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Research article

Improved Kneser-type oscillation criterion for half-linear dynamic equations on time scales

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Abstract: We study the Kneser-type oscillation criterion for a class of second-order half-linear functional dynamic equations on an arbitrary time scale utilizing the integral averaging approach and the Riccati transformation method. The results show an improvement in Kneser-type when compared to some known results. We provide some illustrative examples to demonstrate the significance of our main results.

Keywords: oscillation criteria; Kneser-type; half-linear; dynamic equation; time scales; differential equations
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1. Introduction

The theory of time scales, which unites discrete and continuous analysis was proposed in 1988 by Stefan Hilger, see [1]. Numerous interesting time scales exist, such as $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = \mathbb{R}$, and $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0 \text{ for } q > 1\}$. For a wonderful time-scales calculus introduction and application, see [1–5].

Several researchers from numerous applied fields have taken an interest in the phenomenon of oscillation. The fundamental reason for this is that oscillation has a huge variety of engineering and science applications. References [6–9] point to numerous studies on the oscillation of delay differential equation solutions. Research on second-order delay dynamic equations in [10–14] establishes various oscillation standards for diverse second-order dynamic equations. When comparing the study on advanced oscillation to other areas of research, there are not many publications that deal with this topic specifically [15–19]. We advised the reader to [20–24] in order to have a more understanding of this topic. In dynamical models, deviation and oscillation scenarios are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [20, 21, 25, 26]. Since these fields have significant practical implications, a thorough comprehensive understanding of the mathematical principles supporting them is necessary. Researchers looking for more information can view the papers [27–31]. Therefore, this study is focused on the oscillation of the second-order half-linear functional dynamic equation

$$\left(\alpha_1(s)\left|\varkappa^{\Delta}(s)\right|^{\varrho-1}\varkappa^{\Delta}(s)\right)^{\Delta}+\alpha_2(s)\left|\varkappa(\mu(s))\right|^{\varrho-1}\varkappa(\mu(s))=0. \tag{1.1}$$

On an above-unbounded time scale $\mathbb{T}$, where $s\in[s_0,\infty)_{\mathbb{T}}$, $s_0\geq 0$, $s_0\in\mathbb{T}$; ϱ is a positive real number; $\alpha_1,\alpha_2\in \mathrm{C}_{\mathrm{rd}}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+\right)$ such that

$$\tilde{R}(s):=\int_{s_0}^{s}\frac{\Delta\omega}{\alpha_1^{1/\varrho}(\omega)}\to\infty \quad \text{as } s\to\infty; \tag{1.2}$$

and $\mu\in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$ is a nondecreasing rd-continuous function such that $\lim_{s\to\infty}\mu(s)=\infty$.

By a solution of the Eq (1.1), we mean a nontrivial real-valued function $\varkappa\in \mathrm{C}^1_{\mathrm{rd}}[s_x,\infty)_{\mathbb{T}}$, $s_\varkappa\in[s_0,\infty)_{\mathbb{T}}$ such that $\alpha_1(s)\left|\varkappa^{\Delta}\right|^{\varrho-1}\varkappa^{\Delta}\in \mathrm{C}^1_{\mathrm{rd}}[s_\varkappa,\infty)_{\mathbb{T}}$ and $\varkappa$ satisfies (1.1) on $[s_\varkappa,\infty)_{\mathbb{T}}$, where C_{rd} is the set of right-dense continuous functions. Let $s_1\geq s_0$ be a given initial point, ϕ be a given initial rd-continuous function on $[\mu^*(s_1),s_1]$ with $\mu^*(s_1)=\min_{s\geq s_1}\{\mu(s),s\}$ and β be a given initial constant. An initial value problem (1.1) with any initial conditions $\varkappa(s)=\phi(s)$ for $s\in[\mu^*(s_1),s_1]$ and $\varkappa^{\Delta}(s)=\beta$, has a solution which exists on the whole interval $[s_0,\infty)_{\mathbb{T}}$, see [2, 32]. If a solution $\varkappa$ of (1.1) is neither eventually positive nor eventually negative, it is known as oscillatory; if it is neither, it is called nonoscillatory. We shall not take into account the solutions that vanish in a neighborhood around infinity. The following shows the differential oscillation results related to (1.1) oscillation results on time scales. It offers an extensive summary of the important contributions that this work has made. Since Sturm's major contribution to the literature, Euler differential equations and their many generalizations have been an essential component of oscillation theory. The second-order Euler equation is one of the most well-known and often used that is

$$\varkappa''(s)+\frac{\lambda}{s^2}\varkappa(s)=0,\ \lambda>0, \tag{1.3}$$

which is oscillatory if and only if

$$\lambda>\frac{1}{4}.$$

Among the most important oscillation criteria of second-order differential equations are Kneser-type [33], which used Sturmian comparison methods, and the oscillatory behavior of (1.3) to investigate that

$$\varkappa''(s)+\alpha_2(s)\varkappa(s)=0, \tag{1.4}$$

is oscillatory if

$$\liminf_{s\to\infty}s^2\alpha_2(s)>\frac{1}{4}. \tag{1.5}$$

Many works that deduce Kneser-type criteria for different types of differential equations have been produced. Here, some of these works, see [34–36]:

(I) The linear equation

$$(\alpha_1(s)\varkappa'(s))' + \alpha_2(s)\varkappa(s) = 0 \tag{1.6}$$

oscillates if

$$\liminf_{s\to\infty} \alpha_1(s)\tilde{R}^2(s)\alpha_2(s) > \frac{1}{4}. \tag{1.7}$$

(II) The half-linear equation

$$\left(|\varkappa'(s)|^{\varrho-1}\varkappa'(s)\right)' + \alpha_2(s)|\varkappa(s)|^{\varrho-1}\varkappa(s) = 0 \tag{1.8}$$

oscillates if

$$\liminf_{s\to\infty} s^{\varrho+1}\alpha_2(s) > \left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1}. \tag{1.9}$$

We note that the Euler equation

$$\left(|\varkappa'(s)|^{\varrho-1}\varkappa'(s)\right)' + \frac{\lambda}{s^{\varrho+1}}|\varkappa(s)|^{\varrho-1}\varkappa(s) = 0,\ \alpha_2 > 0 \tag{1.10}$$

has a solution that is nonoscillatory $\varkappa(s) = s^{\varrho/(\varrho+1)}$ if $\lambda = \left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1}$. That is to say, for all solutions of the Eq (1.10), the constant $\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1}$ is a lower bound of oscillation, for all solutions of (1.10).

Recently, Hassan et al. [37] found some interesting Kneser-type criteria for oscillation for Eq (1.1) as follows:

Theorem 1.1 ([37]). *Assume that* $l := \liminf_{s\to\infty} \frac{s}{\sigma(s)} > 0$ *and*

$$\liminf_{s\to\infty} \frac{s\varphi^{\varrho}(s)\alpha_2(s)}{\alpha_1(s)} > \frac{1}{l^{\varrho(\varrho+1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1}, \tag{1.11}$$

where $\varphi(s) := \min\{s, \mu(s)\}$, *then all solutions of Eq* (1.1) *oscillate.*

It should be mentioned that this study was significantly influenced by the works made by [33–35, 37]. The present research aims to conclude some sharp Kneser-type oscillation conditions for (1.1) with $\mu(s) \le s$ and $\mu(s) \ge s$.

2. An oscillation criterion of (1.1) when $\mu(s) \le s$

This section focuses on Kneser-type oscillation of (1.1) with $\mu(s) \le s$.

First, we introduce an important lemma that as a fundamental role in establishing our results.

Lemma 2.1. *[28, Lemma 2.2] Assume that*

$$\varkappa(s) > 0,\ \varkappa^{\Delta}(s) > 0,\ \left[\alpha_1(s)\ \left|\varkappa^{\Delta}(s)\right|^{\varrho-1}\varkappa^{\Delta}(s)\right]^{\Delta} < 0 \text{ on } [s_0,\infty)_{\mathbb{T}}.$$

Then

$$\left(\frac{\varkappa(s)}{s-s_0}\right)^{\Delta} < 0, \text{ and } \varkappa(s) \ge (s-s_0)\varkappa^{\Delta}(s), \text{ on } (s_0,\infty)_{\mathbb{T}}.$$

Theorem 2.1. *For* $0 < \varrho \leq 1$. *If* $l := \liminf_{s\to\infty} \dfrac{s}{\sigma(s)} > 0$ *and*

$$\Lambda := \liminf_{s\to\infty} \frac{s^{1-\varrho}\sigma^{\varrho}(s)\,\mu^{\varrho}(s)\,\alpha_2(s)}{\alpha_1(s)} > \frac{1}{l^{\varrho(1-\varrho)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1}, \tag{2.1}$$

then all solutions of Eq (1.1) *oscillate.*

Proof. Assume to the contrary that Eq (1.1) has a nonoscillatory solution $\varkappa$ on $[s_0,\infty)_{\mathbb{T}}$. Without loss of generality, we let $\varkappa(\mu(s)) > 0$ for $s \in [s_0,\infty)_{\mathbb{T}}$. By using Lemma 2.1, there exists $s_1 \in [s_0,\infty)_{\mathbb{T}}$ such that for $s \geq s_1$,

$$\varkappa^{\Delta}(s) > 0,\ \left(\frac{\varkappa(s)}{s-s_0}\right)^{\Delta} < 0,\ \varkappa(s) \geq (s-s_0)\,\varkappa^{\Delta}(s),\ \text{and}\ \left[\alpha_1(s)\ \left|\varkappa^{\Delta}(s)\right|^{\varrho-1}\varkappa^{\Delta}(s)\right]^{\Delta} < 0. \tag{2.2}$$

Let

$$\Omega(s) \ := \ \alpha_1(s)\left(\frac{\varkappa^{\Delta}(s)}{\varkappa(s)}\right)^{\varrho}. \tag{2.3}$$

We have

$$\begin{aligned}
\Omega^{\Delta}(s) &= \left(\frac{1}{\varkappa^{\varrho}(s)}\left[\alpha_1(s)\left(\varkappa^{\Delta}(s)\right)^{\varrho}\right]\right)^{\Delta} \\
&= \frac{1}{\varkappa^{\varrho}(s)}\left[\alpha_1(s)\left(\varkappa^{\Delta}(s)\right)^{\varrho}\right]^{\Delta} - \frac{(\varkappa^{\varrho}(s))^{\Delta}}{\varkappa^{\varrho}(s)\varkappa^{\varrho}(\sigma(s))}\left[\alpha_1(s)\left(\varkappa^{\Delta}(s)\right)^{\varrho}\right]^{\sigma} \\
&\overset{(1.1)}{=} -\alpha_2(s)\left(\frac{\varkappa(\mu(s))}{\varkappa(s)}\right)^{\varrho} - \frac{(\varkappa^{\varrho}(s))^{\Delta}}{\varkappa^{\varrho}(s)}\Omega^{\sigma}(s).
\end{aligned} \tag{2.4}$$

Pötzsche chain rule ([2, Theorem 1.90]) application yields

$$\begin{aligned}
\frac{(\varkappa^{\varrho}(s))^{\Delta}}{\varkappa^{\varrho}(s)} &= \frac{\varrho}{\varkappa^{\varrho}(s)}\int_0^1 [(1-h)\,\varkappa(s) + h\varkappa^{\sigma}(s)]^{\varrho-1}\,\mathrm{d}h\,\varkappa^{\Delta}(s) \\
&\geq \varrho\left(\frac{\varkappa(s)}{\varkappa^{\sigma}(s)}\right)^{1-\varrho}\frac{\varkappa^{\Delta}(s)}{\varkappa(s)} \\
&= \varrho\left(\frac{\varkappa(s)}{\varkappa^{\sigma}(s)}\right)^{1-\varrho}\left(\frac{\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho}.
\end{aligned}$$

Hence,

$$\Omega^{\Delta}(s) \leq -\alpha_2(s)\left(\frac{\varkappa(\mu(s))}{\varkappa(s)}\right)^{\varrho} - \varrho\Omega^{\sigma}(s)\left(\frac{\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho}\left(\frac{\varkappa(s)}{\varkappa^{\sigma}(s)}\right)^{1-\varrho}.$$

Assume $0 < \kappa < 1$ is arbitrary. By using the fact that $\left(\dfrac{\varkappa(s)}{s-s_0}\right)^{\Delta} < 0$, there is $s_\kappa \in [s_1,\infty)_{\mathbb{T}}$ such that for $s \in [s_\kappa,\infty)_{\mathbb{T}}$,

$$\Omega^{\Delta}(s) \leq -\kappa^{\varrho}\alpha_2(s)\left(\frac{\mu(s)}{s}\right)^{\varrho} - \varrho\kappa^{1-\varrho}\Omega^{\sigma}(s)\left(\frac{\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho}\left(\frac{s}{\sigma(s)}\right)^{1-\varrho}. \tag{2.5}$$

Define

$$\Omega_* := \liminf_{s\to\infty} \frac{s^{\varrho}\Omega(s)}{\alpha_1(s)}. \tag{2.6}$$

It is obvious that $\Omega_* \in (0,\infty)$ due to (2.2) and (2.3). For any $\varepsilon \in (0,1)$, there exists $S \in [s_\kappa,\infty)_{\mathbb{T}}$ such that for $s \in [S,\infty)_{\mathbb{T}}$,

$$\frac{s^{1-\varrho}\sigma^{\varrho}(s)\mu^{\varrho}(s)\alpha_2(s)}{\alpha_1(s)} \geq \varepsilon\Lambda,\ \frac{s}{\sigma(s)} \geq \varepsilon l,\ \text{and}\ \frac{s^{\varrho}\Omega(s)}{\alpha_1(s)} \geq \varepsilon\Omega_*. \tag{2.7}$$

Therefore,

$$\begin{aligned}
\Omega^{\Delta}(s) &\leq -\varepsilon\kappa^{\varrho}\Lambda\frac{\alpha_1(s)}{t\sigma^{\varrho}(s)} - \varrho\kappa^{1-\varrho}\alpha_1^{-\frac{1}{\varrho}}(s)\left(\frac{s}{\sigma(s)}\right)^{1-\varrho}\Omega^{1/\varrho}(s)\,\Omega^{\sigma}(s)\\
&= -\varepsilon\kappa^{\varrho}\Lambda\frac{\alpha_1(s)}{s\sigma^{\varrho}(s)} - \varrho\kappa^{1-\varrho}\left(\frac{s}{\sigma(s)}\right)^{1-\varrho}\frac{\alpha_1^{\sigma}(s)}{s\sigma^{\varrho}(s)}\left(\frac{t^{\varrho}\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho}\left(\frac{s^{\varrho}\Omega(s)}{\alpha_1(s)}\right)^{\sigma}\\
&\leq -\varepsilon\kappa^{\varrho}\Lambda\frac{\alpha_1(s)}{s\sigma^{\varrho}(s)} - \varrho\kappa^{1-\varrho}\varepsilon^{2-\varrho+1/\varrho}l^{1-\varrho}\Omega_*^{1+1/\varrho}\frac{\alpha_1(s)}{s\sigma^{\varrho}(s)}\\
&= -\left(\varepsilon\kappa^{\varrho}\frac{\Lambda}{\varrho} + \kappa^{1-\varrho}\varepsilon^{2-\varrho+1/\varrho}l^{1-\varrho}\Omega_*^{1+1/\varrho}\right)\frac{\varrho\alpha_1(s)}{s\sigma^{\varrho}(s)}.
\end{aligned}$$

Integrating from s to v, we obtain

$$\begin{aligned}
\Omega(v) - \Omega(s) &\leq -\left(\varepsilon\kappa^{\varrho}\frac{\Lambda}{\varrho} + \kappa^{1-\varrho}\varepsilon^{2-\varrho+1/\varrho}l^{1-\varrho}\Omega_*^{1+1/\varrho}\right)\int_s^v \frac{\varrho\alpha_1(\omega)}{\omega\sigma^{\varrho}(\omega)}\Delta\omega\\
&\leq -\alpha_1(s)\left(\varepsilon\kappa^{\varrho}\frac{\Lambda}{\varrho} + \kappa^{1-\varrho}\varepsilon^{2-\varrho+1/\varrho}l^{1-\varrho}\Omega_*^{1+1/\varrho}\right)\int_s^v \frac{\varrho}{\omega\sigma^{\varrho}(\omega)}\Delta\omega.
\end{aligned}$$

Since $\Omega > 0$ and letting $v \to \infty$, we have

$$\Omega(s) \geq \alpha_1(s)\left(\varepsilon\kappa^{\varrho}\frac{\Lambda}{\varrho} + \kappa^{1-\varrho}\varepsilon^{2-\varrho+1/\varrho}l^{1-\varrho}\Omega_*^{1+1/\varrho}\right)\int_s^{\infty} \frac{\varrho}{\omega\sigma^{\varrho}(\omega)}\Delta\omega. \tag{2.8}$$

Applying the Pötzsche chain rule ([2, Theorem 1.90]) yields that

$$(\omega^{\varrho})^{\Delta} = \varrho\int_0^1 [(1-h)\,\omega + h\sigma(\omega)]^{\varrho-1}\,\mathrm{d}h \leq \varrho\omega^{\varrho-1}. \tag{2.9}$$

From the quotient rule and (2.9), we obtain

$$\left(\frac{-1}{\omega^{\varrho}}\right)^{\Delta} = \frac{(\omega^{\varrho})^{\Delta}}{\omega^{\varrho}\sigma^{\varrho}(\omega)} \leq \frac{\varrho}{\omega\sigma^{\varrho}(\omega)}. \tag{2.10}$$

Substituting (2.10) into (2.8), we deduce that

$$\begin{aligned}\Omega(s) &\geq \alpha_1(s)\left(\varepsilon\kappa^{\varrho}\frac{\Lambda}{\varrho}+\kappa^{1-\varrho}\varepsilon^{2-\varrho+1/\varrho}l^{1-\varrho}\Omega_*^{1+1/\varrho}\right)\int_s^{\infty}\left(\frac{-1}{\omega^{\varrho}}\right)^{\Delta}\Delta\omega\\ &= \frac{\alpha_1(s)}{s^{\varrho}}\left(\varepsilon\kappa^{\varrho}\frac{\Lambda}{\varrho}+\kappa^{1-\varrho}\varepsilon^{2-\varrho+1/\varrho}l^{1-\varrho}\Omega_*^{1+1/\varrho}\right).\end{aligned}$$

Hence,

$$\varepsilon\kappa^{\varrho}\Lambda \leq \varrho\frac{s^{\varrho}\Omega(s)}{\alpha_1(s)}-\varrho\kappa^{1-\varrho}\varepsilon^{2-\varrho+1/\varrho}l^{1-\varrho}\Omega_*^{1+1/\varrho}.$$

Taking lim inf of every side as $s \to \infty$, we obtain

$$\varepsilon\kappa^{\varrho}\Lambda \leq \varrho\Omega_*-\varrho\kappa^{1-\varrho}\varepsilon^{2-\varrho+1/\varrho}l^{1-\varrho}\Omega_*^{1+\frac{1}{\varrho}}.$$

Due to ε and κ being arbitrary, we obtain

$$\Lambda \leq \varrho\Omega_*-\varrho l^{1-\varrho}\Omega_*^{1+\frac{1}{\varrho}}.$$

Let

$$Y=\varrho l^{1-\varrho},\quad X=\varrho,\quad \text{and}\quad U=\Omega_*.$$

Using the inequality

$$XU-YU^{1+\frac{1}{\varrho}} \leq \frac{\varrho^{\varrho}}{(\varrho+1)^{\varrho+1}}\frac{X^{\varrho+1}}{Y^{\varrho}},\quad X,Y>0, \tag{2.11}$$

we have,

$$\Lambda \leq \frac{1}{l^{\varrho(1-\varrho)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1},$$

which provides the contradiction with (2.1). □

Theorem 2.2. *Assume that* $\varrho \geq 1$. *If* $l := \liminf_{s\to\infty}\frac{s}{\sigma(s)} > 0$ *and*

$$ß := \liminf_{s\to\infty}\frac{\sigma(s)\mu^{\varrho}(s)\alpha_2(s)}{\alpha_1(s)} > \frac{1}{l^{\varrho(\varrho-1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1}, \tag{2.12}$$

then all solutions of Eq (1.1) *oscillate.*

Proof. Assume to the contrary that Eq (1.1) has a nonoscillatory solution $\varkappa$ on $[s_0,\infty)_{\mathbb{T}}$. Without loss of generality, we let $\varkappa(\mu(s)) > 0$ for $s \in [s_0,\infty)_{\mathbb{T}}$. As the proof of Theorem 2.1, there is $s_{\kappa} \in [s_1,\infty)_{\mathbb{T}}$, $s_1 \in [s_0,\infty)_{\mathbb{T}}$, such that for $s \in [s_{\kappa},\infty)_{\mathbb{T}}$,

$$\Omega^{\Delta}(s) = -\kappa^{\varrho}\alpha_2(s)\left(\frac{\mu(s)}{s}\right)^{\varrho}-\frac{(\varkappa^{\varrho}(s))^{\Delta}}{\varkappa^{\varrho}(s)}\Omega^{\sigma}(s),$$

where $\Omega(s)$ is defined by (2.3). By the Pötzsche chain rule ([2, Theorem 1.90]), we have

$$\frac{(\varkappa^{\varrho}(s))^{\Delta}}{\varkappa^{\varrho}(s)} = \frac{\varrho}{\varkappa^{\varrho}(s)}\int_0^1[(1-h)\varkappa(s)+h\varkappa^{\sigma}(s)]^{\varrho-1}\,\mathrm{d}h\,\varkappa^{\Delta}(s) \geq \varrho\frac{\varkappa^{\Delta}(s)}{\varkappa(s)} = \varrho\left(\frac{\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho}.$$

Thus,

$$
\begin{aligned}
\Omega^{\Delta}(s) &\leq -\kappa^{\varrho}\alpha_2(s)\left(\frac{\mu(s)}{s}\right)^{\varrho} - \varrho\Omega^{\sigma}(s)\left(\frac{\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho} \\
&= -\kappa^{\varrho}\alpha_2(s)\left(\frac{\mu(s)}{s}\right)^{\varrho} - \varrho\frac{\alpha_1^{\sigma}(s)}{s\sigma^{\varrho}(s)}\left(\frac{s^{\varrho}\Omega(s)}{\alpha_1(s)}\right)^{\sigma}\left(\frac{s^{\varrho}\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho}. \qquad (2.13)
\end{aligned}
$$

For any $\varepsilon \in (0,1)$, there exists $S \in [s_{\kappa}, \infty)_{\mathbb{T}}$ such that for $s \in [S, \infty)_{\mathbb{T}}$,

$$
\frac{\sigma(s)\mu^{\varrho}(s)\alpha_2(s)}{\alpha_1(s)} \geq \varepsilon ß, \ \frac{s}{\sigma(s)} \geq \varepsilon l, \text{ and } \frac{s^{\varrho}\Omega(s)}{\alpha_1(s)} \geq \varepsilon\Omega_*, \qquad (2.14)
$$

where Ω_* is defined by (2.6). Substituting (2.14) into (2.13), we have

$$
\begin{aligned}
\Omega^{\Delta}(s) &\leq -\varepsilon\kappa^{\varrho}ß\frac{\alpha_1(s)}{s^{\varrho}\sigma(s)} - \varrho(\varepsilon\Omega_*)^{1+1/\varrho}\frac{\alpha_1^{\sigma}(s)}{s\sigma^{\varrho}(s)} \\
&\leq -\left(\varepsilon\kappa^{\varrho}\frac{ß}{\varrho} + (\varepsilon\Omega_*)^{1+1/\varrho}\left(\frac{s}{\sigma(s)}\right)^{\varrho-1}\right)\frac{\varrho\alpha_1(s)}{s^{\varrho}\sigma(s)} \\
&\leq -\left(\varepsilon\kappa^{\varrho}\frac{ß}{\varrho} + \varepsilon^{\varrho+1/\varrho}l^{\varrho-1}\Omega_*^{1+1/\varrho}\right)\frac{\varrho\alpha_1(s)}{t^{\varrho}\sigma(s)}. \qquad (2.15)
\end{aligned}
$$

Integrating (2.15) from s to v, we obtain

$$
\begin{aligned}
\Omega(v) - \Omega(s) &\leq -\left(\varepsilon\kappa^{\varrho}\frac{ß}{\varrho} + \varepsilon^{\varrho+1/\varrho}l^{\varrho-1}\Omega_*^{1+1/\varrho}\right)\int_s^v \frac{\varrho\alpha_1(\omega)}{\omega^{\varrho}\sigma(\omega)}\Delta\omega \\
&\leq -\alpha_1(s)\left(\varepsilon\kappa^{\varrho}\frac{ß}{\varrho} + \varepsilon^{\varrho+1/\varrho}l^{\varrho-1}\Omega_*^{1+1/\varrho}\right)\int_s^v \frac{\varrho}{\omega^{\varrho}\sigma(\omega)}\Delta\omega.
\end{aligned}
$$

Since $\Omega > 0$ and passing to the limit as $v \to \infty$, we obtain

$$
-\Omega(s) \leq -\alpha_1(s)\left(\varepsilon\kappa^{\varrho}\frac{\mathbf{ß}}{\varrho} + \varepsilon^{\varrho+1/\varrho}l^{\varrho-1}\Omega_*^{1+1/\varrho}\right)\int_s^{\infty} \frac{\varrho}{\omega^{\varrho}\sigma(\omega)}\Delta\omega. \qquad (2.16)
$$

Applying the Pötzsche chain rule ([2, Theorem 1.90]), we obtain

$$
(\omega^{\varrho})^{\Delta} = \varrho\int_0^1 [(1-h)\omega + h\sigma(\omega)]^{\varrho-1}\mathrm{d}h \leq \varrho\sigma^{\varrho-1}(\omega).
$$

From the quotient rule and (2.9), we obtain

$$
\left(\frac{-1}{\omega^{\varrho}}\right)^{\Delta} = \frac{(\omega^{\varrho})^{\Delta}}{\omega^{\varrho}\sigma^{\varrho}(\omega)} \leq \frac{\varrho}{\omega^{\varrho}\sigma(\omega)}.
$$

Hence,

$$
-\Omega(s) \leq -\alpha_1(s)\left(\varepsilon\kappa^{\varrho}\frac{ß}{\varrho} + \varepsilon^{\varrho+1/\varrho}l^{\varrho-1}\Omega_*^{1+1/\varrho}\right)\int_s^{\infty}\left(\frac{-1}{\omega^{\varrho}}\right)^{\Delta}\Delta\omega = -\frac{\alpha_1(s)}{s^{\varrho}}\left(\varepsilon\kappa^{\varrho}\frac{ß}{\varrho} + \varepsilon^{\varrho+1/\varrho}l^{\varrho-1}\Omega_*^{1+1/\varrho}\right).
$$

Then

$$\varepsilon\kappa^{\varrho}ß \leq \varrho\left[\frac{s^{\varrho}\Omega(s)}{\alpha_1(s)} - \varepsilon^{\varrho+1/\varrho}l^{\varrho-1}\Omega_*^{1+1/\varrho}\right].$$

Taking the lim inf as $s \to \infty$, yields

$$\varepsilon\kappa^{\varrho}ß \leq \varrho\Omega_* - \varrho\varepsilon^{\varrho+1/\varrho}l^{\varrho-1}\Omega_*^{1+1/\varrho}.$$

Since ε and κ are arbitrary, we achieve

$$ß \leq \varrho\Omega_* - \varrho l^{\varrho-1}\Omega_*^{1+1/\varrho}.$$

By the inequality (2.11), we arrive at

$$ß \leq \frac{1}{l^{\varrho(\varrho-1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1}.$$

It provides us with the contradiction in (2.12). □

3. Oscillation criterion to (1.1) with $\mu(s) \geq s$

This section focuses on the Kneser-type oscillation criterion to Eq (1.1) when $\mu(s) \geq s$.

Theorem 3.1. *For* $0 < \varrho \leq 1$. *If* $l := \liminf_{s\to\infty} \dfrac{s}{\sigma(s)} > 0$ *and*

$$\liminf_{s\to\infty} \frac{s\sigma^{\varrho}(s)\alpha_2(s)}{\alpha_1(s)} > \frac{1}{l^{\varrho(1-\varrho)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1}, \tag{3.1}$$

then all solutions of Eq (1.1) *oscillate.*

Proof. Assume to the contrary that Eq (1.1) has a nonoscillatory solution $\varkappa$ on $[s_0,\infty)_{\mathbb{T}}$. Without loss of generality, we let $\varkappa(s) > 0$ for $s \in [s_0,\infty)_{\mathbb{T}}$. As shown in the proof of Theorem 2.1, there is $s_1 \in [s_0,\infty)_{\mathbb{T}}$ such that for $s \in [s_1,\infty)_{\mathbb{T}}$,

$$\Omega^{\Delta}(s) \leq -\alpha_2(s)\left(\frac{\varkappa(\mu(s))}{\varkappa(s)}\right)^{\varrho} - \varrho\Omega^{\sigma}(s)\left(\frac{\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho}\left(\frac{\varkappa(s)}{\varkappa^{\sigma}(s)}\right)^{1-\varrho}.$$

By (2.2), we have for $s \in [s_1,\infty)_{\mathbb{T}}$,

$$\frac{\varkappa(\mu(s))}{\varkappa(s)} \geq 1. \tag{3.2}$$

Hence,

$$\Omega^{\Delta}(s) \leq -\alpha_2(s) - \varrho\Omega^{\sigma}(s)\left(\frac{\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho}\left(\frac{\varkappa(s)}{\varkappa^{\sigma}(s)}\right)^{1-\varrho}.$$

The remainder of the proof is omitted because it can be proved similarly as in Theorem 2.1. □

Theorem 3.2. *For* $\varrho \geq 1$. *If* $l := \liminf_{s\to\infty} \dfrac{s}{\sigma(s)} > 0$ *and*

$$\liminf_{s\to\infty} \frac{s^{\varrho}\sigma(s)\alpha_2(s)}{\alpha_1(s)} > \frac{1}{l^{\varrho(\varrho-1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1}, \tag{3.3}$$

then all solutions of Eq (1.1) *oscillate.*

Proof. Assume to the contrary that Eq (1.1) has a nonoscillatory solution $\varkappa$ on $[s_0,\infty)_{\mathbb{T}}$. Without loss of generality, we let $\varkappa(s)>0$ for $s\in[s_0,\infty)_{\mathbb{T}}$. As shown in the proof of Theorems 2.1 and 2.2, there is $s_1\in[s_0,\infty)_{\mathbb{T}}$ such that for $s\in[s_1,\infty)_{\mathbb{T}}$,

$$\Omega^{\Delta}(s)\le -\alpha_2(s)\left(\frac{\varkappa(\mu(s))}{\varkappa(s)}\right)^{\varrho}-\varrho\Omega^{\sigma}(s)\left(\frac{\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho}.$$

By (3.2), we have for $s\in[s_1,\infty)_{\mathbb{T}}$,

$$\Omega^{\Delta}(s)\le -\alpha_2(s)-\varrho\Omega^{\sigma}(s)\left(\frac{\Omega(s)}{\alpha_1(s)}\right)^{1/\varrho}.$$

The rest of the proof is omitted because it is similar to that of Theorem 2.2. □

4. Examples

The next examples illustrate how the theoretical concepts presented in this work can be applied.

Example 4.1. *The generalized Euler second-order dynamic equations:*

(1) For $0<\varrho\le 1$,

$$\left(\alpha_1(s)\left|\varkappa^{\Delta}(s)\right|^{\varrho-1}\varkappa^{\Delta}(s)\right)^{\Delta}+\delta\frac{\alpha_1(s)}{s\sigma^{\varrho}(s)}\left|\varkappa(s)\right|^{\varrho-1}\varkappa(s)=0$$

and

$$\left(\alpha_1(s)\left|\varkappa^{\Delta}(s)\right|^{\varrho-1}\varkappa^{\Delta}(s)\right)^{\Delta}+\delta\frac{\alpha_1(s)}{s\sigma^{\varrho}(s)}\left|\varkappa(\sigma(s))\right|^{\varrho-1}\varkappa(\sigma(s))=0;$$

(2) For $\varrho\ge 1$,

$$\left(\alpha_1(s)\left|\varkappa^{\Delta}(s)\right|^{\varrho-1}\varkappa^{\Delta}(s)\right)^{\Delta}+\delta\frac{\alpha_1(s)}{s^{\varrho}\sigma(s)}\left|\varkappa(s)\right|^{\varrho-1}\varkappa(s)=0$$

and

$$\left(\alpha_1(s)\left|\varkappa^{\Delta}(s)\right|^{\varrho-1}\varkappa^{\Delta}(s)\right)^{\Delta}+\delta\frac{\alpha_1(s)}{s^{\varrho}\sigma(s)}\left|\varkappa(\sigma(s))\right|^{\varrho-1}\varkappa(\sigma(s))=0,$$

where $\delta>0$ *is a constant and* $l:=\liminf_{s\to\infty}\dfrac{s}{\sigma(s)}>0$*, oscillate if* $\delta>\dfrac{1}{l^{\varrho|\varrho-1|}}\left(\dfrac{\varrho}{\varrho+1}\right)^{\varrho+1}$ *by using Theorems 3.1 and 3.2 respectively.*

- *It is worth noting the following:*

(i) If $\mathbb{T}=\mathbb{R}$*, the Euler equation for* $\varrho>m\ge 0$,

$$\left(s^{m}\left|\varkappa'(s)\right|^{\varrho-1}\varkappa'(s)\right)'+\delta\left(\frac{\varrho-m}{\varrho}\right)^{\varrho+1}\frac{1}{s^{\varrho+1-m}}\left|\varkappa(s)\right|^{\varrho-1}\varkappa(s)=0 \tag{4.1}$$

has a nonoscillatory solution $\varkappa(s)=\left(\left(\frac{\varrho}{\varrho-m}\right)^{\varrho}s^{\varrho-m}\right)^{1/(\varrho+1)}$ *if* $\delta=\left(\dfrac{\varrho}{\varrho+1}\right)^{\varrho+1}$*. That is to say, the constant* $\left(\dfrac{\varrho}{\varrho+1}\right)^{\varrho+1}$ *is the lower bound of oscillation for all solutions of the Eq* (4.1).

(ii) If $\alpha_1(s) = 1$ and $\varrho = 1$, the Euler second-order dynamic equations

$$\varkappa^{\Delta\Delta}(s) + \frac{\delta}{s\sigma(s)}\varkappa(s) = 0$$

and

$$\varkappa^{\Delta\Delta}(s) + \frac{\delta}{s\sigma(s)}\varkappa(\sigma(s)) = 0,$$

oscillate if $\delta > \frac{1}{4}$. This condition is known to be the optimal one for the second-order Euler differential equation

$$\varkappa''(s) + \frac{\delta}{s^2}\varkappa(s) = 0.$$

Example 4.2. *Consider a second-order half-linear sublinear delay dynamic equation*

$$\left[\sqrt{s}\frac{\varkappa^{\Delta}(s)}{\sqrt{\left|\varkappa^{\Delta}(s)\right|}}\right]^{\Delta} + \frac{\delta}{3\sqrt{3\sigma(s)\mu(s)}}\frac{\varkappa(\mu(s))}{\sqrt{|\varkappa(\mu(s))|}} = 0 \tag{4.2}$$

where $\delta > 0$ is a constant. It is evident that (1.2) *holds since*

$$\int_{\omega_0}^{\infty}\frac{\Delta\omega}{\alpha_1^{\frac{1}{\varrho}}(\omega)} = \int_{\omega_0}^{\infty}\frac{\Delta\omega}{\omega} = \infty,$$

by [4, Example 5.60]. Also,

$$\liminf_{s\to\infty}\frac{s^{1-\varrho}\sigma^{\varrho}(s)\mu^{\varrho}(s)\delta(s)}{\alpha_1(s)} = \frac{\delta}{3\sqrt{3}}.$$

Therefore, Theorem 2.1 implies that all solutions of Eq (4.2) *oscillate if $\delta > \frac{1}{\sqrt[4]{l}}$.*

5. Conclusions

(1) The results of this study are applicable to all time scales without any restriction conditions, such as: $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = \mathbb{R}$, and $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0 \text{ for } q > 1\}$.

(2) We concluded in this paper some sharp Kneser-type oscillation conditions for the second-order half-linear functional dynamic equation when $\mu(s) \leq s$ and $\mu(s) \geq s$. The results reveal an improvement in Kneser-type when compared to some known outcomes, as described below:

(i) Let $\mu(s) \leq s$ and $0 < \varrho \leq 1$. By virtue of

$$\frac{s^{1-\varrho}\sigma^{\varrho}(s)\mu^{\varrho}(s)\alpha_2(s)}{\alpha_1(s)} \geq \frac{s\mu^{\varrho}(s)\alpha_2(s)}{\alpha_1(s)}$$

and

$$\frac{1}{l^{\varrho(\varrho-1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1} < \frac{1}{l^{\varrho(\varrho+1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1} \quad \text{for } 0 < l < 1,$$

Theorem 2.1 improves Theorem 1.1 (condition (2.1) improves (1.11)).

(ii) Let $\mu(s) \leq s$ and $\varrho \geq 1$. Since

$$\frac{\sigma(s)\mu^{\varrho}(s)\alpha_2(s)}{\alpha_1(s)} \geq \frac{s\mu^{\varrho}(s)\alpha_2(s)}{\alpha_1(s)}$$

and

$$\frac{1}{l^{\varrho(\varrho-1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1} < \frac{1}{l^{\varrho(\varrho+1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1} \quad \text{for } 0 < l < 1,$$

Theorem 2.2 improves Theorem 1.1 (condition (2.12) improves (1.11)).

(iii) Let $\mu(s) \geq s$ and $0 < \varrho \leq 1$. By virtue of

$$\frac{s\sigma^{\varrho}(s)\alpha_2(s)}{\alpha_1(s)} \geq \frac{s^{\varrho+1}\alpha_2(s)}{\alpha_1(s)}$$

and

$$\frac{1}{l^{\varrho(\varrho-1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1} < \frac{1}{l^{\varrho(\varrho+1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1} \quad \text{for } 0 < l < 1,$$

Theorem 3.1 improves Theorem 1.1 (condition (3.1) improves (1.11)).

(iv) Let $\mu(s) \geq s$ and $\varrho \geq 1$. Since

$$\frac{s^{\varrho}\sigma(s)\alpha_2(s)}{\alpha_1(s)} \geq \frac{s^{\varrho+1}\alpha_2(s)}{\alpha_1(s)}$$

and

$$\frac{1}{l^{\varrho(\varrho-1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1} < \frac{1}{l^{\varrho(\varrho+1)}}\left(\frac{\varrho}{\varrho+1}\right)^{\varrho+1} \quad \text{for } 0 < l < 1,$$

Theorem 3.2 improves Theorem 1.1 (condition (3.3) improves (1.11)).

(3) It would be interesting to consider the Kneser-type oscillation criterion of all solutions of Eq (1.1) under the condition

$$\int_{s_0}^{\infty} \frac{\Delta\omega}{\alpha_1^{1/\varrho}(\omega)} < \infty.$$

Author contributions

Taher S. Hassan: Supervision,writing—original draft, writing—review editing, Formal analysis, Resources, and Investigation; Amir Abdel Menaem, Formal analysis, Resources; Hasan Nihal Zaidi, Formal analysis, Resources; Khalid Alenzi, Formal analysis, Resources; Bassant M. El-Matary: writing—review editing, Validation, Formal analysis, Resources, and Investigation. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that they have no competing interests.

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