AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 29370–29385.
DOI: 10.3934/math.20241423
Received: 02 September 2024
Revised: 25 September 2024
Accepted: 05 October 2024
Published: 17 October 2024

Research article

Some results for the family of holomorphic functions associated with the Babalola operator and combination binomial series

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Abstract: In this paper, we define a new class $\mathcal{R}^{m,n,\sigma}_{t,\delta,\upsilon}(\mathcal{A},\mathcal{B})$ of holomorphic functions in the open unit disk defined connected with the combination binomial series and Babalola operator using the differential subordination with Janowski-type functions. Using the well-known Carathéodory's inequality for function with real positive parts and the Keogh-Merkes and Ma-Minda's in equalities, we determined the upper bound for the first two initial coefficients of the Taylor-Maclaurin power series expansion. Then, we found an upper bound for the Fekete-Szegö functional for the functions in this family. Further, a similar result for the first two coefficients and for the Fekete-Szegő inequality have been done the function $\mathcal{G}^{-1}$ when $\mathcal{G} \in \mathcal{R}^{m,n,\sigma}_{t,\delta,\upsilon}(\mathcal{A},\mathcal{B})$. Next, for the functions of these newly defined family we determine coefficient estimates, distortion bounds, radius problems, and the radius of starlikeness and close-to-convexity. The novelty of the results is that we were able to investigate basic properties of these new classes of functions using simple methods and these classes are connected with the new convolution operator and the Janowski functions.

Keywords: holomorphic functions; convolution; starlike and convex functions; Fekete-Szegö functional; subordination; binomial series; Babalola operator; Janowski function
Mathematics Subject Classification: 30C45, 30C80, 30C20

1. Introduction

Let $\mathbb{A}$ stand for the collection of functions $\mathcal{G}$ of the type

$$\mathcal{G}(\xi) = \xi + \sum_{j=2}^{\infty} a_j \xi^j, \tag{1.1}$$

that are holomorphic in the open unit disk $\Lambda := \{\xi \in \mathbb{C} : |\xi| < 1\}$ of the complex plane, and let $\mathfrak{S}$ indicate the subclass of functions of $\mathbb{A}$ which are *univalent* in Λ. For functions $\mathcal{G} \in \mathbb{A}$ given by (1.1) and $\mathcal{H} \in \mathbb{A}$ given by $\mathcal{H}(\zeta) = \zeta + \sum_{j=2}^{\infty} b_j \zeta^j$, we define the convolution product (or Hadamard) of $\mathcal{G}$ and $\mathcal{H}$ by

$$(\mathcal{G} * \mathcal{H})(\zeta) = (\mathcal{H} * \mathcal{G})(\zeta) = \zeta + \sum_{j=2}^{\infty} a_j b_j \zeta^j, \quad \zeta \in \Lambda. \tag{1.2}$$

Let $\mathcal{G}$ and $\mathcal{F}$ be two holomorphic functions in Λ. The function $\mathcal{G}$ is said to be subordinated to $\mathcal{F}$ if there are *Schwarz function* $w(\xi)$, that is, holomorphic in Λ with $w(0) = 0$ and $|w(\xi)| < 1$, $\xi \in \Lambda$, such as $\mathcal{G}(\xi) = \mathcal{F}(w(\xi))$ for all $\xi \in \Lambda$. This subordination notion is indicated by

$$\mathcal{G} \prec \mathcal{F} \quad \text{or} \quad \mathcal{G}(\xi) \prec \mathcal{F}(\xi).$$

If the function $\mathcal{F}$ is univalent in Λ, then we have the inclusion equivalence

$$\mathcal{G}(\xi) \prec \mathcal{F}(\xi) \Leftrightarrow \mathcal{G}(0) = \mathcal{F}(0) \quad \text{and} \quad \mathcal{G}(\Lambda) \subset \mathcal{F}(\Lambda).$$

The subfamilies of $\mathfrak{S}$ which are the *starlike* and the *convex* function in Λ defined by

$$\mathfrak{S}^* := \left\{ \mathcal{G} \in \mathbb{A} : Re\frac{\xi \mathcal{G}'(\xi)}{\mathcal{G}(\xi)} > 0,\ \xi \in \Lambda \right\} \tag{1.3}$$

and

$$\mathfrak{C} := \left\{ \mathcal{G} \in \mathbb{A} : Re\frac{(\xi \mathcal{G}'(\xi))'}{\mathcal{G}'(\xi)} > 0,\ \xi \in \Lambda \right\}, \tag{1.4}$$

respectively. Equivalently, we have

$$\mathfrak{S}^*(\varphi) = \left\{ \mathcal{G} \in \mathbb{A} : \frac{\xi \mathcal{G}'(\xi)}{\mathcal{G}(\xi)} \prec \varphi(\xi) \right\}, \quad \mathfrak{C}(\varphi) = \left\{ \mathcal{G} \in \mathbb{A} : \frac{(\xi \mathcal{G}'(\xi))'}{\mathcal{G}'(\xi)} \prec \varphi(\xi) \right\},$$

where

$$\varphi(\xi) = \frac{1 + \xi}{1 - \xi}. \tag{1.5}$$

Janowski defined in [4] the extended function family $\mathfrak{S}^*[\mathcal{A}, \mathcal{B}]$ of starlike functions called the *Janwoski class of functions* as follows: A function $\mathcal{G} \in \mathbb{A}$ is in the family $\mathfrak{S}^*[\mathcal{A}, \mathcal{B}]$ if

$$\frac{\xi \mathcal{G}'(\xi)}{\mathcal{G}(\xi)} \prec \frac{1 + \mathcal{A}\xi}{1 + \mathcal{B}\xi} \quad (-1 \leq \mathcal{B} < \mathcal{A} \leq 1).$$

The above subordination could be written as

$$\frac{\xi\mathcal{G}'(\xi)}{\mathcal{G}(\xi)} = \frac{1+\mathcal{A}p(\xi)}{1+\mathcal{B}p(\xi)} \quad (-1 \le \mathcal{B} < \mathcal{A} \le 1), \tag{1.6}$$

where an analytical function with a real positive part in Λ is denoted by $p(\xi)$.

The Janowski convex and Janowski starlike functions are obtained by reducing the above-described classes to the requirement $-1 \le \mathcal{B} < \mathcal{A} \le 1$. For the special cases $\mathcal{A} := 1-2\alpha$ and $\mathcal{B} := -1$, where $0 \le \alpha < 1$, we obtain the families, namely the family of starlike and convex functions of order α $(0 \le \alpha < 1)$ previously defined by Robertson in [6], and considered respectively by

$$\begin{aligned} \mathfrak{S}^*(\alpha) \;:&= \left\{ \mathcal{G} \in \mathbb{A} : Re\frac{\xi\mathcal{G}'(\xi)}{\mathcal{G}(\xi)} > \alpha,\ \xi \in \Lambda \right\}, \\ \mathfrak{C}(\alpha) \;:&= \left\{ \mathcal{G} \in \mathbb{A} : Re\frac{(\xi\mathcal{G}'(\xi))'}{\mathcal{G}'(\xi)} > \alpha,\ \xi \in \Lambda \right\}. \end{aligned}$$

Babalola defined the operator $\mathcal{I}_\upsilon^\rho : \mathbb{A} \to \mathbb{A}$ as

$$\mathcal{I}_\upsilon^\sigma \mathcal{G}(\zeta) = \left(\rho_\sigma * \rho_{\sigma,\upsilon}^{-1} * \mathcal{G}\right)(\zeta), \tag{1.7}$$

where

$$\rho_{\sigma,\upsilon}(\zeta) = \frac{\zeta}{(1-\zeta)^{\sigma-\upsilon+1}}, \quad \sigma-\upsilon+1>0, \quad \rho_\sigma = \rho_{\sigma,0},$$

and $\rho_{\sigma,\upsilon}^{-1}$ is

$$\left(\rho_{\sigma,\upsilon} * \rho_{\sigma,\upsilon}^{-1}\right)(\zeta) = \frac{\zeta}{1-\zeta} \quad (\sigma,\upsilon \in \mathbb{N} = \{1,2,3,...\}).$$

For $\mathcal{G} \in \mathbb{A}$, then (1.7) is equivalent to

$$\mathcal{I}_\upsilon^\sigma \mathcal{G}(\zeta) = \zeta + \sum_{j=2}^{\infty} \left[\tfrac{\Gamma(\sigma+j)}{\Gamma(\sigma+1)} \cdot \tfrac{(\sigma-\upsilon)!}{(\sigma+j-\upsilon-1)!}\right] a_j\, \zeta^j.$$

Making use the binomial series

$$(1-\delta)^t = \sum_{i=0}^{t} \binom{t}{i} (-1)^i\ \delta^i \quad (t \in \mathbb{N}),$$

for $\mathcal{G} \in \mathbb{A}$, El-Deeb [3] introduced the linear differential operator as follows:

$$\mathcal{D}_{m,\delta,\upsilon}^{\sigma,0}\mathcal{G}(\zeta) = \mathcal{G}(\zeta),$$

$$\begin{aligned} \mathcal{D}_{t,\delta,\upsilon}^{\sigma,1}\mathcal{G}(\zeta) &= \mathcal{D}_{t,\delta,\upsilon}^{\sigma}\mathcal{G}(\zeta) = (1-\delta)^t\, \mathcal{I}_\upsilon^\sigma\mathcal{G}(\zeta) + [1-(1-\delta)^t]\, \zeta \left(\mathcal{I}_\upsilon^\sigma\mathcal{G}\right)'(\zeta) \\ &= \zeta + \sum_{j=2}^{\infty} [1+(j-1)\, c^t(\delta)] \left[\tfrac{\Gamma(\sigma+j)}{\Gamma(\sigma+1)} \cdot \tfrac{(\sigma-\upsilon)!}{(\sigma+j-\upsilon-1)!}\right] a_j\, \zeta^j \\ &. \end{aligned}$$

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\begin{aligned}
\mathcal{D}^{\sigma,n}_{t,\delta,\upsilon}\mathcal{G}(\zeta) &= \mathcal{D}^{\sigma}_{t,\delta,\upsilon}\left(\mathcal{D}^{\sigma,n-1}_{t,\delta,\upsilon}\mathcal{G}(\zeta)\right)\\
&= (1-\delta)^t\,\mathcal{D}^{\sigma,n-1}_{t,\delta,\upsilon}\mathcal{G}(\zeta) + \left[1-(1-\delta)^t\right]\zeta\left(\mathcal{D}^{\sigma,n-1}_{t,\delta,\upsilon}\mathcal{G}(\zeta)\right)'\\
&= \zeta + \sum_{j=2}^{\infty}\left[1+(j-1)\,c^t(\delta)\right]^n\left[\tfrac{\Gamma(\sigma+j)}{\Gamma(\sigma+1)}\cdot\tfrac{(\sigma-\upsilon)!}{(\sigma+j-\upsilon-1)!}\right]a_j\,\zeta^j\\
&= \zeta + \sum_{j=2}^{\infty}\psi^n_j\left[\tfrac{\Gamma(\sigma+j)}{\Gamma(\sigma+1)}\cdot\tfrac{(\sigma-\upsilon)!}{(\sigma+j-\upsilon-1)!}\right]a_j\,\zeta^j, \qquad (1.8)\\
&\quad (\delta>0;\ t,\sigma,\upsilon\in\mathbb{N};\ n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}),
\end{aligned}
$$

where

$$\psi^n_j = \left[1+(j-1)\,c^t(\delta)\right]^n, \tag{1.9}$$

and

$$c^t(\delta) = \sum_{i=1}^{t}\binom{t}{i}(-1)^{i+1}\ \delta^i \qquad (t\in\mathbb{N}).$$

From (1.8), we obtain that

$$c^t(\delta)\,\zeta\,\left(\mathcal{D}^{\sigma,n}_{t,\delta,\upsilon}\mathcal{G}(\zeta)\right)' = \mathcal{D}^{\sigma,n+1}_{t,\delta,\upsilon}\mathcal{G}(\zeta) - \left[1-c^t(\delta)\right]\mathcal{D}^{\sigma,n}_{t,\delta,\upsilon}\mathcal{G}(\zeta). \tag{1.10}$$

In this article using the El-Deeb operator defined in (1.8), we define a new sub-family of $\mathbb{A}$:

$$\mathcal{R}^{m,n,\sigma}_{t,\delta,\upsilon}(\mathcal{A},\mathcal{B}) = \left\{\mathcal{G}\in\mathbb{A}:\frac{\mathcal{D}^{\sigma,m}_{t,\delta,\upsilon}\mathcal{G}(\zeta)}{\mathcal{D}^{\sigma,n}_{t,\delta,\upsilon}\mathcal{G}(\zeta)} \prec \frac{1+\mathcal{A}\xi}{1+\mathcal{B}\xi}\right\}, \tag{1.11}$$

where $-1\le\mathcal{A}<\mathcal{B}\le 1$; $\delta>0$; $t,\sigma,\upsilon\in\mathbb{N}$ and $n,m\in\mathbb{N}_0$, that will lead us to the study of Fekete-Szegö problem. Further, coefficient estimates, characteristic properties and partial sums results will be established.

Specializing the values of $\mathcal{A}$ and $\mathcal{B}$ one can obtain the particular cases

(i) $\mathcal{R}^{m,n,\sigma}_{t,\delta,\upsilon}(1-2\alpha,-1) =: \mathcal{W}^{m,n,\sigma}_{t,\delta,\upsilon}(\alpha) = \left\{\mathcal{G}\in\mathbb{A}: Re\left(\frac{\mathcal{D}^{\sigma,m}_{t,\delta,\upsilon}\mathcal{G}(\zeta)}{\mathcal{D}^{\sigma,n}_{t,\delta,\upsilon}\mathcal{G}(\zeta)}\right)>\alpha,\ (0\le\alpha<1)\right\}$;

and

(ii) $\mathcal{R}^{m,n,\sigma}_{t,\delta,\upsilon}(1,-1) =: \mathcal{F}^{m,n,\sigma}_{t,\delta,\upsilon} = \left\{\mathcal{G}\in\mathbb{A}: Re\left(\frac{\mathcal{D}^{\sigma,m}_{t,\delta,\upsilon}\mathcal{G}(\zeta)}{\mathcal{D}^{\sigma,n}_{t,\delta,\upsilon}\mathcal{G}(\zeta)}\right)>0\right\}$.

2. The Fekete-Szegö functional bounds for the class $\mathcal{R}^{m,n,\sigma}_{t,\delta,\upsilon}(\mathcal{A},\mathcal{B})$

To solve the Fekete-Szegö type inequality for $\mathcal{G}\in\mathcal{R}^{m,n,\sigma}_{t,\delta,\upsilon}(\mathcal{A},\mathcal{B})$ we will use the next results (the first part is due to Carathéodory [1]):

Lemma 1. *[1, 5] If $P(\xi) = 1 + p_1\xi + p_2\xi^2 + \cdots \in \mathcal{P}$ where $\mathcal{P}$ the class of holomorphic functions with positive real part in Λ, with $P(0) = 1$, then*

$$|p_n| \leq 2, \ n \geq 1, \tag{2.1}$$

and for the complex number $\mu \in \mathbb{C}$ we have

$$\left|p_2 - \mu p_1^2\right| \leq 2 \max\{1; |1 - 2\mu|\}. \tag{2.2}$$

If μ is a real parameter, then

$$\left|p_2 - \mu p_1^2\right| \leq \begin{cases} -4\mu + 2, & \text{if} \quad \mu \leq 0, \\ 2 & \text{if} \quad 0 \leq \mu \leq 1, \\ 4\mu - 2 & \text{if} \quad \mu \geq 1. \end{cases} \tag{2.3}$$

When $\mu > 1$ or $\mu < 0$, equality (2.3) *holds true if and only if $P_1(\xi) = \dfrac{1+\xi}{1-\xi}$ or one of its rotations. When $0 < \mu < 1$, the equality* (2.3) *holds if and only if $P_2(\xi) = \dfrac{1+\xi^2}{1-\xi^2}$ or one of its rotations. When $\mu = 0$, equality* (2.3) *holds if and only if*

$$P_3(\xi) = \left(\frac{1+c}{2}\right)\frac{1+\xi}{-\xi+1} + \left(\frac{1-c}{2}\right)\frac{-\xi+1}{1+\xi} \quad (0 \leq c \leq 1)$$

or one of its rotations. When $\mu = 1$, the equality (2.3) *holds true if $P(\xi)$ is a reciprocal of one of the functions such that the equality holds true in the case when $\mu = 0$.*

Theorem 1. *If $\mathcal{G} \in \mathbb{A}$ defined as* (1.1)*, belongs to $\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A}, \mathcal{B})$, then*

$$|a_2| \leq \frac{(\sigma - \upsilon + 1)(\mathcal{A} - \mathcal{B})}{(\sigma + 1)|\psi_2^m - \psi_2^n|}, \tag{2.4}$$

$$|a_3| \leq \frac{(\sigma - \upsilon + 1)(\sigma - \upsilon + 2)(\mathcal{A} - \mathcal{B})}{(\sigma + 1)(\sigma + 2)|\psi_2^m - \psi_2^n|} \times \max\left\{1; \left|-\mathcal{B} + \frac{(\mathcal{A} - \mathcal{B})\left(\psi_2^{n+m} - \psi_2^{2n}\right)}{\left(\psi_2^m - \psi_2^n\right)^2}\right|\right\}, \tag{2.5}$$

and for a complex number τ, we have

$$\left|a_3 - \tau a_2^2\right| \leq \frac{(\sigma - \upsilon + 1)(\sigma - \upsilon + 2)(\mathcal{A} - \mathcal{B})}{(\sigma + 1)(\sigma + 2)\left|\psi_3^m - \psi_3^n\right|} \max\{1; |\Omega(\tau, \sigma, \upsilon, \mathcal{A}, \mathcal{B})|\}, \tag{2.6}$$

where

$$\Omega(\tau, \sigma, \upsilon, \mathcal{A}, \mathcal{B}) = 1 - 2\Theta(\tau, \sigma, \upsilon, \mathcal{A}, \mathcal{B}),$$

$$\Theta(\tau, \sigma, \upsilon, \mathcal{A}, \mathcal{B}) = \frac{1}{2}\left(1 + \mathcal{B} - \frac{(\mathcal{A} - \mathcal{B})\left(\psi_2^{n+m} - \psi_2^{2n}\right)}{\left(\psi_2^m - \psi_2^n\right)^2} + \frac{\tau(\mathcal{A} - \mathcal{B})(\sigma + 2)(\sigma - \upsilon + 1)\left(\psi_3^m - \psi_3^n\right)}{(\sigma + 1)(\sigma - \upsilon + 2)\left(\psi_2^m - \psi_2^n\right)^2}\right), \tag{2.7}$$

and ψ_j^n is given by (1.9)*.*

Proof. We will show that the relations (2.4)–(2.6) and (2.16) hold true for $\mathcal{G} \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$. If $\mathcal{G} \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$, then

$$\frac{\mathcal{D}_{t,\delta,\upsilon}^{\sigma,m}\mathcal{G}(\zeta)}{\mathcal{D}_{t,\delta,\upsilon}^{\sigma,n}\mathcal{G}(\zeta)} \prec \frac{1+\mathcal{A}\xi}{1+\mathcal{B}\xi}$$

which yields

$$\frac{\mathcal{D}_{t,\delta,\upsilon}^{\sigma,m}\mathcal{G}(\zeta)}{\mathcal{D}_{t,\delta,\upsilon}^{\sigma,n}\mathcal{G}(\zeta)} \prec \frac{1+\mathcal{A}w(\xi)}{1+\mathcal{B}w(\xi)} = G(w(\xi)), \quad (-1 \le \mathcal{B} < \mathcal{A} \le 1). \tag{2.8}$$

Since we can write $w(\xi)$ as

$$w(\xi) = \frac{1-h(\xi)}{1+h(\xi)} = \frac{p_1\xi + p_2\xi^2 + p_3\xi^3 + \dots}{2 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \dots},$$

where $h(\xi) \in \mathcal{P}$ and have the form $h(\xi) = 1 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \dots$, so

$$G(w(\xi)) = 1 + \frac{1}{2}(\mathcal{A}-\mathcal{B})\,p_1\xi + \frac{(\mathcal{A}-\mathcal{B})}{4}\left[2p_2 - (1+\mathcal{B})\,p_1^2\right]\xi^2 + \dots, \tag{2.9}$$

and therefore

$$\begin{aligned}\frac{\mathcal{D}_{t,\delta,\upsilon}^{\sigma,m}\mathcal{G}(\zeta)}{\mathcal{D}_{t,\delta,\upsilon}^{\sigma,n}\mathcal{G}(\zeta)} &= 1 + \frac{(\sigma+1)}{(\sigma-\upsilon+1)}\left(\psi_2^m - \psi_2^n\right)a_2\zeta \\ &+ \left(\frac{(\sigma+1)(\sigma+2)}{(\sigma-\upsilon+1)(\sigma-\upsilon+2)}\left(\psi_3^m - \psi_3^n\right)a_3\right. \\ &\left. - \frac{(\sigma+1)^2}{(\sigma-\upsilon+1)^2}\left(\left(\psi_2^{n+m} - \psi_2^{2n}\right)\right)a_2^2\right)\zeta^2 + \dots.\end{aligned} \tag{2.10}$$

If we compare the first coefficients of (2.9) and (2.10), we get

$$a_2 = \frac{(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}{2(\sigma+1)\left(\psi_2^m - \psi_2^n\right)}p_1, \tag{2.11}$$

$$a_3 = \frac{(\sigma-\upsilon+1)(\sigma-\upsilon+2)(\mathcal{A}-\mathcal{B})}{2(\sigma+1)(\sigma+2)\left(\psi_3^m - \psi_3^n\right)} \times \left(p_2 - \frac{p_1^2}{2}\left[\frac{1+\mathcal{B}}{\mathcal{A}-\mathcal{B}} - \left(\frac{(\mathcal{A}-\mathcal{B})\left(\psi_2^{n+m} - \psi_2^{2n}\right)}{\left(\psi_2^m - \psi_2^n\right)^2}\right)\right]\right) \tag{2.12}$$

and by using (2.1) in (2.11) and (2.2) in (2.12), we get

$$|a_2| \le \frac{(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}{(\sigma+1)\left|\psi_2^m - \psi_2^n\right|}, \tag{2.13}$$

$$|a_3| \le \frac{(\sigma-\upsilon+1)(\sigma-\upsilon+2)(\mathcal{A}-\mathcal{B})}{(\sigma+1)(\sigma+2)\left|\psi_3^m - \psi_3^n\right|} \times$$

$$\max\left\{1;\left|-\mathcal{B}+\frac{(\mathcal{A}-\mathcal{B})\left(\psi_2^{n+m}-\psi_2^{2n}\right)}{\left(\psi_2^m-\psi_2^n\right)^2}\right|\right\}. \tag{2.14}$$

For a complex nubmer τ, and from (2.11) together with (2.12), we have

$$\left|a_3-\tau a_2^2\right|=\frac{(\sigma-\upsilon+1)(\sigma-\upsilon+2)(\mathcal{A}-\mathcal{B})}{2(\sigma+1)(\sigma+2)\left(\psi_3^m-\psi_3^n\right)}\left|p_2-\Theta(\tau,\sigma,\upsilon,\mathcal{A},\mathcal{B})\,p_1^2\right|, \tag{2.15}$$

where $\Theta(\tau,\sigma,\upsilon,\mathcal{A},\mathcal{B})$ is denoted by (2.7). Now, we apply Lemma 1 to (2.15) and obtain the required results. □

Theorem 2. *If the function $\mathcal{G}\in\mathbb{A}$ defined as* (1.1) *belongs to $\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$, then for any real parameter τ we obtain*

$$\left|a_3-\tau a_2^2\right|\le\frac{(\sigma-\upsilon+1)(\sigma-\upsilon+2)(\mathcal{A}-\mathcal{B})}{2(\sigma+1)(\sigma+2)\left|\psi_3^m-\psi_3^n\right|}\begin{cases}1-2\Theta(\tau,\sigma,\upsilon,\mathcal{A},\mathcal{B}), & \text{if } \tau<\varphi_1,\\ 1, & \text{if } \varphi_1\le\tau\le\varphi_2,\\ 2\Theta(\tau,\sigma,\upsilon,\mathcal{A},\mathcal{B})-1, & \text{if } \tau>\varphi_2,\end{cases} \tag{2.16}$$

where $\Theta(\tau,\sigma,\upsilon,\mathcal{A},\mathcal{B})$ is given by (2.7),

$$\varphi_1=\frac{(\sigma+1)(\sigma-\upsilon+2)\left(\psi_2^m-\psi_2^n\right)^2}{(\mathcal{A}-\mathcal{B})(\sigma+2)(\sigma-\upsilon+1)\left(\psi_3^m-\psi_3^n\right)}\times\left(-1-\mathcal{B}+\frac{(\mathcal{A}-\mathcal{B})\left(\psi_2^{n+m}-\psi_2^{2n}\right)}{\left(\psi_2^m-\psi_2^n\right)^2}\right),$$

and

$$\varphi_2=\frac{(\sigma+1)(\sigma-\upsilon+2)\left(\psi_2^m-\psi_2^n\right)^2}{(\mathcal{A}-\mathcal{B})(\sigma+2)(\sigma-\upsilon+1)\left(\psi_3^m-\psi_3^n\right)}\times\left(1-\mathcal{B}+\frac{(\mathcal{A}-\mathcal{B})\left(\psi_2^{n+m}-\psi_2^{2n}\right)}{\left(\psi_2^m-\psi_2^n\right)^2}\right).$$

Proof. The proof can be produced directly by making use of Lemma 1 in (2.15), so we choose to omit it. □

3. The coefficient inequalities for $\mathcal{G}^{-1}\in\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{D},\mathcal{E})$

The "Koebe one quarter theorem" [2] ensures that the image of Λ under each univalent function $\mathcal{G}\in\mathbb{A}$ consists a disk of radius $\frac{1}{4}$. Thus each univalent function $\mathcal{G}$ has an inverse $\mathcal{G}^{-1}$ satisfying

$$\mathcal{G}^{-1}(\mathcal{G}(\xi))=\xi,\ (\xi\in\Lambda)\quad\text{and}\quad\mathcal{G}\left(\mathcal{G}^{-1}(w)\right)=w,\ \left(|w|<r_0(\mathcal{G}),\ r_0(\mathcal{G})\ge\frac{1}{4}\right).$$

A function $\mathcal{G}\in\mathbb{A}$ is called bi-univalent in Λ if both $\mathcal{G}$ and $\mathcal{G}^{-1}$ are univalent in Λ. We mention that the collection of bi-univalent functions defined in the unit disk Λ is not empty. For example, the functions ξ, $\frac{\xi}{1-\xi}$, $-\log(1-\xi)$ and $\frac{1}{2}\log\frac{1+\xi}{1-\xi}$ are members of bi-univalent function family, however the Koebe function is not a member.

Theorem 3. *If $\mathcal{G}\in\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$ and the inverse function of $\mathcal{G}$ is $\mathcal{G}^{-1}(w)=w+\sum_{j=2}^{\infty}d_jw^j$, then*

$$|d_2|\ \le\ \frac{(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}{(\sigma+1)\left|\psi_2^m-\psi_2^n\right|} \tag{3.1}$$

$$|d_3| \leq \frac{(\sigma-\upsilon+1)(\sigma-\upsilon+2)(\mathcal{A}-\mathcal{B})}{(\sigma+1)(\sigma+2)\left|\psi_3^m-\psi_3^n\right|}\max\{1;|2\Theta(2,\sigma,\upsilon,\mathcal{A},\mathcal{B})-1|\}, \tag{3.2}$$

and for any $\mu \in \mathbb{C}$, we have

$$\left|d_3-\mu d_2^2\right| \leq \frac{(\sigma-\upsilon+1)(\sigma-\upsilon+2)(\mathcal{A}-\mathcal{B})}{(\sigma+1)(\sigma+2)\left|\psi_3^m-\psi_3^n\right|}\times$$
$$\max\left\{1;\left|2\Theta(2,\sigma,\upsilon,\mathcal{A},\mathcal{B})+\mu\frac{(\mathcal{A}-\mathcal{B})(\sigma+2)(\sigma-\upsilon+1)\left(\psi_3^m-\psi_3^n\right)}{(\sigma+1)(\sigma-\upsilon+2)\left(\psi_2^m-\psi_2^n\right)^2}-1\right|\right\},$$

where $\Theta(2,\sigma,\upsilon,\mathcal{A},\mathcal{B})$ given by (2.7).

Proof. Since

$$\mathcal{G}^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \tag{3.3}$$

is the inverse of the function $\mathcal{G}$, it can be seen that

$$\xi = \mathcal{G}^{-1}(\mathcal{G}(\xi)) = \mathcal{G}\left(\mathcal{G}^{-1}(\xi)\right), \ |\xi| < r_0(\mathcal{G}). \tag{3.4}$$

From (1.1) and (3.4), we obtain that

$$\xi = \mathcal{G}^{-1}\left(\xi + \sum_{n=2}^{\infty} a_n \xi^n\right), \ |\xi| < r_0(\mathcal{G}). \tag{3.5}$$

Therefore from (3.4) and (3.5) we get

$$\xi + (a_2+d_2)\xi^2 + (a_3+2a_2d_2+d_3)\xi^3 + \cdots = \xi, \ |\xi| < r_0(\mathcal{G}). \tag{3.6}$$

Equating the corresponding coefficients of the relation (3.6), we conclude that

$$d_2 = -a_2, \tag{3.7}$$
$$d_3 = 2a_2^2 - a_3. \tag{3.8}$$

First, from the relations (2.11) and (3.7) we have

$$d_2 = -\frac{(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}{2(\sigma+1)\left(\psi_2^m-\psi_2^n\right)}p_1. \tag{3.9}$$

To find $|d_3|$, from (3.8) we have

$$|d_3| = |a_3 - 2a_2^2|.$$

Hence, by using (2.15) for real $\tau = 2$ we deduce that

$$|d_3| = \left|a_3 - 2a_2^2\right|$$
$$= \frac{(\sigma-\upsilon+1)(\sigma-\upsilon+2)(\mathcal{A}-\mathcal{B})}{2(\sigma+1)(\sigma+2)\left|\psi_3^m-\psi_3^n\right|}\left|p_2-\Theta(2,\sigma,\upsilon,\mathcal{A},\mathcal{B})p_1^2\right|$$

$$= \frac{(\sigma - \upsilon + 1)(\sigma - \upsilon + 2)(\mathcal{A} - \mathcal{B})}{(\sigma + 1)(\sigma + 2)\left|\psi_3^m - \psi_3^n\right|} \max\{1; |2\Theta(2, \sigma, \upsilon, \mathcal{A}, \mathcal{B})| - 1\}, \tag{3.10}$$

where $\Theta(2, \sigma, \upsilon, \mathcal{A}, \mathcal{B})$ given by (2.7). For any complex number μ, a simple computation gives us that

$$\begin{aligned} d_3 - \mu d_2^2 &= \frac{(\sigma - \upsilon + 1)(\sigma - \upsilon + 2)(\mathcal{A} - \mathcal{B})}{2(\sigma + 1)(\sigma + 2)\left(\psi_3^m - \psi_3^n\right)} \left(p_2 - \Theta(2, \sigma, \upsilon, \mathcal{A}, \mathcal{B})\, p_1^2\right) \\ &- \mu \frac{[(\sigma - \upsilon + 1)(\mathcal{A} - \mathcal{B})]^2}{\left[2(\sigma + 1)\left(\psi_2^m - \psi_2^n\right)\right]^2} p_1^2. \\ &= \frac{(\sigma - \upsilon + 1)(\sigma - \upsilon + 2)(\mathcal{A} - \mathcal{B})}{2(\sigma + 1)(\sigma + 2)\left(\psi_3^m - \psi_3^n\right)} \times \\ &\left(p_2 - \frac{p_1^2}{2}\left[2\Theta(2, \sigma, \upsilon, \mathcal{A}, \mathcal{B}) + \mu \frac{(\mathcal{A} - \mathcal{B})(\sigma + 2)(\sigma - \upsilon + 1)\left(\psi_3^m - \psi_3^n\right)}{(\sigma + 1)(\sigma - \upsilon + 2)\left(\psi_2^m - \psi_2^n\right)^2}\right]\right). \end{aligned} \tag{3.11}$$

By taking modulus on both sides of (3.11) and applying Lemma 1 and (2.1), we find that

$$\begin{aligned} \left|d_3 - \mu d_2^2\right| &\le \frac{(\sigma - \upsilon + 1)(\sigma - \upsilon + 2)(\mathcal{A} - \mathcal{B})}{(\sigma + 1)(\sigma + 2)\left|\psi_3^m - \psi_3^n\right|} \times \\ &\max\left\{1; \left|2\Theta(2, \sigma, \upsilon, \mathcal{A}, \mathcal{B}) + \mu \frac{(\mathcal{A} - \mathcal{B})(\sigma + 2)(\sigma - \upsilon + 1)\left(\psi_3^m - \psi_3^n\right)}{(\sigma + 1)(\sigma - \upsilon + 2)\left(\psi_2^m - \psi_2^n\right)^2} - 1\right|\right\}, \end{aligned}$$

and this completes our proof. □

4. Characterization properties

By applying the techniques introduced by Silverman in [7], we will introduce some characteristic properties of the functions $\mathcal{G} \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A}, \mathcal{B})$ such as partial sums results, necessary and sufficient conditions, radii of close-to-convexity, distortion bounds, radii of starlikeness and convexity.

Theorem 4. *If $\mathcal{G} \in \mathbb{A}$ and be defined as* (1.1) *belongs to $\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A}, \mathcal{B})$, then*

$$\sum_{j=2}^{\infty} \left((1 - \mathcal{B})\psi_j^m + (\mathcal{A} - 1)\psi_j^n\right)\left(\tfrac{\sigma + 1}{(\sigma - \upsilon + 1)}\right)\left|a_j\right| \le (\mathcal{A} - \mathcal{B}), \tag{4.1}$$

where ψ_j^n given by (1.9).

Proof. Letting $\mathcal{G} \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A}, \mathcal{B})$, by (1.11) we deduce that

$$\frac{\mathcal{D}_{t,\delta,\upsilon}^{\sigma,m}\mathcal{G}(\zeta)}{\mathcal{D}_{t,\delta,\upsilon}^{\sigma,n}\mathcal{G}(\zeta)} = \frac{1 + \mathcal{A}w(\xi)}{1 + \mathcal{B}w(\xi)}, \ \xi \in \Lambda, \tag{4.2}$$

where $w(\xi)$ is a Schwarz function, or equivalently

$$\left|\frac{\mathcal{D}_{t,\delta,\upsilon}^{\sigma,m}\mathcal{G}(\zeta) - \mathcal{D}_{t,\delta,\upsilon}^{\sigma,n}\mathcal{G}(\zeta)}{\mathcal{A}\mathcal{D}_{t,\delta,\upsilon}^{\sigma,n}\mathcal{G}(\zeta) - \mathcal{B}\mathcal{D}_{t,\delta,\upsilon}^{\sigma,m}\mathcal{G}(\zeta)}\right| < 1, \ \xi \in \Lambda.$$

Thus, the above relation leads us to

$$
\begin{aligned}
&\left|\frac{\mathcal{D}^{\sigma,m}_{t,\delta,\upsilon}\mathcal{G}(\zeta)-\mathcal{D}^{\sigma,n}_{t,\delta,\upsilon}\mathcal{G}(\zeta)}{\mathcal{A}\mathcal{D}^{\sigma,n}_{t,\delta,\upsilon}\mathcal{G}(\zeta)-\mathcal{B}\mathcal{D}^{\sigma,m}_{t,\delta,\upsilon}\mathcal{G}(\zeta)}\right| \\
= &\left|\frac{\sum\limits_{j=2}^{\infty}\left(\psi_j^m-\psi_j^n\right)\left(\frac{\sigma+1}{(\sigma-\upsilon+1)}\right)a_j\xi^j}{(\mathcal{A}-\mathcal{B})\xi+\sum\limits_{j=2}^{\infty}\left(\mathcal{A}\psi_j^n-\mathcal{B}\psi_j^m\right)\left(\frac{\sigma+1}{(\sigma-\upsilon+1)}\right)a_j\xi^j}\right| \\
\le &\frac{\sum\limits_{j=2}^{\infty}\left(\psi_j^m-\psi_j^n\right)\left(\frac{\sigma+1}{(\sigma-\upsilon+1)}\right)\left|a_j\right|r^{j-1}}{(\mathcal{A}-\mathcal{B})-\sum\limits_{j=2}^{\infty}\left(\mathcal{A}\psi_j^n-\mathcal{B}\psi_j^m\right)\left(\frac{\sigma+1}{(\sigma-\upsilon+1)}\right)\left|a_j\right|r^{j-1}}<1,
\end{aligned}
$$

and taking $|\xi|=r\to 1^-$ simple computation yields (4.1). □

Example 1. *For*

$$
\mathcal{G}(\xi)=\xi+\sum_{j=2}^{\infty}\frac{(\mathcal{A}-\mathcal{B})}{(1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n}\left(\tfrac{\sigma-\upsilon+1}{\sigma+1}\right)\ell_j\xi^j,\ \xi\in\Lambda,
$$

such that $\sum\limits_{j=2}^{\infty}\ell_j=1$, *we get*

$$
\begin{aligned}
&\sum_{j=2}^{\infty}\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)\left(\tfrac{\sigma+1}{\sigma-\upsilon+1}\right)\left|a_j\right| \\
= &\sum_{j=2}^{\infty}\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)\left(\tfrac{\sigma+1}{\sigma-\upsilon+1}\right)\times \\
&\frac{(\mathcal{A}-\mathcal{B})}{(1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n}\left(\tfrac{\sigma-\upsilon+1}{\sigma+1}\right)\ell_j \\
= &(\mathcal{A}-\mathcal{B})\sum_{j=2}^{\infty}\ell_j=(\mathcal{A}-\mathcal{B}).
\end{aligned}
$$

Then $\mathcal{G}\in\mathcal{R}^{m,n,\sigma}_{t,\delta,\upsilon}(\mathcal{A},\mathcal{B})$, *and we note that the inequality* (4.1) *is sharp.*

Corollary 1. *Let* $\mathcal{G}\in\mathcal{R}^{m,n,\sigma}_{t,\delta,\upsilon}(\mathcal{A},\mathcal{B})$ *given by* (1.1). *Then*

$$
\left|a_j\right|\le\frac{(\mathcal{A}-\mathcal{B})}{(1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n}\left(\tfrac{\sigma-\upsilon+1}{\sigma+1}\right),\ \text{for}\ j\ge 2, \tag{4.3}
$$

where ψ_j^n *is defined by* (1.9). *The approximation is sharp for the function*

$$
\mathcal{G}_*(\xi):=\xi-\frac{(\mathcal{A}-\mathcal{B})}{(1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n}\left(\tfrac{\sigma-\upsilon+1}{\sigma+1}\right)\xi^j,\ \ j\ge 2. \tag{4.4}
$$

Theorem 5. *If* $\mathcal{G} \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$*, then*

$$r - \frac{(\mathcal{A}-\mathcal{B})}{(1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n}(\tfrac{\sigma-\upsilon+1}{\sigma+1})r^2 \leq |\mathcal{G}(\eta)| \leq r + \frac{(\mathcal{A}-\mathcal{B})}{(1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n}(\tfrac{\sigma-\upsilon+1}{\sigma+1})r^2. \tag{4.5}$$

For the function defined by

$$\widehat{\mathcal{G}}(\xi) := \xi - \frac{(\mathcal{A}-\mathcal{B})}{(1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n}(\tfrac{\sigma-\upsilon+1}{\sigma+1})\xi^2, \ |\xi| = r < 1, \tag{4.6}$$

the approximation is sharp.

Proof. For $|\xi| = r < 1$ we have

$$|\mathcal{G}(\xi)| = \left|\xi + \sum_{j=2}^{\infty} a_j \xi^j\right| \leq |\xi| + \sum_{j=2}^{\infty} a_j |\xi|^j = r + \sum_{j=2}^{\infty} a_j |r|^j .$$

Moreover, since for $|\xi| = r < 1$ we get $r^j < r^2$ for all $j \geq 2$, the above relation implies that

$$|\mathcal{G}(\xi)| \leq r + r^2 \sum_{j=2}^{\infty} \left|a_j\right| . \tag{4.7}$$

Similarly, we get

$$|\mathcal{G}(\xi)| \geq r - r^2 \sum_{j=2}^{\infty} \left|a_j\right| . \tag{4.8}$$

From the relation (4.1) we have

$$\sum_{j=2}^{\infty} \left((1-\mathcal{B})\psi_j^m + (\mathcal{A}-1)\psi_j^n\right)(\tfrac{\sigma+1}{\sigma-\upsilon+1})\left|a_j\right| \leq (\mathcal{A}-\mathcal{B}),$$

but

$$((1-\mathcal{B})\psi_2^m + (\mathcal{A}-1)\psi_2^n)(\tfrac{\sigma+1}{\sigma-\upsilon+1})\sum_{j=2}^{\infty}\left|a_j\right| \leq \sum_{j=2}^{\infty} \left((1-\mathcal{B})\psi_j^m + (\mathcal{A}-1)\psi_j^n\right)(\tfrac{\sigma+1}{\sigma-\upsilon+1})\left|a_j\right| \leq (\mathcal{A}-\mathcal{B}).$$

Therefore,

$$\sum_{j=2}^{\infty} a_j \leq \frac{(\frac{\sigma-\upsilon+1}{\sigma+1})(\mathcal{A}-\mathcal{B})}{(1-\mathcal{B})\psi_2^m + (\mathcal{A}-1)\psi_2^n}, \tag{4.9}$$

and by using (4.9) in (4.7) and (4.8) we get the desired result. □

The next distortion theorem for the family $\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$ could be similarly obtained:

Theorem 6. *If* $\mathcal{G} \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$*, then*

$$1 - \frac{2(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}{(\sigma+1)\left((1-\mathcal{B})\psi_2^m+(\mathcal{A}-1)\psi_2^n\right)}r \leq |\mathcal{G}'(\xi)| \leq 1 + \frac{2(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}{(\sigma+1)\left((1-\mathcal{B})\psi_2^m+(\mathcal{A}-1)\psi_2^n\right)}r.$$

The equality holds if the function is $\widehat{\mathcal{G}}$ *given by* (4.6).

Proof. Since the proof is quite analogous with those of Theorem 5, so it will be omitted. □

The next result deals with the fact that a convex combination of functions of the class $\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$ belongs to the same class, as follows:

Theorem 7. *Let $\mathcal{G}_i \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$ given by*

$$\mathcal{G}_i(\xi) = \xi + \sum_{j=2}^{\infty} a_{i,j}\xi^j, \ i = 1,2,3,\ldots,m. \tag{4.10}$$

Then $H \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$, where

$$H(\xi) := \sum_{i=1}^{m} c_i\mathcal{G}_i(\xi), \ and \ \sum_{i=1}^{m} c_i = 1. \tag{4.11}$$

Proof. By Theorem 4 we have

$$\sum_{j=2}^{\infty}\left((1-\mathcal{B})\psi_j^m + (\mathcal{A}-1)\psi_j^n\right)\left(\tfrac{\sigma+1}{\sigma-\upsilon+1}\right)\left|a_j\right| \leq (\mathcal{A}-\mathcal{B}),$$

and,

$$H(\xi) = \sum_{i=1}^{m} c_i\left(\xi + \sum_{j=2}^{\infty} a_{i,j}\xi^j\right) = \xi + \sum_{j=2}^{\infty}\left(\sum_{i=1}^{m} c_i a_{i,j}\right)\xi^j.$$

Therefore

$$\begin{aligned}
&\sum_{j=2}^{\infty}\left((1-\mathcal{B})\psi_j^m + (\mathcal{A}-1)\psi_j^n\right)\left(\tfrac{\sigma+1}{\sigma-\upsilon+1}\right)\left|\sum_{i=1}^{m} c_i a_{i,j}\right| \\
\leq\ & \sum_{i=1}^{m}\left[\sum_{j=2}^{\infty}\left((1-\mathcal{B})\psi_j^m + (\mathcal{A}-1)\psi_j^n\right)\left(\tfrac{\sigma+1}{\sigma-\upsilon+1}\right)\left|a_{i,j}\right|\right]c_i \\
=\ & \sum_{i=1}^{m}(\mathcal{A}-\mathcal{B})c_i = (\mathcal{A}-\mathcal{B})\sum_{i=1}^{m} c_i = (\mathcal{A}-\mathcal{B}),
\end{aligned}$$

thus $H(\xi) \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$. □

Regarding the arithmetic means of the functions of the family $\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$ the next result holds:

Theorem 8. *If $\mathcal{G}_i \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$ are given by* (4.10)*, then*

$$\mathcal{G}(\xi) := \xi + \frac{1}{k}\sum_{j=2}^{\infty}\left(\sum_{i=1}^{k} a_{i,j}\xi^j\right) \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B}). \tag{4.12}$$

Where $\mathcal{G}$ is the arithmetic mean of $\mathcal{G}_i$, $i = 1,2,3,\ldots,k$.

Proof. From the definition relation (4.12) we get

$$\mathcal{G}(\xi)=\frac{1}{k}\sum_{i=1}^{k}f_i(\xi)=\frac{1}{k}\sum_{i=1}^{k}\left(\xi+\sum_{j=2}^{\infty}a_{i,j}\xi^j\right)=\xi+\sum_{j=2}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}a_{i,j}\right)\xi^j,$$

and to prove that $\mathcal{G}(\xi)\in\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$, according to the Theorem 4 it is sufficient to prove that

$$\sum_{j=2}^{\infty}\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)\left(\tfrac{\sigma+1}{\sigma-\upsilon+1}\right)\left(\frac{1}{k}\sum_{i=1}^{k}\left|a_{i,j}\right|\right)\leq(\mathcal{A}-\mathcal{B}).$$

A simple computation shows that

$$\begin{aligned}&\sum_{j=2}^{\infty}\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)\left(\tfrac{\sigma+1}{\sigma-\upsilon+1}\right)\left(\frac{1}{k}\sum_{i=1}^{k}\left|a_{i,j}\right|\right)\\ =\;&\frac{1}{k}\sum_{i=1}^{k}\left(\sum_{j=2}^{\infty}\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)\left(\tfrac{\sigma+1}{\sigma-\upsilon+1}\right)\left|a_{i,j}\right|\right)\\ \leq\;&\frac{1}{k}\sum_{i=1}^{k}(\mathcal{A}-\mathcal{B})=(\mathcal{A}-\mathcal{B}).\end{aligned}$$

Therefore $\mathcal{G}\in\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$. □

Theorem 9. *If $\mathcal{G}\in\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$, then $\mathcal{G}$ is a starlike functions of order ϑ $(0\leq\vartheta<1)$, $|\xi|<r_1^*$,*

$$r_1^*=\inf_{j\geq2}\left(\frac{(1-\vartheta)\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)(\sigma+1)}{(j-\vartheta)(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}\right)^{\frac{1}{j-1}}.$$

The equality holds for $\mathcal{G}$ given in (4.4).

Proof. Let $\mathcal{G}\in\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$. We see that $\mathcal{G}$ is a starlike functions of order ϑ, if

$$\left|\frac{\xi\mathcal{G}'(\xi)}{\mathcal{G}(\xi)}-1\right|<1-\vartheta.$$

By simple calculation, we deduce

$$\sum_{j=2}^{\infty}\left(\frac{j-\vartheta}{1-\vartheta}\right)\left|a_j\right||\xi|^{j-1}<1. \tag{4.13}$$

Since $\mathcal{G}\in\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$, from (4.1) we get

$$\sum_{j=2}^{\infty}\frac{\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)(\sigma+1)}{(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}\left|a_j\right|<1. \tag{4.14}$$

The relation (4.13) will holds true if

$$\sum_{j=2}^{\infty}\left(\frac{j-\vartheta}{1-\vartheta}\right)\left|a_j\right||\xi|^{j-1} < \sum_{j=2}^{\infty}\frac{\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)(\sigma+1)}{(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}\left|a_j\right|,$$

which implies that

$$|\xi|^{j-1} < \left(\frac{(1-\vartheta)\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)(\sigma+1)}{(j-\vartheta)(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}\right),$$

or, equivalently

$$|\xi| < \left(\frac{(1-\vartheta)\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)(\sigma+1)}{(j-\vartheta)(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}\right)^{\frac{1}{j-1}},$$

which yields the starlikeness of the family. □

Theorem 10. *If $\mathcal{G}\in\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$, then $\mathcal{G}$ is a close-to-convex function of order ϑ $(0\leq\vartheta<1)$, $|\xi|<r_2^*$,*

$$r_2^* = \inf_{j\geq 2}\left(\frac{(1-\vartheta)(\sigma+1)\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)}{j(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}\right)^{\frac{1}{j-1}}.$$

Proof. Let $\mathcal{G}\in\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$. If $\mathcal{G}$ is close-to-convex function of order ϑ, then we find that

$$|\mathcal{G}'(\xi)-1| < 1-\vartheta,$$

that is

$$\sum_{j=2}^{\infty}\frac{j}{1-\vartheta}\left|a_j\right||\xi|^{j-1} < 1. \tag{4.15}$$

Since $\mathcal{G}\in\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$, by (4.1) we have

$$\sum_{j=2}^{\infty}\frac{(\sigma+1)\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)}{(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}\left|a_j\right| < 1. \tag{4.16}$$

The relation (4.13) will holds true if

$$\sum_{j=2}^{\infty}\frac{j}{1-\vartheta}\left|a_j\right||\xi|^{j-1} < \sum_{j=2}^{\infty}\frac{(\sigma+1)\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)}{(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}\left|a_j\right|,$$

which implies that

$$|\xi|^{j-1} < \left(\frac{(1-\vartheta)(\sigma+1)\left((1-\mathcal{B})\psi_j^m+(\mathcal{A}-1)\psi_j^n\right)}{j(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})}\right),$$

or, equivalently

$$|\xi| < \left(\frac{(1-\vartheta)(\sigma+1)\left((1-\mathcal{B})\psi_j^m + (\mathcal{A}-1)\psi_j^n\right)}{j(\sigma-\upsilon+1)(\mathcal{A}-\mathcal{B})} \right)^{\frac{1}{j-1}},$$

which yields the desired result. □

5. Conclusions

In this paper, we introduced a new class $\mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$ of holomorphic functions defined in the open unit disk, which is connected to the combination of the Binomial series and the Babalola operator. We employed differential subordination involving Janowski-type functions to investigate these properties. Utilizing well-established results, such as Carathéodory's inequality for functions with real positive parts, as well as the Keogh-Merkes and Ma-Minda inequalities, we established upper bounds for the first two initial coefficients of the Taylor-Maclaurin power series expansion. Additionally, we derived an upper bound for the Fekete-Szegő functional for functions within this family.

We also extended our findings to include similar results for the first two coefficients and for the Fekete-Szegő inequality for functions $\mathcal{G}^{-1}$ when $\mathcal{G} \in \mathcal{R}_{t,\delta,\upsilon}^{m,n,\sigma}(\mathcal{A},\mathcal{B})$. Furthermore, we determined coefficient estimates, distortion bounds, radius problems, and the radius of starlikeness and close-to-convexity for these newly defined functions.

Author contributions

Kholood M. Alsager: Conceptualization, validation, formal analysis, investigation, supervision; Sheza M. El-Deeb: Methodology, formal analysis, investigation; Ala Amourah: Methodology, validation, writing-original draft; Jongsuk Ro: Writing-original draft, writing-review. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government(MSIT) (No. NRF-2022R1A2C2004874). This work was supported by the Korea Institute of Energy Technology Evaluation and Planning(KETEP) and the Ministry of Trade, Industry & Energy(MOTIE) of the Republic of Korea (No. 20214000000280).

Conflict of interest

The authors declare no conflict of interest.

References

1. C. Carathéodory, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen, *Math. Ann.,* **64** (1907), 95–115. https://doi.org/10.1007/BF01449883

2. P. L. Duren, *Univalent functions,* Grundlehren der Mathematischen Wissenschaften, Band 259, New York, Berlin, Heidelberg and Tokyo, Springer-Verlag, 1983.

3. S. M. El-Deeb, A. A. Lupaş, Coefficient estimates for the functions with respect to symmetric conjugate points connected with the combination Binomial series and Babalola operator and Lucas polynomials, *Fractal Fract.*, **6** (2022), 1–10.

4. W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Pol. Math.*, **23** (1970), 159–177. https://doi.org/10.1086/150300

5. W. Ma, D. Minda, *A unified treatment of some special classes of univalent functions,* Proceedings of the Conference on Complex Analysis (Tianjin, Peoples Republic of China; June 19-23, 1992), (Li, Z.; Ren, F.; Yang, L. and Zhang, S. eds), pp. 157–169, International Press, Cambridge, Massachusetts, 1994.

6. M. S. Robertson, Certain classes of starlike functions, *Michigan Math. J.*, **32** (1985), 135–140.

7. H. Silverman, Univalent functions with negative coefficients, *Proc. Am. Math. Soc.*, **51** (1975), 109–116. https://doi.org/10.1090/S0002-9939-1975-0369678-0