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Research article

Frequentist and Bayesian approach for the generalized logistic lifetime model with applications to air-conditioning system failure times under joint progressive censoring data

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Abstract: Based on joint progressive Type-II censored data, we examined the statistical inference of the generalized logistic distribution with different shape and scale parameters in this research. Wherever possible, we explored maximum likelihood estimators for unknown parameters within the scope of the joint progressive censoring scheme. Bayesian inferences for these parameters were demonstrated using a Gamma prior under the squared error loss function and the linear exponential loss function. It was important to note that obtaining Bayes estimators and the corresponding credible intervals was not straightforward; thus, we recommended using the Markov Chain Monte Carlo method to compute them. We performed real-world data analysis for demonstrative purposes and ran Monte Carlo simulations to compare the performance of all the suggested approaches.

Keywords: generalized logistic distribution; maximum likelihood estimation; loss function; Bayesian estimation; Markov chain Monte Carlo; joint progressive censoring scheme
Mathematics Subject Classification: 62F10, 62N05

1. Introduction

This paper presents a novel framework for statistical inference using the generalized logistic (GL) lifetime model under joint progressive censoring schemes (join PCS), which has direct applications to real-world reliability analysis. The innovation of this work lies in the integration of both frequentist and Bayesian approaches, where unlike previous studies that focus on either one of these methods, this paper uniquely combines maximum likelihood estimation (MLE) and Bayesian inference with Gamma priors under different loss functions, such as squared error loss (SEL) and linear exponential (LINEX) loss. This dual approach offers a robust and comprehensive statistical estimation framework, allowing for a thorough comparison of both methods in practical applications. The introduction of the joint progressive censoring scheme within the GL model enables the simultaneous analysis of failure times in two populations, thereby enhancing the efficiency of data collection and parameter estimation. This approach overcomes the limitations of traditional censoring methods by allowing failures in multiple populations, making it more applicable to real-world situations where testing time is constrained. To demonstrate the practicality of the proposed methods, the paper applies them to the analysis of air-conditioning system failure times, illustrating how the GL distribution can be effectively used to assess system reliability, optimize maintenance schedules, and reduce unexpected downtimes. Moreover, the paper showcases the use of Markov Chain Monte Carlo (MCMC) methods for Bayesian estimation, which addresses the computational challenges of obtaining closed-form solutions for complex models. This is especially important for practitioners, as it offers a reliable approach for Bayesian estimation in scenarios involving censored data. Finally, the paper provides extensive Monte Carlo simulations to validate the performance of the proposed estimators, showcasing the robustness of both frequentist and Bayesian methods across different sample sizes and censoring schemes. This comprehensive simulation study allows for empirical comparisons between estimation methods, offering valuable insights into their relative efficiency under various conditions.

Based on the usual logistic distribution of the difference between two independent Gumbel-distributed random variables, a GL distribution is proposed. The GL distribution is one of three generalized versions of the standard logistic distribution, according to Balakrishnan and Leung [1]. More focus has been on determining the parameters of the GL distribution for practical applications. With respect to $\rho > 0$ and $\eta > 0$, the cumulative distribution function (CDF) of the two-parameter GL distribution is as follows:

$$F(x) = \frac{1}{(1 + e^{-\frac{x}{\eta}})^{\rho}}, \qquad -\infty < x < \infty, \tag{1.1}$$

and, additionally, the probability density function (PDF) that corresponds to it is given as follows:

$$f(x) = \frac{\rho e^{-\frac{x}{\eta}}}{\eta(1 + e^{-\frac{x}{\eta}})^{\rho+1}}, \qquad -\infty < x < \infty. \tag{1.2}$$

In this case, the shape and scale parameters are denoted by ρ and η, respectively. When $\eta < 1$, the GL distribution exhibits negative skewness; when $\eta > 1$, it shows positive skewness; and when $\eta = 1$, the distribution is symmetric. The PDF in Eq (1.2) is noted for being log-concave, unimodal, and effective in modeling data that is skewed either to the left or to the right. Extensive research on the GL distribution can be found in references Johnson and Kotz [2], Asgharzadeh [3], Alkasasbeh and Raqab [4], Gupta and Kundu [5], Li et al. [6], and Alkasasbeh et al. [7].

The PDF, CDF, survival function, and hazard rate function of the GL distribution are shown for a range of parameter values for η and ρ in Figures 1, 2, 3, 4, and 5.

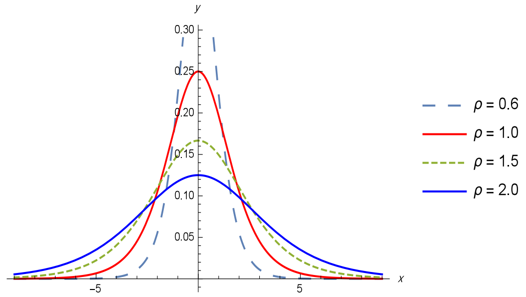


Figure 1. Shows the PDF of the GL distribution for various values of the parameter ρ, with $\eta = 1$, illustrating its symmetric nature.

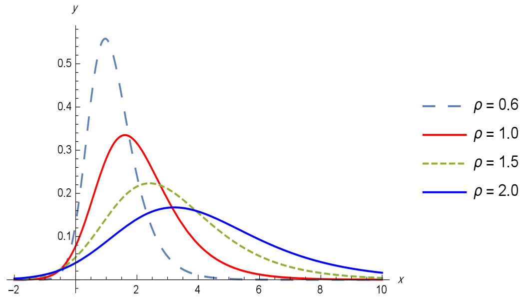


Figure 2. Shows the PDF of the GL distribution for various values of the parameter ρ, with $\eta > 1$, indicating positive skewness.

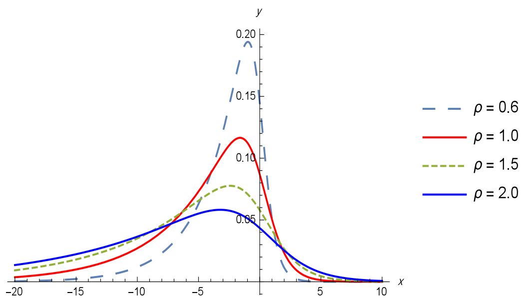


Figure 3. Shows the PDF of the GL distribution for various values of the parameter ρ, with $\eta > 1$, indicating negative skewness.

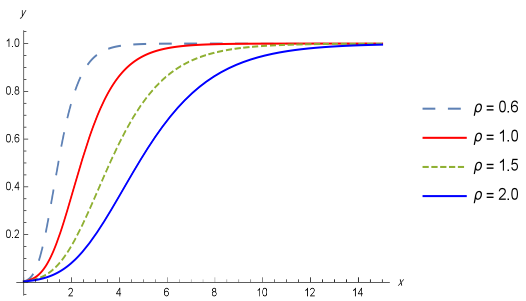


Figure 4. Shows the CDF of the GL distribution for various values of the parameter ρ.

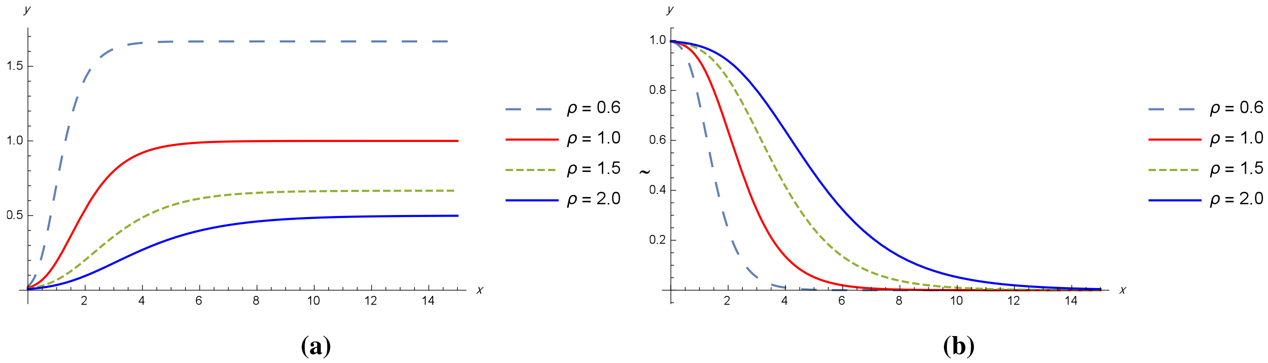


Figure 5. (a) Shows the hazard rate function of the GL distribution for various values of the parameter ρ. (b) Shows the survival function of the GL distribution for various values of the parameter ρ.

The findings from the paper offer several managerial implications and insights that can enhance decision-making. The use of the GL lifetime model enables managers to better assess the reliability of air-conditioning systems, leading to more informed maintenance schedules and a reduction in unexpected downtimes. By employing both MLE and Bayesian approaches, managers can utilize real-world data for data-driven decisions, which enhances the robustness of parameter estimates and provides more reliable predictions about system performance. The analysis also offers valuable insights for optimizing resource allocation for maintenance and repairs by identifying critical failure points and their probabilities, allowing managers to prioritize interventions with the highest return on investment. Furthermore, understanding the uncertainty in parameter estimates through credible intervals (CRIs) helps in assessing the risk associated with various operational strategies, thereby improving risk management practices. Additionally, these findings can inform strategic planning for product development and enhancements by highlighting factors that influence system failures, enabling managers to focus on improving design and manufacturing processes to extend product longevity.

Given the cost and time constraints in many real-world scenarios, it is challenging to obtain lifetime data for any product. Therefore, it is crucial to employ experimental censoring. A wide range of censoring techniques have been extensively researched. Both Type-I and Type-II censoring schemes prohibit the withdrawal of experimental units during the trial. However, the progressive Type-II censoring technique, which allows for the withdrawal of some units during the experiment, was explained by Balakrishnan and Aggarwala [8].

There are various problems when censoring techniques are applied to a specific group. Even with progressive Type-II censoring, which permits the removal of certain data, obtaining sufficient observations is still expensive. Furthermore, an experiment with a single population cannot provide evidence for population dependence and interaction, which we are interested in. Rasouli and Balakrishnan [9] proposed joint PCS as a solution to largely address these issues. To reduce the time needed to process the same amount of data by half, the joint PCS technique allows failures to occur in two populations. This feature also enables the comparison of failure times between the two populations under the same conditions.

Populations A_1 and A_2 have m and n units, respectively, at the initial stage. An experiment based on life testing utilizing these two populations provided the joint PCS. The expected number of failures (δ) is decided ahead of time. The time points of failure should be indicated as $\chi_1, \ldots, \chi_\delta$. At each time point, s_i surviving units will be randomly removed from population A_1, and t_i units will be randomly removed from population A_2. Thus, a total of $R_i = s_i + t_i$ units are eliminated at the time of the i-th failure. Additionally, it is necessary to provide a second set of random variables called $\omega_1, \ldots, \omega_\delta$, the values of which can only be 1 or 0.

Here,

$$\omega_i = \begin{cases} 1 & \text{if } \chi_i \text{ is taken from population } A_1, \\ 0 & \text{if } \chi_i \text{ is taken from population } A_2. \end{cases}$$

Assume $((\chi_1, \omega_1, s_1), \cdots, (\chi_\delta, \omega_\delta, s_\delta)$ is the censored sample. In this case, the number of failures from population A_1 is represented by $\delta_1 = \sum_{i=1}^{\delta} \omega_i$. Likewise, the number of failures from population A_2 is represented by $\delta_2 = \sum_{i=1}^{\delta}(1 - \omega_i) = \delta - \delta_1$.

In the research community, joint PCS has attracted a lot of attention. Numerous authors have explored joint PCS and related inference techniques in the literature. Many academics have explored a range of approaches and heterogeneous lifetime models in different applications. Please see the publications by [10–15].

The joint PCS offers several unique features that set it apart from traditional censoring methods. Unlike Type-I and Type-II censoring schemes, which prohibit the withdrawal of experimental units during the trial, joint PCS allows for the removal of certain units as failures occur. This flexibility helps manage resources more effectively and reduces costs associated with data collection. Additionally, joint PCS enables the simultaneous analysis of two populations, facilitating a comparison of failure times under the same experimental conditions. This is particularly useful for understanding interactions and dependencies between different groups, a capability often unavailable in single-population studies. By allowing unit withdrawal, joint PCS can streamline the data collection process, enabling researchers to focus on the most relevant data points and obtain a sufficient number of observations without incurring excessive costs. Furthermore, the scheme supports the application of advanced statistical methods, such as MLE and Bayesian inference, which can provide more accurate estimates of failure times. This study employs these methods to derive parameter estimates under the GL distribution, thereby enhancing the robustness of the results.

The study aims to achieve several key objectives related to the statistical analysis of the GL distribution, particularly under a joint PCS. The primary focus is on statistical inference, where MLE for unknown parameters is conducted alongside Bayesian estimation using both gamma and non-informative priors, demonstrating the applicability of these methods in analyzing life distributions for two populations. The research also evaluates the performance of the proposed estimates by employing LINEX loss and SEL functions, using Monte Carlo simulations to compare the effectiveness of Bayesian estimates against alternative estimators. Additionally, a real-world data analysis is included to illustrate the practical application of the theoretical results. Extensive simulation studies assess the robustness of the derived estimators across various sample sizes and failure counts, ensuring the reliability of the proposed methods. The paper presents several novel contributions to the field of statistical inference for lifetime models, including a comprehensive examination of the GL distribution under a joint progressive censoring scheme, which allows for more accurate modeling of failure times in engineering applications, such as air-conditioning systems where data may be censored due to operational constraints. Moreover, the study combines Bayesian and frequentist estimation techniques, offering a dual perspective on parameter estimation and enabling a more robust analysis by leveraging the strengths of both methodologies. The use of Monte Carlo simulations provides empirical evidence of the effectiveness of the proposed estimators under various conditions, further enhancing the reliability of the findings. Collectively, these objectives and contributions advance the understanding of the GL lifetime model and its practical applications in real-world scenarios.

Below is an outline of the paper's structure: The main goal of Section 2 is to determine the confidence intervals (CIs) and MLEs for the unknown values. In Section 3, we construct Bayes estimates of the lifetime GL distribution, taking into account the LINEX loss and SEL functions. The examination of an actual dataset and the simulation results are presented in Section 4. Finally, in Section 5, we provide a summary of the work.

2. Traditional estimation

2.1. MLE

Assume that the items $X_1, \cdots, X_m$ belong to population A_1, while the items $Y_1, \cdots, Y_n$ are from population A_2. The observed data is $(\chi_1, \omega_1, s_1, t_1), \cdots, (\chi_\delta, \omega_\delta, s_\delta, t_\delta)$ with a specified joint progressive Type-II censoring scheme $(R_1, \cdots, R_\delta)$. In the following pages, the censoring data group $(\chi_1, \omega_1, s_1, t_1), \cdots, (\chi_\delta, \omega_\delta, s_\delta, t_\delta)$ is designated as "data". The likelihood function is expressed as follows:

$$L(\rho_1, \rho_2, \eta_1, \eta_2|data) = \prod_{i=1}^{\delta} \Big[[f(\chi_i)]^{\omega_i}[g(\chi_i)]^{1-\omega_i}\Big][\bar{F}(\chi_i)]^{s_i}[\bar{G}(\chi_i)]^{t_i}, \tag{2.1}$$

$\chi_1 \leq \chi_2 \leq \ldots \leq \chi_\delta$, $\bar{F} = 1 - F$, $\bar{G} = 1 - G$, and $\sum_{i=1}^{\delta} s_i + \sum_{i=1}^{\delta} t_i = \sum_{i=1}^{\delta} R_i$.

Applying the CDF and PDF derived from Eqs (1.1) and (1.2), respectively, to the likelihood equation provided in (2.1) yields the following result.

$$\begin{aligned} L(\rho_1, \rho_1, \eta_1, \eta_2|data) =& \rho_1^{\delta_1}\eta_1^{-\delta_1}\rho_2^{\delta_2}\eta_2^{-\delta_2} e^{\frac{-1}{\eta_1}\sum_{i=1}^{\delta}\chi_i\omega_i} e^{\frac{-1}{\eta_2}\sum_{i=1}^{\delta}\chi_i(1-\omega_i)} e^{(-\rho_1-1)\sum_{i=1}^{\delta}\omega_i \ln\left[1+e^{\frac{-\chi_i}{\eta_1}}\right]} \\ &\times e^{(-\rho_2-1)\sum_{i=1}^{\delta}(1-\omega_i)\ln\left[1+e^{\frac{-\chi_i}{\eta_2}}\right]} e^{\sum_{i=1}^{\delta} s_i \ln\left[1-\left[1+e^{\frac{-\chi_i}{\eta_1}}\right]^{-\rho_1}\right]} \\ &\times e^{\sum_{i=1}^{\delta} t_i \ln\left[1-\left[1+e^{\frac{-\chi_i}{\eta_2}}\right]^{-\rho_2}\right]}. \end{aligned} \tag{2.2}$$

The log-likelihood function, denoted as $\ell(\rho_1, \rho_2, \eta_1, \eta_2 \mid \text{data}) = \ln L(\rho_1, \rho_2, \eta_1, \eta_2 \mid \text{data})$, is provided as follows:

$$\begin{aligned} \ell(\rho_1, \rho_2, \eta_1, \eta_2|data) =& \delta_1 \ln\rho_1 - \delta_1 \ln\eta_1 + \delta_2 \ln\rho_2 - \delta_2 \ln\eta_2 - \frac{1}{\eta_1}\sum_{i=1}^{\delta}\chi_i\omega_i - \frac{1}{\eta_2}\sum_{i=1}^{\delta}\chi_i(1-\omega_i) \\ &+ (-\rho_1 - 1)\sum_{i=1}^{\delta}\omega_i \ln[1 + e^{\frac{-\chi_i}{\eta_1}}] + (-\rho_2 - 1)\sum_{i=1}^{\delta}(1-\omega_i)\ln[1 + e^{\frac{-\chi_i}{\eta_2}}] \\ &+ \sum_{i=1}^{\delta} s_i \ln\Big[1 - [1 + e^{\frac{-\chi_i}{\eta_1}}]^{-\rho_1}\Big] + \sum_{i=1}^{\delta} t_i \ln\Big[1 - [1 + e^{\frac{-\chi_i}{\eta_2}}]^{-\rho_2}\Big]. \end{aligned} \tag{2.3}$$

To determine the normal equations for the unknown parameters ρ_1, ρ_2, η_1, and η_2, we partially differentiate Eq (2.3) and set the resulting expressions to zero. Solving the equations $\hat{\rho_1}$, $\hat{\rho_2}$, $\hat{\eta_1}$, and $\hat{\eta_2}$

provides the estimators for each of the parameters.

$$\frac{\delta_1}{\rho_1} - \sum_{i=1}^{\delta} \omega_i \ln\left[1 + e^{\frac{-\chi_i}{\eta_1}}\right] + \sum_{i=1}^{\delta}\left(\frac{s_i[1 + e^{\frac{-\chi_i}{\eta_1}}]^{-\rho_1} \ln\left[1 + e^{\frac{-\chi_i}{\eta_1}}\right]}{\left[1 - [1 + e^{\frac{-\chi_i}{\eta_1}}]^{-\rho_1}\right]}\right) = 0, \tag{2.4}$$

$$\frac{\delta_2}{\rho_2} - \sum_{i=1}^{\delta} (1-\omega_i) \ln\left[1 + e^{\frac{-\chi_i}{\eta_2}}\right] + \sum_{i=1}^{\delta}\left(\frac{t_i[1 + e^{\frac{-\chi_i}{\eta_2}}]^{-\rho_2} \ln\left[1 + e^{\frac{-\chi_i}{\eta_2}}\right]}{\left[1 - [1 + e^{\frac{-\chi_i}{\eta_2}}]^{-\rho_2}\right]}\right) = 0, \tag{2.5}$$

$$\frac{-\delta_1}{\eta_1} + \frac{1}{\eta_1^2}\sum_{i=1}^{\delta} \omega_i\chi_i + (-\rho_1 - 1)\sum_{i=1}^{\delta}\left(\frac{\omega_i e^{\frac{-\chi_i}{\eta_1}} \frac{\chi_i}{\eta_1^2}}{[1 + e^{\frac{-\chi_i}{\eta_1}}]}\right) + \sum_{i=1}^{\delta}\left(\frac{s_i\rho_1[1 + e^{\frac{-\chi_i}{\eta_1}}]^{-\rho_1-1} e^{\frac{-\chi_i}{\eta_1}} \frac{\chi_i}{\eta_1^2}}{\left[1 - [1 + e^{\frac{-\chi_i}{\eta_1}}]^{-\rho_1}\right]}\right) = 0, \tag{2.6}$$

$$\frac{-\delta_2}{\eta_2} + \frac{1}{\eta_2^2}\sum_{i=1}^{\delta} (1-\omega_i)\chi_i + (-\rho_2 - 1)\sum_{i=1}^{\delta}\left(\frac{(1-\omega_i) e^{\frac{-\chi_i}{\eta_2}} \frac{\chi_i}{\eta_2^2}}{[1 + e^{\frac{-\chi_i}{\eta_2}}]}\right) + \sum_{i=1}^{\delta}\left(\frac{t_i\rho_1[1 + e^{\frac{-\chi_i}{\eta_2}}]^{-\rho_2-1} e^{\frac{-\chi_i}{\eta_2}} \frac{\chi_i}{\eta_2^2}}{\left[1 - [1 + e^{\frac{-\chi_i}{\eta_2}}]^{-\rho_2}\right]}\right) = 0. \tag{2.7}$$

Since there are no closed-form solutions for the system of nonlinear equations, the first partial derivatives of the log-likelihood function with respect to the individual parameters, as displayed in Eqs (2.4), (2.5), (2.6), and (2.7), cannot be solved explicitly. As a result, the corresponding MLEs are evaluated using iterative numerical approximation techniques. We can use a suitable numerical technique, such as the Newton-Raphson iteration method, to obtain these estimates.

In order to provide proof of the existence and uniqueness of the MLE for the GL distribution under joint progressive censoring, a rigorous approach typically involves a theorem that ensures the conditions for both existence and uniqueness of the MLE.

Here's a general outline of how such a theorem can be structured, along with proof concepts:

Theorem 2.1. *Existence and uniqueness of MLE.*

Let $\{X_1, X_2, \ldots, X_n\}$ *be independent random variables from the GL distribution with the PDF:*

$$f(x|\theta) = \frac{\rho e^{-x/\eta}}{\eta(1 + e^{-x/\eta})^{\rho+1}}, \quad -\infty < x < \infty,$$

where $\theta = (\rho, \eta)$ *are the shape and scale parameters, respectively, and* $\rho > 0, \eta > 0$.

Under the following regularity conditions, the MLE of the parameters $\theta = (\rho, \eta)$ *exist and are unique:*

Regularity conditions:

- *Continuity: The likelihood function* $L(\theta)$ *is continuous in the parameter space* Θ.
- *Differentiability: The log-likelihood function* $\ell(\theta)$ *is twice differentiable with respect to* θ.
- *Concavity: The log-likelihood function* $\ell(\theta)$ *is strictly concave in the parameter space* Θ.
- *Identifiability: The model is identifiable, i.e., for distinct parameter values* $\theta_1 \neq \theta_2$, *the probability distributions* $f(x|\theta_1)$ *and* $f(x|\theta_2)$ *are not identical.*
- *Boundedness: The second-order derivatives of the log-likelihood function are bounded, ensuring that the Fisher information matrix is positive definite.*

Proof of existence.

- Continuity: Since the PDF is continuous in the parameters $\theta = (\rho, \eta)$ and the data is observed, the likelihood function $L(\theta)$ is a continuous function of θ. The log-likelihood function $\ell(\theta)$, being the logarithm of the likelihood function, inherits this continuity.
- Regularity conditions: If the second-order derivatives of the log-likelihood function exist and are continuous, the likelihood function satisfies the conditions for MLE existence.

□

Proof of uniqueness.

- Concavity: The log-likelihood function $\ell(\theta)$ is strictly concave if its Hessian matrix (second-order partial derivatives with respect to the parameters) is negative definite. Concavity implies that there is only one maximum of the log-likelihood function.
  In the case of the GL distribution, strict concavity can be shown by verifying that the second derivatives with respect to ρ and η lead to a negative definite Hessian matrix.
- Identifiability: Since the model is identifiable, different parameter values produce different likelihoods, ensuring that no two distinct parameter values maximize the likelihood function simultaneously. This guarantees the uniqueness of the MLE.
- Fisher information matrix: The Fisher information matrix, $I(\theta)$, which is based on the second-order partial derivatives of the log-likelihood function, is positive definite under regularity conditions. This positive definiteness ensures that the MLE is unique by enforcing the strict concavity of the log-likelihood function.

□

2.2. Asymptotic variance covariance matrix

Building the CIs requires the use of the asymptotic variance-covariance matrix. This matrix is obtained by taking the inverse of the Fisher information matrix. The MLE of $\psi = (\rho_1, \rho_2, \eta_1, \eta_2)$ can be represented by $\hat{\psi} = (\hat{\rho_1}, \hat{\rho_2}, \hat{\eta_1}, \hat{\eta_2})$, and the Fisher information matrix $I(\psi)$ is given as follows:

$$I(\psi) = -E\begin{pmatrix} -\frac{\partial^2\ell}{\partial\rho_1^2} & -\frac{\partial^2\ell}{\partial\rho_1\partial\rho_2} & -\frac{\partial^2\ell}{\partial\rho_1\partial\eta_1} & -\frac{\partial^2\ell}{\partial\rho_1\partial\eta_1} \\ -\frac{\partial^2\ell}{\partial\rho_2\partial\rho_1} & -\frac{\partial^2\ell}{\partial\rho_2^2} & -\frac{\partial^2\ell}{\partial\rho_2\partial\eta_1} & -\frac{\partial^2\ell}{\partial\rho_2\partial\eta_2} \\ -\frac{\partial^2\ell}{\partial\eta_1\partial\rho_1} & -\frac{\partial^2\ell}{\partial\eta_1\partial\rho_2} & -\frac{\partial^2\ell}{\partial\eta_1^2} & -\frac{\partial^2\ell}{\partial\eta_1\partial\eta_2} \\ -\frac{\partial^2\ell}{\partial\eta_2\partial\rho_1} & -\frac{\partial^2\ell}{\partial\eta_2\partial\rho_2} & -\frac{\partial^2\ell}{\partial\eta_2\partial\eta_1} & -\frac{\partial^2\ell}{\partial\eta_2^2} \end{pmatrix}. \tag{2.8}$$

We apply the following asymptotic normality result to obtain the confidence interval: $N(0, I^{-1}(\psi)) \to \sqrt{n}(\hat{\psi} - \psi)$. Under certain regularity conditions, $\hat{\psi} = (\hat{\rho_1}, \hat{\rho_2}, \hat{\eta_1}, \hat{\eta_2})$ roughly follows a normal distribution with mean $(\rho_1, \rho_2, \eta_1, \eta_2)$ and covariance matrix $I^{-1}(\rho_1, \rho_2, \eta_1, \eta_2)$. Obtaining the exact mathematical expression for $I(\psi)$ in closed form is challenging. We estimate $I^{-1}(\psi)$ by $I^{-1}(\hat{\psi})$ using the uniqueness property of the MLE to derive the asymptotic confidence intervals (ACIs) for the unknown parameters $(\rho_1, \rho_2, \eta_1, \eta_2)$ as follows:

$$\left(\hat{\rho_1} \pm Z_{\sigma/2}\sqrt{\widehat{var}(\hat{\rho_1})}\right), \left(\hat{\rho_2} \pm Z_{\sigma/2}\sqrt{\widehat{var}(\hat{\rho_2})}\right), \left(\hat{\eta_1} \pm Z_{\sigma/2}\sqrt{\widehat{var}(\hat{\eta_1})}\right) \text{and} \left(\hat{\eta_2} \pm Z_{\sigma/2}\sqrt{\widehat{var}(\hat{\eta_2})}\right), \tag{2.9}$$

where the estimated variances $\text{var}(\hat{\rho_1})$, $\text{var}(\hat{\rho_2})$, $\text{var}(\hat{\eta_1})$, and $\text{var}(\hat{\eta_2})$ are given by the diagonal elements of $I^{-1}(\hat{\psi})$. Additionally, $Z_{\sigma/2}$ represents the percentile of the standard normal distribution corresponding to a right-tail probability of $\sigma/2$. This value is the point on the standard normal distribution (which has a mean of 0 and a standard deviation of 1) where the area under the curve to the right of the point equals $\sigma/2$. In statistical testing, σ typically denotes the significance level, which is the probability of rejecting the null hypothesis when it is true, also known as the Type-I error rate.

3. Bayesian inference

In order to generate a posterior distribution and provide a strong foundation for parameter estimation, Bayesian estimation is a potent statistical technique that combines past knowledge with the available data. While Bayesian estimation incorporates prior distributions that reflect prior information or views about the parameters, standard frequentist approaches only take into account the sample data. When data may be scant or ambiguous, this integration makes it possible to conduct a more thorough examination. Bayesian estimation is very helpful when dealing with the GL distribution because it allows previous knowledge about the distribution parameters to be incorporated, improving the estimates' accuracy and dependability. Bayesian approaches can be tailored to suit different experimental conditions by utilizing alternative censoring schemes and loss functions. This allows for a flexible and sophisticated approach to parameter inference, which can greatly enhance the quality of the statistical inferences made from the data.

3.1. Prior distribution

In Bayesian statistics, the prior distribution and posterior distribution are fundamental concepts that describe our knowledge about a parameter before and after observing data, respectively. The prior distribution represents the initial beliefs or assumptions about the parameter's value before any data is taken into account. This distribution can be based on previous studies, expert knowledge, or other relevant information. Priors can be informative, incorporating substantial previous knowledge, or non-informative, reflecting a lack of specific prior information.

In this section, we provide the Bayesian estimates for the unknown parameters, along with the CRIs, employing the JPog-II scheme as described earlier. Our primary emphasis is on the SEL and LINEX loss functions.

Let's now establish the prior distributions for ρ_1, ρ_2, η_1, and η_2. It is preferable for the model parameters to be independent, ensuring that both prior and posterior densities belong to similar families. These selected priors facilitate analytical handling of the posterior distribution and computational efficiency. A suitable choice for the priors of ρ_1, ρ_2, η_1, and η_2 would be to assume that these four quantities follow independent gamma distributions, denoted as $Gamma(a_i, b_i)$, where i takes values 1, 2, 3, and 4, respectively. The PDFs of these distributions are as follows:

$$\rho_1 \sim Gamma(a_1, b_1),\ \pi_1(\rho_1) \propto \rho_1^{a_1-1} e^{-b_1\rho_1}, \qquad \rho_1 > 0, a_1, b_1 > 0, \tag{3.1}$$

$$\rho_2 \sim Gamma(a_2, b_2),\ \pi_2(\rho_2) \propto \rho_2^{a_2-1} e^{-b_2\rho_2}, \qquad \rho_2 > 0, a_2, b_2 > 0, \tag{3.2}$$

$$\eta_1 \sim Gamma(a_3, b_3),\ \pi_3(\eta_1) \propto \eta_1^{a_3-1} e^{-b_3\eta_1}, \qquad \eta_1 > 0, a_3, b_3 > 0, \tag{3.3}$$

$$\eta_2 \sim Gamma(a_4, b_4),\ \pi_4(\eta_2) \propto \eta_2^{a_4-1} e^{-b_4\eta_2}, \qquad \eta_2 > 0, a_4, b_4 > 0, \tag{3.4}$$

where $\rho_1, \rho_2, \eta_1, \eta_2$ and a_i, and b_i, where $i = 1, 2, 3, 4$, are selected to reflect prior knowledge about ρ_1, ρ_2, η_1, and η_2.

By integrating the prior distributions outlined in Eqs (3.1), (3.2), (3.3), and (3.4), we derive the joint prior density for ρ_1, ρ_2, η_1, and η_2 as follows:

$$\pi(\rho_1,\rho_2,\eta_1,\eta_2) \propto \rho_1^{a_1-1}\rho_2^{a_2-1}\eta_1^{a_3-1}\eta_2^{a_4-1}e^{-b_1\rho_1-b_2\rho_2-b_3\eta_1-b_4\eta_2}. \tag{3.5}$$

3.2. Posterior distribution

The posterior distribution, on the other hand, is the updated probability distribution of the parameter after incorporating the observed data. It combines the prior distribution with the likelihood of the observed data through Bayes' theorem. Mathematically, the posterior distribution is proportional to the product of the prior distribution and the likelihood function. This updating process allows for the integration of new information with existing beliefs, refining the parameter estimates and quantifying uncertainty in a coherent manner. The posterior distribution is central to Bayesian inference, enabling probabilistic statements and predictions about the parameter based on both prior knowledge and empirical data.

The joint posterior density function for ρ_1, ρ_2, η_1, and η_2 can be derived by combining Eqs (2.2) and (3.5) as follows:

$$\begin{aligned}\pi^*(\rho_1,\rho_2,\eta_1,\eta_2|data) \propto\ & \rho_1^{a_1-1}\rho_2^{a_2-1}\eta_1^{a_3-1}\eta_2^{a_4-1}e^{-b_1\rho_1-b_2\rho_2-b_3\eta_1-b_4\eta_2}\\ &\times \rho_1^{\delta_1}\eta_1^{-\delta_1}\rho_2^{\delta_2}\eta_2^{-\delta_2}e^{\frac{-1}{\eta_1}\sum_{i=1}^{\delta}\chi_i\omega_i}e^{\frac{-1}{\eta_2}\sum_{i=1}^{\delta}\chi_i(1-\omega_i)}e^{(-\rho_1-1)\sum_{i=1}^{\delta}\omega_i\ln\left[1+e^{\frac{-\chi_i}{\eta_1}}\right]}\\ &\times e^{(-\rho_2-1)\sum_{i=1}^{\delta}(1-\omega_i)\ln\left[1+e^{\frac{-\chi_i}{\eta_2}}\right]}e^{\sum_{i=1}^{\delta}s_i\ln\left[1-\left[1+e^{\frac{-\chi_i}{\eta_1}}\right]^{-\rho_1}\right]}\\ &\times e^{\sum_{i=1}^{\delta}t_i\ln\left[1-\left[1+e^{\frac{-\chi_i}{\eta_2}}\right]^{-\rho_2}\right]}.\end{aligned} \tag{3.6}$$

Explicit formulas for the marginal posterior distributions are difficult to derive, as illustrated by Eq (3.6). This challenge can be addressed by generating samples from Eq (3.6) using the MCMC method. The conditional posterior density functions for η_1, η_2, ρ_1, and ρ_2 are as follows:

$$\pi_1^*(\rho_1|\rho_2,\eta_1,\eta_2) \propto \rho_1^{\delta_1+a_1-1}e^{-\rho_1\left[b_1+\sum_{i=1}^{\delta}\omega_i\ln\left(1+e^{\frac{-\chi_i}{\eta_1}}\right)\right]}e^{\sum_{i=1}^{\delta}s_i\ln\left[1-\left[1+e^{\frac{-\chi_i}{\eta_1}}\right]^{-\rho_1}\right]}, \tag{3.7}$$

$$\pi_2^*(\rho_2|\rho_1,\eta_1,\eta_2) \propto \rho_2^{\delta_2+a_2-1}e^{-\rho_2\left[b_2+\sum_{i=1}^{\delta}(1-\omega_i)\ln\left(1+e^{\frac{-\chi_i}{\eta_2}}\right)\right]}e^{\sum_{i=1}^{\delta}t_i\ln\left[1-\left[1+e^{\frac{-\chi_i}{\eta_2}}\right]^{-\rho_2}\right]}, \tag{3.8}$$

$$\pi_3^*(\eta_1|\rho_1,\rho_2,\eta_2) \propto \eta_1^{-\delta_1+a_3-1}e^{-b_3\eta_1-\frac{1}{\eta_1}\sum_{i=1}^{\delta}\chi_i\omega_i-\sum_{i=1}^{\delta}\omega_i\ln\left[1+e^{\frac{-\chi_i}{\eta_1}}\right]}e^{\sum_{i=1}^{\delta}s_i\ln\left[1-\left[1+e^{\frac{-\chi_i}{\eta_1}}\right]^{-\rho_1}\right]}, \tag{3.9}$$

$$\pi_4^*(\eta_2|\rho_1,\rho_2,\eta_1) \propto \eta_2^{-\delta_2+a_4-1}e^{-b_4\eta_2-\frac{1}{\eta_2}\sum_{i=1}^{\delta}\chi_i(1-\omega_i)-\sum_{i=1}^{\delta}(1-\omega_i)\ln\left[1+e^{\frac{-\chi_i}{\eta_2}}\right]}e^{\sum_{i=1}^{\delta}t_i\ln\left[1-\left[1+e^{\frac{-\chi_i}{\eta_2}}\right]^{-\rho_2}\right]}. \tag{3.10}$$

It is observed that Eqs (3.7), (3.8), (3.9), and (3.10) present mathematical tractability issues. By utilizing various loss functions, such as SEL and LINEX loss functions, we derive Bayes estimators for the unknown parameters.

3.3. Loss functions

Loss functions are essential tools in Bayesian estimating because they put a number on the cost of estimation errors and help to minimize predicted loss, which powers decision-making. Many loss functions exist, each with a specific purpose and setting in mind.

3.3.1. Squared error loss

A common loss function in statistical estimation and decision theory, especially in the context of Bayesian inference, is the SEL. SEL provides an indication of the size of mistakes in a model's predictions. It is defined as the square of the difference between the estimated and true parameter values. Mathematically, the SEL for an estimate $\hat{\phi}$ of a parameter ϕ is given by:

$$L(\hat{\phi}, \phi) = (\hat{\phi} - \phi)^2.$$

By emphasizing the reduction of significant disparities, this quadratic penalty helps minimize total error because it ensures that larger deviations between the estimate and the true value are penalized more severely than smaller ones. The SEL is popular due to its desirable characteristics, such as differentiability, which facilitates the application of optimization strategies to determine the optimal estimates. It is also mathematically simple. The posterior mean is the best choice rule in Bayesian estimation when using SEL, as it minimizes the expected squared error. This makes the SEL particularly useful when the objective is to obtain an unbiased estimate that is generally closer to the true parameter value in a least-squares sense. However, because SEL is quadratic, it is sensitive to outliers, which can be problematic when robustness to extreme results is crucial.

3.3.2. Linear exponential loss function

The asymmetric LINEX loss function proposed by Varian [16] is utilized in statistical estimation and decision-making. It is distinguished by its ability to represent varying costs associated with both overestimation and underestimation of parameters. The LINEX loss function is more flexible in capturing situations where the penalty for errors is not equal, compared to symmetric loss functions like the SEL.

The LINEX loss function can be expressed mathematically as follows:

$$L(\hat{\phi}, \phi) = e^{\epsilon(\hat{\phi} - \phi)} - \epsilon(\hat{\phi} - \phi) - 1,$$

where ϕ is the true parameter, $\hat{\phi}$ is the estimated parameter, and ϵ is a parameter that regulates the loss function's asymmetry. Overestimation is penalized more severely than underestimation when $\epsilon > 0$, and vice versa when $\epsilon < 0$. This asymmetry is particularly useful in real-world situations where the costs of overestimating and underestimating a parameter have differing consequences. The LINEX loss function is valuable in applications where it is important to consider the varying effects of estimation errors on decision outcomes. For example, in quality control or financial forecasting, underestimation might lead to missed opportunities or subpar performance, while overestimation could result in excessive costs or resource misallocation. By adjusting ϵ, the LINEX loss function can be tailored to account for these fluctuating costs and guide the estimation procedure appropriately. When the LINEX loss function is used in Bayesian estimation, the posterior mode that minimizes the expected LINEX loss is found to be the best choice rule. In domains where error asymmetry significantly impacts decision-making, this approach can lead to estimates that are more useful and contextually appropriate. Overall, the LINEX loss function offers a refined method for parameter estimation and error assessment, making it a valuable alternative to symmetric loss functions.

3.4. MCMC in Bayesian estimation and credible intervals

The MCMC method is a class of algorithms used to sample from complex probability distributions and estimate statistical properties when direct sampling is infeasible. These methods are particularly useful in Bayesian estimation, where deriving the posterior distribution of parameters analytically is challenging. MCMC generates samples by constructing a Markov chain with the desired distribution as its equilibrium distribution, using iterative processes that converge to this distribution. Two popular MCMC algorithms are the Metropolis-Hastings (M-H) algorithm, which proposes new samples based on a proposal distribution and accepts or rejects them according to specific criteria, and the Gibbs sampler, which iteratively samples from the conditional distributions of each parameter given the others (see Metropolis et al. [17] and Hastings [18]). These techniques are invaluable for providing approximate solutions to high-dimensional and complex problems, enabling researchers to estimate posterior distributions, calculate integrals, and make probabilistic inferences in a computationally feasible manner. In this context, credible intervals, the Bayesian counterpart to CIs in frequentist statistics, are calculated using the MCMC method, as mentioned in the paper. These intervals provide a range of values within which the true parameter is believed to lie with a certain probability, based on the posterior distribution. The study highlights that Bayesian estimates and their corresponding credible intervals often outperform MLEs in terms of accuracy and reliability, especially in simulation studies where Bayesian CRIs showed smaller average interval lengths compared to confidence intervals. Additionally, the practical application of Bayesian methods, including CRIs, in analyzing life distributions for two populations underscores their relevance in real-world scenarios.

Theorem 3.1. *Conditions for the successful execution of the M-H algorithm for Bayesian estimation of the GL distribution.*

Let $\pi(\theta|X)$ be the target posterior distribution of the parameters $\theta = (\rho_1, \rho_2, \eta_1, \eta_2)$ of the GL distribution under joint progressive censoring, where X represents the observed failure time data. Let $q(\theta'|\theta)$ be the proposal distribution for generating candidate values θ' in the M-H algorithm. The M-H algorithm generates a sequence of samples $\{\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}\}$, intended to converge to the target posterior distribution $\pi(\theta|X)$. For the M-H algorithm to execute successfully, the following conditions must be satisfied:

Regularity conditions:

- *Aperiodicity: The Markov chain must be aperiodic.*
  Definition: A Markov chain is aperiodic if there is no fixed period such that the chain revisits a state only at fixed intervals.
  Verification: The acceptance probability in the M-H algorithm allows for the possibility of staying at the current state, ensuring aperiodicity. For the GL distribution with joint progressive censoring, the chain can stay at the current state when the proposed state is rejected.
- *Irreducibility: The Markov chain must be irreducible, meaning that it is possible to reach any state $\theta' \in \Theta$ from any other state $\theta \in \Theta$ in a finite number of steps.*
  Definition: Irreducibility ensures that the entire parameter space $\Theta = (\rho_1, \rho_2, \eta_1, \eta_2)$ is explored by the algorithm.
  Verification: For Bayesian estimation of the GL distribution, a proper choice of the proposal distribution $q(\theta'|\theta)$ ensures that any state $\theta' = (\rho_1', \rho_2', \eta_1', \eta_2')$ can be reached from the current state $\theta = (\rho_1, \rho_2, \eta_1, \eta_2)$. Common choices like Gaussian proposals with nonzero variance satisfy

this condition.

- *Detailed balance: The Markov chain must satisfy the detailed balance condition with respect to the target posterior distribution $\pi(\theta|X)$.*
  Definition: Detailed balance ensures that for any two states θ and θ', the following holds:

$$\pi(\theta|X)q(\theta'|\theta)\alpha(\theta'|\theta) = \pi(\theta'|X)q(\theta|\theta')\alpha(\theta|\theta'),$$

  where $\alpha(\theta'|\theta)$ is the acceptance probability given by:

$$\alpha(\theta'|\theta) = \min\left(1, \frac{\pi(\theta'|X)q(\theta|\theta')}{\pi(\theta|X)q(\theta'|\theta)}\right).$$

  Verification: In the context of Bayesian estimation for the GL distribution, the M-H algorithm's acceptance rule is designed to ensure detailed balance. Since the acceptance probability $\alpha(\theta'|\theta)$ is derived from the ratio of the posterior densities and proposal distributions, detailed balance is satisfied, guaranteeing that the posterior distribution is the stationary distribution of the Markov chain.
- *Positivity of proposal density: The proposal density $q(\theta'|\theta)$ must be positive for all $\theta, \theta' \in \Theta$.*
  Definition: This condition ensures that there is a positive probability of proposing any new candidate state θ' from the current state θ.
  Verification: In the Bayesian estimation of the GL distribution, this condition is satisfied by choosing a proposal distribution such as a Gaussian proposal, where $q(\theta'|\theta) > 0$ for all pairs $\theta = (\rho_1, \rho_2, \eta_1, \eta_2)$ and $\theta' = (\rho'_1, \rho'_2, \eta'_1, \eta'_2)$.
- *Convergence to stationary distribution: The Markov chain must converge to the target posterior distribution $\pi(\theta|X)$ as the number of iterations tends to infinity.*
  Definition: Convergence ensures that after a sufficiently large number of iterations, the samples generated by the algorithm are from the target posterior distribution $\pi(\theta|X)$.
  Verification: Convergence is a direct consequence of the Ergodic theorem, which states that an irreducible and aperiodic Markov chain converges to its stationary distribution. In the case of the Bayesian estimation for the GL distribution, if the Markov chain satisfies the conditions of aperiodicity, irreducibility, and detailed balance, it will converge to the target posterior distribution $\pi(\theta|X)$.

Proof outline. Aperiodicity and irreducibility: The ability of the M-H algorithm to either stay at the current state or propose a wide range of values ensures aperiodicity and irreducibility. For Bayesian estimation of the GL distribution, the joint progressive censoring setup allows the exploration of the parameter space Θ, and appropriate choices of proposal distributions ensure that all states can be reached.

Detailed balance: The acceptance probability is constructed to satisfy detailed balance, which implies that the target posterior distribution is the stationary distribution of the Markov chain.

Convergence: The conditions of aperiodicity, irreducibility, and detailed balance ensure that the M-H algorithm converges to the posterior distribution $\pi(\theta|X)$.

M-H algorithm:

- **Step 1:** To begin, use the following initial values: $(\rho_1^{(0)}, \rho_2^{(0)}, \eta_1^{(0)}, \eta_2^{(0)})$, and K is burn-in.
- **Step 2:** Set $j = 1$.

- **Step 3:** The M-H algorithm can be employed to generate $\rho_1^{(j)}$, $\rho_2^{(j)}$, $\eta_1^{(j)}$, and $\eta_2^{(j)}$ using Eqs (3.7), (3.8), (3.9), and (3.10). It is recommended to use normal distributions $N(\rho_1^{(j-1)}, \mathrm{var}(\rho_1))$, $N(\rho_2^{(j-1)}, \mathrm{var}(\rho_2))$, $N(\eta_1^{(j-1)}, \mathrm{var}(\eta_1))$, and $N(\eta_2^{(j-1)}, \mathrm{var}(\eta_2))$ for this purpose.
  (I) Use the corresponding normal distributions to generate the suggested values ρ_1^*, ρ_2^*, η_1^*, and η_2^*.
  (II) Using the steps listed below, determine the probability of acceptance,

$$r_1 = \min\left[1, \frac{\pi_1^*(\rho_1^*|\eta_2^{(j-1)}, \eta_1^{(j-1)}, \rho_2^{(j-1)}, data)}{\pi_1^*(\rho_1^{(j-1)}|\eta_2^{(j-1)}, \eta_1^{(j-1)}, \rho_2^{(j-1)}, data)}\right],$$

$$r_2 = \min\left[1, \frac{\pi_2^*(\rho_2^*|\eta_2^{(j-1)}, \eta_1^{(j-1)}, \rho_1^{(j-1)}, data)}{\pi_2^*(\rho_2^{(j-1)}|\eta_2^{(j-1)}, \eta_1^{(j-1)}, \rho_1^{(j-1)}, data)}\right],$$

$$r_3 = \min\left[1, \frac{\pi_3^*(\eta_1^*|\eta_2^{(j-1)}, \rho_1^j, \rho_2^j, data)}{\pi_3^*(\eta_1^{(j-1)}|\eta_2^{(j-1)}, \rho_1^j, \rho_2^j, data)}\right],$$

$$r_4 = \min\left[1, \frac{\pi_4^*(\eta_2^*|\eta_1^{(j-1)}, \rho_1^j, \rho_2^j, data)}{\pi_4^*(\eta_2^{(j-1)}|\eta_1^{(j-1)}, \rho_1^j, \rho_2^j, data)}\right].$$

  (III) Choose a random variable u from a uniform distribution with values between 0 and 1.
  (IV) Accept the suggestion and update $\rho_1^{(j)}$ to ρ_1^* if $u \leq r_1$; otherwise, keep $\rho_1^{(j)}$ as $\rho_1^{(j-1)}$.
  (V) Accept the suggestion and update $\rho_2^{(j)}$ to ρ_2^* if $u \leq r_2$; otherwise, keep $\rho_2^{(j)}$ as $\rho_2^{(j-1)}$.
  (VI) Accept the suggestion and update $\eta_1^{(j)}$ to η_1^* if $u \leq r_3$; otherwise, keep $\eta_1^{(j)}$ as $\eta_1^{(j-1)}$.
  (VII) Accept the suggestion and update $\eta_2^{(j)}$ to η_2^* if $u \leq r_4$; otherwise, keep $\eta_2^{(j)}$ as $\eta_2^{(j-1)}$.
- **Step 7:** Set j = j + 1.
- **Step 8:** Proceed through Steps 2 through 7 δ times. As a result, the estimated posterior means of $(\rho_1, \rho_2, \eta_1, \eta_2)$, indicated by β under the SEL function, can be computed as follows:

$$\hat{\beta}_{BS} = E[\beta|\underline{x}] = \frac{1}{\delta - K} \sum_{i=k+1}^{\delta} \beta^{(j)}. \tag{3.11}$$

  Lastly, use the LINEX loss function to find the Bayesian estimates of β.

$$\hat{\beta}_{BL} = -\frac{1}{\epsilon} \ln\left[\frac{1}{\delta - K} \sum_{i=k+1}^{\delta} e^{-\epsilon\beta^{(j)}}\right]. \tag{3.12}$$

For more information about Bayesian methods and lifetime data analysis, see Xu et al. [19] and Zhuang et al. [20]. □

4. Numerical study

The aim of this section is to evaluate the effectiveness of the different estimation methods discussed in previous sections. To illustrate this, we analyze a real dataset and conduct a simulated experiment to assess the statistical performance of the estimators under the joint PCS.

4.1. *Real data study*

The data, initially obtained from Proschan [21], shows the air conditioning system failure times (in hours) for airplanes 7913 and 7914. It is assumed that the two datasets are independent and that the failure times are independent within each dataset. Table 1 presents the data below.

Table 1. The air conditioning system failure times (in hours) for airplanes 7913 and 7914.

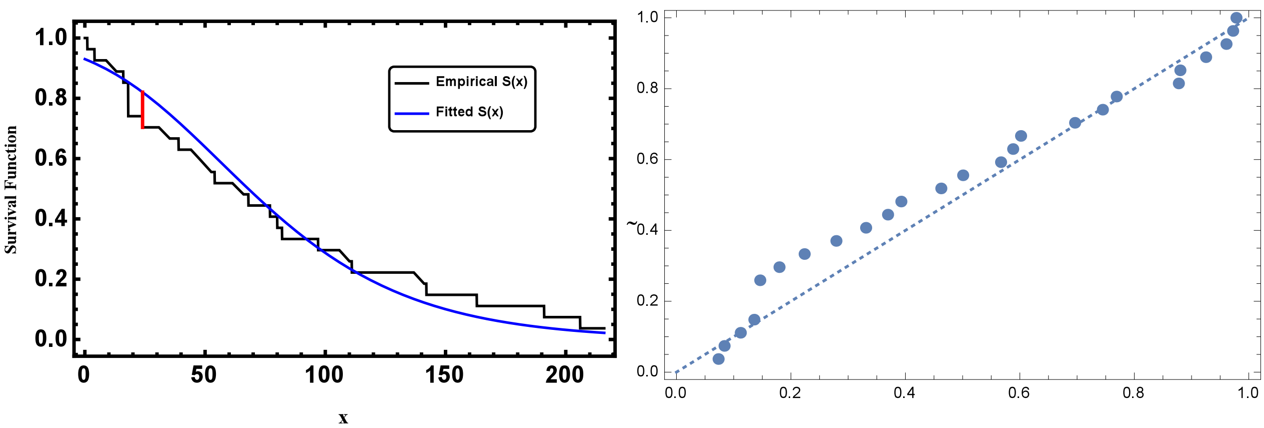
Data I	Airplane 7913	1	4	11	16	18	18	18	24	31	39	46	51	54	63
		68	77	80	82	97	106	111	141	142	163	191	206	216.	
Data II	Airplane 7914	3	5	5	13	14	15	22	22	23	30	36	39	44	46
		50	72	79	88	97	102	139	188	197	210.				

The Kolmogorov-Smirnov (K-S) test results, which are used to determine if the data complies with the GL distribution, are shown in Table 2.

Table 2. K-S test and P-value of air conditioning system failure times for airplanes 7913 and 7914.

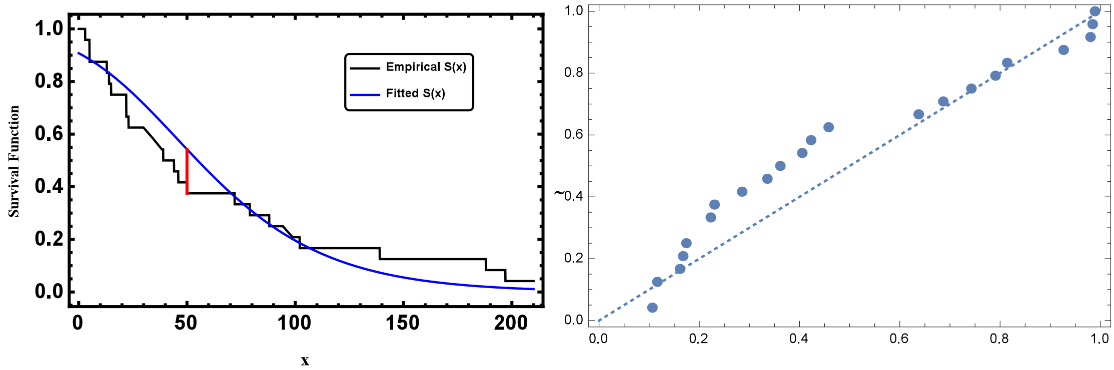
Airplanes	n	K-S	5% Significance	P-value
Airplane 7913	27	0.1166	0.2544	0.8151
Airplane 7914	24	0.1674	0.2693	0.5000

The computed K-S values for the data are clearly smaller than the equivalent values predicted at a significance level of 5%, as shown in Table 2. Notably high p-values have also been observed. Therefore, it makes sense to conclude that the GL distribution is a good model fit for the data. In addition, we have produced fitted $S(x)$ and empirical $S(x)$ for each dataset, as shown in Figures 6 and 7. The GL distribution model fits the data better, as these charts further demonstrate. Figures 8 and 9 present the profile log-likelihood function for the parameters ρ_1, ρ_2, η_1, and η_2, demonstrating that it achieves a single maximum.



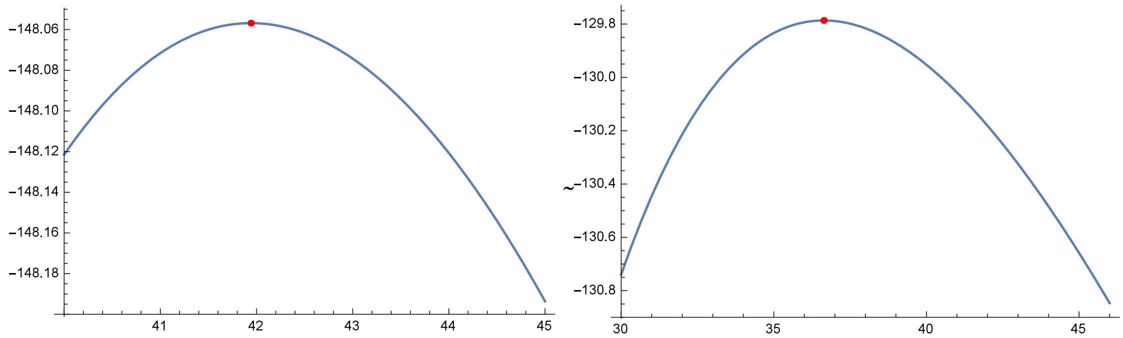
(a) (b)

Figure 6. (a) Plots of the GL distribution's fitted functions for airplane 7913, and (b) GL distribution for airplane 7913 in a probability plot.



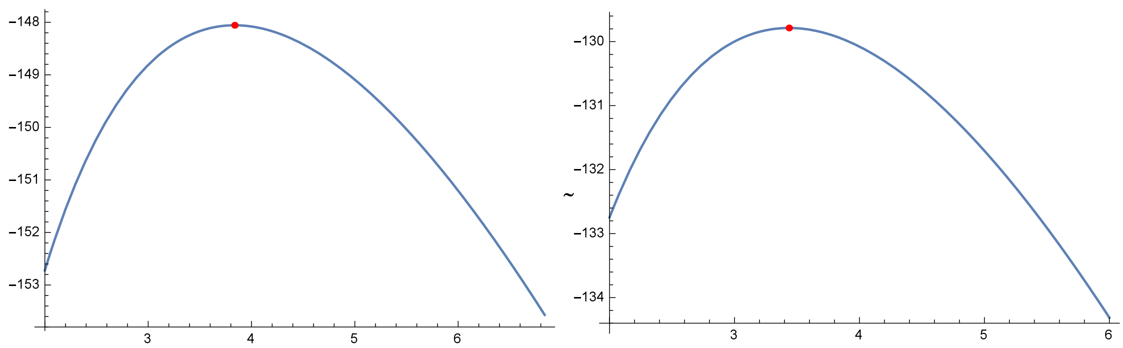
(a) **(b)**

Figure 7. (a) Plots of the GL distribution's fitted functions for airplane 7914, and (b) GL distribution for airplane 7914 in a probability plot.



(a) **(b)**

Figure 8. (a) The log-likelihood function profile for η_1, and (b) the log-likelihood function profile for η_2.



(a) **(b)**

Figure 9. (**a**) The log-likelihood function profile for ρ_1, and (**b**) the log-likelihood function profile for ρ_2.

Using the following censoring strategy, we created a joint PCS sample from the datasets listed above. To implement joint PCS with $r = 20$, set $m = 27$ for the first sample and $n = 24$ for the second.

Below is the definition of the censoring vectors:

$$\begin{aligned} S &= (0,0,4,0,0,3,0,0,0,0,3,0,0,0,6,0,0,0,0,0), \\ R &= (0,0,8,0,0,6,0,0,0,0,6,0,0,0,11,0,0,0,0,0), \\ T &= (0,0,4,0,0,3,0,0,0,0,3,0,0,0,5,0,0,0,0,0). \end{aligned}$$

The datasets created are as follows:

$$\begin{aligned} w &= (1,3,4,5,5,11,13,14,15,16,18,18,18,22,22,23,24,30,31,36), \\ z &= (1,0,1,0,0,1,0,0,0,1,1,1,1,0,0,0,1,0,1,0). \end{aligned}$$

Depending on the type of data utilized in this investigation, we employed the MLE approach to derive estimates for ρ_1, ρ_2, η_1, and η_2. Table 3 presents the corresponding results, whereas Tables 4 and 5 provide the 95% ACIs for ρ_1, ρ_2, η_1, and η_2. To ensure convergence, we omitted the first 1000 iterations as "burn-in" and used the MCMC method for Bayesian estimation, running 12,000 iterations. The hyperparameters a_i and b_i that we have chosen are 10^{-3}, which is close to zero for the prior distributions. Both the SEL and LINEX loss functions were used to construct Bayesian estimates for ρ_1, ρ_2, η_1, and η_2; Table 3 details the associated outcomes. Furthermore, Tables 4 and 5 offer the 95% CRIs for ρ_1, ρ_2, η_1, and η_2.

Table 3. Frequentist and Bayesian estimates for $(\rho_1, \rho_2, \eta_1, \eta_2)$.

Parameters	Frequentist estimates	Bayesian estimates			
		SEL	LINEX		
			$c = -5.0$	$c = 10^{-3}$	$c = 5.0$
ρ_1	4.9530	4.9709	4.9717	4.9709	4.9702
ρ_2	4.7285	4.7286	4.7288	4.7286	4.7285
η_1	25.8396	25.8550	25.8587	25.8550	25.8510
η_2	18.4558	18.4806	18.4821	18.4806	18.4791

Table 4. Interval Bayesian for (ρ_1, ρ_2).

Method	ρ_1			ρ_2		
	Lower	Upper	Length	Lower	Upper	Length
CI	2.5287	7.3773	4.8486	2.2229	7.2341	5.0112
CRI	4.9402	4.9949	0.0546	4.7155	4.7433	0.0277

Table 5. Interval Bayesian for (η_1, η_2).

Method	η_1			η_2		
	Lower	Upper	Length	Lower	Upper	Length
CI	13.2258	38.4533	25.2275	10.5552	26.3565	15.8013
CRI	25.7671	25.9145	0.1473	18.4351	18.5281	0.0929

4.2. Simulation study

To evaluate the effectiveness of the estimation techniques developed in the earlier sections, we conduct simulation tests in this section. For both populations, we investigate a range of sample sizes, such as (m, n) = (15, 20), (30, 40), (40, 60), and a range of failure counts for each sample size, such as $(15, 20), (25, 35), (40, 50)$. For both populations, the parameter values are ρ_1, ρ_2, η_1, and η_2 = $(6.5, 8.5, 5.5, 7.5)$. For each parameter ρ_1, ρ_2, η_1, and η_2, we calculated MLEs and 95% confidence intervals for the parameters ρ_1, ρ_2, η_1, and η_2. After 1000 iterations of this procedure, we computed the mean values of the MLEs and their corresponding lengths. Tables 6, 7, 8 and 9 display the findings.

Table 6. Frequentist and Bayesian estimates, along with the length and corresponding MSE (in bold) of estimates for the parameter ρ_1.

(m, n)	r	Scheme	Frequentist estimates		Bayesian estimates			
			MLE	Length	SEL	LINEX		Length
						$\epsilon = -6$	$\epsilon = 6$	
(15, 20)	15	$(0_{(14)}, 20)$	6.6378	8.6201	6.6596	6.6607	6.6585	0.0693
			0.9990		**0.9255**	**0.9258**	**0.9251**	
	15	$(20, 0_{(14)})$	7.4592	10.9775	7.4414	7.4420	7.4408	0.0493
			0.9201		**0.8862**	**0.8873**	**0.8851**	
(15, 20)	20	$(0_{(19)}, 15)$	7.1963	8.8725	7.2723	7.2761	7.2685	0.1160
			0.8849		**0.7965**	**0.7024**	**0.7906**	
	20	$(15, 0_{(19)})$	7.8856	13.0641	7.9421	7.9532	7.9306	0.2225
			0.8200		**0.6797**	**0.6117**	**0.6466**	
(30, 40)	25	$(0_{(24)}, 45)$	5.3294	4.6483	5.3367	5.3367	5.3366	0.0167
			0.7702		**0.6533**	**0.5532**	**0.5533**	
	25	$(45, 0_{(24)})$	7.5117	10.0567	7.5188	7.5212	7.5165	0.0876
			0.6236		**0.6380**	**0.5429**	**0.5333**	
(30, 40)	35	$(0_{(34)}, 35)$	9.4022	10.5080	9.4329	9.4347	9.4309	0.0854
			0.5226		**0.6019**	**0.5126**	**0.5005**	
	35	$(35, 0_{(34)})$	5.0710	4.9565	5.0685	5.0685	5.0685	0.0014
			0.3419		**0.3293**	**0.3244**	**0.3223**	
(40, 60)	40	$(0_{(39)}, 60)$	5.9177	4.4882	5.9311	5.9314	5.9309	0.0327
			0.3391		**0.3236**	**0.3233**	**0.3220**	
	40	$(60, 0_{(39)})$	8.5844	9.0196	8.5965	8.5967	8.5963	0.0290
			0.3245		**0.2951**	**0.2960**	**0.2943**	
(40, 60)	50	$(0_{(49)}, 50)$	6.6985	5.7894	6.6907	6.6910	6.6904	0.0319
			0.1394		**0.0364**	**0.0365**	**0.0363**	
	50	$(50, 0_{(49)})$	4.1476	3.3121	4.1474	4.1474	4.1474	0.0009
			0.0937		**0.0234**	**0.0347**	**0.0347**	

Furthermore, in the context of Bayesian estimation under SE and LINEX loss functions, we used informative gamma priors for α_1, α_2, γ_1, and γ_2. The hyperparameters $a_i = 5.0$ and $b_i = 7.0$ were specified for $i = 1, 2, 3, 4$. $\epsilon = 6$ signified overestimation, and $\epsilon = -6$ indicated underestimation. Using 18,000 samples, we applied the MCMC technique to obtain 95% CRIs and Bayesian estimates for ρ_1, ρ_2, η_1, and η_2 with 1000 simulations. To ensure convergence, we omitted the first 4000 iterations as "burn-in". The performance of the derived estimators for ρ_1, ρ_2, η_1, and η_2 were assessed using mean squared errors (MSE). We calculated the mean values of the MLEs and their corresponding lengths after 1000 iterations of this method. The findings are shown in Tables 6, 7, 8, and 9.

Table 7. Frequentist and Bayesian estimates, along with the length and corresponding MSE (in bold) of estimates for the parameter ρ_2.

(m, n)	r	Scheme	Frequentist estimates		Bayesian estimates			
			MLE	Length	SEL	LINEX		Length
						$\epsilon = -6$	$\epsilon = 6$	
(15, 20)	15	$(0_{(14)}, 20)$	7.9557	13.7203	7.9969	7.9973	7.9965	0.0418
			0.2963		**0.2531**	**0.2527**	**0.2536**	
	15	$(20, 0_{(14)})$	8.5179	16.5289	8.4225	8.4240	8.4209	0.0841
			0.2103		**0.2006**	**0.2058**	**0.2063**	
(15, 20)	20	$(0_{(19)}, 15)$	6.2977	9.7348	6.2666	6.2672	6.2659	0.0510
			0.1503		**0.1881**	**0.1852**	**0.1910**	
	20	$(15, 0_{(19)})$	8.1496	10.6903	8.1709	8.1719	8.1700	0.0689
			0.1228		**0.1083**	**0.1077**	**0.1069**	
(30, 40)	25	$(0_{(24)}, 45)$	9.1471	11.7626	9.1469	9.1469	9.1469	0.0540
			0.1187		**0.1045**	**0.1035**	**0.1029**	
	25	$(45, 0_{(24)})$	7.2385	7.6991	7.2394	7.2405	7.2384	0.0638
			0.1113		**0.1043**	**0.1033**	**0.1011**	
(30, 40)	35	$(0_{(34)}, 35)$	6.5883	6.3805	6.5378	6.5381	6.5375	0.0408
			0.1025		**0.0984**	**0.0992**	**0.0955**	
	35	$(35, 0_{(34)})$	7.9318	9.0774	7.9282	7.9283	7.9281	0.0214
			0.0988		**0.0940**	**0.0932**	**0.0928**	
(40, 60)	40	$(0_{(39)}, 60)$	9.9721	10.9662	9.9723	9.9712	9.9700	0.0208
			0.0872		**0.0648**	**0.0645**	**0.0644**	
	40	$(60, 0_{(39)})$	8.7826	8.8072	8.8181	8.8199	8.8162	0.0722
			0.0798		**0.0512**	**0.0524**	**0.0500**	
(40, 60)	50	$(0_{(49)}, 50)$	8.4488	7.6291	8.4481	8.4474	8.4470	0.0204
			0.0262		**0.0257**	**0.0255**	**0.0215**	
	50	$(50, 0_{(49)})$	9.0410	10.7017	9.0453	9.0443	9.0432	0.0124
			0.0209		**0.0173**	**0.01355**	**0.0138**	

Table 8. Frequentist and Bayesian estimates, along with the length and corresponding MSE (in bold) of estimates for the parameter η_1.

(m, n)	r	Scheme	Frequentist estimates		Bayesian estimates			
			MLE	Length	SEL	LINEX		Length
						$\epsilon = -6$	$\epsilon = 6$	
(15, 20)	15	$(0_{(14)}, 20)$	6.3874	5.8281	6.3883	6.3885	6.3880	0.0328
			0.7874		**0.7891**	**0.7895**	**0.7886**	
	15	$(20, 0_{(14)})$	4.4801	4.2685	4.4920	4.4923	4.4918	0.0270
			0.6403		**0.6216**	**0.6155**	**0.6145**	
(15, 20)	20	$(0_{(19)}, 15)$	7.6231	5.5054	7.5997	7.5998	7.5995	0.0299
			0.5077		**0.4086**	**0.4093**	**0.4079**	
	20	$(15, 0_{(19)})$	6.1953	6.1765	6.2171	6.2175	6.2167	0.0412
			0.4835		**0.3142**	**0.3147**	**0.3136**	
(30, 40)	25	$(0_{(24)}, 45)$	8.0908	6.2906	8.1544	8.1555	8.1533	0.0658
			0.3120		**0.2658**	**0.2514**	**0.2399**	
	25	$(45, 0_{(24)})$	5.6466	4.5870	5.6237	5.6242	5.6231	0.0443
			0.2215		**0.2153**	**0.2154**	**0.2152**	
(30, 40)	35	$(0_{(34)}, 35)$	5.3897	3.1869	5.3590	5.3595	5.3586	0.0394
			0.2122		**0.1999**	**0.1997**	**0.1800**	
	35	$(35, 0_{(34)})$	6.9059	4.7703	6.8875	6.8879	6.8872	0.0367
			0.2065		**0.1982**	**0.1962**	**0.1942**	
(40, 60)	40	$(0_{(39)}, 60)$	6.5522	3.4617	6.5388	6.5390	6.5385	0.0319
			0.1872		**0.1790**	**0.1796**	**0.1785**	
	40	$(60, 0_{(39)})$	4.9577	3.1105	4.9586	4.9586	4.9585	0.0143
			0.1041		**0.0931**	**0.0928**	**0.0921**	
(40, 60)	50	$(0_{(49)}, 50)$	6.3491	3.4742	6.3508	6.3510	6.3507	0.0259
			0.0821		**0.0723**	**0.0721**	**0.0708**	
	50	$(50, 0_{(49)})$	5.5948	3.0464	5.6032	5.6011	5.6031	0.0165
			0.0584		**0.0206**	**0.0107**	**0.0106**	

Table 9. Frequentist and Bayesian estimates, along with the length and corresponding MSE (in bold) of estimates for the parameter η_2.

(m, n)	r	Scheme	Frequentist estimates		Bayesian estimates			
			MLE	Length	SEL	LINEX		Length
						$\epsilon = -6$	$\epsilon = 6$	
(15, 20)	15	$(0_{(14)}, 20)$	5.9875	6.5316	5.9703	5.9713	5.9692	0.0703
			0.5876		**0.5454**	**0.5369**	**0.5433**	
	15	$(20, 0_{(14)})$	9.8408	10.4470	9.8479	9.8485	9.8473	0.0501
			0.5792		**0.5125**	**0.5153**	**0.5097**	
(15, 20)	20	$(0_{(19)}, 15)$	8.7166	9.2264	8.7245	8.7250	8.7239	0.0512
			0.4802		**0.4993**	**0.5007**	**0.4979**	
	20	$(15, 0_{(19)})$	9.0165	6.2204	8.9867	8.9872	8.9861	0.0510
			0.2996		**0.2102**	**0.2119**	**0.2086**	
(30, 40)	25	$(0_{(24)}, 45)$	5.6502	4.4376	5.6777	5.6778	5.6777	0.0201
			0.2218		**0.2106**	**0.2103**	**0.2009**	
	25	$(45, 0_{(24)})$	8.0368	4.7977	8.0208	8.0216	8.0201	0.0558
			0.2181		**0.2013**	**0.2050**	**0.2005**	
(30, 40)	35	$(0_{(34)}, 35)$	8.2908	5.6586	8.2800	8.2805	8.2795	0.0426
			0.2054		**0.1084**	**0.1091**	**0.1076**	
	35	$(35, 0_{(34)})$	7.2820	4.4473	7.2588	7.2591	7.2586	0.0382
			0.1075		**0.1058**	**0.1058**	**0.1058**	
(40, 60)	40	$(0_{(39)}, 60)$	7.1157	4.9217	7.1271	7.1274	7.1269	0.0334
			0.1055		**0.0839**	**0.0829**	**0.0792**	
	40	$(60, 0_{(39)})$	7.046	4.0895	7.035	7.0355	7.0344	0.0422
			0.0861		**0.0822**	**0.0787**	**0.0766**	
(40, 60)	50	$(0_{(49)}, 50)$	7.1415	3.8964	7.1115	7.1119	7.1111	0.0391
			0.0754		**0.0811**	**0.0654**	**0.0621**	
	50	$(50, 0_{(49)})$	6.0121	3.4268	5.9919	5.9930	5.9909	0.0522
			0.0139		**0.0244**	**0.0182**	**0.0175**	

Many inferences can be made from the data provided above:

1) Comparing Bayesian estimators at $\epsilon = -6$ with the LINEX loss function at $\epsilon = 6$, we find that the former performs better in terms of MSEs than the latter.
2) A closer look at Tables 6, 7, 8, and 9 reveals that, when comparing the average interval lengths of CIs to CRIs, CRIs have the smallest interval lengths, suggesting that Bayesian estimators are superior to MLEs.
3) Remarkably, in terms of having the smallest MSEs, Bayesian estimators utilizing the LINEX loss function outperform those employing the SE loss function.
4) According to Tables 6, 7, 8, and 9, Bayesian estimates often perform better than MLEs in terms of having the smallest MSEs.

5. Conclusions

The research presented in this paper offers significant insights into the statistical inference of the GL distribution under joint progressive censoring. It effectively estimates maximum likelihood for unknown parameters and conducts Bayesian estimation using both gamma and non-informative priors, demonstrating the versatility and applicability of these methods in analyzing life distributions for two populations, thereby enhancing the understanding of failure times in air conditioning systems. The study's use of two loss functions, SEL and LINEX, facilitated a comprehensive evaluation of the proposed estimates, with Monte Carlo simulations revealing that Bayesian estimates, along with their corresponding CRIs, consistently outperformed alternative estimators, underscoring the effectiveness of Bayesian methods. Additionally, the robustness of the derived estimators for parameters ρ_1, ρ_2, η_1, and η_2 was confirmed through simulation studies, with detailed tables illustrating the reliability of these methods across various sample sizes and failure counts. A numerical example further highlighted the practical application of the theoretical findings, emphasizing the relevance of the research in addressing actual failure times. The conclusions not only underscore the importance of both frequentist and Bayesian approaches but also pave the way for future research, providing a comprehensive framework for further investigations into failure time analysis and related fields.

Author contributions

Mustafa M. Hasaballah: Conceptualization, methodology, software, writing—original draft; Oluwafemi Samson Balogun: Validation, formal analysis, investigation; M. E. Bakr: Resources, data curation, writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflict of interest.

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