established the condition to guarantee the invariance of peak point(s).

2. Useful definitions

This section gives basic definitions and/or information needed in the sequel.

Definition 2.1. ([24]) Partition $P = \{[x_0, x_1], [x_1, x_2], \cdots, [x_{i-1}, x_i], \cdots, [x_{n-1}, x_n]\}$ of the interval $[a, b]$ is a finite-ordered set of points in $\boldsymbol{R}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_{n-1} < x_n = b$.

Definition 2.2. ([24]) The *length* of the subinterval $I_i = [x_{i-1}, x_i]$ of a partition

$$P = \{[x_0, x_1], [x_1, x_2], \cdots, [x_{i-1}, x_i], \cdots, [x_{n-1}, x_n]\}$$

of the interval $[a, b]$ is $\delta x_i = x_i - x_{i-1}$.

Definition 2.3. A partition is said to be *uniform* if all its subintervals are of the same length, that is,

$$x_i - x_{i-1} = \alpha \in \boldsymbol{R}, \quad \forall i \in [1, n],$$

where α is a real constant.

Remark 2.4. $P = \{[x_{i-1}, x_i]\}$ is a collection of subintervals $I_i = [x_{i-1}, x_i]$ $(i = 1, 2, 3, \cdots, n)$ of $[a, b]$ which are not disjoint but a partition is supposed to be a disjoint or nonoverlapping collection. Hence, we will give another definition of partition of an interval $[a, b]$ in Section 3.

3. Results

Definition 3.1. The set $P = \{[x_0, x_1], (x_1, x_2], (x_2, x_3], \cdots, (x_{n-1}, x_n]\}$, which is a collection of finite-ordered sets of points in $\boldsymbol{R}$ (uniform subintervals of the interval $I = [a, b]$) such that $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$, is a partition of I.

Remark 3.2. Hence, we can look at $\{(x_{i-1}, x_i]\}_{i=1}^{n}$ as a uniform partition of the interval $I = (x_0, x_n]$. Also, note that the use of uniform partition ensures that αs are sufficiently far from each other so that the multiplying effects are significant and uniform on the data collected. Otherwise, it will produce a biased effect as will be seen in an example given later.

Remark 3.3. Let $I_s = (a_s, b_s] \subset (0, 1]$ be a cell in the uniform partition $\{(x_{i-1}, x_i]\}_{i=1}^{n}$ of $(0, 1]$, where $1 \leq s \leq n$. Then,

(i) $\bigcup I_s = (0, 1]$,
(ii) $I_t \cap I_l = \emptyset$, if $t \neq l$, and
(iii) $|b_s - a_s|$ =constant for all s.

Example 3.4. (i) The set $\{(0, 0.25], (0.25, 0.50], (0.50, 0.75], (0.75, 1]\}$ is a uniform partition of $(0, 1]$.

(ii) The set

$$\{(0, 0.125], (0.125, 0.25], (0.25, 0.375], (0.375, 0.50],$$
$$(0.50, 0.625], (0.625, 0.75], (0.75, 0.875], (0.875, 1]\}$$

of subsets of $(0, 1]$ is a uniform partition of it.

As noted above, all researchers who use the CETD matrix to find a peak point make choices of α in the interval $(0, 1]$ arbitrarily. We have found out that if a set of chosen α are interchanged with another set, without any condition, the peak point in the CETD matrix is not always necessarily the same as can be seen later in our example. We need to establish the condition on the choice of α which will make the peak point invariant.

Remark 3.5. Besides, as shall be required to know later in a part of this work, in a row of a matrix, if we have $[1\ \ -1\ \ \ 1\ \ \ 0\ \ -1\ \ -1]$, the total number of 0 and/or -1 is 4 and the total number of 0 and/or 1 is 3. In this case, most entries are 0 and/or -1. On the other hand, if in a row we have $[1\ \ \ 1\ \ \ 1\ \ \ 0\ \ -1\ \ -1]$, the total number of 0 and/or -1 is 3 and the total number of 0 and/or 1 is 4, in which case most entries are 0 and/or 1.

Theorem 3.6. *Given an ATD matrix, if $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n$ are applied in the increasing order to generate the RTD matrices A^{α_s} such that $\alpha_s \in (a_s, b_s]$ (for $1 \leq s \leq n$) and $\{(x_{i-1}, x_i]\}_{i=1}^{n}$ is a uniform partition of $(0, 1]$, then some e_{ij} which were -1 and 1 become 0 and the total number of 0 and/or 1 in the row with highest sum in the newer RTD matrix is more than the total number of -1 and/or 0.*

Proof. For $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n$, $(\mu_j - \alpha_1\delta_j, \mu_j + \alpha_1\delta_j) \subset (\mu_j - \alpha_2\delta_j, \mu_j + \alpha_2\delta_j) \subset (\mu_j - \alpha_3\delta_j, \mu_j + \alpha_3\delta_j) \subset \cdots \subset (\mu_j - \alpha_n\delta_j, \mu_j + \alpha_n\delta_j)$. This implies that, for $a_{ij} \in (\mu_j - \alpha_1\delta_j, \mu_j + \alpha_1\delta_j)$, $e_{ij} = 0$ for α_1 RTD matrix and also for other RTD matrices obtained by other α_s since $a_{ij} \in (\mu_j - \alpha_s\delta_j, \mu_j + \alpha_s\delta_j)$. Furthermore, for some $a_{ij} \notin (\mu_j - \alpha_1\delta_j, \mu_j + \alpha_1\delta_j)$, we now have such $a_{ij} \in (\mu_j - \alpha_2\delta_j, \mu_j + \alpha_2\delta_j) \subset (\mu_j - \alpha_3\delta_j, \mu_j + \alpha_3\delta_j) \subset \cdots \subset (\mu_j - \alpha_n\delta_j, \mu_j + \alpha_n\delta_j)$ and their corresponding e_{ij} which were, respectively, 1 and -1 in RTD matrix A^{α_1} becoming 0 in RTD matrix $A^{\alpha_2}, A^{\alpha_3}, A^{\alpha_4}, \cdots, A^{\alpha_n}$, so that the total number of 0 and 1 in RTD matrix A^{α_t}, for $2 \leq t \leq n$, have now increased and the total number of 0 and -1 have rather decreased for α_t. □

Remark 3.7. As the practice has commonly been with many authors, choosing most α from a cell of the partition is not appropriate to guarantee the invariance of peak point. For an instance, author choosing $\alpha = 0.06, 0.1, 0.25, 0.75$ in a partition $\{(0, 0.25], (0.25, 0.5], (0.5, 0.75], (0.75, 1]\}$ is not appropriate.

Theorem 3.8. *The shape of the graph (not usually symmetrical) and the peak value of the CETD matrix is NOT INVARIANT if most α_s are chosen from a cell $(a_s, b_s]$ of the uniform partition $\{(x_{i-1}, xi]\}_{i=0}^{n}$ of $(0, 1]$.*

Proof. Let $\{(a_s, b_s]\}$ be a partition of $(0, 1]$ and $\{\alpha_s\}_{s=1}^{n}$ be the α-levels chosen to generate the RTD matrices $A^{\alpha_s} = (e_{ij}^{s})$, where $1 \leq i \leq m$ and $1 \leq j \leq w$. Assume that most α_s are chosen from a particular cell $(a_s, b_s]$. Let

$$B = \sum_{s=1}^{n} A^{\alpha_s} = (\sum_{s=1}^{n} e_{ij}^{s}) = (b_{ij}) \tag{3.1}$$

be the CETD matrix resulting from addition of A^{α_s}. Let k-th row have the highest sum P_B in B. Then, in each A^{α_s} matrix, it is such that

$$\sum_{j=1}^{w} e_{kj}^{s} \geq \sum_{j=1}^{w} e_{ij}^{s} \Rightarrow b_{kj} \geq b_{ij} \Rightarrow \sum_{j=1}^{w} b_{kj} = P_B \geq \sum_{j=1}^{w} b_{ij}, i \neq k. \tag{3.2}$$

Hence, either all or most $e_{kj}^{s} \in A^{\alpha_s}$ are such that $e_{kj}^{s} \geq 0$ and there are fewer $e_{kj}^{s} < 0$. In this case, in the ATD matrix, most a_{kj} are such that $a_{kj} > \mu - \alpha_s \delta$ and fewer $a_{kj} \leq \mu - \alpha_s \delta$. However, in the other rows when $i \neq k$, most $e_{ij}^{s} \in A^{\alpha_s}$ are such that $e_{ij}^{s} = -1$ and $e_{kj} \geq 0$ are fewer; hence, in the ATD matrix, most a_{ij} are such that $a_{ij} < \mu - \alpha_s \delta$ and fewer $a_{ij} \geq \mu - \alpha_s \delta$.

Suppose further that $\{\alpha_t\}_{t=1}^{n}$ are the α-levels chosen to generate the RTD matrices $A^{\alpha_t} = (e_{ij}^{t})$, where $1 \leq i \leq m$ and $1 \leq j \leq w$. Also, suppose that most of the α_t are chosen from a particular $(a_t, b_t]$ such that (without loss of generality) $a_t < b_t < a_s < b_s$, in which case $\alpha_t < \alpha_s$. Then,

$$\mu - \alpha_s \delta < \mu - \alpha_t \delta < \mu + \alpha_t \delta < \mu + \alpha_s \delta. \tag{3.3}$$

If the k-th row retains its peak value $P_{\overline{B}}$ in the resulting CETD matrix

$$\overline{B} = \sum_{t=1}^{n} A^{\alpha_t} = (\sum_{t=1}^{n} e_{ij}^{t}) = (\overline{b}_{ij}), \tag{3.4}$$

then it should be that most $e_{kj}^{t} \in A^{\alpha_t}$ are such that $e_{kj}^{t} \geq 0$. Note that in the ATD matrix we could now have some a_{kj} such that $\mu - \alpha_s \leq a_{kj} < \mu - \alpha_t \delta$, so that some more $e_{kj}^{s} \in A^{\alpha_s}$ which were nonnegative in the RTD matrix A^{α_s} could now become negative in the RTD A^{α_t}. Also, all such a_{kj} such that $\mu + \alpha_t < \mu - \alpha_s \delta \leq a_{kj}$ in RTD matrix A^{α_s} retain the value 1 in the RTD matrix A^{α_t}. Hence, $e_{kj}^{t} \geq 0$ are now fewer in A^{α_t} than in A^{α_s}. So, we can have an $\epsilon \geq 1$ such that

$$P_{\overline{B}} = \sum_{j=1}^{w} \overline{b}_{kj} = \sum_{j=1}^{w} b_{kj} - \epsilon < \sum_{j=1}^{w} b_{kj} = P_B. \tag{3.5}$$

This is a contradiction to the assumption that the CETD matrix $\overline{B}$ retains the peak value obtained in the CETD matrix B. This inevitably distorts the shape of the graph obtained earlier. □

Theorem 3.9. *Given an ATD matrix, if* $\alpha_1 < \alpha_2 < \alpha_3 \cdots < \alpha_n$ *are applied in the increasing order to generate the RTD matrices* A^{α_s} *such that* $\alpha_s \in (a_s, b_s]$ *(for* $1 \leq s \leq n$*) and* $\{(x_{i-1}, xi]\}_{i=0}^{n}$ *is a uniform partition of* $(0, 1]$*, then the row with the highest sum in the resulting CETD matrix retains the maximum sum for different set of* α_s.

Proof. Assume that an RTD matrix is obtained at level α_1 and that the k-th row of the CETD matrix has the maximum sum. Since, in such an RTD matrix, any row $i \neq k$ in which e_{ij} are mostly -1 cannot have the maximum row sum, it certainly has its sum less or equal to each of the other rows in which entries are not mostly -1. Hence, there are two possible cases of the row(s) with maximum sum(s) in it:

Case 1. $e_{kj} = 1$ for most or all entries in row k.

Then,

$$a_{kj} \geq \mu_j + \alpha_1 \delta_j \tag{3.6}$$

for most or all the entries in row k over columns j in the ATD matrix A^{α_1}. If we choose α_2 such that $\alpha_1 < \alpha_2$, then the corresponding a_{kj} in the ATD matrix is such that either

$$\mu_j + \alpha_1\delta_j \leq a_{kj} < \mu_j + \alpha_2\delta_j, \tag{3.7}$$

in which case the corresponding e_{kj}, which was 1 in the RTD matrix A^{α_1} is now 0 in the new RTD matrix A^{α_2}, or

$$a_{kj} \geq \mu_j + \alpha_2\delta_j, \tag{3.8}$$

in which case the corresponding e_{kj} which was 1 in A^{α_1}, still remains 1 in the new RTD matrix A^{α_2}. So, e_{kj} is mostly 1 and 0 in the k-th row of the new RTD matrix A^{α_2}. Note that the inequalities (3.7) and (3.8) hold for all α_s leading to RTD matrices A^{α_s}, for $2 \leq s \leq n$. Let

$$\sum_{s=1}^{n}(e_{ij}^{s}) = (b_{ij}), \tag{3.9}$$

be the component-wise sum of the entries of all the RTD matrices A^{α_s}, where b_{ij} is the entry in the new CETD matrix obtained due to α_s. Then, for b_{ij} in the CETD matrix, we have the sum

$$\sum_j b_{ij} \tag{3.10}$$

of the i-th row in the CETD matrix. Since most entries in row k of each of the RTD matrices are mostly 0 and 1, by Theorems 3.6 and 3.8,

$$\sum_j b_{kj} \geq \sum_j b_{ij}, \ \forall i \neq k, \tag{3.11}$$

in which case row k has the maximum sum in the CETD matrix.

Case 2. When most e_{kj} are 0 and 1.
For $e_{kj} = 0$,

$$a_{kj} \in (\mu_j - \alpha_1\delta_j, \mu_j + \alpha_1\delta_j) \subset (\mu_j - \alpha_2\delta_j, \mu_j + \alpha_2\delta_j)$$

so that corresponding $e_{kj} = 0$ remains the same in the new RTD matrix.
On the other hand, for $e_{kj} = 1$, either

$$\mu_j + \alpha_1\delta_j \leq a_{kj} < \mu_j + \alpha_2\delta_j, \tag{3.12}$$

in which case the corresponding e_{kj} in the new RTD matrix becomes 0, or

$$a_{kj} \geq \mu_j + \alpha_2\delta_j > \mu_j + \alpha_1\delta_j, \tag{3.13}$$

in which case the corresponding e_{kj} in the new RTD matrix becomes 1 so that e_{kj} is mostly 0 and 1 in the k-th row of the ATD matrices. Also, note that the inequalities (3.12) and (3.13) hold for all α_s leading to RTD matrices A^{α_s}, for $2 \leq s \leq n$. Hence, by Theorems 3.6 and 3.8,

$$\sum_j b_{kj} \geq \sum_j b_{ij}, \ \forall i \neq k. \tag{3.14}$$

Thus, row k has the maximum sum in the CETD matrix. □

4. Algorithm for computing CETD matrices

- Uniformly partition the interval $(0, 1]$ into n cells $\{(x_{i-1}, x_i]\}_{i=1}^{n}$ for $n \geq 3$;
- Choose equal number of α_i from each cell $(x_{i-1}, x_i]$ to have the set $\{\alpha_i\}_{i=1}^{n}$;
- Compute the RTD matrix with the $\{\alpha_i\}_{i=1}^{n}$;
- Find the CETD matrix as the sum of all the RTD matrices resulting from $\{\alpha_i\}_{i=1}^{n}$.

5. Illustrative examples

Example 5.1. In this example, three of the four α-levels are chosen close to each other in cell of the uniform partition $\{(0, 0.25], (0.25, 0.5], (0.5, 0.75], (0.75, 1]\}$ to show that this produces a biased effect. Different crimes are represented on the tables with $\mathbf{C}_i, i = 1, 2, 3, 4, 5, 6$. Stealing = $\mathbf{C}_1$, Assault = $\mathbf{C}_2$, Affray = $\mathbf{C}_3$, Abduction = $\mathbf{C}_4$, Burglary = $\mathbf{C}_5$, and Cheating = $\mathbf{C}_6$.

From Tables 1 and 2, we can obtain the RTD matrices for the set of α_s as $0.06, 0.1, 0.25$, and 0.75.

Table 1. The initial RD matrix of crime 5×6.

Years	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$	$\mathbf{C}_6$
1994-1996	21	31	5	5	5	8
1997-2002	32	29	20	8	8	9
2003-2008	45	28	15	10	8	10
2009-2015	46	30	10	12	11	11
2015-2020	44	30	10	11	11	12

Table 2. The ATD matrix of crime 5×6.

Years	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$	$\mathbf{C}_6$
1994-1996	7.00	10.33	1.67	1.67	1.67	2.67
1997-2002	5.33	4.83	3.33	1.33	1.33	1.5
2003-2008	7.5	4.67	2.50	1.67	1.33	1.67
2009-2015	7.67	5.00	1.67	2.00	1. 83	1.83
2015-2020	7.33	5.00	1.67	1.83	1.83	2.00
Average	6.97	5.97	2.17	1.70	1.60	1.93
S.D	0.95	2.44	0.74	0.25	0.25	0.45

The RTD matrix for $\alpha = 0.06$ is

$$\begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ -2 \\ 0 \\ 2 \end{bmatrix}. \tag{5.1}$$

The RTD matrix for $\alpha = 0.1$ is

$$\begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ -2 \\ 0 \\ 2 \end{bmatrix}. \tag{5.2}$$

The RTD matrix for $\alpha = 0.25$ is

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ -1 \\ 1 \\ 1 \end{bmatrix}. \tag{5.3}$$

The RTD matrix for $\alpha = 0.75$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ -1 \\ 2 \\ 1 \end{bmatrix}. \tag{5.4}$$

The CETD matrix is given below

$$\begin{bmatrix} 0 & 4 & -3 & -2 & 3 & 4 \\ -4 & -3 & 4 & -4 & -4 & -4 \\ 3 & -3 & 3 & -2 & -4 & -3 \\ 3 & -3 & -3 & 4 & 4 & 2 \\ 3 & -3 & -3 & 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -15 \\ -6 \\ 3 \\ 6 \end{bmatrix}. \tag{5.5}$$

The graph depicting the peak value of CETD matrix for $\alpha = 0.06, 0.1, 0.25, 0.75$ is in Figure 1.

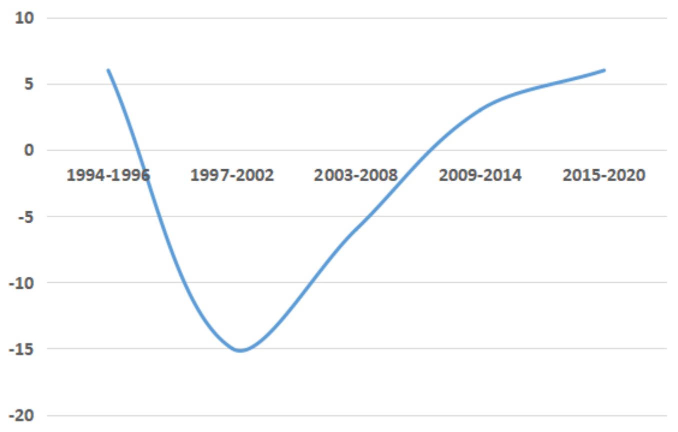

Figure 1. The graph depicting the peak value of CETD matrix for $\alpha = 0.06, 0.1, 0.25, 0.75$.

Remark 5.2. It should be noted that, for the set of α_s as $0.06, 0.1, 0.25$, and 0.75, the column matrix given beside each matrix is the row sum of the matrix. It should also be noted that the effect of choosing most α-levels close to each other from a cell of the partition as pointed out in Remark 3.7 is the reason why matrices (5.1) and (5.2) are the same. As a matter of fact, it is not so much different from (5.3). In what follows, we will choose two different sets of α-level ($\alpha = 0.2, 0.4, 0.6, 0.8$ and $\alpha = 0.25, 0.45, 0.75, 0.9$) so that α_s in each set is from each cell of the partition $\{(0, 0.25], (0.25, 0.5], (0.5, 0.75], (0.75, 1]\}$.

For the set of $\alpha = 0.2, 0.4, 0.6, 0.8$, the RTD matrix for $\alpha = 0.2$ is

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ -1 \\ 0 \\ 1 \end{bmatrix}. \tag{5.6}$$

The RTD matrix for $\alpha = 0.4$ is

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ -1 \\ 2 \\ 1 \end{bmatrix}. \tag{5.7}$$

The RTD matrix for $\alpha = 0.6$ is

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \\ 0 \end{bmatrix}. \tag{5.8}$$

The RTD matrix for $\alpha = 0.8$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ -1 \\ 2 \\ 1 \end{bmatrix}. \tag{5.9}$$

The CETD matrix is given below

$$\begin{bmatrix} 0 & 4 & -3 & 0 & 1 & 4 \\ -4 & -2 & 4 & -4 & -4 & -4 \\ 2 & -2 & 2 & 0 & -4 & -2 \\ 3 & -1 & -3 & 4 & 4 & -1 \\ 1 & -1 & -3 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ -14 \\ -4 \\ 6 \\ 3 \end{bmatrix}. \tag{5.10}$$

The graph depicting the peak value of CETD matrix for $\alpha = 0.2, 0.4, 0.6, 0.8$ is in Figure 2.

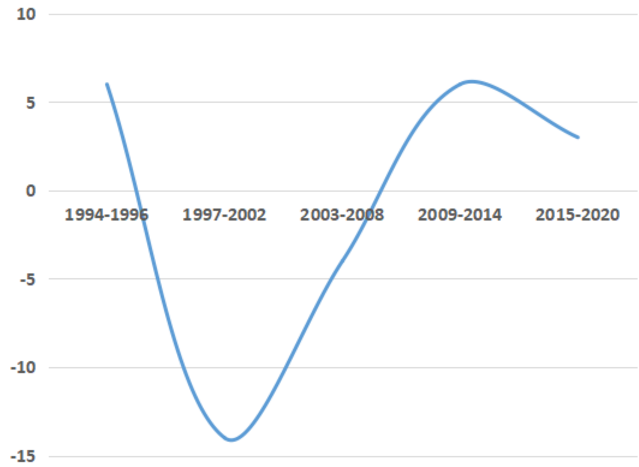

Figure 2. The graph depicting the peak value of CETD matrix for $\alpha = 0.2, 0.4, 0.6, 0.8$.

However, consider $\alpha = 0.25, 0.45, 0.75, 0.9$, which has a similar spread as $\alpha = 0.2, 0.4, 0.6, 0.8$. The CETD matrix with the row sum for $\alpha = 0.25, 0.45, 0.75, 0.9$ is given as

$$\begin{bmatrix} 0 & 4 & -2 & 0 & 1 & 4 \\ -4 & -2 & 4 & -4 & -4 & -4 \\ 2 & -2 & 1 & 0 & -4 & -2 \\ 2 & -1 & -2 & 4 & 4 & 0 \\ 1 & -1 & -2 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ -14 \\ -5 \\ 7 \\ 4 \end{bmatrix}. \tag{5.11}$$

The graph depicting the peak value of the CETD matrix for $\alpha = 0.25, 0.45, 0.75, 0.9$ is given in Figure 3.

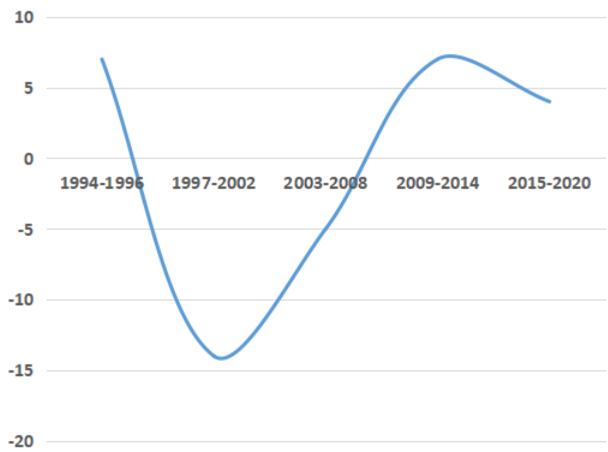

Figure 3. The graph depicting the peak value of CETD matrix for $\alpha = 0.25, 0.45, 0.75, 0.9$.

Remark 5.3. It can be seen from Figures 1 and 2 that the class having the peak values and the shapes of the graphs obtained from the row sums of the CETD matrices using different set of α_s are different.

However, considering another set of α_s, $0.25, 0.45, 0.75, 0.9$, chosen similarly as $\alpha = 0.2, 0.4, 0.6, 0.8$, respectively, from the same cell, the class (which in some cases can be age-group) with the peak value in Figure 3 is the same as the class with the peak value in Figure 2. As a matter of fact, the CETD matrix (5.5) has its peaks at row 1 and row 5, but CETD matrix (5.10) has its peaks at row 1 and row 4 (i.e., a change also occurred in peak point). However, CETD matrix (5.11) has its peaks at row 1 and row 4, and the graph (Figure 3) retains the same shape as the graph (Figure 2) of CETD matrix (5.10) because they have relatively the same spread of α_s.

Example 5.4. The attribute openness of the example in [11] will be used now. We apply our method and show that the class having the peak value is invariant under our condition.

From Tables 3 and 4, we can obtain the RTD matrices for a set of α_s as $0.2, 0.32, 0.57$ and 0.8.

Table 3. Initial RD matrix for openness of order 7×6.

Age	$\mathbf{O}_1$	$\mathbf{O}_2$	$\mathbf{O}_3$	$\mathbf{O}_4$	$\mathbf{O}_5$	$\mathbf{O}_6$
10-17	22	23	19	25	27	16
18-24	22	24	20	21	18	13
25-29	22	24	24	21	21	24
30-39	17	17	14	19	23	20
40-49	11	20	20	12	23	18
50-59	7	13	14	5	21	13
60-75	4	13	8	4	20	8

Table 4. The ATD matrix of openness of order 7×6.

Age	$\mathbf{O}_1$	$\mathbf{O}_2$	$\mathbf{O}_3$	$\mathbf{O}_4$	$\mathbf{O}_5$	$\mathbf{O}_6$
10-17	2.75	2.88	2.38	3.13	3.38	2.00
18-24	3.14	3.43	2.86	3.00	2.57	1.86
25-29	4.40	4.80	4.80	4.20	4.20	4.80
30-39	1.70	1.70	1.40	1.90	2.30	2.00
40-49	1.10	2.00	2.00	1.20	2.30	1.80
50-59	0.70	1.30	1.40	0.50	2.10	1.30
60-75	0.25	0.81	0.50	0.25	1.25	0.50
Average	2.01	2.42	2.19	2.03	2.59	2.04
Standard Deviation	1.49	1.38	1.38	1.48	0.95	1.33

The RTD matrix for $\alpha = 0.2$ is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \\ -4 \\ -4 \\ -6 \\ -6 \end{bmatrix}. \tag{5.12}$$

The RTD matrix for $\alpha = 0.32$ is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \\ 6 \\ -2 \\ -2 \\ -6 \\ -6 \end{bmatrix}. \tag{5.13}$$

The RTD matrix for $\alpha = 0.57$ is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 6 \\ -1 \\ -1 \\ -4 \\ -6 \end{bmatrix}. \tag{5.14}$$

The RTD matrix for $\alpha = 0.8$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \\ 0 \\ -3 \\ -6 \end{bmatrix}. \tag{5.15}$$

The CETD matrix with the row sum is given below

$$\begin{bmatrix} 2 & 2 & 0 & 3 & 4 & 0 \\ 3 & 3 & 2 & 3 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ -1 & -2 & -3 & 0 & -1 & 0 \\ -3 & -1 & 0 & -2 & -1 & 0 \\ -4 & -4 & -3 & -4 & -2 & -2 \\ -4 & -4 & -4 & -4 & -4 & -4 \end{bmatrix}\begin{bmatrix} 11 \\ 11 \\ 24 \\ -7 \\ -7 \\ -19 \\ -24 \end{bmatrix}. \tag{5.16}$$

The graph depicting the peak value of CETD matrix for $\alpha = 0.2, 0.32, 0.57, 0.8$ is in Figure 4.

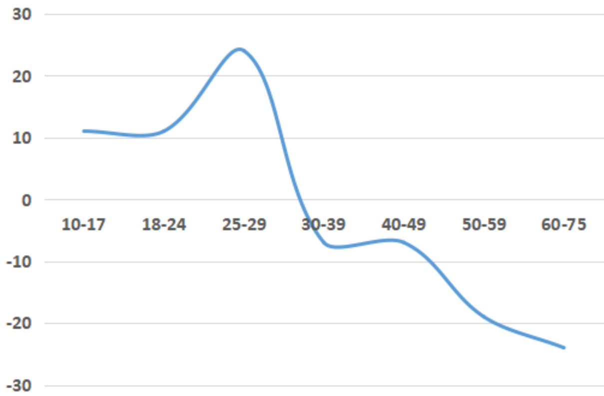

Figure 4. The graph depicting the peak value of CETD matrix for $\alpha = 0.2, 0.32, 0.57, 0.8$.

If we choose another set of α_s, say, $0.25, 0.45, 0.68$, and 0.85, the RTD matrix for $\alpha = 0.25$ is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \\ -3 \\ -4 \\ -6 \\ -6 \end{bmatrix}. \tag{5.17}$$

The RTD matrix for $\alpha = 0.45$ is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 6 \\ -2 \\ -2 \\ -6 \\ -6 \end{bmatrix}. \tag{5.18}$$

The RTD matrix for $\alpha = 0.68$ is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 6 \\ 0 \\ 0 \\ -3 \\ -6 \end{bmatrix}. \tag{5.19}$$

The RTD matrix for $\alpha = 0.85$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \\ 0 \\ 0 \\ -2 \\ -6 \end{bmatrix}. \tag{5.20}$$

The CETD matrix with the row sum is given below

$$\begin{bmatrix} 2 & 1 & 0 & 3 & 3 & 0 \\ 3 & 3 & 2 & 2 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ 0 & -2 & -2 & 0 & -1 & 0 \\ -2 & -1 & 0 & -2 & -1 & 0 \\ -4 & -3 & -2 & -4 & -2 & -2 \\ -4 & -4 & -4 & -4 & -4 & -4 \end{bmatrix} \begin{bmatrix} 9 \\ 10 \\ 24 \\ -5 \\ -6 \\ -17 \\ -24 \end{bmatrix}. \tag{5.21}$$

The graph depicting the peak value of CETD matrix for $\alpha = 0.25, 0.45, 0.68, 0.85$ is in Figure 5.

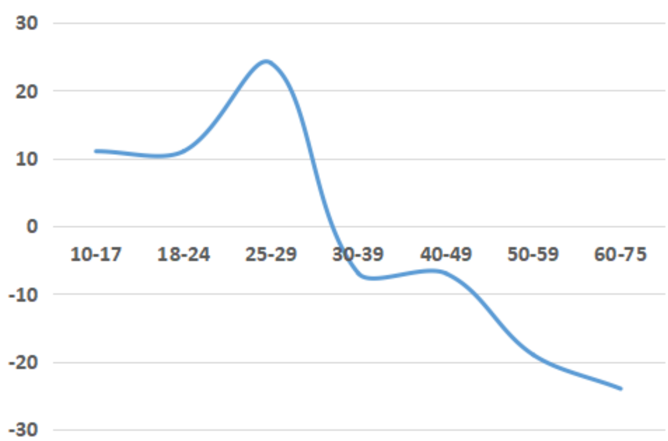

Figure 5. The graph depicting the peak value of CETD matrix for $\alpha = 0.25, 0.45, 0.68, 0.85$.

Remark 5.5. It can be seen that Figures 4 and 5 have the same shape and peak points because the spread of α is the same. In what follows, we will like to show that, even when the age groups are refined, the class(es) having the peak vale(s) in the refinement is/are invariant under our condition.

Example 5.6. Tables 5 and 6 in this example are, respectively, the refinements of Tables 1 and 2.

Table 5. The initial RD matrix of crime of order 7×6.

Years	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$	$\mathbf{C}_6$
1994-1996	21	31	5	5	5	8
1997-2002	32	29	20	8	8	9
2003-2005	19	19	8	4	4	6
2006-2008	26	9	7	6	4	4
2009-2011	22	18	4	5	6	5
2012-2015	30	14	6	7	6	6
2016-2020	38	28	10	10	9	12

Table 6. The ATD matrix of crime of order 7×6.

Years	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_4$	$\mathbf{C}_5$	$\mathbf{C}_6$
1994-1996	7.00	10.33	1.67	1.67	1.67	2.67
1997-2002	5.33	4.83	3.33	1.33	1.33	1.5
2003-2005	6.33	6.33	2.67	1.33	1.33	2.00
2006-2008	8.67	3.00	2.33	2.00	1.33	1.33
2009-2011	7.33	6.00	1.33	1.67	2.00	1.67
2012-2015	7.50	3.50	1.50	1.75	1.50	1.50
2016-2020	7.60	5.60	2.00	2.00	1.80	2.40
Average	7.11	5.66	2.12	1.68	1.57	1.87
S.D	1.05	2.41	0.71	0.28	0.27	0.51

From Tables 5 and 6, we can obtain the RTD matrices for the set of α_s as $0.2, 0.4, 0.7$, and 0.8. The RTD matrix for $\alpha = 0.2$ is

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 0 & -1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \\ 0 \\ -2 \\ 4 \end{bmatrix}. \tag{5.22}$$

The RTD matrix for $\alpha = 0.4$ is

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -2 \\ -1 \\ 0 \\ -3 \\ 4 \end{bmatrix}. \tag{5.23}$$

The RTD matrix for $\alpha = 0.7$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ -2 \\ -1 \\ 0 \\ -3 \\ 3 \end{bmatrix}. \tag{5.24}$$

The RTD matrix for $\alpha = 0.8$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -2 \\ -1 \\ 0 \\ -2 \\ 3 \end{bmatrix}. \tag{5.25}$$

The CETD matrix with the row sum is given below

$$\begin{bmatrix} 0 & 4 & -2 & 0 & 1 & 4 \\ -4 & -1 & 4 & -4 & -4 & -3 \\ -3 & 1 & 3 & -4 & -4 & 1 \\ 4 & -4 & 1 & 4 & -4 & -4 \\ 1 & 0 & -4 & 0 & 4 & -1 \\ 1 & -4 & -4 & 1 & -1 & -3 \\ 2 & 0 & 0 & 4 & 4 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ -12 \\ -6 \\ -3 \\ 0 \\ -10 \\ 14 \end{bmatrix}. \tag{5.26}$$

The graph depicting the peak value of CETD matrix for $\alpha = 0.2, 0.4, 0.7, 0.8$ is in Figure 6.

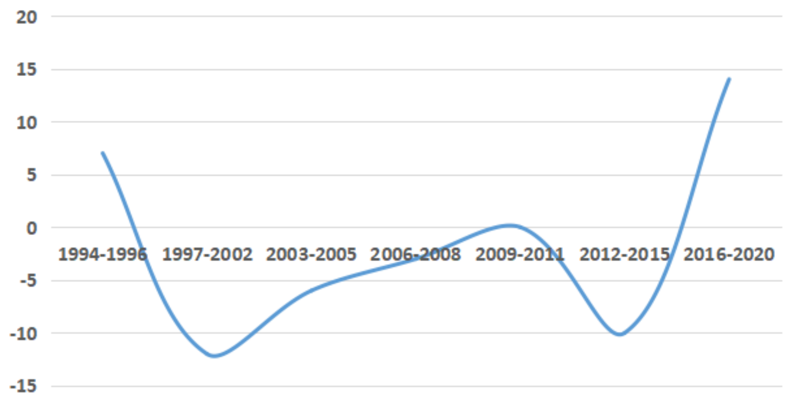

Figure 6. The graph depicting the peak value of CETD matrix for $\alpha = 0.2, 0.4, 0.7, 0.8$.

If we choose another set of α_s, say, 0.1, 0.3, 0.6, and 0.78, the RTD matrix for $\alpha = 0.1$ is

$$\begin{bmatrix} -1 & 1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 1 \\ -2 \\ 3 \end{bmatrix}. \tag{5.27}$$

The RTD matrix for $\alpha = 0.3$ is

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \\ -2 \\ 0 \\ -1 \\ -2 \\ 4 \end{bmatrix}. \tag{5.28}$$

The RTD matrix for $\alpha = 0.6$ is

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ -2 \\ -1 \\ 0 \\ -3 \\ 3 \end{bmatrix}. \tag{5.29}$$

The RTD matrix for $\alpha = 0.78$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ -2 \\ -1 \\ 0 \\ -2 \\ 3 \end{bmatrix}. \tag{5.30}$$

The CETD matrix with the row sum is given below

$$\begin{bmatrix} -1 & 4 & -3 & 0 & 2 & 4 \\ -4 & -2 & 4 & -4 & -4 & -3 \\ -3 & 1 & 3 & -4 & -4 & 1 \\ 4 & -4 & 2 & 4 & -4 & -4 \\ 1 & 1 & -4 & 0 & 4 & -2 \\ 2 & -4 & -4 & 1 & -1 & -3 \\ 2 & 0 & -1 & 4 & 4 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ -13 \\ -6 \\ -2 \\ 0 \\ -9 \\ 13 \end{bmatrix}. \tag{5.31}$$

The graph depicting the peak value of CETD matrix for $\alpha = 0.1, 0.3, 0.6, 0.78$ is in Figure 7.

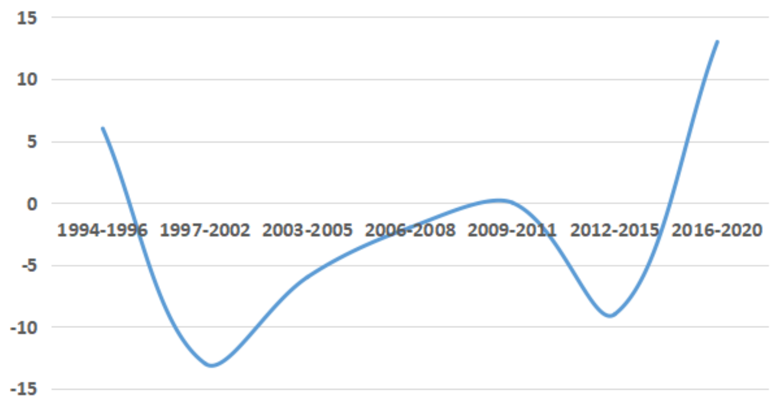

Figure 7. The graph depicting the peak value of CETD matrix for $\alpha = 0.1, 0.3, 0.6, 0.78$.

Remark 5.7. It can be seen that Figures 6 and 7 have the same shape and pick points becuse the spread of α is the same.

6. Discussion of findings

In Example 2, we have used our data to construct the RD matrix (Table 1) and the ATD matrix (Table 2). Considering the uniform partition $\{(0, 0.25], (0.25, 0.5], (0.5, 0.75], (0.75, 1]\}$ and choosing α-levels $0.06, 0.1, 0.25, 0.75$, it is obvious that $0.06, 0.1, 0.25$ are from the cell $(0, 0.25]$ of the partition. This CETD has its highest row sums in rows 1 and 5. These classes (or rows) depicting the peak values are not guaranteed when another set of α-levels are arbitrarily chosen as in the case of $\alpha = 0.2, 0.4, 0.6, 0.8$, where the CETD matrix has its highest row sum in rows 1 and 4. Hence, the experiment does not guarantee consistency in the class depicting peak values for arbitrary choice of α-levels. Meanwhile, still using the same tables, considering the levels $\alpha = 0.2, 0.4, 0.6, 0.8$, where at most one α-level was chosen from each cell of the partition, the class depicting peak values are in rows 1 and 4 of the CETD matrix. When another set of α-levels are chosen such that only at most one α-level is chosen from each cell of the partition as in the case of $\alpha = 0.25, 0.45, 0.75, 0.9$, the CETD matrix also has the groups depicting peak values in rows 1 and 4. Comparing Figures 1–3, it can be concluded that Figures 2 and 3 have maintained the same classes for their peak values.

Repeating this method for an existing example, using Tables 3 and 4 and two different sets of α-levels, chosen according to our rule, namely, $\alpha = 0.2, 0.32, 0.57, 0.8$ and $\alpha = 0.25, 0.45, 0.68, 0.85$, it can be seen in Figures 4 and 5 that these sets of α-levels maintain peak values within the same age group. Besides, when Tables 1 and 2 are refined, the two sets of α-levels chosen according to our rule, namely, $\alpha = 0.1, 0.3, 0.6, 0.78$ and $\alpha = 0.2, 0.4, 0.7, 0.8$, maintain the peak at the same age group.

7. Conclusions

This paper has established the condition which guarantees that the peak point in a CETD matrix is not affected by the changes in the set $\{\alpha_s\}$. Hence, given that the interval $(0, 1]$ has a uniform partition into n cells, the set of n α-levels should be chosen such that, at most, one α-level in the set is chosen from each cell of the partition. If the experiment is repeated for as many sets of α-levels as preferred, the class depicting the peak value will not change.

8. Area of further research

In the next research, rather than devote much time and computational rigor to finding RTD matrices for different α, it is worth investigating if there is a single algorithm that excuses the steps of using different α to obtain RTD matrices. Also, it will be more appropriate to develop algorithms to obtain RTD matrices whose entries are truly in the range $[0, 1]$ and not merely $\{-1, 0, 1\}$. This idea can now be used in analyzing behaviors and characteristics which are age or time dependent in military personnel, agriculture, and logistics management, just to mention a few.

Author contributions

Babatunde Oluwaseun Onasanya: Conceptualization, formal analysis; Yuming Feng: Resources; Aishat Omobolanle Ilesanmi: Data curation; Babatunde Oluwaseun Onasanya and Aishat Omobolanle Ilesanmi: Writing-original draft; Dongfang Yan: Writing-review & editing, supervision; Hanyin Zhang, Dongfang Yan and Yuming Feng: Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflict of interests.

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