Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 29386–29403.
DOI: 10.3934/math.20241424
Received: 22 July 2024
Revised: 08 October 2024
Accepted: 11 October 2024
Published: 17 October 2024

Research article

Generalized Lie n-derivations on generalized matrix algebras

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Abstract: Let $\mathcal{G}$ be a generalized matrix algebra. We show that under certain conditions, each generalized Lie n-derivation associated with a linear map on $\mathcal{G}$ is a sum of a generalized derivation and a central map vanishing on all $(n-1)$-th commutators and is also a sum of a generalized inner derivation and a Lie n-derivation. As an application, generalized Lie n-derivations on von Neumann algebras are characterized.

Keywords: generalized Lie n-derivation; Lie n-derivation; generalized matrix algebra
Mathematics Subject Classification: 47B47, 47C15

1. Introduction

Let $\mathcal{A}$ be a unital algebra over a unital commutative ring R with the center $Z(\mathcal{A})$. Recall that a linear map G on $\mathcal{A}$ is called a *derivation* if $G(xy) = G(x)y + xG(y)$ for each $x, y \in \mathcal{A}$, G is a *generalized derivation* if there exists a linear map D on $\mathcal{A}$ such that $G(xy) = G(x)y+xD(y) = D(x)y+xG(y)$ for each $x, y \in \mathcal{A}$. Let $[x, y] = xy-yx$ denote the commutator or the Lie product of $x, y \in \mathcal{A}$. Define the sequence of polynomials: $p_1(x) = x$ and $p_n(x_1, \ldots, x_n) = [p_{n-1}(x_1, \ldots, x_{n-1}), x_n]$ for each $x_1, \ldots, x_n \in \mathcal{A}$. The polynomial $p_n(x_1, \ldots, x_n)$ is called the $(n-1)$-*th commutator*, where $n \geq 2$ is an integer. A linear map D on $\mathcal{A}$ is a *Lie n-derivation* if

$$D(p_n(x_1, \ldots, x_n)) = \sum_{i=1}^{n} p_n(x_1, \ldots, x_{i-1}, D(x_i), x_{i+1}, \ldots, x_n)$$

for each $x_1, \ldots, x_n \in \mathcal{A}$. In particular, every Lie 2-derivation (resp. Lie 3-derivation) is called a *Lie derivation* (resp. *Lie triple derivation*). During the past two decades, many scholars have studied the structure of Lie n-derivations and achieved remarkable results. In this paper, we restrict our attention to the generalized form of Lie n-derivations. A linear map G on $\mathcal{A}$ is a *generalized Lie n-derivation associated with L* if

$$G(p_n(x_1,\ldots,x_n)) = p_n(G(x_1),\ldots,x_n) + \sum_{i=2}^{n} p_n(x_1,\ldots,L(x_i),\ldots,x_n) \tag{1.1}$$

for each $x_1,\ldots,x_n \in \mathcal{A}$, where L is a linear map on $\mathcal{A}$. In particular, if $n = 2$ (resp. $n = 3$), G is the generalized Lie derivation (resp. generalized Lie triple derivation) associated with L; if $G = L$, G is the classical Lie n-derivation; and if $L = 0$, G is the Lie n-centralizer.

Bennis et al. [7] studied another generalized version of Lie derivations, which is defined as follows: A linear map G on $\mathcal{A}$ is a *Lie generalized derivation* if there exists a linear map D on $\mathcal{A}$ such that

$$G([x,y]) = G(x)y - G(y)x + xD(y) - yD(x)$$

for each $x, y \in \mathcal{A}$. However, these two generalized versions of Lie derivations are not equivalent, and here we focus on the first one.

A generalized Lie n-derivation G on $\mathcal{A}$ is *proper* if $G = d + \tau$, where $d : \mathcal{A} \to \mathcal{A}$ is a generalized derivation and $\tau : \mathcal{A} \to Z(\mathcal{A})$ is a linear map vanishing on all $(n-1)$-th commutators of $\mathcal{A}$. In the recent past, the evaluation of conditions under which a generalized Lie n-derivation is proper has attracted the attention of many researchers. Lin [13] proved that each generalized Lie n-derivation on triangular algebras is proper under suitable assumptions. Jabeen [11] provided some conditions under which each generalized Lie n-derivation on generalized matrix algebras is proper. Feng and Qi [10] showed that each generalized Lie n-derivation on von Neumann algebras without central summands of type I_1 is proper. Benkovič [5] stated that under certain assumptions every generalized Lie n-derivation G on unital algebras $\mathcal{A}$ with a nontrivial idempotent is of the form

$$G(x) = \lambda x + \delta(x), \tag{1.2}$$

for each $x \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})$ and δ is a Lie n-derivation on $\mathcal{A}$.

However, the precondition of the afore-mentioned works is that L in (1.1) is an associated Lie n-derivation. In this paper, we relax this assumption by considering L to be merely a linear map. Note that for any linear map $L : \mathcal{A} \to Z(\mathcal{A})$, if $G = 0$, then L satisfies (1.1), which does not necessarily imply that L is a Lie n-derivation [6]. Consequently, the task of characterizing (1.1) when L is a linear map presents a complex and meaningful challenge that calls for new methodologies to address.

Meanwhile, Benkovič [6] also pointed out that every generalized Lie n-derivation G associated with a linear map L on triangular algebras is of the form (1.2) under some conditions. Motivated by Benkovič's work, we aim to describe generalized Lie n-derivations on generalized matrix algebras when L is a linear map by using a method different from [6].

2. Main theorem

As preliminaries, we introduce some notations about generalized matrix algebras that play an important role in the proof of our main result.

Let A and B be two unital algebras over a unital commutative ring R with units e and f, respectively. A Morita context consists of A, B, two bimodules (A, B)-bimodule M and (B, A)-bimodule N, and two bimodule homomorphisms called the bilinear pairings $\Phi_{MN} : M \otimes_B N \to A$ and $\Psi_{NM} : N \otimes_A M \to B$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} M \otimes_B N \otimes_A M & \xrightarrow[\Phi_{MN} \otimes I_M]{} & A \otimes_A M \\ \Big\downarrow{\scriptstyle I_M \otimes \Psi_{NM}} & & \Big\downarrow{\scriptstyle \cong} \\ M \otimes_B B & \xrightarrow{\cong} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} N \otimes_A M \otimes_B N & \xrightarrow[\Psi_{NM} \otimes I_N]{} & B \otimes_B M \\ \Big\downarrow{\scriptstyle I_N \otimes \Phi_{MN}} & & \Big\downarrow{\scriptstyle \cong} \\ N \otimes_A A & \xrightarrow{\cong} & N \end{array}$$

If $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then $\mathcal{G} = \mathcal{G}(A, M, N, B) = \begin{pmatrix} A & M \\ N & B \end{pmatrix} = \Big\{ x = \begin{pmatrix} a & m \\ t & b \end{pmatrix} \mid a \in A, m \in M, t \in N, b \in B \Big\}$ forms an algebra under matrix-like addition and multiplication, where at least one of the two bimodules M and N is distinct from zero. Such an algebra is called a *generalized matrix algebra*. All associative algebras with nontrivial idempotents are isomorphic to generalized matrix algebras. In particular, when $M = 0$ or $N = 0$, $\mathcal{G}$ is the triangular algebra. We further assume that M is a faithful (A, B)-bimodule, and N is a faithful (B, A)-bimodule.

The center of $\mathcal{G}$ is

$$Z(\mathcal{G}) = \Big\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid am = mb, na = bn \ \text{ for each } m \in M, n \in N \Big\}.$$

Define two projections $\pi_A : \mathcal{G} \to A$ and $\pi_B : \mathcal{G} :\to B$ by $\pi_A(x) = a$ and $\pi_B(x) = b$, where $x = \begin{pmatrix} a & m \\ t & b \end{pmatrix} \in \begin{pmatrix} A & M \\ N & B \end{pmatrix} = \mathcal{G}$. Moreover, $\pi_A(Z(\mathcal{G})) \subseteq Z(A)$ and $\pi_B(Z(\mathcal{G})) \subseteq Z(B)$. It follows from [14, Claim 1] that there exists a unique algebra isomorphism φ from $\pi_A(Z(\mathcal{G}))$ to $\pi_B(Z(\mathcal{G}))$ such that $am = m\varphi(a)$ and $\varphi(a)n = na$ for each $a \in Z(A), m \in M, n \in N$. Hence, for each $m \in M$, if $am = mb$, then $a + b \in Z(\mathcal{G})$, where $a \in A$ and $b \in B$. For more information about generalized matrix algebras, see [18].

For each $x \in \mathcal{G}$, we consider the following condition:

$$[x, \mathcal{G}] \subseteq Z(\mathcal{G}) \Rightarrow x \in Z(\mathcal{G}). \tag{2.1}$$

Some specific examples of unital algebras satisfying the condition (2.1) are commutative algebras, triangular algebras, matrix algebras, and prime algebras.

We are in a position to give the following theorem.

Theorem 2.1. *Let $\mathcal{G} = \mathcal{G}(A, M, N, B)$ be a unital $(n-1)$-torsion-free generalized matrix algebra, where $n \geq 3$ is an integer. Assume that*

(i) $Z(A) = \pi_A(Z(\mathcal{G}))$ and $Z(B) = \pi_B(Z(\mathcal{G}))$;
(ii) A or B does not contain nonzero central ideals;
(iii) A or B satisfies the condition (2.1);
(iv) For each $m \in M$ and $t \in N$, the condition $mN = 0 = Nm$ implies $m = 0$, $Mt = 0 = tM$ implies $t = 0$.

Suppose that G and L are linear maps on $\mathcal{G}$. Then G and L satisfy

$$G(p_n(x_1, x_2, \ldots, x_n)) = p_n(G(x_1), x_2, \ldots, x_n) + \sum_{i=2}^{n} p_n(x_1, \ldots, L(x_i), \ldots, x_n)$$

for each $x_1, x_2, \ldots, x_n \in \mathcal{G}$ *if and only if* $G = D + \tau$ *and* $L = H + \gamma$*, where* D *is a generalized derivation associated with a derivation* H*,* τ *and* γ *are linear maps from* $\mathcal{G}$ *into* $Z(\mathcal{G})$*, and* τ *vanishes on each* $(n-1)$*-th commutator.*

The sufficiency is obvious, the necessity can be realized via a series of lemmas. By direct calculation, we have the following lemma.

Lemma 2.2. *For each* $x \in \mathcal{G}$*, we have*

$$\begin{aligned} p_n(x, e, \ldots, e) &= (-1)^{n-1} exf + fxe, \\ p_n(x, f, \ldots, f) &= (-1)^{n-1} fxe + exf. \end{aligned} \tag{2.2}$$

Lemma 2.3. $\begin{pmatrix} eL(e)e & 0 \\ 0 & fL(e)f \end{pmatrix} \in Z(\mathcal{G})$ *and* $\begin{pmatrix} eL(f)e & 0 \\ 0 & fL(f)f \end{pmatrix} \in Z(\mathcal{G})$.

Proof. Let $m \in M$. Applying (2.2) yields

$$\begin{aligned} G((-1)^{n-1}m) &= G(p_n(m, e, \ldots, e)) \\ &= p_n(G(m), e, \ldots, e) + \sum_{i=2}^{n} p_n(m, e, \ldots, \underbrace{L(e)}_{ith-place}, \ldots, e) \\ &= (-1)^{n-1} eG(m)f + fG(m)e + (n-1)((-1)^{n-2} e[m, L(e)]f + f[m, L(e)]e) \\ &= (-1)^{n-1} eG(m)f + fG(m)e + (n-1)(-1)^{n-2} e[m, L(e)]f. \end{aligned} \tag{2.3}$$

Multiplying e from the left side and f from the right side of (2.3), thus $mL(e)f = eL(e)m$. Then $\begin{pmatrix} eL(e)e & 0 \\ 0 & fL(e)f \end{pmatrix} \in Z(\mathcal{G})$. Similarly, one can obtain $\begin{pmatrix} eL(f)e & 0 \\ 0 & fL(f)f \end{pmatrix} \in Z(\mathcal{G})$. □

In the sequel, we define linear maps $\varphi : \mathcal{G} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{G}$ by

$$\varphi(x) = G(x) - [x, eL(e)f - fL(e)e]$$

and

$$\psi(x) = L(x) - [x, eL(e)f - fL(e)e]$$

for each $x \in \mathcal{G}$. It is easy to check that

$$\varphi(p_n(x_1, x_2, \ldots, x_n)) = p_n(\varphi(x_1), x_2, \ldots, x_n) + \sum_{i=2}^{n} p_n(x_1, \ldots, \psi(x_i), \ldots, x_n)$$

for each $x_1, x_2, \ldots, x_n \in \mathcal{G}$.

Lemma 2.4. $\varphi(e), \varphi(f) \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ *and* $\psi(e), \psi(f) \in Z(\mathcal{G})$.

Proof. By a simple calculation, we have

$$\psi(e) = L(e) - [e, eL(e)f - fL(e)e] = \begin{pmatrix} eL(e)e & 0 \\ 0 & fL(e)f \end{pmatrix} \in Z(\mathcal{G}). \tag{2.4}$$

On account of $[e,1]=0=[G(e),1]$ and (2.2), one can see that

$$\begin{aligned}0&=G(p_n(e,1,e,\ldots,e))\\&=p_n(G(e),1,e,\ldots,e)+p_n(e,L(1),e,\ldots,e)+\sum_{i=3}^{n}p_n(e,1,e,\ldots,\underbrace{L(e)}_{ith-place},\ldots,e)\\&=p_n(e,L(1),e,\ldots,e)\\&=(-1)^{n-2}e[e,L(1)]f+f[e,L(1)]e\\&=(-1)^{n-2}eL(1)f-fL(1)e.\end{aligned}\tag{2.5}$$

Multiplying e from the left and f from the right of (2.5), one can conclude that $eL(1)f=0$. Similarly, $fL(1)e=0$. In view of Lemma 2.3, we have $L(1)=\begin{pmatrix}eL(1)e & 0\\ 0 & fL(1)f\end{pmatrix}=\begin{pmatrix}e(L(e)+L(f))e & 0\\ 0 & f(L(e)+L(f))f\end{pmatrix}\in Z(\mathcal{G})$. By (2.4), we obtain $\psi(f)=\psi(1)-\psi(e)=L(1)-\psi(e)\in Z(\mathcal{G})$.

It follows from $\psi(e)\in Z(\mathcal{G})$ that

$$\begin{aligned}0&=\varphi(p_n(f,e,\ldots,e))\\&=p_n(\varphi(f),e,\ldots,e)+\sum_{i=2}^{n}p_n(f,\ldots,\underbrace{\psi(e)}_{ith-place},\ldots,e)\\&=(-1)^{n-1}e\varphi(f)f+f\varphi(f)e.\end{aligned}\tag{2.6}$$

Now observe that $e\varphi(f)f=0$ and $f\varphi(f)e=0$, and hence $\varphi(f)\in\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$. Applying the similar calculation as above, we have $\varphi(e)\in\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$. □

Lemma 2.5. *$\varphi(M)\subseteq M$ and $\varphi(N)\subseteq N$, there exist linear maps $k_{12}:M\to Z(\mathcal{G})$ and $k_{21}:N\to Z(\mathcal{G})$ such that $\psi(M)-k_{12}(M)\subseteq M$ and $\psi(N)-k_{21}(N)\subseteq N$.*

Proof. For each $m\in M$, since $\psi(e)\in Z(\mathcal{G})$ and (2.2), we obtain

$$\begin{aligned}(-1)^{n-1}\varphi(m)&=\varphi(p_n(m,e,\ldots,e))\\&=p_n(\varphi(m),e,\ldots,e)+\sum_{i=2}^{n}p_n(m,\ldots,\underbrace{\psi(e)}_{ith-place},\ldots,e)\\&=(-1)^{n-1}e\varphi(m)f+f\varphi(m)e.\end{aligned}\tag{2.7}$$

Multiplying e and f from both sides of (2.7), respectively, one can obtain

$$e\varphi(m)e=0 \text{ and } f\varphi(m)f=0.\tag{2.8}$$

If n is even, it follows from (2.7) that $f\varphi(m)e=0$.

If n is odd, for each $m, m', m'' \in M$, by $[m, m'] = 0$ and $\psi(f) \in Z(\mathcal{G})$, one can see that

$$\begin{aligned}
0 &= \varphi(p_n(m, m', m'', f, \ldots, f)) \\
&= p_n(\varphi(m), m', m'', f, \ldots, f) + p_n(m, \psi(m'), m'', f, \ldots, f) \\
&= e[[\varphi(m), m'] + [m, \psi(m')], m'']f + (-1)^{n-3} f[[\varphi(m), m'] + [m, \psi(m')], m'']e \\
&= e[[\varphi(m), m'] + [m, \psi(m')], m'']f \\
&= e([\varphi(m), m'] + [m, \psi(m')])m'' - m''([\varphi(m), m'] + [m, \psi(m')])f.
\end{aligned}$$

Hence, we arrive at

$$\begin{pmatrix} e([\varphi(m), m'] + [m, \psi(m')])e & 0 \\ 0 & f([\varphi(m), m'] + [m, \psi(m')])f \end{pmatrix} \in Z(\mathcal{G}). \tag{2.9}$$

It follows from (2.9) that

$$e([\varphi(m), m'] + [m, \psi(m')])e \in Z(A), \quad f([\varphi(m), m'] + [m, \psi(m')])f \in Z(B).$$

In addition, by $[m, m'] = 0$ and $\psi(f) \in Z(\mathcal{G})$, we have

$$\begin{aligned}
[m, \psi(m')] &= p_n(m, f, \ldots, f, \psi(m')) \\
&= \varphi(p_n(m, f, \ldots, f, m')) - p_n(\varphi(m), f, \ldots, f, m') \\
&= -p_n(\varphi(m), f, \ldots, f, m') \\
&= -[(-1)^{n-2} f\varphi(m)e + e\varphi(m)f, m'] \\
&= (-1)^{n-1}[f\varphi(m)e, m'] \\
&= [f\varphi(m)e, m'].
\end{aligned} \tag{2.10}$$

Combining (2.8), (2.10) and $[e\varphi(m)f, m'] = 0$, we have $[f\varphi(m)e, m'] = [\varphi(m), m'] = [m, \psi(m')]$. According to (2.9), we have

$$\begin{aligned}
Z(\mathcal{G}) &\ni e([\varphi(m), m'] + [m, \psi(m')])e + f([\varphi(m), m'] + [m, \psi(m')])f \\
&= 2(e[\varphi(m), m']e + f[\varphi(m), m']f) \\
&= 2(f\varphi(m)m' - m'\varphi(m)e) \\
&= 2([f\varphi(m)e, m']).
\end{aligned}$$

Therefore,

$$[f\varphi(m)e, m'] \in Z(\mathcal{G}). \tag{2.11}$$

Hence $f\varphi(m)eM \subseteq Z(B)$ and $Mf\varphi(m)e \subseteq Z(A)$. Without loss of generality, we assume that A does not contain nonzero central ideals. Since $Mf\varphi(m)e$ is a central ideal of A, we get $Mf\varphi(m)e = 0$ and then $f\varphi(m)eM = 0$ by (2.11). In view of condition (iv), we obtain $f\varphi(m)e = 0$ for each $m \in M$. According to (2.8), $\varphi(M) \subseteq M$.

For each $m \in \mathcal{A}$, it follows from $\psi(e) \in Z(\mathcal{G})$ and $\varphi(f) \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ that

$$(-1)^{n-1}\varphi(m) = \varphi(p_n(f, m, e, \ldots, e))$$

$$
\begin{aligned}
&= p_n(\varphi(f), m, e, \ldots, e) + p_n(f, \psi(m), e, \ldots, e) \\
&= (-1)^{n-2} e[\varphi(f), m]f + f[\varphi(f), m]e + (-1)^{n-2} e[f, \psi(m)]f + f[f, \psi(m)]e \\
&= (-1)^{n-2} e\varphi(f)m - (-1)^{n-2} m\varphi(f)f - (-1)^{n-2} e\psi(m)f + f\psi(m)e. \qquad (2.12)
\end{aligned}
$$

Multiplying f from left and e by right of (2.12) and using the relation $\varphi(M) \subseteq M$, we arrive at

$$f\psi(m)e = (-1)^{n-1} f\varphi(m)e = 0.$$

This leads to $\psi(M) \subseteq \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.

Moreover, for each $m, m' \in M$, $\psi(e) \in Z(\mathcal{G})$ and $\varphi(M) \subseteq M$ imply that

$$
\begin{aligned}
0 &= \varphi(p_n(m', m, e, \ldots, e)) \\
&= p_n(\varphi(m'), m, e, \ldots, e) + p_n(m', \psi(m), e, \ldots, e) \\
&= p_n(m', \psi(m), e, \ldots, e) \\
&= (-1)^{n-2} e[m', \psi(m)]f + f[m', \psi(m)]e \\
&= (-1)^{n-2} m'\psi(m)f - (-1)^{n-2} e\psi(m)m'.
\end{aligned}
$$

Therefore, $\begin{pmatrix} e\psi(m)e & 0 \\ 0 & f\psi(m)f \end{pmatrix} \in Z(\mathcal{G})$. Define a linear map $k_{12} : M \to Z(\mathcal{G})$ by $k_{12}(m) = \psi(m) - e\psi(m)f = e\psi(m)e + f\psi(m)f$ for each $m \in M$. Then $\psi(m) - k_{12}(m) = e\psi(m)f \in M$.

In a similar manner, we obtain $\varphi(N) \subseteq N$, and there exists a linear map $k_{21} : N \to Z(\mathcal{G})$ such that $\psi(N) - k_{21}(N) \subseteq N$. □

Lemma 2.6. *There exist linear maps $\tau_1 : A \to Z(\mathcal{G})$, $\tau_2 : B \to Z(\mathcal{G})$, $\gamma_1 : A \to Z(\mathcal{G})$ and $\gamma_2 : B \to Z(\mathcal{G})$ such that $\varphi(A) - \tau_1(A) \subseteq A$, $\varphi(B) - \tau_2(B) \subseteq B$, $\psi(A) - \gamma_1(A) \subseteq A$ and $\psi(B) - \gamma_2(B) \subseteq B$.*

Proof. For each $a \in A$, in view of $[a, f] = 0$, $\psi(f) \in Z(\mathcal{G})$ and (2.2), we have

$$
\begin{aligned}
0 &= \varphi(p_n(a, f, \ldots, f)) \\
&= p_n(\varphi(a), f, \ldots, f) + \sum_{i=2}^{n} p_n(a, f, \ldots, \underbrace{\psi(f)}_{ith-place}, \ldots, f) \\
&= (-1)^{n-1} f\varphi(a)e + e\varphi(a)f.
\end{aligned}
$$

It follows that $e\varphi(a)f = 0 = f\varphi(a)e$. Hence $\varphi(a) \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Furthermore, by using $\varphi(f) \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\psi(e) \in Z(\mathcal{G})$, we have

$$
\begin{aligned}
0 &= \varphi(p_n(f, a, e, \ldots, e)) \\
&= p_n(\varphi(f), a, e, \ldots, e) + p_n(f, \psi(a), e, \ldots, e) \\
&= (-1)^{n-2} e[\varphi(f), a]f + f[\varphi(f), a]e + (-1)^{n-2} e[f, \psi(a)]f + f[f, \psi(a)]e \\
&= (-1)^{n-1} a\varphi(f)f + f\varphi(f)a + (-1)^{n-1} e\psi(a)f + f\psi(a)e
\end{aligned}
$$

$$= (-1)^{n-1} e\psi(a)f + f\psi(a)e.$$

This implies $e\psi(a)f = f\psi(a)e = 0$. Hence

$$\psi(a) = e\psi(a)e + f\psi(a)f \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{2.13}$$

Therefore, $\varphi(a) \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\psi(a) \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then $\varphi(b) \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\psi(b) \in \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ can be proved analogously.

In addition, for each $a \in A, m \in M$ and $b \in B$, using $[a, b] = 0$ together with $\psi(f) \in Z(\mathcal{G})$, we have

$$\begin{aligned}
0 &= \varphi(p_n(a, b, m, f, \ldots, f)) \\
&= p_n(\varphi(a), b, m, f, \ldots, f) + p_n(a, \psi(b), m, f, \ldots, f) \\
&= (-1)^{n-3} f[[\varphi(a), b] + [a, \psi(b)], m]e + e[[\varphi(a), b] + [a, \psi(b)], m]f \\
&= e[[\varphi(a), b] + [a, \psi(b)], m]f \\
&= e([\varphi(a), b] + [a, \psi(b)])m - m([\varphi(a), b] + [a, \psi(b)])f.
\end{aligned}$$

This implies that

$$\begin{pmatrix} e([\varphi(a), b] + [a, \psi(b)])e & 0 \\ 0 & f([\varphi(a), b] + [a, \psi(b)])f \end{pmatrix} \in Z(\mathcal{G}). \tag{2.14}$$

Besides,

$$\begin{aligned}
\begin{pmatrix} e([\varphi(a), b] + [a, \psi(b)])e & 0 \\ 0 & f([\varphi(a), b] + [a, \psi(b)])f \end{pmatrix} &= \begin{pmatrix} e[a, \psi(b)]e & 0 \\ 0 & f[\varphi(a), b]f \end{pmatrix} \\
&= \begin{pmatrix} [a, e\psi(b)e] & 0 \\ 0 & [f\varphi(a)f, b)] \end{pmatrix}.
\end{aligned}$$

It follows from (2.14) that

$$\begin{pmatrix} [a, e\psi(b)e] & 0 \\ 0 & [f\varphi(a)f, b)] \end{pmatrix} \in Z(\mathcal{G}). \tag{2.15}$$

Multiplying (2.15) from both sides by f, we arrive at $[f\varphi(a)f, b] \in Z(B)$. The condition (2.1) leads to $f\varphi(a)f \in Z(B)$. There exists a unique $z \in Z(\mathcal{G})$ such that $f\varphi(a)f = fz$. Therefore,

$$\varphi(a) = e\varphi(a)e + f\varphi(a)f = e\varphi(a)e + fz = (e\varphi(a)e - ez) + z.$$

Define a linear map $\tau_1 : A \to Z(\mathcal{G})$ by $\tau_1(a) = z$. Then

$$\varphi(a) - \tau_1(a) = e\varphi(a)e - ez \in A.$$

By $f\varphi(a)f \in Z(B)$ and (2.15), we have $e\psi(b)e \in Z(A)$. There exists a unique $z' \in Z(\mathcal{G})$ such that

$$\psi(b) = e\psi(b)e + f\psi(b)f = ez' + f\psi(b)f = z' + (f\psi(b)f - fz').$$

We can also define a linear map $\gamma_2 : B \to Z(\mathcal{G})$ by $\gamma_2(b) = z'$. Then

$$\psi(b) - \gamma_2(b) = f\psi(b)f - fz' \in B.$$

Next, we prove that τ_1 and γ_2 are unique. Suppose that $\varphi(a) = \tau_1(a) + ez = \tau_1''(a) + ez''$, which implies that $\tau_1(a) - \tau_1''(a) = ez'' - ez \in A \cap Z(\mathcal{G}) = \{0\}$. Hence $\tau_1 = \tau_1''$. A similar proof yields that γ_2 is unique.

Similarly, there exist linear maps $\tau_2 : B \to Z(\mathcal{G})$ and $\gamma_1 : A \to Z(\mathcal{G})$ such that $\varphi(B) - \tau_2(B) \subseteq B$, $\psi(A) - \gamma_1(A) \subseteq A$.

□

Now, for each $x = \begin{pmatrix} a & m \\ t & b \end{pmatrix} \in \begin{pmatrix} A & M \\ N & B \end{pmatrix} = \mathcal{G}$, define linear maps $d : \mathcal{G} \to \mathcal{G}$, $h : \mathcal{G} \to \mathcal{G}$, $\tau : \mathcal{G} \to Z(\mathcal{G})$ and $\gamma : \mathcal{G} \to Z(\mathcal{G})$ by

$$\begin{aligned} \tau(x) = \tau_1(a) + \tau_2(b), &\quad d(x) = \varphi(x) - \tau(x), \\ \gamma(x) = \gamma_1(a) + \gamma_2(b) + k_{12}(m) + k_{21}(t), &\quad h(x) = \psi(x) - \gamma(x). \end{aligned}$$

By Lemmas 2.5 and 2.6, it follows that

$$\begin{aligned} d(A) \subseteq A,\ d(M) = \varphi(M) \subseteq M,\ d(N) = \varphi(N) \subseteq N,\ d(B) \subseteq B, \\ h(A) \subseteq A,\ h(M) \subseteq M,\ h(N) \subseteq N,\ h(B) \subseteq B. \end{aligned}$$

Lemma 2.7. *d is a generalized derivation associated with a derivation h on $\mathcal{G}$.*

Proof. We divide the proof into the following six claims:
Claim 1: For each $a \in A$, $m \in M$, $t \in N$ and $b \in B$,

$$\begin{aligned} d(am) &= h(a)m + ad(m) = d(a)m + ah(m), \\ d(bt) &= h(b)t + bd(t) = d(b)t + bh(t), \\ d(mb) &= h(m)b + md(b) = d(m)b + mh(b), \\ d(ta) &= h(t)a + td(a) = d(t)a + th(a). \end{aligned}$$

Next, we prove only the first equation, and the others can be proven in a similar way. Since τ and γ are linear maps from $\mathcal{G}$ into $Z(\mathcal{G})$, and $\psi(f) \in Z(\mathcal{G})$, we have

$$\begin{aligned} d(am) = \varphi(am) &= -\varphi(p_n(m, a, f, \ldots, f)) \\ &= -p_n(\varphi(m), a, f, \ldots, f) - p_n(m, \psi(a), f, \ldots, f) \\ &= -p_n(d(m) + \tau(m), a, f, \ldots, f) - p_n(m, h(a) + \gamma(a), f, \ldots, f) \\ &= -p_n(d(m), a, f, \ldots, f) - p_n(m, h(a), f, \ldots, f) \\ &= h(a)m + ad(m). \end{aligned}$$

In addition,

$$d(am) = \varphi(am) = \varphi(p_n(a, m, f, \ldots, f))$$

$$\begin{aligned} &= p_n(\varphi(a), m, f, \ldots, f) + p_n(a, \psi(m), f, \ldots, f) \\ &= p_n(d(a), m, f, \ldots, f) + p_n(a, h(m), f, \ldots, f) \\ &= d(a)m + ah(m). \end{aligned}$$

Claim 2: For each $a, a' \in A$ and $b, b' \in B$,

$$h(aa') = h(a)a' + ah(a'), \quad d(aa') = h(a)a' + ad(a'),$$
$$h(bb') = h(b)b' + bh(b'), \quad d(bb') = h(b)b' + bd(b').$$

By Claim 1, for each $a, a' \in A, m \in M$, one can obtain

$$d(aa'm) = h(aa')m + aa'd(m) \tag{2.16}$$
$$= d(aa')m + aa'h(m) \tag{2.17}$$

and

$$d(aa'm) = h(a)a'm + ad(a'm)$$
$$= h(a)a'm + ah(a')m + aa'd(m) \tag{2.18}$$
$$= h(a)a'm + ad(a')m + aa'h(m). \tag{2.19}$$

Comparing (2.16) with (2.18) and (2.17) with (2.19), respectively, we have

$$(h(aa') - h(a)a' - ah(a'))m = 0, \quad (d(aa') - h(a)a' - ad(a'))m = 0,$$

for each $m \in M$. It follows that $h(aa') = h(a)a' + ah(a')$ and $d(aa') = h(a)a' + ad(a')$. Similarly, we can prove $h(bb') = h(b)b' + bh(b')$ and $d(bb') = h(b)b' + bd(b')$ for each $b, b' \in B$.
Claim 3: For each $m \in M$ and $t \in N$,

$$d(mt) = h(m)t + md(t) = d(m)t + mh(t),$$
$$d(tm) = h(t)m + td(m) = d(t)m + th(m).$$

Let $m \in M$ and $t \in N$. Since τ and γ are linear maps from $\mathcal{G}$ into $Z(\mathcal{G})$, and $\psi(f) \in Z(\mathcal{G})$, it follows that

$$\varphi(p_n(m, f, \ldots, f, t)) = p_n(\varphi(m), f, \ldots, f, t) + p_n(m, f, \ldots, f, \psi(t)).$$

Then

$$d([m, t]) + \tau([m, t]) = [\varphi(m), t] + [m, \psi(t)] = [d(m), t] + [m, h(t)].$$

This leads to

$$\begin{pmatrix} d(m)t + mh(t) - d(mt) & 0 \\ 0 & d(tm) - td(m) - h(t)m \end{pmatrix} = \tau([m, t]) \in Z(\mathcal{G}).$$

Multiplying e and f from both sides of the above equation, respectively, we find that $d(m)t + mh(t) - d(mt) = e\tau([m, t]) \in Z(A)$ and $d(tm) - td(m) - h(t)m = f\tau([m, t]) \in Z(B)$. Without loss of generality, we assume that A does not contain nonzero central ideals. Set

$$\varepsilon(m, t) := d(mt) - d(m)t - mh(t) \in Z(A).$$

Therefrore, for each $a \in A$, $m \in M$, and $t \in N$,

$$\begin{aligned}\varepsilon(am, t) &= d(amt) - d(am)t - amh(t)\\ &= h(a)mt + ad(mt) - h(a)mt - ad(m)t - amh(t)\\ &= ad(mt) - ad(m)t - amh(t)\\ &= a\varepsilon(m, t),\end{aligned}$$

which leads that $A\varepsilon(m, t)$ is a central ideal of A. Hence, $\varepsilon(m, t) = 0$. Thus $d(mt) = d(m)t + mh(t)$. Moreover, $d(tm) = h(t)m + td(m)$. Using the same computational method on relation

$$\varphi(p_n(t, f, \ldots, f, m)) = p_n(\varphi(t), f, \ldots, f, m) + p_n(t, f, \ldots, f, \psi(m)),$$

we obtain $d(mt) = h(m)t + md(t)$ and $d(tm) = d(t)m + th(m)$ for each $m \in M$ and $t \in N$.
Claim 4: For each $m \in M$ and $t \in N$,

$$h(mt) = h(m)t + mh(t), \quad h(tm) = h(t)m + th(m).$$

For each $m, m' \in M$ and $t \in N$, on account of Claim 3, we arrive at

$$\begin{aligned}d(mtm') &= h(m)tm' + md(tm')\\ &= h(m)tm' + mh(t)m' + mtd(m')\end{aligned} \tag{2.20}$$

and

$$d(mtm') = h(mt)m' + mtd(m'). \tag{2.21}$$

Comparing (2.20) with (2.21), we obtain $(h(mt) - h(m)t - mh(t))m' = 0$ for each $m' \in M$. Hence $h(mt) = h(m)t + h(t)$. Similarly, $h(tm) = h(t)m + th(m)$.
Claim 5: For each $a \in A$, $m \in M$, $t \in N$ and $b \in B$,

$$\begin{aligned}h(am) = h(a)m + ah(m), &\quad h(mb) = h(m)b + mh(b),\\ h(ta) = h(t)a + ah(t), &\quad h(bt) = h(b)t + bh(t).\end{aligned}$$

Next, we will only prove the first equation, while the other equations can be proven using similar methods. For each $a \in A, m \in M, 0 \neq t \in N$, it follows from Claim 4 that

$$h(amt) = h(am)t + amh(t), \tag{2.22}$$

$$h(amt) = h(a)mt + ah(mt) = h(a)mt + ah(m)t + amh(t). \tag{2.23}$$

Comparing (2.22) with (2.23), we can obtain $(h(am) - h(a)m - ah(m))t = 0$. Besides,

$$d(tam) = d(t)am + th(am), \tag{2.24}$$

$$d(tam) = d(ta)m + tah(m) = d(t)am + th(a)m + tah(m). \tag{2.25}$$

Hence, (2.24) and (2.25) imply that $t(h(am) - h(a)m - ah(m)) = 0$ for each $t \in N$. Condition (iv) forces that $h(am) = h(a)m + ah(m)$ for each $a \in A$ and $m \in M$.

Claim 6: For each $a, a' \in A$ and $b, b' \in B$,

$$d(aa') = h(a)a' + ad(a') = d(a)a' + ah(a')$$
$$d(bb') = h(b)b' + bd(b') = d(b)b' + bh(b').$$

In view of Claims 1 and 3, for each $a, a' \in A$ and $m \in N$, we have

$$\begin{aligned} d(aa'm) &= d(a)a'm + ah(a'm) \\ &= d(a)a'm + ah(a')m + aa'h(m). \end{aligned} \tag{2.26}$$

Comparing (2.17) with (2.26), $(d(aa') - d(a)a' - ah(a'))m = 0$ for each $m \in M$. It follows that $d(aa') = d(a)a' + ah(a')$. Combining with Claim 2, we have $d(aa') = h(a)a' + ad(a') = d(a)a' + ah(a')$. Making similar discussion, we get $d(bb') = h(b)b' + bd(b') = d(b)b' + bh(b')$, for each $b, b' \in B$.

Thus $d(xy) = h(x)y + xd(y) = d(x)y + xh(y)$ and $h(xy) = h(x)y + xh(y)$ for each $x, y \in \mathcal{G}$, i.e., h is a derivation and d is a generalized derivation associated with h. □

Proof of Theorem 2.1. Since τ and γ are linear maps from $\mathcal{G}$ into $Z(\mathcal{G})$, for each $x_i \in \mathcal{G}$ $(i = 1, \dots, n)$, by the lemmas 2.2–2.7, we have

$$\begin{aligned} \tau(p_n(x_1, x_2, \dots, x_n)) &= \varphi(p_n(x_1, x_2, \dots, x_n)) - d(p_n(x_1, x_2, \dots, x_n)) \\ &= p_n(\varphi(x_1), x_2, \dots, x_n) + p_n(x_1, \psi(x_2), \dots, x_n) \\ &\quad + \dots + p_n(x_1, \dots, \psi(x_n)) - p_n(d(x_1), x_2, \dots, x_n) \\ &\quad - p_n(x_1, h(x_2), \dots, x_n) \dots - p_n(x_1, x_2, \dots, h(x_n)) \\ &= 0. \end{aligned}$$

Moreover, for each $x \in \mathcal{G}$, define maps $D, H : \mathcal{G} \to \mathcal{G}$ as:

$$D(x) = d(x) + [x, eL(e)f - fL(e)e], \quad H(x) = h(x) + [x, eL(e)f - fL(e)e].$$

Obviously, D is a generalized derivation associated with H, and H is also a derivation on $\mathcal{G}$. Then

$$\begin{aligned} G(x) &= \varphi(x) + [x, eL(e)f - fL(e)e] \\ &= d(x) + \tau(x) + [x, eL(e)f - fL(e)e] \\ &= D(x) + \tau(x) \end{aligned}$$

and

$$\begin{aligned} L(x) &= L(x) + [x, eL(e)f - fL(e)e] \\ &= h(x) + \gamma(x) + [x, eL(e)f - fL(e)e] \\ &= H(x) + \gamma(x). \end{aligned}$$

The proof is completed.

□

In the following, we investigate the relation of generalized inner derivations, Lie n-derivations, and generalized Lie n-derivations. Let us start with strong generalized Lie n-derivations. Let $\mathcal{A}$ be a unital algebra. A linear map G on $\mathcal{A}$ is called a *strong generalized Lie n-derivation* if G is the sum of a generalized inner derivation and a Lie n-derivation. Recall that a linear map I on $\mathcal{A}$ is called a *generalized inner derivation* if $I(x) = mx + xm'$ for each $x \in \mathcal{A}$, where m and m' are fixed elements of $\mathcal{A}$. It is obvious that every generalized derivation on $\mathcal{A}$ is the sum of a derivation and a generalized inner derivation of the form $I(x) = \lambda x$ for every $x \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})$.

In particular, if $n = 2$, Adrabi et al. [2] investigated strong generalized Lie derivations and generalized Lie derivations on bounded quiver algebras associated with a finite acyclic quiver. Furthermore, Bennis et al. [7] gave a complete description of the relation between generalized Lie derivations and strong generalized Lie derivations on unital algebras with nontrivial idempotents and trivial extension algebras. In the sequel, we present a fact.

Lemma 2.8. *Let $\mathcal{A}$ be a unital algebra. If each Lie n-derivation on $\mathcal{A}$ is proper, then the following assertions are equivalent:*

(1) G is a proper generalized Lie n-derivation on $\mathcal{A}$;
(2) G is a strong generalized Lie n-derivation, that is, $G = I + \delta$, where δ is a Lie n-derivation on $\mathcal{A}$ and I is a generalized inner derivation on $\mathcal{A}$ of the form $I = \lambda x$ for every $x \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})$.

Proof. (1) $\Longrightarrow$ (2) Let G be a proper generalized Lie n-derivation on $\mathcal{A}$. Then $G = d + \tau$, where d is a generalized derivation on $\mathcal{A}$ and $\tau : \mathcal{A} \to Z(\mathcal{A})$ is a linear map vanishing on all $(n-1)$-th commutators on $\mathcal{A}$. In addition, $d = h + I$, where h is a derivation on $\mathcal{A}$ and I is a generalized inner derivation of the form $I(x) = \lambda x$ for every $x \in \mathcal{A}$ with $\lambda \in Z(\mathcal{A})$. Hence $G = I + h + \tau$. Clearly $\delta := h + \tau$ is a Lie n-derivation, thus G is a strong generalized Lie derivation.

(2) $\Longrightarrow$ (1) If $G = I + \delta$, where δ is a Lie n-derivation on $\mathcal{A}$ and I is a generalized inner derivation on $\mathcal{A}$ of the form $I = \lambda x$ for every $x \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})$. Since every Lie n-derivation δ on $\mathcal{A}$ is proper, then $\delta = h + \tau$, where h is a derivation and $\tau : \mathcal{A} \to Z(\mathcal{A})$ is a linear map vanishing on all $(n-1)$-th commutators on $\mathcal{A}$. Therefore, $G = d + \tau$, where $d := I + h$ is a generalized derivation and $\tau : \mathcal{A} \to Z(\mathcal{A})$ is a linear map vanishing on all $(n-1)$-th commutators on $\mathcal{A}$. □

Corollary 2.9. *Let $\mathcal{G} = \mathcal{G}(A, M, N, B)$ be a unital $(n-1)$-torsion-free generalized matrix algebra, where $n \geq 3$ is an integer. Assume that*

(i) $Z(A) = \pi_A(Z(\mathcal{G}))$ and $Z(B) = \pi_B(Z(\mathcal{G}))$;
(ii) A or B does not contain nonzero central ideals;
(iii) A or B satisfies the condition (2.1);
(iv) For each $m \in M$ and $t \in N$, the condition $mN = 0 = Nm$ implies $m = 0$, $Mt = 0 = tM$ implies $t = 0$.

Suppose that G and L are linear maps on $\mathcal{G}$ satisfying

$$G(p_n(x_1, x_2, \ldots, x_n)) = p_n(G(x_1), x_2, \ldots, x_n) + \sum_{i=2}^{n} p_n(x_1, \ldots, L(x_i), \ldots x_n)$$

for each $x_1, x_2, \ldots, x_n \in \mathcal{G}$, then $G = I + \delta$, where δ is a Lie n-derivation on $\mathcal{G}$ and I is a generalized inner derivation on $\mathcal{G}$.

Proof. Since every Lie n-derivation on generalized matrix algebras is proper [17], by Lemma 2.8, every generalized Lie n-derivation associated with a linear map on generalized matrix algebras is a strong generalized Lie n-derivation under the conditions (i)–(iv). □

3. Applications

In particular, if $G = L$ in (1.1), G is a Lie n-derivation. In recent years, many scholars have studied the conditions under which every Lie n-derivation is proper on generalized matrix algebras [17], unital algebras with a nontrivial idempotent [8], von Neumann algebras without central summands of type I_1 [1], and so on. Here, we limit our attention to some applications of Theorem 2.1.

Let A be a unital algebra and $\mathcal{M}_s(A)$ be the algebra of all $s \times s$ matrices over A, where $s \geq 2$ is an integer. Then $\mathcal{M}_s(A)$ is a generalized matrix algebra with the form $\begin{pmatrix} A & \mathcal{M}_{1\times(s-1)}(A) \\ \mathcal{M}_{(s-1)\times 1}(A) & \mathcal{M}_{(s-1)\times(s-1)}(A) \end{pmatrix}$. Note that $Z(\mathcal{M}_s(A)) = Z(A) \cdot I_s$, where I_s is the unit of $\mathcal{M}_s(A)$. In addtion, $\mathcal{M}_s(A)$ does not contain nonzero central ideals [9, Lemma 1] and satisfies the conditions (iii) (see [4, Example 5.6]) and (iv) (see [17, Lemma 1]) of Theorem 2.1. As a consequence of Theorem 2.1, the following corollary holds.

Corollary 3.1. *Let A be a $(n-1)$-torsion-free unital algebra and $\mathcal{M}_s(A)$ be a full matrix algebra with $s \geq 3$. Suppose that G and L are linear maps on $\mathcal{M}_s(A)$. Then G and L satisfy*

$$G(p_n(x_1, x_2, \ldots, x_n)) = p_n(G(x_1), x_2, \ldots, x_n) + \sum_{i=2}^{n} p_n(x_1, \ldots, L(x_i), \ldots x_n)$$

for each $x_1, x_2, \ldots, x_n \in \mathcal{M}_s(A)$ if and only if $G = D + \tau, L = H + \gamma$, where D is a generalized derivation associated with a derivation H, τ and γ are linear maps from $\mathcal{M}_s(A)$ into $Z(\mathcal{M}_s(A))$, and τ vanishes on each $(n-1)$-th commutator.

Theorem 3.2. *Let $\mathcal{A}$ be a von Neumann algebra. Suppose that G is a generalized Lie n-derivation associated with a linear map L on $\mathcal{A}$. Then $G = d + \tau$ and $L = h + \gamma$, where d is a generalized derivation associated with a derivation h, τ and γ are linear maps from $\mathcal{A}$ into $Z(\mathcal{A})$, and τ vanishes on each $(n-1)$-th commutator.*

Proof. For every von Neumann algebra $\mathcal{A}$, we consider the central projection $z_0 := \sup\{z \in \mathcal{P}(Z(\mathcal{A})) : z\mathcal{A} \subset Z(\mathcal{A})\}$. It is clear that

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1,$$

where $\mathcal{A}_0 := z_0\mathcal{A} = z_0 Z(\mathcal{A})$ is a commutative von Neumann algebra and $\mathcal{A}_1 := (1 - z_0)\mathcal{A} = z_1\mathcal{A}$ is a von Neumann algebra without central summands of type I_1.

For each $x \in \mathcal{A}$, we obtain

$$\begin{aligned} G(x) &= z_1 G(z_1 x) + z_0 G(z_1 x) + z_1 G(z_0 x) + z_0 G(z_0 x), \\ L(x) &= z_1 L(z_1 x) + z_0 L(z_1 x) + z_1 L(z_0 x) + z_0 L(z_0 x). \end{aligned}$$

First, we show that $G_1(x) := z_0 G(z_1 x)$, $G_2(x) := z_1 G(z_0 x)$, and $G_3(x) := z_0 G(z_0 x)$ are linear maps from $\mathcal{A}$ to $Z(\mathcal{A})$ vanishing on each $(n-1)$-th commutator, and $L_1(x) := z_0 L(z_1 x)$, $L_2(x) := z_1 L(z_0 x)$, and $L_3(x) := z_0 L(z_0 x)$ are linear maps from $\mathcal{A}$ to $Z(\mathcal{A})$.

It is clear that $G_1(x) = z_0G(z_1x) \in z_0\mathcal{A} \subset Z(\mathcal{A})$ and $F_1(x) = z_0L(z_1x) \in Z(\mathcal{A})$. For each $x_1, x_2, \ldots, x_n \in \mathcal{A}$, $z_1p_n(x_1, x_2, \ldots, x_n) = p_n(z_1x_1, z_1x_2, \ldots, z_1x_n)$. By $z_0z_1 = 0$, we have

$$\begin{aligned} G_1(p_n(x_1, x_2, \ldots, x_n)) &= z_0G(z_1p_n(x_1, x_2, \ldots, x_n)) = z_0G(p_n(z_1x_1, z_1x_2, \ldots, z_1x_n)) \\ &= z_0(p_n(G(z_1x_1), z_1x_2, \ldots, z_1x_n) + \sum_{i=2}^{n} p_n(z_1x_1, \ldots, L(z_1x_i), \ldots, z_1x_n)) \\ &= 0. \end{aligned}$$

For each $x, x_i \in \mathcal{A}$ $(1 \le i \le n)$, by $z_0x \in Z(\mathcal{A})$, we have

$$p_{n+1}(G(z_0x), x_1, \ldots, x_n) = G(p_{n+1}(z_0x, x_1, \ldots, x_n)) - \sum_{i=1}^{n}(z_0x, x_1, \ldots, L(x_i), \ldots, x_n) = 0,$$

$$\begin{aligned} p_{n+1}(x_1, L(z_0x), x_2, \ldots, x_n) &= G(p_{n+1}(x_1, z_0x, x_2, \ldots, x_n)) - p_{n+1}(G(x_1), z_0x, x_2, \ldots, x_n)) \\ &\quad - \sum_{i=2}^{n}(x_1, z_0x, \ldots, L(x_i), \ldots, x_n) = 0. \end{aligned}$$

It follows from [8, Remark 2.1] that

$$p_{n+1}(G(z_0x), \mathcal{A}, \ldots, \mathcal{A}) = 0 \Longrightarrow p_n(G(z_0x), \mathcal{A}, \ldots, \mathcal{A}) = 0 \cdots \Longrightarrow [G(z_0x), \mathcal{A}] = 0,$$
$$p_{n+1}(\mathcal{A}, L(z_0x), \mathcal{A}, \ldots, \mathcal{A}) = 0 \Longrightarrow p_n(\mathcal{A}, L(z_0x), \mathcal{A}, \ldots, \mathcal{A}) = 0 \cdots \Longrightarrow [\mathcal{A}, L(z_0x)] = 0,$$

i.e., $G(z_0x) \in Z(\mathcal{A})$ and $L(z_0x) \in Z(\mathcal{A})$. Thus $G_2(x) = z_1G(z_0x) \in Z(\mathcal{A})$ and $L_2(x) = z_1L(z_0x) \in Z(\mathcal{A})$. Moreover, for each $x_1, x_2, \ldots, x_n \in \mathcal{A}$, by $z_0x_i \in Z(\mathcal{A})$, we have

$$G_2(p_n(x_1, x_2, \ldots, x_n)) = z_1G(z_0p_n(x_1, x_2, \ldots, x_n)) = z_1G(p_n(z_0x_1, z_0x_2, \ldots, z_0x_n)) = 0.$$

Similarly, G_3 is a linear map from $\mathcal{A}$ to $Z(\mathcal{A})$ vanishing on each $(n-1)$-th commutator, and L_3 is a linear map from $\mathcal{A}$ to $Z(\mathcal{A})$.

Next we prove that $\widetilde{G} := z_1G$ is a generalized Lie n-derivation associated with $\widetilde{L} := z_1L$ on $\mathcal{A}_1$. Since G is a generalized Lie n-derivation associated with a linear map L on $\mathcal{A}$ for each $y_1, y_2, \ldots, y_n \in \mathcal{A}_1$, we have

$$\begin{aligned} \widetilde{G}(p_n(y_1, y_2, \ldots, y_n)) &= z_1G(z_1p_n(y_1, y_2, \ldots, y_n)) = z_1G(p_n(z_1y_1, z_1y_2, \ldots, z_1y_n)) \\ &= z_1p_n(G(z_1y_1), z_1y_2, \ldots, z_1y_n) + \sum_{i=2}^{n} z_1p_n(z_1y_1, \ldots, L(z_1y_i), \ldots, z_1y_n) \\ &= p_n(\widetilde{G}(y_1), y_2, \ldots, y_n) + \sum_{i=2}^{n} p_n(y_1, \ldots, \widetilde{L}(y_i), \ldots, y_n). \end{aligned}$$

Then $\widetilde{G}$ is a generalized Lie n-derivation associated with $\widetilde{L}$ on $\mathcal{A}_1$.

Let $e \in \mathcal{A}_1$ be a projection and $f = 1 - e$. Denote $A = e\mathcal{A}_1e$, $M = e\mathcal{A}_1f$, $N = f\mathcal{A}_1e$ and $B = f\mathcal{A}_1f$, then $\mathcal{A}_1 = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$. Besides, by [15, Lemma 5], we have that $Z(A) = eZ(\mathcal{A}_1)e$ and $Z(B) = fZ(\mathcal{A}_1)f$. Moreover, $\mathcal{A}_1$ satisfies (ii), (iii) (see [8, Cortollary 3.14]) and (iv) (see [16, Lemma 1]). Then $\mathcal{A}_1$

satisfies the conditions of Theorem 2.1. Therefore, $z_1G = \widetilde{G} = d_1 + \tau_1$ and $z_1L = \widetilde{L} = h_1 + \gamma_1$, where d_1 is a generalized derivation associated with a derivation h_1 on $\mathcal{A}_1$, τ_1 and γ_1 are linear maps from $\mathcal{A}_1$ to $Z(\mathcal{A}_1)$, and τ_1 vanishes on each $(n-1)$-th commutator of $\mathcal{A}_1$.

Finally, for each $x \in \mathcal{A}$, it is enough to show that $d(x) := d_1(z_1x)$ is a generalized derivation associated with a derivation $h(x) := h_1(z_1x)$ on $\mathcal{A}$, $\tau(x) := \tau_1(z_1x)$ and $\gamma(x) := \gamma_1(z_1x)$ are linear maps from $\mathcal{A}$ to $Z(\mathcal{A})$, and τ vanishes on each $(n-1)$-th commutator on $\mathcal{A}$. For each $x, y \in \mathcal{A}$, we have

$$\begin{aligned}
d(xy) &= d_1(z_1xy) = d_1(z_1xz_1y) \\
&= d_1(z_1x)(z_1y) + z_1xh_1(z_1y) = d_1(z_1x)y + xh_1(z_1y) \\
&= d(x)y + xh(y) \\
&= h_1(z_1x)(z_1y) + z_1xd_1(z_1y) = h_1(z_1x)y + xd_1(z_1y) \\
&= h(x)y + xd(y), \\
h(xy) &= h_1(z_1xy) = h_1(z_1xz_1y) = h_1(z_1x)(z_1y) + z_1xh_1(z_1y) \\
&= h_1(z_1x)y + xh_1(z_1y) = h(x)y + xh(y), \\
\tau(x) &= \tau_1(z_1x) \in Z(\mathcal{A}_1) \subset Z(\mathcal{A}), \\
\gamma(x) &= \gamma_1(z_1x) \in Z(\mathcal{A}_1) \subset Z(\mathcal{A}), \\
\tau(p_n(x_1, x_2, \ldots, x_n)) &= \tau_1(z_1p_n(x_1, x_2, \ldots, x_n)) = \tau_1(p_n(z_1x_1, z_1x_2, \ldots, z_1x_n)) = 0.
\end{aligned}$$

Thus, for each $x \in \mathcal{A}$,

$$\begin{aligned}
G(x) &= d(x) + (\tau(x) + G_1(x) + G_2(x) + G_3(x)), \\
L(x) &= h(x) + (\gamma(x) + L_1(x) + L_2(x) + L_3(x)),
\end{aligned}$$

where d is a generalized derivation associated with a derivation h on $\mathcal{A}$, $\tau + G_1 + G_2 + G_3$ and $\gamma + L_1 + L_2 + L_3$ are linear maps from $\mathcal{A}$ to $Z(\mathcal{A})$, and $\tau + G_1 + G_2 + G_3$ vanishes on each $(n-1)$-th commutator. Hence G is a proper generalized Lie n-derivation. □

4. Conclusions

In this paper, we give a proper description of generalized Lie n-derivations on generalized matrix algebras under certain conditions. However, it is challenging to further relax the conditions of Theorem 2.1 or to find a more straightforward approach to prove the Theorem 2.1.

Author contributions

Shan Li: Writing–original draft, writing–review & editing, funding acquisition; Kaijia Luo: Writing–original draft, writing–review & editing; Jiankui Li: Writing–original draft, writing–review & editing, funding acquisition. All authors are contributed equally. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

This work is partially supported by National Science Foundation of China (NSFC) (No. 11871021, 12326374) and Changzhou Science and Technology Program (No. CJ20235062). And the authors would like to thank the reviewers for their insightful comments that helped to improve the results in this paper.

Conflict of interest

All authors declare that they have no conflicts of interest.

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