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General and optimal decay rates for a system of wave equations with damping and a coupled source term

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Abstract: In this article, we aim to investigate the decay characteristics of a system consisting of two viscoelastic wave equations with Dirichlet boundary conditions, where the dispersion term and nonlinear weak damping term are taken into account. Under appropriate conditions, we establish both general and optimal decay results. This work generalizes and improves earlier results in the literature.

Keywords: general decay; coupled equations; Lyapunov function; relaxation function
Mathematics Subject Classification: 35L05, 35L20

1. Introduction

In this paper, we study the following system of nonlinear viscoelastic equations:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + |u_t|^{m-2}u_t = f_1(u,v), & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt} - \Delta v - \Delta v_{tt} + \int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + |v_t|^{r-2}v_t = f_2(u,v), & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain of R^n $(n \geq 1)$ with a smooth boundary $\partial\Omega$, g_1 and g_2 denote the kernel of the memory term and the index number $m, r \geq 2$, u and v represent the transverse displacements of waves, and f_1 and f_2 are given functions to be specified later.

To motivate our work, we recall some results related to our work. For a single viscoelastic wave equation of the form

$$u_{tt} - \Delta u - \omega\Delta u_t + \mu u_t = u \ln|u|, (x,t) \in \Omega \times (0,\infty),$$

Xu and Lian [1] studied the above initial boundary value problem at three different initial energy levels, they proved the local existence of a weak solution, and in the framework of a potential well, they showed the global existence and energy decay of the solution with sub-critical initial energy. Then, through a scaling technique, they parallelly extended all the results for the subcritical case to the critical case. A similar result was also obtained in [2, 3]. In [4], Xu et al. considered the following initial boundary value problem

$$u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t + u_t = u|u|^{p-1}, \text{ in } \Omega \times (0,\infty), \tag{1.2}$$

they obtained the invariant sets, and proved the existence and nonexistence of a global solution. Moreover, they got a finite time blow-up result for certain solutions under a high energy case. In [5, 6], Messaoudi proved, by using the perturbed energy method and under the supposition that $g'(t) \le -\xi(t)g(t)$, that the solution energy was a general decay, not necessarily of exponential or polynomial type. Under the same assumptions on a relaxation function, Liu [7] studied a viscoelastic wave equation with the dispersion term and proved that the decay rate of the solution energy was similar to that of the relaxation function. Furthermore, Messaoudi and Al-Khulaifi [8] established a general and optimal decay of the solution energy where the relaxation function satisfies $g'(t) \le -\xi(t)g^{\theta}(t)$, $1 \le \theta < \frac{3}{2}$. In [9], the author established the optimal explicit and general energy decay results under the condition $g'(t) \le -\xi(t)H(g(t))$, where H is an increasing and convex function near the origin and ξ is a non-increasing function.

Subsequently, the above methods used in the single viscoelastic wave equation were extended to coupled wave systems. For example, Li [10] considered the following wave system:

$$\begin{cases} |u_t|^{\rho}u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + |u_t|^{m-1}u_t = f_1(u,v), & (x,t) \in \Omega \times (0,+\infty), \\ |v_t|^{\rho}v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + |v_t|^{r-1}v_t = f_2(u,v), & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset R^n$ $(n \ge 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $0 < \rho$, $m, r \ge 1$. Under the following assumptions on the relaxation functions: $g_i(t) \le -\xi_i(t)g_i(t), t \ge 0, i = 1, 2$. Based on the potential well method and the perturbed energy method, the author obtained a general decay result of the solution. Furthermore, in [11], the author got some sufficient conditions on initial data such that the solution blows up in finite time at arbitrarily high initial energy. In the same direction and in the case of $\rho = 0$, Liu et al. [12] proved that the solution energy was an exponential decay or polynomial decay. Later, this result was furthered by Said-Houari [13], who demonstrated that the solution energy was a general decay. In the same nature, Liu [14] considered the following system:

$$\begin{cases} |u_t|^{\rho}u_{tt} - \Delta u - \gamma_1\Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + f(u,v) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ |v_t|^{\rho}v_{tt} - \Delta v - \gamma_2\Delta v_{tt} + \int_0^t h(t-\tau)\Delta v(\tau)d\tau + k(u,v) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in \Omega, \end{cases}$$

where $\gamma_1, \gamma_2 \ge 0$ are constans, ρ is a real number, $0 < \rho$, and $\Omega \subset R^n$ $(n \ge 1)$ is a bounded domain with a smooth boundary $\partial\Omega$. By exploiting the perturbed energy method, the author proved that the solution energy was an exponential and polynomial decay.

As far as we know, the decay property for the coupled system (1.1) has not been considered. Inspired by the literature [8], our aim in this paper is to extend some existing results for a single equation to the case of a couple viscoelastic wave system (1.1), while handling the additional difficulty caused by the nonlinear weak damping term and coupled source term. In this article, the energy decay result is new and the assumptions on the relaxation functions are weak. Then, the purpose of this paper is to study the decay property of solution energy for the system (1.1) by modifying the method used in the above literatures. Namely, under a certain class of relaxation functions and initial data, by using some inequalities and constructing a suitable Lyapunov function, we establish a general decay result for the system (1.1). Moreover, without restrictive conditions, we also obtain the optimal polynomial decay rate which seldom appears in previous literatures.

This article is organized as follows: In Section 2, some materials needed for our work are presented. In Section 3, we show the global existence of a solution and establish the general and optimal decay rates of the solution for the system (1.1).

2. Preliminaries

In this part, some theorems and lemmas needed for our work are given. First, we make the following assumptions:

(A1) $g_i : (0, +\infty) \to (0, +\infty)(i = 1, 2)$ are non-increasing C^1 functions satisfying

$$g_i(0) > 0, \quad 1 - \int_0^\infty g_1(\tau)d\tau = l_1 > 0, \quad 1 - \int_0^\infty g_2(\tau)d\tau = l_2 > 0.$$

(A2) There exists two positive differentiable functions $\xi_i(t) : [0, +\infty) \to (0, +\infty)$ such that

$$g_i'(t) \le -\xi_i(t) g_i^\gamma(t), \quad t \ge 0, \quad 1 \le \gamma < \frac{3}{2}, \quad i = 1, 2,$$

and the function $\xi_i(t)$ satisfies

$$\xi_i'(t) \le 0, \quad \int_0^{+\infty} \xi_i(t)dt = +\infty, \quad \forall\, t > 0.$$

(A3) For the functions $f_1$andf_2, we assume that

$$f_1(u, v) := [|u + v|^{2(p+1)}(u + v) + |u|^p u|v|^{p+2}],$$
$$f_2(u, v) := [|u + v|^{2(p+1)}(u + v) + |v|^p v|u|^{p+2}].$$

It is easy to verify that

$$u f_1(u, v) + v f_2(u, v) = 2(p + 2)F(u, v),$$
$$u_t f_1(u, v) + v_t f_2(u, v) = \tfrac{d}{dt}F(u, v),$$

where

$$F(u, v) = \frac{1}{2(p + 2)}[|u + v|^{2(p+2)} + 2|uv|^{p+2}].$$

(A4) For the nonlinearity, we assume that

$$\begin{cases} 1 < p < +\infty,\ n = 1, 2, & 1 < p \le \frac{n}{n-2},\ n \ge 3, \\ 2 \le m, r < +\infty,\ n = 1, 2, & 2 \le m, r \le \frac{2n}{n-2},\ n \ge 3. \end{cases}$$

For our work, we introduce the following functionals:

$$J(t) := \frac{1}{2}(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) + \frac{1}{2}(1 - \int_0^t g_1(\tau)d\tau)\|\nabla u\|_2^2 + \frac{1}{2}(1 - \int_0^t g_2(\tau)d\tau)\|\nabla v\|_2^2 + \frac{1}{2}[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] - \int_\Omega F(u, v)dx,$$

$$I(t) := \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + (1 - \int_0^t g_1(\tau)d\tau)\|\nabla u\|_2^2 + (1 - \int_0^t g_2(\tau)d\tau)\|\nabla v\|_2^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) - 2(p + 2)\int_\Omega F(u, v)dx,$$

$$E(t) := \frac{1}{2}(\|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) + \frac{1}{2}(1 - \int_0^t g_1(\tau)d\tau)\|\nabla u\|_2^2 - \int_\Omega F(u, v)dx + \frac{1}{2}[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] + \frac{1}{2}(1 - \int_0^t g_2(\tau)d\tau)\|\nabla v\|_2^2,$$

where

$$(g_1 \circ \nabla u)(t) = \int_0^t g_1(t - \tau)\|\nabla u(t) - \nabla u(\tau)\|_2^2 d\tau,$$

$$(g_2 \circ \nabla v)(t) = \int_0^t g_2(t - \tau)\|\nabla v(t) - \nabla v(\tau)\|_2^2 d\tau.$$

The following result is concerned with local existence and uniqueness of weak solutions to the system (1.1). We can easily obtain it by using the Faedo-Galerkin approximation methods and the Banach contraction mapping principle, which is similar to [15] with slight modification, and the process of the proof is standard, so we omit it here.

Theorem 2.1 ([15], Theorem 2.1)**.** Suppose that $(A1)$, $(A2)$, and $(A4)$ hold and initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ are given. Then there exists a unique local weak solution (u, v) of problem (1.1) defined in $[0, T]$ for some $T > 0$ small enough.

Lemma 2.2. Assume that $(A1)$–$(A4)$ hold. Let (u, v) be the solution of the system (1.1). Then the energy functional $E(t)$ is non-increasing. In addition, we get the following energy inequality:

$$\frac{dE(t)}{dt} \le -\|u_t\|_m^m - \|v_t\|_r^r + \frac{1}{2}(g_1' \circ \nabla u)(t) + \frac{1}{2}(g_2' \circ \nabla v)(t) \le 0, \quad \forall t \ge 0. \tag{2.1}$$

Proof. Multiplying the first two equations in system (1.1) by u_t and v_t, respectively, integrating over Ω, and then adding them up, we can get

$$\begin{aligned}
&\frac{d}{dt}\Big[\frac{1}{2}\Big(\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\Big)-\int_\Omega F(u,v)dx\Big]\\
&=-\|u_t\|_m^m-\|v_t\|_r^r+\int_0^t g_1(t-\tau)\int_\Omega \nabla u_t(t)\cdot\nabla u(\tau)dxd\tau\\
&\quad+\int_0^t g_2(t-\tau)\int_\Omega \nabla v_t(t)\cdot\nabla v(\tau)dxd\tau,
\end{aligned}\tag{2.2}$$

where

$$\begin{aligned}
&\int_0^t g_1(t-\tau)\int_\Omega \nabla u_t(t)\cdot\nabla u(\tau)dxd\tau\\
=&\int_0^t g_1(t-\tau)\int_\Omega \nabla u_t(t)\cdot\big[\nabla u(\tau)-\nabla u(t)\big]dxd\tau+\int_0^t g_1(t-\tau)\int_\Omega \nabla u_t(t)\cdot\nabla u(t)dxd\tau\\
=&-\frac{1}{2}\int_0^t g_1(t-\tau)\Big(\frac{d}{dt}\int_\Omega|\nabla u(\tau)-\nabla u(t)|^2dx\Big)d\tau+\int_0^t g_1(\tau)\Big(\frac{d}{dt}\frac{1}{2}\int_\Omega|\nabla u(t)|^2dx\Big)d\tau\\
=&-\frac{1}{2}\frac{d}{dt}\Big[\int_0^t g_1(t-\tau)\int_\Omega|\nabla u(\tau)-\nabla u(t)|^2dxd\tau\Big]+\frac{1}{2}\frac{d}{dt}\Big[\int_0^t g_1(\tau)\int_\Omega|\nabla u(t)|^2dxd\tau\Big]\\
&+\frac{1}{2}\int_0^t g_1'(t-\tau)\int_\Omega|\nabla u(\tau)-\nabla u(t)|^2dxd\tau-\frac{1}{2}g_1(t)\int_\Omega|\nabla u(t)|^2dx.
\end{aligned}\tag{2.3}$$

Similarly, we have

$$\begin{aligned}
&\int_0^t g_2(t-\tau)\int_\Omega \nabla v_t(t)\cdot\nabla v(\tau)dxd\tau\\
=&-\frac{1}{2}\frac{d}{dt}\Big[\int_0^t g_2(t-\tau)\int_\Omega|\nabla v(\tau)-\nabla v(t)|^2dxd\tau\Big]+\frac{1}{2}\frac{d}{dt}\Big[\int_0^t g_2(\tau)\int_\Omega|\nabla v(t)|^2dxd\tau\Big]\\
&+\frac{1}{2}\int_0^t g_2'(t-\tau)\int_\Omega|\nabla v(\tau)-\nabla v(t)|^2dxd\tau-\frac{1}{2}g_2(t)\int_\Omega|\nabla v(t)|^2dx.
\end{aligned}\tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2) yields

$$\begin{aligned}
&\frac{d}{dt}\Big[\frac{1}{2}(\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2)+\frac{1}{2}(1-\int_0^t g_1(\tau)d\tau)\|\nabla u\|_2^2+\frac{1}{2}(1-\int_0^t g_2(\tau)d\tau)\|\nabla v\|_2^2\\
&\quad+\frac{1}{2}[(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)]-\int_\Omega F(u,v)dx\Big]\\
&=-\|u_t\|_m^m-\|v_t\|_r^r+\frac{1}{2}[(g_1'\circ\nabla u)(t)+(g_2'\circ\nabla v)(t)]-\frac{1}{2}g_1(t)\|\nabla u\|_2^2-\frac{1}{2}g_2(t)\|\nabla v\|_2^2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
E'(t)&=-\|u_t\|_m^m-\|v_t\|_r^r+\frac{1}{2}[(g_1'\circ\nabla u)(t)+(g_2'\circ\nabla v)(t)]-\frac{1}{2}g_1(t)\|\nabla u\|_2^2-\frac{1}{2}g_2(t)\|\nabla v\|_2^2\\
&\le-\|u_t\|_m^m-\|v_t\|_r^r+\frac{1}{2}[(g_1'\circ\nabla u)(t)+(g_2'\circ\nabla v)(t)]\le 0.
\end{aligned}\tag{2.5}$$

This completes the proof. □

Now, we establish the global existence theorem.

Lemma 2.3. Assume the assumptions $(A1)$–$(A4)$ hold and the initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} \beta = \eta\Big[\frac{2(p+2)}{p+1}E(0)\Big]^{p+1} < 1, \\ I(0) = I(u_0, v_0) > 0, \end{cases} \tag{2.6}$$

and then we have

$$I(t) = I(u(t), v(t)) > 0, \quad t \in [0, T_m]. \tag{2.7}$$

Proof. Due to $I(u_0) > 0$, then by continuity of $I(t)$ about variable t, there exists a maximal time $0 < T^* < T$ such that

$$I(t) \geq 0, \quad t \in [0, T^*],$$

which implies that $\forall t \in [0, T^*]$,

$$\begin{aligned}
J(t) &= \frac{1}{2}(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) + \frac{1}{2}(1 - \int_0^t g_1(\tau)d\tau)\|\nabla u\|_2^2 + \frac{1}{2}(1 - \int_0^t g_2(\tau)d\tau)\|\nabla v\|_2^2 \\
&\quad + \frac{1}{2}\big[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\big] - \int_\Omega F(u, v)dx \\
&= \frac{1}{2(p+2)}I(t) + \frac{p+1}{2(p+2)}\Big[(1 - \int_0^t g_1(\tau)d\tau)\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \\
&\quad + (1 - \int_0^t g_2(\tau)d\tau)\|\nabla v\|_2^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\Big] \\
&\geq \frac{p+1}{2(p+2)}\Big[(1 - \int_0^t g_1(\tau)d\tau)\|\nabla u\|_2^2 + (g_1 \circ \nabla u)(t) + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \\
&\quad + (1 - \int_0^t g_2(\tau)d\tau)\|\nabla v\|_2^2 + (g_2 \circ \nabla v)(t)\Big].
\end{aligned} \tag{2.8}$$

By (A1) and (2.1), we see that

$$\begin{aligned}
l_1\|\nabla u\|_2^2 + l_2\|\nabla v\|_2^2 &\leq (1 - \int_0^t g_1(\tau)d\tau)\|\nabla u\|_2^2 + (1 - \int_0^t g_2(\tau)d\tau)\|\nabla v\|_2^2 \\
&\leq \frac{2(p+2)}{p+1}J(t) \leq \frac{2(p+2)}{p+1}E(t) \leq \frac{2(p+2)}{p+1}E(0).
\end{aligned} \tag{2.9}$$

By (A3), (2.25), and (2.6), we can deduce that

$$\begin{aligned}
2(p+2)\int_\Omega F(u, v)dx &\leq \eta(l_1\|\nabla u\|_2^2 + l_2\|\nabla v\|_2^2)^{p+2} \\
&= \eta(l_1\|\nabla u\|_2^2 + l_2\|\nabla v\|_2^2)^{p+1}(l_1\|\nabla u\|_2^2 + l_2\|\nabla v\|_2^2) \\
&\leq \eta\Big[\frac{2(p+2)}{p+1}E(0)\Big]^{p+1}(l_1\|\nabla u\|_2^2 + l_2\|\nabla v\|_2^2), \quad \forall t \in [0, T^*].
\end{aligned} \tag{2.10}$$

From (2.6) and (2.10), we infer that

$$
\begin{aligned}
&2(p+2)\int_\Omega F(u,v)dx\\
&\le \beta(l_1\|\nabla u\|_2^2 + l_2\|\nabla v\|_2^2)\\
&\le \beta(1-\int_0^t g_1(\tau)d\tau)\|\nabla u\|_2^2 + \beta(1-\int_0^t g_2(\tau)d\tau)\|\nabla v\|_2^2\\
&\le (1-\int_0^t g_1(\tau)d\tau)\|\nabla u\|_2^2 + (1-\int_0^t g_2(\tau)d\tau)\|\nabla v\|_2^2.
\end{aligned}
$$

Therefore,

$$I(t) > 0, \quad \forall t \in [0, T^*].$$

By repeating these steps, T^* is extended to T_m. □

Theorem 2.4. Assume that $(A1)$–$(A4)$ hold, the initial data $(u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$, $(v_0, v_1) \in H_0^1(\Omega)\times L^2(\Omega)$, and satisfies (2.6). Then the solution is bounded and global in time.

Proof. Applying Lemmas 2.2 and 2.3, we can obtain

$$
\begin{aligned}
E(0) \ge E(t) &= J(t) + \frac{1}{2}\Big(\|u_t\|_2^2 + \|v_t\|_2^2\Big)\\
&\ge \frac{p+1}{2(p+2)}\Big(l_1\|\nabla u\|_2^2 + l_2\|\nabla v\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\Big),
\end{aligned}
\tag{2.11}
$$

which implies that the solution of system (1.1) is global and bounded. □

Lemma 2.5 ([16]). *Suppose that $g \in C[0,\infty]$, $\omega \in L^1_{loc}(0,\infty)$, and $0 \le \theta \le 1$; and then we have that*

$$\int_0^t |g(\tau)\omega(\tau)|d\tau \le \Big\{\int_0^t |g(\tau)|^{1-\theta}|\omega(\tau)|d\tau\Big\}^{\frac{1}{\sigma+1}}\Big\{\int_0^t |g(\tau)|^{1+\frac{\theta}{\sigma}}|\omega(\tau)|d\tau\Big\}^{\frac{\sigma}{\sigma+1}}. \tag{2.12}$$

Lemma 2.6 ([8]). *Assume that g_i satisfies $(A1)$ and $(A2)$, for $i = 1, 2$, and then*

$$\int_0^{+\infty} \xi_i(t)g_i^{1-\theta}(t)dt < +\infty, \quad \forall\ 0 \le \theta < 2-\gamma. \tag{2.13}$$

Proof. From $(A1)$ and $(A2)$, we can obtain

$$\xi_i(t)g_i^{1-\theta}(t) = \xi_i(t)g_i^{\gamma}(t)g_i^{1-\theta-\gamma}(t) \le -g_i'(t)g_i^{1-\theta-\gamma}(t). \tag{2.14}$$

Integrating (2.14) over $(0,+\infty)$ and using the condition $0 \le \theta < 2-\gamma$, we can deduce that

$$\int_0^{+\infty} \xi_i(t)g_i^{1-\theta}(t)dt \le -\int_0^{+\infty} g_i'(t)g_i^{1-\theta-\gamma}(t)dt = \frac{-g_i^{2-\theta-\gamma}(t)}{2-\theta-\gamma}\Big|_0^{+\infty} < +\infty. \tag{2.15}$$

□

Lemma 2.7 ([17]). Suppose that the conditions $(A1)$–$(A4)$, (2.6), and (2.11) hold, $u \in L^\infty(0,T;H_0^1(\Omega))$, and g is a continuous function. For $0<\theta<1$, there exists $C>0$ such that

$$(g_1 \circ \nabla u)(t) \le C\Big\{\Big(\int_0^t g_1^{1-\theta}(\tau)d\tau\Big)E(0)\Big\}^{\frac{\gamma-1}{\gamma-1+\theta}}\Big[(g_1^\gamma \circ \nabla u)\Big]^{\frac{\theta}{\gamma-1+\theta}}(t). \tag{2.16}$$

Proof. From the hypothesis on u and Lemma 2.7, we can obtain

$$\begin{aligned}(g_1 \circ \nabla u)(t) &= \int_\Omega\int_0^t g_1(t-\tau)|\nabla u(t)-\nabla u(\tau)|^2 d\tau dx\\ &\le \Big\{\int_\Omega\int_0^t g_1^{1-\theta}(t-\tau)|\nabla u(t)-\nabla u(\tau)|^2 d\tau dx\Big\}^{\frac{\gamma-1}{\gamma-1+\theta}}\Big\{\int_\Omega\int_0^t g_1^{\gamma}(t-\tau)|\nabla u(t)-\nabla u(\tau)|^2 d\tau dx\Big\}^{\frac{\theta}{\gamma-1+\theta}}.\end{aligned} \tag{2.17}$$

Now, for $0<\theta<1$ and the conditions (2.6) and (2.11), we have

$$\begin{aligned}\int_\Omega\int_0^t g_1^{1-\theta}(t-\tau)|\nabla u(t)-\nabla u(\tau)|^2 d\tau dx &= \int_0^t g_1^{1-\theta}(t-\tau)\int_\Omega |\nabla u(t)-\nabla u(\tau)|^2 dx d\tau\\ &\le C\int_0^t g_1^{1-\theta}(\tau)d\tau\|u\|_2^2\\ &\le C\int_0^t g_1^{1-\theta}(\tau)d\tau E(0).\end{aligned} \tag{2.18}$$

The proof is now complete. □

By taking $\theta = \frac{1}{2}$, we have

$$(g_1 \circ \nabla u)(t) \le C\Big\{\int_0^t g_1^{\frac{1}{2}}(\tau)d\tau\Big\}^{\frac{2\gamma-2}{2\gamma-1}}\Big[(g_1^\gamma \circ \nabla u)\Big]^{\frac{1}{2\gamma-1}}(t). \tag{2.19}$$

Similarly, we have

$$(g_2 \circ \nabla v)(t) \le C\Big\{\int_0^t g_2^{1/2}(\tau)d\tau\Big\}^{\frac{2\gamma-2}{2\gamma-1}}\Big[(g_2^\gamma \circ \nabla v)\Big]^{\frac{1}{2\gamma-1}}(t). \tag{2.20}$$

The following lemmas are crucial for studying the decay of a solution.

Lemma 2.8 ([8]). Assume that $g_i(t)$ satisfies $(A1)$ and $(A2)$, and (u,v) is the solution of (1.1), and then we obtain

$$\begin{aligned}&\xi_1(t)(g_1 \circ \nabla u)(t) \le C[-E'(t)]^{\frac{1}{2\gamma-1}},\\ &\xi_2(t)(g_2 \circ \nabla v)(t) \le C[-E'(t)]^{\frac{1}{2\gamma-1}}.\end{aligned} \tag{2.21}$$

Proof. Multiplying both sides of (2.19) by $\xi_1(t)$ and using $(A2)$, (2.1), and (2.13), we can deduce that

$$\begin{aligned}&\xi_1(t)(g_1 \circ \nabla u)(t)\\ &\le C\xi_1^{\frac{2\gamma-2}{2\gamma-1}}(t)\Big[\int_0^t g_1^{\frac{1}{2}}(\tau)d\tau\Big]^{\frac{2\gamma-2}{2\gamma-1}}\xi_1^{\frac{1}{2\gamma-1}}(t)(g_1^\gamma \circ \nabla u)^{\frac{1}{2\gamma-1}}(t)\\ &\le C\Big[\int_0^t \xi_1(\tau)g_1^{\frac{1}{2}}(\tau)d\tau\Big]^{\frac{2\gamma-2}{2\gamma-1}}(\xi_1 g_1^\gamma \circ \nabla u)^{\frac{1}{2\gamma-1}}(t)\end{aligned}$$

$$\leq C\Big[\int_0^t \xi_1(\tau)g_1^{\frac{1}{2}}(\tau)d\tau\Big]^{\frac{2\gamma-2}{2\gamma-1}}(-g_1'\circ\nabla u)^{\frac{1}{2\gamma-1}}(t)$$
$$\leq C[-E'(t)]^{\frac{1}{2\gamma-1}}. \tag{2.22}$$

Similarly, we have

$$\xi_2(t)(g_2\circ\nabla v)(t)\leq C[-E'(t)]^{\frac{1}{2\gamma-1}}. \tag{2.23}$$

□

Lemma 2.9 ([18], Lemma 4.2)**.** There exist two positive constants γ_1 and γ_2 such that

$$\int_\Omega |f_i(u,v)|^2dx\leq\gamma_i(l_1\|\nabla u\|_2^2+l_2\|\nabla v\|_2^2)^{2p+3},\quad i=1,2. \tag{2.24}$$

Lemma 2.10 ([18], Lemma 3.2)**.** Assume that (A4) holds. Then there exists $\eta>0$ such that, for any $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$, we obtain

$$\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\leq\eta(l_1\|\nabla u\|_2^2+l_2\|\nabla v\|_2^2)^{p+2}. \tag{2.25}$$

3. The decay result

In this section, we state and prove the decay result for the global solutions. For obtaining the general decay rate estimate, let us consider the following functionals:

$$L(t):=ME(t)+\varepsilon\chi(t)+\zeta(t), \tag{3.1}$$

where M and ε are positive constants and

$$\chi(t):=\int_\Omega u_tudx+\int_\Omega v_tvdx+\int_\Omega\nabla u\nabla u_tdx+\int_\Omega\nabla v\nabla v_tdx, \tag{3.2}$$
$$\zeta(t):=\int_\Omega(\Delta u_t-u_t)\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx$$
$$+\int_\Omega(\Delta v_t-v_t)\int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx. \tag{3.3}$$

Lemma 3.1. For M large enough and ε small enough, we have that the following relation

$$\nu_1L(t)\leq E(t)\leq\nu_2L(t) \tag{3.4}$$

holds, where ν_1 and ν_2 are two positive constants.

Proof. Applying the Hölder inequality, Young inequality, Sobolev embedding theorem, and (2.9), for $\forall\delta>0$, we have

$$\int_\Omega u_tudx\leq\|u_t\|_2\|u\|_2\leq\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|u\|_2^2\leq\frac{1}{2}\|u_t\|_2^2+\frac{C_*^2}{2}\|\nabla u\|_2^2,$$
$$\int_\Omega\nabla u_t\nabla udx\leq\|\nabla u_t\|_2\|\nabla u\|_2\leq\frac{1}{2}\|\nabla u_t\|_2^2+\frac{1}{2}\|\nabla u\|_2^2,$$

$$
\begin{aligned}
&\int_\Omega \Delta u_t \int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx\\
&\quad \le \int_\Omega \nabla u_t \int_0^t g_1(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau dx\\
&\quad \le \delta\|\nabla u_t\|_2^2 + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t g_1(t-\tau)|\nabla u(t)-\nabla u(\tau)|d\tau\Big)^2 dx\\
&\quad \le \delta\|\nabla u_t\|_2^2 + \frac{1-l_1}{4\delta}(g_1\circ\nabla u)(t),
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega u_t \int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx\\
&\quad \le \delta\|u_t\|_2^2 + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau\Big)^2 dx\\
&\quad \le \delta\|u_t\|_2^2 + \frac{(1-l_1)C_*^2}{4\delta}(g_1\circ\nabla u)(t).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int_\Omega v_t v dx &\le \frac{1}{2}\|v_t\|_2^2 + \frac{C_*^2}{2}\|\nabla v\|_2^2,\\
\int_\Omega \nabla v_t \nabla v dx &\le \frac{1}{2}\|\nabla v_t\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2,\\
\int_\Omega \Delta v_t \int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx &\le \delta\|\nabla v_t\|_2^2 + \frac{1-l_2}{4\delta}(g_2\circ\nabla v)(t),\\
\int_\Omega v_t \int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx &\le \delta\|v_t\|_2^2 + \frac{(1-l_2)C_*^2}{4\delta}(g_2\circ\nabla v)(t).
\end{aligned}
$$

When M is large enough and ε is small enough, we arrive at

$$
\begin{aligned}
L(t) \le\ & ME(t) + \frac{\varepsilon}{2}\|u_t\|_2^2 + \frac{\varepsilon C_*^2}{2}\|\nabla u\|_2^2 + \frac{\varepsilon}{2}\|v_t\|_2^2 + \frac{\varepsilon C_*^2}{2}\|\nabla v\|_2^2\\
& + \frac{\varepsilon}{2}\|\nabla u_t\|_2^2 + \frac{\varepsilon}{2}\|\nabla u\|_2^2 + \frac{\varepsilon}{2}\|\nabla v_t\|_2^2 + \frac{\varepsilon}{2}\|\nabla v\|_2^2\\
& + \delta\|\nabla u_t\|_2^2 + \frac{1-l_1}{4\delta}(g_1\circ\nabla u)(t) + \delta\|u_t\|_2^2 + \frac{(1-l_1)C_*^2}{4\delta}(g_1\circ\nabla u)(t)\\
& + \delta\|\nabla v_t\|_2^2 + \frac{1-l_2}{4\delta}(g_2\circ\nabla v)(t) + \delta\|v_t\|_2^2 + \frac{(1-l_2)C_*^2}{4\delta}(g_2\circ\nabla v)(t)\\
\le\ & (\frac{M}{2}+\frac{\varepsilon}{2}+\delta)\|u_t\|_2^2 + (\frac{M}{2}+\frac{\varepsilon}{2}+\delta)\|v_t\|_2^2 + (\frac{M}{2}+\frac{\varepsilon}{2})\|\nabla u_t\|_2^2 + (\frac{M}{2}+\frac{\varepsilon}{2})\|\nabla v_t\|_2^2 - M\int_\Omega F(u.v)dx\\
& + \Big[\frac{M}{2}(1-\int_0^t g_1(\tau)d\tau) + \frac{\varepsilon C_*^2}{2} + \frac{\varepsilon}{2}\Big]\|\nabla u\|_2^2 + \Big[\frac{M}{2}(1-\int_0^t g_2(\tau)d\tau) + \frac{\varepsilon C_*^2}{2} + \frac{\varepsilon}{2}\Big]\|\nabla v\|_2^2\\
& + \Big[\frac{M}{2} + \frac{1-l_1}{4\delta}(1+C_*^2)\Big](g_1\circ\nabla u)(t) + \Big[\frac{M}{2} + \frac{1-l_2}{4\delta}(1+C_*^2)\Big](g_2\circ\nabla v)(t)\\
\le\ & \frac{1}{\nu_1}E(t).
\end{aligned}
$$

Analogously, we have

$$\begin{aligned}
L(t) \geq & (\frac{M}{2}-\frac{\varepsilon}{2}-\delta)\|u_t\|_2^2+(\frac{M}{2}-\frac{\varepsilon}{2}-\delta)\|v_t\|_2^2+(\frac{M}{2}-\frac{\varepsilon}{2})\|\nabla u_t\|_2^2+(\frac{M}{2}-\frac{\varepsilon}{2})\|\nabla v_t\|_2^2-M\int_\Omega F(u.v)dx \\
& +\Big[\frac{M}{2}(1-\int_0^t g_1(\tau)d\tau)-\frac{\varepsilon C_*^2}{2}-\frac{\varepsilon}{2}\Big]\|\nabla u\|_2^2+\Big[\frac{M}{2}(1-\int_0^t g_2(\tau)d\tau)-\frac{\varepsilon C_*^2}{2}-\frac{\varepsilon}{2}\Big]\|\nabla v\|_2^2 \\
& +\Big[\frac{M}{2}-\frac{1-l_1}{4\delta}(1+C_*^2)\Big](g_1\circ\nabla u)(t)+\Big[\frac{M}{2}-\frac{1-l_2}{4\delta}(1+C_*^2)\Big](g_2\circ\nabla v)(t) \\
\geq & \frac{1}{\nu_2}E(t).
\end{aligned}$$

□

Lemma 3.2. Under the assumptions $(A1)$–$(A4)$, let (u, v) be the solution of system (1.1). Then the functional

$$\chi(t)=\int_\Omega u_t u dx+\int_\Omega v_t v dx+\int_\Omega \nabla u\nabla u_t dx+\int_\Omega \nabla v\nabla v_t dx \tag{3.5}$$

satisfies

$$\begin{aligned}
\chi'(t) \leq & \|u_t\|_2^2+\|v_t\|_2^2+2(p+2)\int_\Omega F(u,v)dx+(\frac{C_*^2}{4\alpha}-\frac{l_1}{2})\|\nabla u\|_2^2 \\
& +\frac{1-l_1}{2l_1}(g_1\circ\nabla u)(t)+(\frac{C_*^2}{4\alpha}-\frac{l_2}{2})\|\nabla v\|_2^2+\frac{1-l_2}{2l_2}(g_2\circ\nabla v)(t) \\
& +\Big[1+\alpha C_*^{2m-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{m-2}\Big]\|\nabla u_t\|_2^2+\Big[1+\alpha C_*^{2r-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{r-2}\Big]\|\nabla v_t\|_2^2.
\end{aligned} \tag{3.6}$$

Proof. Taking a time derivative of (3.5) and applying Eq (1.1), we can deduce

$$\begin{aligned}
\chi'(t)= & \int_\Omega u_{tt}u dx+\|u_t\|_2^2+\int_\Omega v_{tt}v dx+\|v_t\|_2^2 \\
& +\int_\Omega \nabla u\nabla u_{tt}dx+\|\nabla u_t\|_2^2+\int_\Omega \nabla v\nabla v_{tt}dx+\|\nabla v_t\|_2^2 \\
= & \|u_t\|_2^2-\|\nabla u\|_2^2+\|\nabla u_t\|_2^2+\int_\Omega\int_0^t g_1(t-\tau)\nabla u(\tau)d\tau\nabla u(t)d\tau dx \\
& +\|v_t\|_2^2-\|\nabla v\|_2^2+\|\nabla v_t\|_2^2+\int_\Omega\int_0^t g_2(t-\tau)\nabla v(\tau)d\tau\nabla v(t)d\tau dx \\
& -\int_\Omega |u_t|^{m-2}u_t u dx+\int_\Omega u f_1 dx-\int_\Omega |v_t|^{r-2}v_t u dx+\int_\Omega v f_2 dx.
\end{aligned} \tag{3.7}$$

We now estimate the third term on the right-hand side of (3.7), yielding

$$\begin{aligned}
& \int_\Omega \nabla u(t)\int_0^t g_1(t-\tau)\nabla u(\tau)d\tau dx \\
& \leq \frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2}\int_\Omega\Big(\int_0^t g_1(t-\tau)\nabla u(\tau)d\tau\Big)^2 dx \\
& \leq \frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2}\int_\Omega\Big(\int_0^t g_1(t-\tau)(|\nabla u(\tau)-\nabla u(t)|+|\nabla u(t)|)d\tau\Big)^2 dx,
\end{aligned} \tag{3.8}$$

at present. We estimate the second term in the right-hand side of (3.8), for $\forall\ \eta_1 > 0$, and we arrive at

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t g_1(t-\tau)(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau \Big)^2 dx \\
&\le \int_\Omega \Big(\int_0^t g_1(t-\tau)|\nabla u(\tau) - \nabla u(t)| d\tau \Big)^2 dx + \int_\Omega \Big(\int_0^t g_1(t-\tau)|\nabla u(t)| d\tau \Big)^2 dx \\
&\quad + 2 \int_\Omega \Big(\int_0^t g_1(t-\tau)|\nabla u(\tau) - \nabla u(t)| d\tau \Big)\Big(\int_0^t g_1(t-\tau)|\nabla u(t)| d\tau \Big) dx \\
&\le (1 + \frac{1}{\eta_1}) \int_\Omega \Big(\int_0^t g_1(t-\tau)|\nabla u(\tau) - \nabla u(t)| d\tau \Big)^2 dx \\
&\quad + (1+\eta_1) \int_\Omega \Big(\int_0^t g_1(t-\tau)|\nabla u(t)| d\tau \Big)^2 dx \\
&\le (1 + \frac{1}{\eta_1})(1-l_1)(g_1 \circ \nabla u)(t) + (1+\eta_1)(1-l_1)^2 \|\nabla u\|_2^2.
\end{aligned} \tag{3.9}$$

Inserting (3.9) into (3.8), we get

$$\begin{aligned}
&\int_\Omega \nabla u(t) \int_0^t g_1(t-\tau) \nabla u(\tau) d\tau dx \\
&\quad \le \frac{1}{2}(1 + (1+\eta_1)(1-l_1)^2)\|\nabla u\|_2^2 + \frac{1}{2}(1 + \frac{1}{\eta_1})(1-l_1)(g_1 \circ \nabla u)(t).
\end{aligned} \tag{3.10}$$

Similarly, for $\forall\ \eta_2 > 0$, we have

$$\begin{aligned}
&\int_\Omega \nabla v(t) \int_0^t g_2(t-\tau) \nabla v(\tau) d\tau dx \\
&\quad \le \frac{1}{2}(1 + (1+\eta_2)(1-l_2)^2)\|\nabla v\|_2^2 + \frac{1}{2}(1 + \frac{1}{\eta_2})(1-l_2)(g_2 \circ \nabla v)(t).
\end{aligned} \tag{3.11}$$

For the fourth term in the right-hand side of (3.7), for $\forall \alpha > 0$, we can get

$$\begin{aligned}
\int_\Omega |u_t|^{m-2} u_t u dx &\le \alpha \|u_t\|_{2m-2}^{2m-2} + \frac{1}{4\alpha}\|u\|_2^2 \\
&\le \alpha C_*^{2m-2} \|\nabla u_t\|_2^{2m-2} + \frac{C_*^2}{4\alpha}\|\nabla u\|_2^2 \\
&\le \alpha C_*^{2m-2} \Big(\frac{2(p+2)E(0)}{p+1} \Big)^{m-2} \|\nabla u_t\|_2^2 + \frac{C_*^2}{4\alpha}\|\nabla u\|_2^2.
\end{aligned} \tag{3.12}$$

Similarly, we can get

$$\int_\Omega |v_t|^{r-2} v_t v dx \le \alpha C_*^{2r-2} \Big(\frac{2(p+2)E(0)}{p+1} \Big)^{r-2} \|\nabla v_t\|_2^2 + \frac{C_*^2}{4\alpha}\|\nabla v\|_2^2. \tag{3.13}$$

Inserting (3.10)–(3.13) into (3.7), and choosing

$$\eta_1 = l_1/(1-l_1), \quad \eta_2 = l_2/(1-l_2),$$

we can deduce

$$
\begin{aligned}
\chi'(t) \le\ & \|u_t\|_2^2 + \|v_t\|_2^2 + 2(p+2)\int_\Omega F(u,v)dx + (\frac{C_*^2}{4\alpha} - \frac{l_1}{2})\|\nabla u\|_2^2 \\
&+ \frac{1-l_1}{2l_1}(g_1 \circ \nabla u)(t) + (\frac{C_*^2}{4\alpha} - \frac{l_2}{2})\|\nabla v\|_2^2 + \frac{1-l_2}{2l_2}(g_2 \circ \nabla v)(t) \\
&+ \Big[1 + \alpha C_*^{2m-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{m-2}\Big]\|\nabla u_t\|_2^2 + \Big[1 + \alpha C_*^{2r-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{r-2}\Big]\|\nabla v_t\|_2^2. \qquad (3.14)
\end{aligned}
$$

□

Lemma 3.3. Under the assumptions $(A1)$–$(A4)$, letting (u, v) be the solution of (1.1), then the functional

$$
\begin{aligned}
\zeta(t) =\ & \int_\Omega (\Delta u_t - u_t)\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx \\
&+ \int_\Omega (\Delta v_t - v_t)\int_0^t g_2(t-\tau)\big(v(t)-v(\tau)\big)d\tau dx \qquad (3.15)
\end{aligned}
$$

satisfies

$$
\begin{aligned}
\zeta'(t) \le\ & \delta\Big(1 + 2(1-l_1)^2 + \gamma_1 l_1\Big(\frac{2(p+2)}{p+1}E(0)\Big)^{2(p+1)}\Big)\|\nabla u\|_2^2 + (1-l_1)\Big(2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta}\Big)(g_1 \circ \nabla u)(t) \\
&- \frac{g_1(0)}{4\delta}(1 + C_*^2)(g_1' \circ \nabla u)(t) + \Big[\delta - \int_0^t g_1(\tau)d\tau + \delta C_*^{2m-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{m-2}\Big]\|\nabla u_t\|_2^2 \\
&- (\int_0^t g_1(\tau)d\tau - \delta))\|u_t\|_2^2 + \delta\Big(1 + 2(1-l_2)^2 + \gamma_2 l_2\Big(\frac{2(p+2)}{p+1}E(0)\Big)^{2(p+1)}\Big)\|\nabla v\|_2^2 \\
&+ (1-l_2)\Big(2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta}\Big)(g_2 \circ \nabla v)(t) + \Big[\delta - \int_0^t g_2(\tau)d\tau + \delta C_*^{2r-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{r-2}\Big]\|\nabla v_t\|_2^2 \\
&- \frac{g_2(0)}{4\delta}(1 + C_*^2)(g_2' \circ \nabla v)(t) - (\int_0^t g_2(\tau)d\tau - \delta))\|v_t\|_2^2. \qquad (3.16)
\end{aligned}
$$

Proof. Taking the derivative of $\zeta(t)$ with respect to variable t, we have

$$
\begin{aligned}
\zeta'(t) =\ & \int_\Omega (\Delta u_{tt} - u_{tt})\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx \\
&+ \int_\Omega (\Delta u_t - u_t)\int_0^t g_1'(t-\tau)(u(t)-u(\tau))d\tau dx \\
&- \Big(\int_0^t g_1(\tau)d\tau\Big)\|\nabla u_t\|_2^2 - \Big(\int_0^t g_1(\tau)d\tau\Big)\|u_t\|_2^2 \\
&+ \int_\Omega (\Delta v_{tt} - v_{tt})\int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx \\
&+ \int_\Omega (\Delta v_t - v_t)\int_0^t g_1'(t-\tau)(v(t)-v(\tau))d\tau dx \\
&- \Big(\int_0^t g_2(\tau)d\tau\Big)\|\nabla v_t\|_2^2 - \Big(\int_0^t g_2(\tau)d\tau\Big)\|v_t\|_2^2. \qquad (3.17)
\end{aligned}
$$

Inserting Eq (1.1) into (3.17), we get

$$
\begin{aligned}
\zeta'(t) = & -\int_\Omega \Delta u \int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx \\
& + \int_\Omega \Big(\int_0^t g_1(t-\tau)\Delta u(\tau)d\tau\Big)\Big(\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau\Big)dx \\
& + \int_\Omega |u_t|^{m-2}u_t \int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx \\
& - \int_\Omega f_1(u,v) \int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx \\
& + \int_\Omega \Delta u_t \int_0^t g_1'(t-\tau)(u(t)-u(\tau))d\tau dx \\
& - \int_\Omega u_t \int_0^t g_1'(t-\tau)(u(t)-u(\tau))d\tau dx \\
& - \Big(\int_0^t g_1(\tau)d\tau\Big)\|\nabla u_t\|_2^2 - \Big(\int_0^t g_1(\tau)d\tau\Big)\|u_t\|_2^2 \\
& - \int_\Omega \Delta v \int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx \\
& + \int_\Omega \Big(\int_0^t g_2(t-\tau)\Delta v(\tau)d\tau\Big)\Big(\int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau\Big)dx \\
& + \int_\Omega |v_t|^{r-2}v_t \int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx \\
& - \int_\Omega f_2(u,v) \int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx \\
& + \int_\Omega \Delta v_t \int_0^t g_2'(t-\tau)(v(t)-v(\tau))d\tau dx \\
& - \int_\Omega v_t \int_0^t g_2'(t-\tau)(v(t)-v(\tau))d\tau dx \\
& - \Big(\int_0^t g_1(\tau)d\tau\Big)\|\nabla u_t\|_2^2 - \Big(\int_0^t g_1(\tau)d\tau\Big)\|u_t\|_2^2. \qquad (3.18)
\end{aligned}
$$

We now estimate the terms in the right-hand side of (3.18), for $\forall\ \delta > 0$, and we can deduce

$$
\begin{aligned}
\int_\Omega \Delta u \int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx \quad & \le \int_\Omega \nabla u \int_0^t g_1(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau dx \\
& \le \delta\|\nabla u\|_2^2 + \frac{1-l_1}{4\delta}(g_1 \circ \nabla u)(t), \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
& \int_\Omega \Big(\int_0^t g_1(t-\tau)\Delta u(\tau)d\tau\Big)\Big(\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau\Big)dx \\
& \quad \le \int_\Omega \Big(\int_0^t g_1(t-\tau)\nabla u(\tau)d\tau\Big)\Big(\int_0^t g_1(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\Big)dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \delta \int_\Omega \Big(\int_0^t g_1(t-\tau)\nabla u(\tau) d\tau\Big)^2 dx + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t g_1(t-\tau)(\nabla u(t) - \nabla u(\tau)) d\tau\Big)^2 dx \\
&\le \delta \int_\Omega \Big(\int_0^t g_1(t-\tau)(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau\Big)^2 dx \\
&\quad + \frac{1}{4\delta}\int_0^t g_1(t-\tau) ds \int_\Omega \int_0^t g_1(t-\tau)|\nabla u(t) - \nabla u(\tau)|^2 d\tau dx \\
&\le 2\delta \int_\Omega \Big(\int_0^t g_1(t-\tau)|\nabla u(\tau) - \nabla u(t)| d\tau\Big)^2 dx \\
&\quad + 2\delta(1-l_1)^2 \int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta}(1-l_1)(g_1 \circ \nabla u)(t) \\
&\le (2\delta + \frac{1}{4\delta})(1-l_1)(g_1 \circ \nabla u)(t) + 2\delta(1-l_1)^2 \|\nabla u\|_2^2,
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
&\int_\Omega |u_t|^{m-2}u_t \int_0^t g_1(t-\tau)(u(t)-u(\tau)) d\tau dx \\
&\quad \le \delta \|u_t\|_{2m-2}^{2m-2} + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t g_1(t-\tau)(u(t)-u(\tau)) d\tau\Big)^2 dx \\
&\quad \le \delta C_*^{2m-2} \|\nabla u_t\|_2^{2m-2} + \frac{(1-l_1)C_*^2}{4\delta} \int_\Omega \int_0^t g_1(t-\tau)(\nabla u(t) - \nabla u(\tau))^2 d\tau dx \\
&\quad \le \delta C_*^{2m-2} \Big(\frac{2(p+2)E(0)}{p+1}\Big)^{m-2} \|\nabla u_t\|_2^2 + \frac{(1-l_1)C_*^2}{4\delta}(g_1 \circ \nabla u)(t).
\end{aligned} \tag{3.21}
$$

For the fourth term, it follows from Lemma 2.9 that

$$
\begin{aligned}
&\int_\Omega f_1(u,v) \int_0^t g_1(t-\tau)(u(t)-u(\tau)) d\tau dx \\
&\quad \le \delta \int_\Omega f_1^2(u,v) dx + \frac{(1-l_1)C_*^2}{4\delta}(g_1 \circ \nabla u)(t) \\
&\quad \le \delta \gamma_1 (l_1 \|\nabla u\|_2^2 + l_2 \|\nabla v\|_2^2)^{2p+3} + \frac{(1-l_1)C_*^2}{4\delta}(g_1 \circ \nabla u)(t) \\
&\quad \le \delta \gamma_1 (l_1 \|\nabla u\|_2^2 + l_2 \|\nabla v\|_2^2)^{2(p+1)} (l_1 \|\nabla u\|_2^2 + l_2 \|\nabla v\|_2^2) + \frac{(1-l_1)C_*^2}{4\delta}(g_1 \circ \nabla u)(t) \\
&\quad \le \delta \gamma_1 \Big(\frac{2(p+2)}{p+1}E(0)\Big)^{2(p+1)} (l_1 \|\nabla u\|_2^2 + l_2 \|\nabla v\|_2^2) + \frac{(1-l_1)C_*^2}{4\delta}(g_1 \circ \nabla u)(t),
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
&\int_\Omega \Delta u_t \int_0^t g_1'(t-\tau)(u(t)-u(\tau)) d\tau dx \\
&\quad \le \int_\Omega \nabla u_t \int_0^t g_1'(t-\tau)(\nabla u(t) - \nabla u(\tau)) d\tau dx \\
&\quad \le \delta \|\nabla u_t\|_2^2 + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t g_1'(t-\tau)|\nabla u(t) - \nabla u(\tau)| d\tau\Big)^2 dx \\
&\quad \le \delta \|\nabla u_t\|_2^2 - \frac{g_1(0)}{4\delta}(g_1' \circ \nabla u)(t),
\end{aligned} \tag{3.23}
$$

$$\begin{aligned}
&\int_\Omega u_t \int_0^t g_1'(t-\tau)(u(t)-u(\tau))d\tau dx \\
&\leq \delta\|u_t\|_2^2 + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t g_1'(t-\tau)|u(t)-u(\tau)|d\tau\Big)^2 dx \\
&\leq \delta\|u_t\|_2^2 - \frac{g_1(0)C_*^2}{4\delta}(g_1'\circ\nabla u)(t). \qquad (3.24)
\end{aligned}$$

Similarly, we have

$$\int_\Omega \Delta v \int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx \leq \delta\|\nabla v\|_2^2 + \frac{1-l_2}{4\delta}(g_2\circ\nabla v)(t), \qquad (3.25)$$

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t g_2(t-\tau)\Delta v(\tau)d\tau\Big)\Big(\int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau\Big)dx \\
&\leq (2\delta + \frac{1}{4\delta})(1-l_2)(g_2\circ\nabla v)(t) + 2\delta(1-l_2)^2\|\nabla v\|_2^2, \qquad (3.26)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega |v_t|^{r-2}v_t \int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx \\
&\leq \delta C_*^{2r-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{r-2}\|\nabla v_t\|_2^2 + \frac{(1-l_2)C_*^2}{4\delta}(g_2\circ\nabla v)(t), \qquad (3.27)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega f_2(u,v)\int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx \\
&\leq \delta\gamma_2\Big(\frac{2(p+2)}{p+1}E(0)\Big)^{2(p+1)}(l_1\|\nabla u\|_2^2 + l_2\|\nabla v\|_2^2) + \frac{(1-l_2)C_*^2}{4\delta}(g_2\circ\nabla v)(t), \qquad (3.28)
\end{aligned}$$

$$\int_\Omega \Delta v_t \int_0^t g_2'(t-\tau)(v(t)-v(\tau))d\tau dx \leq \delta\|\nabla v_t\|_2^2 - \frac{g_2(0)}{4\delta}(g_2'\circ\nabla v)(t), \qquad (3.29)$$

$$\int_\Omega v_t \int_0^t g_2'(t-\tau)(v(t)-v(\tau))d\tau dx \leq \delta\|v_t\|_2^2 - \frac{g_2(0)C_*^2}{4\delta}(g_2'\circ\nabla v)(t). \qquad (3.30)$$

Combining (3.18)–(3.30), we can deduce that

$$\begin{aligned}
\zeta'(t) \leq\ & \delta\Big(1 + 2(1-l_1)^2 + \gamma_1 l_1\Big(\frac{2(p+2)}{p+1}E(0)\Big)^{2(p+1)}\Big)\|\nabla u\|_2^2 + (1-l_1)\Big(2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta}\Big)(g_1\circ\nabla u)(t) \\
& - \frac{g_1(0)}{4\delta}(1+C_*^2)(g_1'\circ\nabla u)(t) + \Big[\delta - \int_0^t g_1(\tau)d\tau + \delta C_*^{2m-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{m-2}\Big]\|\nabla u_t\|_2^2 \\
& - (\int_0^t g_1(\tau)d\tau - \delta))\|u_t\|_2^2 + \delta\Big(1 + 2(1-l_2)^2 + \gamma_2 l_2\Big(\frac{2(p+2)}{p+1}E(0)\Big)^{2(p+1)}\Big)\|\nabla v\|_2^2 \\
& + (1-l_2)\Big(2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta}\Big)(g_2\circ\nabla v)(t) + \Big[\delta - \int_0^t g_2(\tau)d\tau + \delta C_*^{2r-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{r-2}\Big]\|\nabla v_t\|_2^2 \\
& - \frac{g_2(0)}{4\delta}(1+C_*^2)(g_2'\circ\nabla v)(t) - (\int_0^t g_2(\tau)d\tau - \delta))\|v_t\|_2^2. \qquad (3.31)
\end{aligned}$$

□

Theorem 3.4. Let initial data $(u_0, u_1), (v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given and satisfy (2.6). Assume that $(A1)$–$(A4)$ hold. Then, for each $t_0 > 0$, there exist strictly positive constants k_1, k_2, and k_3 such that the solution of system (1.1) satisfies, for all $t \geq t_0$,

$$E(t) \leq k_1 e^{-k_2 \int_{t_0}^{t} \xi(\tau) d\tau}, \quad \gamma = 1, \tag{3.32}$$

$$E(t) \leq k_3 \Big[\frac{1}{1 + \int_{t_0}^{t} \xi^{2\gamma-1}(\tau) d\tau} \Big]^{\frac{1}{2\gamma-2}}, \quad 1 < \gamma < \frac{3}{2}, \tag{3.33}$$

where $\xi(t) := \min\{\xi_1(t), \xi_2(t)\}$.

Proof. Taking a derivative of (3.1), we can obtain

$$L'(t) = ME'(t) + \varepsilon\chi'(t) + \zeta'(t). \tag{3.34}$$

Since g_1 and g_2 are continuous and $g_i(0) > 0$, then there exists $t \geq t_0 > 0$ such that

$$\int_0^t g_1(\tau) d\tau \geq \int_0^{t_0} g_1(\tau) d\tau = g_0 > 0, \tag{3.35}$$

and

$$\int_0^t g_2(\tau) d\tau \geq \int_0^{t_0} g_2(\tau) d\tau = g_0 > 0. \tag{3.36}$$

By using (2.1), (3.6), (3.16), and (3.34), we arrive at

$$\begin{aligned}
L'(t) \leq & -(g_0 - \delta - \varepsilon)\|u_t\|_2^2 - (g_0 - \delta - \varepsilon)\|v_t\|_2^2 \\
& - \Big[(\frac{l_1}{2} - \frac{C_*^2}{4\alpha})\varepsilon - \delta\Big(1 + 2(1 - l_1)^2 + \gamma_1 l_1 \Big(\frac{2(p+2)}{p+1} E(0)\Big)^{2(p+1)}\Big) \Big] \|\nabla u\|_2^2 \\
& - \Big[(\frac{l_2}{2} - \frac{C_*^2}{4\alpha})\varepsilon - \delta\Big(1 + 2(1 - l_2)^2 + \gamma_2 l_2 \Big(\frac{2(p+2)}{p+1} E(0)\Big)^{2(p+1)}\Big) \Big] \|\nabla v\|_2^2 \\
& + \Big[\frac{M}{2} - \frac{g_1(0)}{4\delta}(1 + C_*^2) \Big] (g_1' \circ \nabla u)(t) + \Big[\frac{M}{2} - \frac{g_2(0)}{4\delta}(1 + C_*^2) \Big] (g_2' \circ \nabla v)(t) \\
& + (1 - l_1)\Big[\frac{\varepsilon}{2l_1} + 2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta} \Big] (g_1 \circ \nabla u)(t) + (1 - l_2)\Big[\frac{\varepsilon}{2l_2} + 2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta} \Big] (g_2 \circ \nabla v)(t) \\
& - \Big[g_0 - \delta - \delta C_*^{2m-2} \Big(\frac{2(p+2)E(0)}{p+1} \Big)^{m-2} - \varepsilon\Big(1 + \alpha C_*^{2m-2} \Big(\frac{2(p+2)E(0)}{p+1} \Big)^{m-2}\Big) \Big] \|\nabla u_t\|_2^2 \\
& - \Big[g_0 - \delta - \delta C_*^{2r-2} \Big(\frac{2(p+2)E(0)}{p+1} \Big)^{r-2} - \varepsilon\Big(1 + \alpha C_*^{2r-2} \Big(\frac{2(p+2)E(0)}{p+1} \Big)^{r-2}\Big) \Big] \|\nabla v_t\|_2^2 \\
& - M\|u_t\|_m^m - M\|v_t\|_r^r + 2(p+2)\varepsilon \int_\Omega F(u, v) dx.
\end{aligned} \tag{3.37}$$

At this point, we choose ε and δ small enough such that

$$\begin{aligned}
A_1 :=& g_0 - \delta - \varepsilon > 0, \\
A_2 :=& (\frac{l_1}{2} - \frac{C_*^2}{4\alpha})\varepsilon - \delta\Big(1 + 2(1 - l_1)^2 + \gamma_1 l_1 \Big(\frac{2(p+2)}{p+1} E(0)\Big)^{2(p+1)}\Big) > 0,
\end{aligned}$$

$$A_3 :=(\frac{l_2}{2} - \frac{C_*^2}{4\alpha})\varepsilon - \delta\Big(1 + 2(1 - l_2)^2 + \gamma_2 l_2\Big(\frac{2(p+2)}{p+1}E(0)\Big)^{2(p+1)}\Big) > 0,$$
$$A_4 :=g_0 - \delta - \delta C_*^{2m-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{m-2} - \varepsilon\Big[1 + \alpha C_*^{2m-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{m-2}\Big] > 0,$$
$$A_5 :=g_0 - \delta - \delta C_*^{2r-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{r-2} - \varepsilon\Big[1 + \alpha C_*^{2r-2}\Big(\frac{2(p+2)E(0)}{p+1}\Big)^{r-2}\Big] > 0.$$

Once ε and δ are fixed, we choose ital "M" sufficiently large such that

$$A_6 :=\frac{M}{2} - \frac{g_1(0)}{4\delta}(1 + C_*^2) > 0,$$
$$A_7 :=\frac{M}{2} - \frac{g_2(0)}{4\delta}(1 + C_*^2) > 0.$$

Hence, the inequality (3.37) can be changed to

$$\begin{aligned} L'(t) \le& -A_1\|u_t\|_2^2 - A_1\|v_t\|_2^2 - A_2\|\nabla u\|_2^2 - A_3\|\nabla v\|_2^2 - A_4\|\nabla u_t\|_2^2 - A_5\|\nabla v_t\|_2^2 \\ &+ A_6(g_1' \circ \nabla u)(t) + A_7(g_2' \circ \nabla v)(t) - M\|u_t\|_m^m - M\|v_t\|_r^r + 2(p+2)\varepsilon\int_\Omega F(u,v)dx \\ &+ (1-l_1)\Big[\frac{\varepsilon}{2l_1} + 2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta}\Big](g_1 \circ \nabla u)(t) + (1-l_2)\Big[\frac{\varepsilon}{2l_2} + 2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta}\Big](g_2 \circ \nabla v)(t). \end{aligned} \tag{3.38}$$

Then, there exist two positive constants m_1 and m_2 such that

$$L'(t) \le -m_1 E(t) + m_2[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla u)(t)], \quad \forall\, t \ge t_0. \tag{3.39}$$

We let $\xi(t) := \min\{\xi_1(t), \xi_2(t)\}$, since $\xi_1(t)$ and $\xi_2(t)$ are non-increasing and non-negative functions, and then the function $\xi(t)$ is a non-increasing and non-negative function. In the case when $\gamma = 1$, by multiplying both sides of (3.39) by $\xi(t)$ and using $(A2)$ and (2.1), we can deduce that

$$\begin{aligned} \xi(t)L'(t) &\le -m_1\xi(t)E(t) + m_2\xi(t)(g_1 \circ \nabla u)(t) + m_2\xi(t)(g_2 \circ \nabla v)(t) \\ &\le -m_1\xi(t)E(t) + m_2\xi_1(t)(g_1 \circ \nabla u)(t) + m_2\xi_2(t)(g_2 \circ \nabla v)(t) \\ &\le -m_1\xi(t)E(t) + m_2(\xi_1 g_1 \circ \nabla u)(t) + m_2(\xi_2 g_2 \circ \nabla v)(t) \\ &\le -m_1\xi(t)E(t) - m_2(g_1' \circ \nabla u)(t) - m_2(g_2' \circ \nabla v)(t) \\ &\le -m_1\xi(t)E(t) - m_2E'(t), \quad \forall\, t \ge t_0. \end{aligned} \tag{3.40}$$

We let $F(t) := \xi(t)L(t) + m_2E(t)$, which is equivalent to $E(t)$, since the function $L(t)$ is also equivalent to $E(t)$ and $E(t) \ge 0$, $\xi'(t) \le 0$. Then we can deduce that $\xi'(t)L(t) \le 0$. Then from (3.40), for some $m_3, m_4 > 0$, we can arrive at

$$\begin{aligned} F'(t) &= \xi'(t)L(t) + \xi(t)L'(t) + m_2E'(t) \\ &\le \xi(t)L'(t) + m_2E'(t) \\ &\le -m_3\xi(t)E(t) \\ &\le -m_4\xi(t)F(t). \end{aligned} \tag{3.41}$$

A simple integration of (3.41) leads to

$$F(t) \leq F(t_0)e^{-m_4 \int_{t_0}^{t} \xi(\tau)d\tau}, \quad \forall\, t \geq t_0. \tag{3.42}$$

In the case when $1 < \gamma < \frac{3}{2}$, we again consider (3.40) and use Lemma 2.8 to get, for $C_1 > 0$,

$$\begin{aligned} \xi(t)L'(t) &\leq -m_1\xi(t)E(t) + m_2\xi(t)(g_1 \circ \nabla u)(t) + m_2\xi(t)(g_2 \circ \nabla v)(t) \\ &\leq -m_1\xi(t)E(t) + m_2\xi_1(t)(g_1 \circ \nabla u)(t) + m_2\xi_2(t)(g_2 \circ \nabla v)(t) \\ &\leq -m_1\xi(t)E(t) + C_1[-E'(t)]^{\frac{1}{2\gamma-1}}, \quad \forall\, t \geq t_0. \end{aligned} \tag{3.43}$$

Multiplying both sides of (3.43) by $\xi^{\sigma}(t)E^{\sigma}(t)$, where $\sigma = 2\gamma - 2$, then applying the Young inequality, we can infer that

$$\begin{aligned} \xi^{\sigma+1}(t)E^{\sigma}(t)L'(t) &\leq -m_1\xi^{\sigma+1}(t)E^{\sigma+1}(t) + C_1(\xi E)^{\sigma}(t)[-E'(t)]^{\frac{1}{\sigma+1}}, \\ &\leq -m_1\xi^{\sigma+1}(t)E^{\sigma+1}(t) + C_1\Big[\varepsilon(\xi E)^{\sigma+1}(t) - C_\varepsilon E'(t)\Big] \\ &= -(m_1 - C_1\varepsilon)(\xi E)^{\sigma+1}(t) - C_1C_\varepsilon E'(t), \quad \forall t \geq t_0. \end{aligned} \tag{3.44}$$

We choose $\varepsilon < \frac{m_1}{C_1}$ such that $m_5 := m_1 - C_1\varepsilon > 0$ and thanks to $\xi'(t) \leq 0$ and $E'(t) \leq 0$, we can deduce that

$$\begin{aligned} (\xi^{\sigma+1}E^{\sigma}L)'(t) &= (\sigma+1)\xi^{\sigma}(t)\xi'(t)E^{\sigma}(t)L(t) + \sigma\xi^{\sigma+1}(t)E^{\sigma-1}(t)E'(t)L(t) + \xi^{\sigma+1}(t)E^{\sigma}(t)L'(t) \\ &\leq \xi^{\sigma+1}(t)E^{\sigma}(t)L'(t) \\ &\leq -m_5(\xi E)^{\sigma+1}(t) - C_1C_\varepsilon E'(t), \quad \forall t \geq t_0. \end{aligned} \tag{3.45}$$

Then we have

$$(\xi^{\sigma+1}E^{\sigma}L + C_1C_\varepsilon E)'(t) \leq -m_5(\xi E)^{\sigma+1}(t).$$

Let $G := \xi^{\sigma+1}E^{\sigma}L + C_1C_\varepsilon E \sim E$. Then we deduce that

$$G'(t) \leq -m_5(\xi E)^{\sigma+1}(t) \leq -m_6\xi^{2\gamma-1}G^{2\gamma-1}, \quad \forall t \geq t_0,\ m_6 > 0.$$

By integrating over (t_0, t) and applying the condition that $G \sim E$, we arrive at

$$E(t) \leq K\Big[\frac{1}{1 + \int_{t_0}^{t} \xi^{2\gamma-1}(\tau)d\tau}\Big]^{\frac{1}{2\gamma-2}}, \quad \forall t \geq t_0. \tag{3.46}$$

This completes the proof. □

4. Conclusions

In this paper, we consider a system of two viscoelastic wave equations with Dirichlet boundary conditions. By constructing a suitable Lyapunov function, we establish a general decay result. Moreover, without restrictive conditions, we also obtain the optimal polynomial decay result.

Author contributions

Qian Li: Conceptualization, methodology, writing original draft, writing and editing, formal analysis, funding acquisition; Yanyuan Xing: methodology, writing and editing, formal analysis supervision, funding acquisition. Both authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflicts of interest.

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