Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 29220–29234.
DOI: 10.3934/math.20241416
Received: 21 August 2024
Revised: 27 September 2024
Accepted: 08 October 2024
Published: 15 October 2024

Research article

Contact CR δ-invariant: an optimal estimate for Sasakian statistical manifolds

Aliya Naaz Siddiqui[1], **Meraj Ali Khan**[2] **and Amira Ishan**[3,*]

[1] Division of Mathematics, School of Basic Sciences, Galgotias University, Greater Noida, Uttar Pradesh 203201, India

[2] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box-65892, Riyadh 11566, Saudi Arabia

[3] Department of Mathematics, College of Science, Taif University, Taif 21944, Saudi Arabia

* **Correspondence:** Email: a.ishan@tu.edu.sa

Abstract: Chen (1993) developed the theory of δ-invariants to establish novel necessary conditions for a Riemannian manifold to allow a minimal isometric immersion into Euclidean space. Later, Siddiqui et al. (2024) derived optimal inequalities involving the CR δ-invariant for a generic statistical submanifold in a holomorphic statistical manifold of constant holomorphic sectional curvature. In this work, we extend the study of such optimal inequality to the contact CR δ-invariant on contact CR-submanifolds in Sasakian statistical manifolds of constant ϕ-sectional curvature. This paper concludes with a summary and final remarks.

Keywords: statistical manifolds; geometric inequality; generic submanifolds; δ-invariant
Mathematics Subject Classification: 53C05, 49K35, 62B10

1. Introduction

The study of statistical manifolds lies at the intersection of differential geometry and information theory (see [1, 17]), where the geometry of parameter spaces of statistical models is examined through the lens of Riemannian and affine geometry. These manifolds, equipped with structures such as the Fisher information metric and connections, provide a rich framework for understanding various aspects of statistical inference and information processing. The significance of statistical manifolds extends to numerous applications, including machine learning, information theory, and theoretical physics, making them an essential topic of study in modern mathematics. The exponential family is a class of probability distributions that is commonly used in statistics and information theory. This family provides a rich example of a statistical manifold that can help illustrate the key concepts of Riemannian metrics and affine connections.

Sasakian geometry [24] has emerged as a fundamental area in differential geometry, characterized by its deep connections to Kähler and contact geometry. Sasakian manifolds are contact metric manifolds that exhibit a rich structure, allowing them to be seen as odd-dimensional counterparts to Kähler manifolds. These manifolds find applications in diverse areas, such as string theory, CR geometry, and even in the study of certain types of foliations. Furuhata et al. introduced the statistical counterpart of Sasakian manifolds in [8, 10]. They continued their study in [11], examining Sasakian statistical manifolds from the perspective of warped products within statistical geometry, and also explored the concept of invariant submanifolds in the same ambient space. Kazan et al. [13] investigated Sasakian statistical manifolds with semi-symmetric metric connections, providing examples to support their findings. Lee et al. [14] established optimal inequalities for submanifolds in the same ambient space, expressed in terms of Casorati curvatures with a pair of affine connection and its conjugate affine connection. Uddin et al. [21] introduced the concept of nearly Sasakian statistical structures, presenting a non-trivial example and discussing different classes of submanifolds, including invariant and anti-invariant, within these manifolds. More recently, a new notion of mixed 3-Sasakian statistical manifolds was investigated in [16].

In [5], Chen introduced the CR δ-invariant for CR-submanifolds of Kähler manifolds and established a sharp inequality involving this invariant for anti-holomorphic warped product submanifolds in complex space forms. Drawing inspiration from this research, Al-Solamy et al. [2, 3] gave an optimal inequality for this invariant specifically for anti-holomorphic submanifolds in complex space forms. This inequality was extended by proving an optimal inequality for the contact CR δ-invariant on contact CR-submanifolds in Sasakian space forms [15]. Siddiqui et al. [19] developed equivalent inequalities for this invariant in the context of a generic submanifold in trans-Sasakian generalized Sasakian space forms. More recently, Siddiqui et al. [20] gave two optimal inequalities involving the CR δ-invariant for generic statistical submanifolds in holomorphic statistical manifolds of constant holomorphic sectional curvature.

Building on the aforementioned findings, this paper explores the generic submanifolds in a Sasakian statistical manifold. We then derive an optimal inequality for the contact CR δ-invariant on contact CR-submanifolds in a Sasakian statistical manifold of constant ϕ-sectional curvature.

2. Preliminaries

Definition 2.1. Let (M, G) be a Riemannian manifold, where G is a Riemannian metric. A pair $(\tilde{D}, G)$ is called a statistical structure on M if

(1) $\tilde{D}$ is of torsion-free, and
(2) The Codazzi equation: $(\tilde{D}_X G)(Y, Z) = (\tilde{D}_Y G)(X, Z)$ holds for any $X, Y, Z \in \Gamma(TM)$.

Here $\tilde{D}$ is an affine connection on M. A manifold equipped with such a statistical structure is referred to as a statistical manifold.

For an affine connection $\tilde{D}$ on (M, G), the dual connection $\tilde{D}*$ of $\tilde{D}$ with respect to G is defined by the formula

$$XG(Y, Z) = G(\tilde{D}_X Y, Z) + G(Y, \tilde{D}*_X Z).$$

We denote $\tilde{D}^0$ as the Levi-Civita connection of G, which satisfies the relation: $2\tilde{D}^0 = \tilde{D} + \tilde{D}*$.

Given a statistical structure $(\tilde{D}, G)$ on M, the statistical curvature tensor field $\tilde{\mathcal{R}} \in \Gamma(TM^{(1,3)})$ is defined as

$$\tilde{\mathcal{R}} = \frac{1}{2}[\tilde{R} + \tilde{R}*].$$

For a point $q \in M$ and a plane $\mathcal{L} = X \wedge Y$ spanned by orthonormal vectors $X, Y \in T_qM$, the statistical sectional curvature $\tilde{S}^{\tilde{D},\tilde{D}*}$ of $(M, \tilde{D}, G)$ for $X \wedge Y$ is defined as

$$\tilde{S}^{\tilde{D},\tilde{D}*}(X \wedge Y) = G(\tilde{\mathcal{R}}(X, Y)Y, X).$$

For two statistical manifolds (N, D, g) and $(M, \tilde{D}, G)$, an immersion $h : N \to M$, h is called a statistical immersion if the statistical structure induced by h from $(\tilde{D}, G)$ coincides with (D, g).

For any $X, Y \in \Gamma(TN)$, the corresponding Gauss formulas [22] are

$$\begin{aligned} \tilde{D}_X Y &= D_X Y + B(X, Y), \\ \tilde{D}*_X Y &= D*_X Y + B*(X, Y), \end{aligned}$$

where B and $B*$ are symmetric and bilinear, called the imbedding curvature tensors of N in M for $\tilde{D}$ and D, respectively. Next, we have the linear transformations A and $A*$ defined by

$$\begin{aligned} g(A_V X, Y) &= G(B*(X, Y), V), \\ g(A*_V X, Y) &= G(B(X, Y), V), \end{aligned}$$

for any $V \in \Gamma(TN^\perp)$. Further, in [22] the corresponding Weingarten formulas are as follows:

$$\begin{aligned} \tilde{D}_X V &= D^\perp_X V - A_V X, \\ \tilde{D}*_X V &= D*^\perp_X V - A*_V X, \end{aligned}$$

where the connections $D^\perp$ and $D*^\perp$ are Riemannian dual connections with respect to the induced metric on $\Gamma(TN^\perp)$.

The mean curvature vector field of a r-dimensional statistical submanifold (N, D, g) in any statistical manifold $(M, \tilde{D}, G)$ with respect to both affine connections is as follows:

$$H = \frac{1}{r} trace_G(B), \quad H* = \frac{1}{r} trace_G(B*).$$

We respectively symbolize the Riemannian curvature tensors of $\tilde{D}$ (respectively, $\tilde{D}*$) and D (respectively, $D*$) by $\tilde{R}$ (respectively, $\tilde{R}*$) and R (respectively, $R*$). Then, the corresponding Gauss equations for conjugate affine connection are given by [22]

$$\tilde{R}_{X,Y,Z,W} = R_{X,Y,Z,W} + G(B(X, Z), B*(YW)) - G(B*(X, W), B(Y, Z)), \tag{2.1}$$

$$\tilde{R}*_{X,Y,Z,W} = R*_{X,Y,Z,W} + G(B*(X, Z), B(YW)) - G(B(X, W), B*(Y, Z)), \tag{2.2}$$

where $\tilde{R}_{X,Y,Z,W} = G(\tilde{R}(X, Y)Z, W)$ and $\tilde{R}*_{X,Y,Z,W} = G(\tilde{R}*(X, Y)Z, W)$. Thus, we have the Gauss formula for both affine connections:

$$2\tilde{\mathcal{R}}_{X,Y,Z,W} = 2\mathcal{R}_{X,Y,Z,W} + G(B(X, Z), B*(Y, W)) - G(B*(X, W), B(Y, Z))$$

$$+G(B*(X,Z),B(Y,W))-G(B(X,W),B*(Y,Z)), \tag{2.3}$$

where $2\mathcal{R}=R+R*$.

The Codazzi equation for both affine connections:

$$\begin{aligned} 2\tilde{\mathcal{R}}_{X,Y,Z,V} &= G(((\tilde{D}_XB)(Y,Z)),V)-G((\tilde{D}_YB)(X,Z),V) \\ &\quad +G((\tilde{D}*_XB*)(Y,Z),V)-G((\tilde{D}*_YB*)(X,Z),V). \end{aligned} \tag{2.4}$$

Definition 2.2. A quadruplet $(\tilde{D},G,\phi,\xi)$ is called a Sasakian statistical structure on M if the following formula holds:

$$K_X\phi Y=-\phi K_XY,$$

where $K_XY=\tilde{D}_XY-\tilde{D}^0_XY$ satisfies $K_XY=K_YX$ and $G(K_XY,Z)=G(Y,K_XZ)$.

Theorem 2.3. *Let $(\tilde{D},G)$ be a statistical structure and (G,ϕ,ξ) an almost contact metric structure on M. Then $(\tilde{D},G,\phi,\xi)$ is a Sasakian statistical structure on M if and only if*

$$\begin{aligned} \tilde{D}_X(\phi Y)-\phi\tilde{D}*_XY &= G(\xi,Y)X-G(X,Y)\xi \\ \tilde{D}_X\xi &= \phi X+G(\tilde{D}_X\xi,\xi)\xi. \end{aligned}$$

Let $(M,\tilde{D},G,\phi,\xi)$ be a Sasakian statistical manifold, and $c\in\mathbb{R}$. The Sasakian statistical structure is said to be of constant ϕ-sectional curvature c if

$$\begin{aligned} \tilde{\mathcal{R}}(X,Y)Z &= \frac{c+3}{4}\{G(Y,Z)X-G(X,Z)Y\}+\frac{c-1}{4}\{G(\phi Y,Z)\phi X \\ &\quad -G(\phi X,Z)\phi Y-2G(\phi X,Y)\phi Z-G(Y,\xi)G(Z,\xi)X \\ &\quad +G(X,\xi)G(Z,\xi)Y+G(Y,\xi)G(Z,X)\xi-G(X,\xi)G(Z,Y)\xi\}, \end{aligned}$$

holds for $X,Y,Z\in\Gamma(TM)$.

Definition 2.4. Let (N,D,g) be a statistical manifold in a Sasakian statistical manifold $(M,\tilde{D},G,\phi,\xi)$. For $X,Y\in\Gamma(TN)$.

(1) N is said to be doubly totally contact umbilical, if

$$B(X,Y)=\Big[g(X,Y)-\eta(X)\eta(Y)\Big]V^{\perp}+\eta(X)B(Y,\xi)+\eta(Y)B(X,\xi),$$

and

$$B*(X,Y)=\Big[g(X,Y)-\eta(X)\eta(Y)\Big]V^{\perp}+\eta(X)B*(Y,\xi)+\eta(Y)B*(X,\xi),$$

where $V^{\perp}$ represents any vector field normal to N.

(2) N is said to be doubly totally contact geodesic if $V^{\perp}=0$, that is,

$$B(X,Y)=\eta(X)B(Y,\xi)+\eta(Y)B(X,\xi),$$

and

$$B*(X,Y)=\eta(X)B*(Y,\xi)+\eta(Y)B*(X,\xi).$$

Definition 2.5. A statistical submanifold (N, D, g) in a Sasakian statistical manifold $(M, \tilde{D}, G, \phi, \xi)$ is called a generic statistical submanifold (or simply generic submanifold) in M if

(1) $\phi T_q N^{\perp} \subset T_q N$, $q \in N$, and
(2) ξ is tangent to N.

The tangent space $T_q N$ at any point $q \in N$ is decomposed as

$$T_q N = \mathcal{H}_q N \oplus \phi T_q N^{\perp},$$

where $\mathcal{H}_q N$ denotes the orthogonal complement of $\phi T_q N^{\perp}$. Consequently, we have

$$\phi \mathcal{H}_q N = \mathcal{H}_q N \backslash \{\xi\}.$$

3. On generic statistical submanifolds

Lemma 3.1. *Let (N, D, g) be a generic submanifold in a Sasakian statistical manifold $(M, \tilde{D}, G, \phi, \xi)$. Then we have*

$$A_{\mathbf{F}Y} Z = A_{\mathbf{F}Z} Y, \quad A*_{\mathbf{F}Y} Z = A*_{\mathbf{F}Z} Y,$$

for $Y, Z \in \Gamma(\phi T N^{\perp})$.

Proof. For $X, Y \in \Gamma(TN)$, we know

$$\tilde{D}_X \phi Y - \phi \tilde{D}*_X Y = \eta(Y) X - g(X, Y)\xi.$$

Since $\phi X = \mathbf{P}X + \mathbf{F}X$

$$D_X \mathbf{P}Y + B(X, \mathbf{P}Y) + D^{\perp}_X \mathbf{F}Y - A_{\mathbf{F}Y} X - \phi D*_X Y - \phi B*(X, Y) = \eta(Y) X - g(X, Y)\xi.$$

$$D_X \mathbf{P}Y + B(X, \mathbf{P}Y) + D^{\perp}_X \mathbf{F}Y - A_{\mathbf{F}Y} X - \mathbf{P}D*_X Y - \mathbf{F}D*_X Y - \phi B*(X, Y) = \eta(Y) X - g(X, Y)\xi.$$

In comparison, we have

$$\phi B*(X, Y) = D_X \mathbf{P}Y - \mathbf{P}D*_X Y - A_{\mathbf{F}Y} X - \eta(Y) X + g(X, Y)\xi,$$

and

$$B(X, \mathbf{P}Y) \quad = \quad -D^{\perp}_X \mathbf{F}Y + \mathbf{F}D*_X Y.$$

In particular, for $Y, Z \in \Gamma(\phi T N^{\perp})$

$$G(\phi B*(X, Y), Z) = g(D_X \mathbf{P}Y, Z) - g(\mathbf{P}D*_X Y, Z) - g(A_{\mathbf{F}Y} X, Z) - \eta(Y) g(X, Z) + \eta(Z) g(X, Y).$$

We notice that $\mathbf{P}\phi T_q N^{\perp} = 0$ and $\phi \mathbf{P} T_q N \subset \mathcal{H}_q N$. So, we have

$$G(\phi B*(X, Y), Z) = -g(A_{\mathbf{F}Y} X, Z),$$

which can be reduced further as

$$g(A_{\mathbf{F}Z} Y, X) = g(X, A_{\mathbf{F}Y} Z),$$

that is, $A_{\mathbf{F}Y} Z = A_{\mathbf{F}Z} Y$. Similarly, one can show that $A*_{\mathbf{F}Y} Z = A*_{\mathbf{F}Z} Y$. □

Lemma 3.1 implies the following result:

Proposition 3.2. *Let (N, D, g) be a generic submanifold in a Sasakian statistical manifold $(M, \tilde{D}, G, \phi, \xi)$ of codimension greater than 1. If N is doubly totally contact umbilical, then N is doubly totally contact geodesic.*

Proposition 3.3. *Let $(M(c), \tilde{D}, G, \phi, \xi)$ be a $(2s+1)$-dimensional Sasakian statistical manifold of constant ϕ-sectional curvature c and (N, D, g) be an $(r+1)$-dimensional generic submanifold in $M(c)$, with $r > s$ and $r \geq 3$. If N is of codimension greater than 1 and totally contact umbilical, then $c = -3$.*

Proof. When M is not a statistical hypersurface, that is, N is of codimension greater than or equal to 2, then Proposition 3.2 implies that B and $B*$ of N are of the following form:

$$B(Y,Z) = g(Y,\xi)\mathbf{F}Z + g(Z,\xi)\mathbf{F}Y, \quad B*(Y,Z) = g(Y,\xi)\mathbf{F}Z + g(Z,\xi)\mathbf{F}Y.$$

The covariant derivatives of B and B^* are defined as

$$\begin{aligned}(\tilde{D}_X B)(Y,Z) &= D^{\perp}_X(B(Y,Z)) - B(D_X Y, Z) - B(Y, D_X Z)\\ &= g(Y, D*_X\xi)\mathbf{F}Z + g(Z, D*_X\xi)\mathbf{F}Y + g(Y,\xi)D^{\perp}_X\mathbf{F}Z\\ &\quad + g(Z,\xi)D^{\perp}_X\mathbf{F}Y - g(Z,\xi)\mathbf{F}D_X Y - g(Y,Z)\mathbf{F}D_X Z,\end{aligned}$$

and

$$\begin{aligned}(\tilde{D}*_X B*)(Y,Z) &= D*_X(B*(Y,Z)) - B*(D*_X Y, Z) - B*(Y, D*_X Z)\\ &= g(Y, D_X\xi)\mathbf{F}Z + g(Z, D_X\xi)\mathbf{F}Y + g(Y,\xi)D*^{\perp}_X\mathbf{F}Z\\ &\quad + g(Z,\xi)D*^{\perp}_X\mathbf{F}Y - g(Z,\xi)\mathbf{F}D*_X Y - g(Y,Z)\mathbf{F}D*_X Z.\end{aligned}$$

So, we derive

$$(\tilde{D}_X B)(Y,Z) = g(Y,\mathbf{P}X)\mathbf{F}Z + g(Z,\mathbf{P}X)\mathbf{F}Y + g(D*_X\xi,\xi)g(Y,\xi)\mathbf{F}Z + g(D*_X\xi,\xi)g(Z,\xi)\mathbf{F}Y,$$

$$(\tilde{D}_Y B)(X,Z) = g(X,\mathbf{P}Y)\mathbf{F}Z + g(Z,\mathbf{P}Y)\mathbf{F}X + g(D*_Y\xi,\xi)g(X,\xi)\mathbf{F}Z + g(D*_Y\xi,\xi)g(Z,\xi)\mathbf{F}X,$$

$$(\tilde{D}*_X B*)(Y,Z) = g(Y,\mathbf{P}X)\mathbf{F}Z + g(Z,\mathbf{P}X)\mathbf{F}Y + g(D_X\xi,\xi)g(Y,\xi)\mathbf{F}Z + g(D_X\xi,\xi)g(Z,\xi)\mathbf{F}Y,$$

and

$$(\tilde{D}*_Y B*)(X,Z) = g(X,\mathbf{P}Y)\mathbf{F}Z + g(Z,\mathbf{P}Y)\mathbf{F}X + g(D_Y\xi,\xi)g(X,\xi)\mathbf{F}Z + g(D_Y\xi,\xi)g(Z,\xi)\mathbf{F}X.$$

Further, we use these expressions in the Codazzi equation (2.4) as

$$\begin{aligned}2\tilde{\mathcal{R}}_{X,Y,Z,V} &= G(((\tilde{D}_X B)(Y,Z)), V) - G((\tilde{D}_Y B)(X,Z), V)\\ &\quad + G((\tilde{D}*_X B*)(Y,Z), V) - G((\tilde{D}*_Y B*)(X,Z), V)\\ &\quad \frac{c-1}{4}\Big[g(\mathbf{P}Y,Z)\mathbf{F}X - g(\mathbf{P}X,Z)\mathbf{F}Y - 2g(\mathbf{P}X,Y)\mathbf{F}Z\Big]\\ &= 2g(Y,\mathbf{P}X)\mathbf{F}Z + g(Z,\mathbf{P}X)\mathbf{F}Y - g(Z,\mathbf{P}Y)\mathbf{F}X\end{aligned}$$

$$+g(Y,\xi)\Big[g(D_X\xi,\xi)+g(D*_X\xi,\xi)\Big]+g(X,\xi)\Big[g(D_Y\xi,\xi)+g(D*_Y\xi,\xi)\Big].$$

From which we arrive at

$$\frac{c+3}{4}\Big[-g(\mathbf{P}Y,Z)\mathbf{F}X+g(\mathbf{P}X,Z)\mathbf{F}Y+2g(\mathbf{P}X,Y)\mathbf{F}Z\Big]$$
$$+g(Y,\xi)\Big[g(D_X\xi,\xi)+g(D*_X\xi,\xi)\Big]+g(X,\xi)\Big[g(D_Y\xi,\xi)+g(D*_Y\xi,\xi)\Big]=0.$$

For $Y\in\mathcal{H}_qN$, we put $Z=\mathbf{P}Y$. Then $\mathbf{F}Y=0$ and $\mathbf{FP}Y=0$. Thus, we conclude that $c=-3$. □

4. A geometric inequality

Following the analogy of Chen's CR δ-invariant, Mihai et al. [15] defined the contact CR δ-invariant for an odd-dimensional contact CR-submanifold in a Sasakian space form. Here, we define the statistical Chen's CR δ-invariant on a $(r+1)$-dimensional contact CR-submanifold (N,D,g) in the $(2s+1)$-dimensional Sasakian statistical manifold $(M(c),\tilde{D},G,\phi,\xi)$ of constant ϕ-sectional curvature c as follows:

$$\delta^{D,D*}(\mathcal{D})(q)=scal^{D,D*}(q)-scal^{D,D*}(\mathcal{D}_q),$$

where $scal^{D,D*}$ and $scal^{D,D*}(\mathcal{D})$ denote the scalar curvature of N and the scalar curvature of the invariant distribution $\mathcal{D}\subset TN$, respectively.

Orthonormal frames on differentiable manifolds provide a powerful tool for simplifying complex geometric and physical problems. They offer a structured way to understand local properties of the manifold (metric tensors), aid in defining connections and curvature, and play a crucial role in both theoretical and applied contexts like Riemannian and Lorentzian geometry. If $\dim(\mathcal{D})=2\alpha+1$ and $\dim(\mathcal{D}^\perp)=\beta$ and let $\{v_0=\xi,v_1,v_2,\cdots,v_r\}$ be an orthonormal frame on N such that $\{v_0,v_1,\cdots,v_{2\alpha}\}$ are tangent to $\mathcal{D}$ and $\{v_{2\alpha+1},\cdots,v_\beta\}$ are tangent to $\mathcal{D}^\perp$. Then the partial mean curvature vectors $H1$ and $H2$ (respectively, $H1*$ and $H2*$ for both affine connections) of N are given by

$$H1=\frac{1}{2\alpha+1}\sum_{I=0}^{2\alpha}B(v_I,v_I),\quad H2=\frac{1}{\beta}\sum_{a=2\alpha+1}^{2\alpha+\beta}B(v_a,v_a),$$
$$H1*=\frac{1}{2\alpha+1}\sum_{I=0}^{2\alpha}B(v_I,v_I),\quad H2*=\frac{1}{\beta}\sum_{a=2\alpha+1}^{2\alpha+\beta}B(v_a,v_a).$$

A contact CR-submanifold (N,D,g) of a Sasakian manifold $(M,\tilde{D},G,\phi,\xi)$ is said to be doubly minimal if $H=H*=0$. Likewise, it is referred to as doubly $\mathcal{D}$-minimal or doubly $\mathcal{D}^\perp$-minimal if $H1=H1*=0$ or $H2=H2*=0$, respectively.

According to the definition of the contact CR δ-invariant, we have

$$\delta^{D,D*}(\mathcal{D}) = \sum_{a=2\alpha+1}^{2\alpha+\beta}S^{D,D*}(\xi,v_a)+\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}S^{D,D*}(v_I,v_a)$$
$$+\frac{1}{2}\sum_{2\alpha+1\leq a\neq b\leq 2\alpha+\beta}S^{D,D*}(v_a,v_b)$$

$$
\begin{aligned}
&= \Big(\frac{c+3}{4}\Big)\Big(\sum_{a=2\alpha+1}^{2\alpha+\beta} g(v_a,v_a)\Big) - \Big(\frac{c-1}{4}\Big)\Big(\sum_{a=2\alpha+1}^{2\alpha+\beta} g(v_a,v_a)\Big) \\
&\quad + \frac{\beta(4\alpha+\beta-1)}{2}\Big(\frac{c+3}{4}\Big) \\
&\quad + \sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}[G(B(v_I,v_I),B*(v_a,v_a)) - G(B(v_I,v_a),B*(v_I,v_a))] \\
&\quad + \sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}[G(B*(v_I,v_I),B(v_a,v_a)) - G(B*(v_I,v_a),B(v_I,v_a))] \\
&\quad + \frac{1}{2}\sum_{a,b=2\alpha+1}^{2\alpha+\beta}[G(B(v_a,v_a),B*(v_b,v_b)) - G(B(v_a,v_b),B*(v_a,v_b))] \\
&\quad + \frac{1}{2}\sum_{a,b=2\alpha+1}^{2\alpha+\beta}[G(B*(v_a,v_a),B(v_b,v_b)) - G(B*(v_a,v_b),B(v_a,v_b))].
\end{aligned}
$$

We use $B*(X,\xi) = B(X,\xi) = 0$ and obtain

$$
\begin{aligned}
\delta^{D,D*}(\mathcal{D}) &= \beta\Big(1 + \frac{(4\alpha+\beta-1)(c+3)}{8}\Big) \\
&\quad + \sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}[G(B(v_I,v_I),B*(v_a,v_a)) - G(B(v_I,v_a),B*(v_I,v_a))] \\
&\quad + \sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}[G(B*(v_I,v_I),B(v_a,v_a)) - G(B*(v_I,v_a),B(v_I,v_a))] \\
&\quad + \frac{1}{2}\sum_{a,b=2\alpha+1}^{2\alpha+\beta}[G(B(v_a,v_a),B*(v_b,v_b)) - G(B(v_a,v_b),B*(v_a,v_b))] \\
&\quad + \frac{1}{2}\sum_{a,b=2\alpha+1}^{2\alpha+\beta}[G(B*(v_a,v_a),B(v_b,v_b)) - G(B*(v_a,v_b),B(v_a,v_b))].
\end{aligned}
$$

Given that $2B^0 = B + B*$, we deduce

$$4H^{02} = H^2 + H*^2 + 2G(H, H*)$$

and

$$4B^{02}_{\mathcal{D}^\perp} = B^2_{\mathcal{D}^\perp} + B*^2_{\mathcal{D}^\perp} + 2G(B_{\mathcal{D}^\perp}, B*_{\mathcal{D}^\perp}).$$

So, we have

$$
\begin{aligned}
\delta^{D,D*}(\mathcal{D}) &= \beta\Big(1 + \frac{(4\alpha+\beta-1)(c+3)}{8}\Big) \\
&\quad +4\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}[G(B^0(v_I,v_I),B^0(v_a,v_a)) - G(B^0(v_I,v_a),B^0(v_I,v_a))]
\end{aligned}
$$

$$-\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}[G(B(v_I,v_I),B(v_a,v_a))-G(B(v_I,v_a),B(v_I,v_a))]$$
$$-\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}[G(B*(v_I,v_I),B*(v_a,v_a))-G(B*(v_I,v_a),B*(v_I,v_a))]$$
$$+2\sum_{a,b=2\alpha+1}^{2\alpha+\beta}[G(B^0(v_a,v_a),B^0(v_b,v_b))-G(B^0(v_a,v_b),B^0(v_a,v_b))]$$
$$-\frac{1}{2}\sum_{a,b=2\alpha+1}^{2\alpha+\beta}[G(B(v_a,v_a),B(v_b,v_b))-G(B(v_a,v_b),B(v_a,v_b))]$$
$$-\frac{1}{2}\sum_{a,b=2\alpha+1}^{2\alpha+\beta}[G(B*(v_a,v_a),B*(v_b,v_b))-G(B*(v_a,v_b),B*(v_a,v_b))].$$

From [15], we have

$$\begin{aligned}&4\Big[\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}G(B^0(v_I,v_I),B^0(v_a,v_a))+\frac{1}{2}\sum_{a,b=2\alpha+1}^{2\alpha+\beta}G(B^0(v_a,v_a),B^0(v_b,v_b))\\&-\frac{1}{2}G(B^0(v_a,v_b),B^0(v_a,v_b))\Big]\\=\;&2(2\alpha+\beta+1)^2H^{02}-2(2\alpha+1)^2H1^{02}-2B^{02}_{\mathcal{D}^\perp}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&-\Big[\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}G(B(v_I,v_I),B(v_a,v_a))+\frac{1}{2}\sum_{a,b=2\alpha+1}^{2\alpha+\beta}G(B(v_a,v_a),B(v_b,v_b))\\&-\frac{1}{2}G(B(v_a,v_b),B(v_a,v_b))\Big]\\=\;&-\frac{(2\alpha+\beta+1)^2}{2}H^2+\frac{(2\alpha+1)^2}{2}H1^2+\frac{1}{2}B^2_{\mathcal{D}^\perp},\end{aligned}$$

and

$$\begin{aligned}&-\Big[\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}G(B*(v_I,v_I),B*(v_a,v_a))+\frac{1}{2}\sum_{a,b=2\alpha+1}^{2\alpha+\beta}G(B*(v_a,v_a),B*(v_b,v_b))\\&-\frac{1}{2}G(B*(v_a,v_b),B*(v_a,v_b))\Big]\\=\;&-\frac{(2\alpha+\beta+1)^2}{2}H*^2+\frac{(2\alpha+1)^2}{2}H1*^2+\frac{1}{2}B*^2_{\mathcal{D}^\perp}.\end{aligned}$$

So, we derive

$$\begin{aligned}\delta^{D,D*}(\mathcal{D}) &= \beta\Big(1+\frac{(4\alpha+\beta-1)(c+3)}{8}\Big)\\ &+2(2\alpha+\beta+1)^2H^{02}-\frac{(2\alpha+\beta+1)^2}{2}(H^2+H*^2)\\ &-2(2\alpha+1)^2H1^{02}+\frac{(2\alpha+1)^2}{2}(H1^2+H1*^2)\\ &-2B^{02}_{\mathcal{D}^\perp}+\frac{1}{2}(B^2_{\mathcal{D}^\perp}+B*^2_{\mathcal{D}^\perp})\\ &-\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}[4B^{02}_{Ia}-B^2_{Ia}-B*^2_{Ia}].\end{aligned}$$

In the final equation, we use the following inequalities for both affine connections (analogous to those obtained for the Levi-Civita connection in [15]):

$$B^2_{\mathcal{D}^\perp}\geq\frac{3\beta^2}{\beta+2}H2^2,\quad B*^2_{\mathcal{D}^\perp}\geq\frac{3\beta^2}{\beta+2}H2*^2$$

with equality holds if and only if the following conditions are met:

(1) $B^a_{aa}=3B^a_{bb}$ and $B*^a_{aa}=3B*^a_{bb}$, for $2\alpha+1\leq a\neq b\leq 2\alpha+\beta$;
(2) $B^a_{bc}=0$ and $B*^a_{bc}=0$ for $a,b,c\in\{2\alpha+1,\cdots,2\alpha+\beta\}$, $a\neq b\neq c$.

Thus, we find that

$$\begin{aligned}\delta^{D,D*}(\mathcal{D}) &\geq \beta\Big(1+\frac{(4\alpha+\beta-1)(c+3)}{8}\Big)\\ &+2(2\alpha+\beta+1)^2H^{02}-\frac{(2\alpha+\beta+1)^2}{2}(H^2+H*^2\\ &-2(2\alpha+1)^2H1^{02}+\frac{(2\alpha+1)^2}{2}(H1^2+H1*^2)\\ &-2B^{02}_{\mathcal{D}^\perp}+\frac{3\beta^2}{2(\beta+2)}(H2^2+H2*^2)\\ &-\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}[4B^{02}_{Ia}-B^2_{Ia}-B*^2_{Ia}].\end{aligned}\tag{4.1}$$

By drawing on the analogy with [3] and Lemma 3.1, we obtain the following inequalities:

$$\begin{aligned}\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}(B^2_{Ia}+B*^2_{Ia})+\frac{(2\alpha+1)^2}{2}(H1^2+H1*^2)+\frac{3\beta^2}{4(\beta+2)}(H2^2+H2*^2)\\ \geq\frac{3\beta^2}{4(\beta+2)}(H2^2+H2*^2).\end{aligned}\tag{4.2}$$

By substituting (4.2) into (4.1), we obtain

$$\begin{aligned} & \delta^{D,D*}(\mathcal{D}) - \beta\Big(1 + \frac{(4\alpha+\beta-1)(c+3)}{8}\Big) \\ & -2(2\alpha+\beta+1)^2 H^{02} + 2(2\alpha+1)^2 H1^{02} + 2B^{02}_{\mathcal{D}^\perp} \\ & +4\sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta} B^{02}_{Ia} \\ \geq\; & \frac{(2\alpha+1)^2}{2}(H1^2 + H1*^2) - \frac{(2\alpha+\beta+1)^2}{2}(H^2 + H*^2) \\ & +\frac{3\beta^2}{2(\beta+2)}(H2^2 + H2*^2) + \sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta}(B^2_{Ia} - B*^2_{Ia}) \\ \geq\; & \frac{3\beta^2}{2(\beta+2)}(H2^2 + H2*^2) - \frac{(2\alpha+\beta+1)^2}{2}(H^2 + H*^2). \end{aligned} \tag{4.3}$$

On the other hand, we have the following relation for the contact CR δ-invariant $\delta^0(\mathcal{D})$ of N with respect to Levi-Civita connection, given by [15]

$$\begin{aligned} \delta^0(\mathcal{D}) \;=\; & \frac{(2\alpha+\beta+1)^2}{2}H^{02} + \beta\Big(1 + (4\alpha+\beta-1)\Big)\frac{c+3}{8} \\ & -\frac{(2\alpha+1)^2}{2}H1^{02} - \sum_{I=1}^{2\alpha}\sum_{a=2\alpha+1}^{2\alpha+\beta} B^{02}(v_I, v_a) - \frac{1}{2}B^{02}_{\mathcal{D}^\perp}. \end{aligned} \tag{4.4}$$

Putting (4.4) into (4.3), we obtain

$$\begin{aligned} & \delta^{D,D*}(\mathcal{D}) + 3\beta\Big(1 + \frac{(4\alpha+\beta-1)(c+3)}{8}\Big) - 4\delta^0(\mathcal{D}) \\ \geq\; & \frac{3\beta^2}{2(\beta+2)}(H2^2 + H2*^2) - \frac{(2\alpha+\beta+1)^2}{2}(H^2 + H*^2). \end{aligned}$$

Hence, we have:

Theorem 4.1. *Let $(M(c), \tilde{D}, G, \phi, \xi)$ be a $(2s+1)$-dimensional Sasakian statistical manifold of constant ϕ-sectional curvature c and (N, D, g) be a $(r+1)$-dimensional generic submanifold in $M(c)$, with* $\dim(\mathcal{D}) = 2\alpha + 1$ *and* $\dim(\mathcal{D}^\perp) = \beta$*. Then*

$$\begin{aligned} \delta^{D,D*}(\mathcal{D}) \;\geq\; & 4\delta^0(\mathcal{D}) - 3\beta\Big(1 + \frac{(4\alpha+\beta-1)(c+3)}{8}\Big) \\ & +\frac{3\beta^2}{2(\beta+2)}(H2^2 + H2*^2) - \frac{(r+1)^2}{2}(H^2 + H*^2). \end{aligned} \tag{4.5}$$

Furthermore, the equality in (4.5) holds identically if and only if the following conditions are met:

(1) N is doubly $\mathcal{D}$-minimal,

(2) *N is mixed totally geodesic with respect to both affine connections, and*

(3) *there exists an orthonormal frame* $\{v_{2\alpha+1}, v_{2\alpha+2}, \cdots, v_{2\alpha+\beta}\}$ *of* $\mathcal{D}^{\perp}$ *such that*

   (a) $B^a_{aa} = 3B^a_{bb}$ *and* $B*^a_{aa} = 3B*^a_{bb}$, *for* $2\alpha + 1 \leq a \neq b \leq 2\alpha + \beta$,

   (b) $B^a_{bc} = 0$ *and* $B*^a_{bc} = 0$ *for* $a, b, c \in \{2\alpha + 1, \cdots, 2\alpha + \beta\}$, $a \neq b \neq c$.

It is evident that the equality case of (4.3) holds identically when N is doubly $\mathcal{D}$-minimal, and mixed totally geodesic with respect to both affine connections. It is worth noting that the equality in (4.5) holds identically if and only if the three conditions from Theorem 4.1 are met.

5. Conclusion and Remarks

(1) In the early 1990s, the renowned author B.-Y. Chen introduced the concept of δ-invariants (see [4, 6, 7]) to address an open question concerning minimal immersions proposed by S.S. Chern in the 1960s, as well as to explore applications of the well-known Nash embedding theorem. Chen specifically defined the CR δ-invariant for anti-holomorphic submanifolds in complex space forms. Building on this work, we extended this study to the statistical version of contact CR δ-invariant.

(2) In fact, Furuhata et al. introduced a novel notion of U sectional curvature for statistical manifolds $(M, \tilde{D}, G)$ in [12] as follows:

$$\begin{aligned} \tilde{S}^U(X \wedge Y) &= G(U(X, Y)Y, X) \\ &= 2G(\tilde{R}^0(X, Y)Y, X) - G(\tilde{\mathcal{R}}(X, Y)Y, X) \\ &= (2\tilde{S}^0 - \tilde{S}^{\tilde{D},\tilde{D}*})(X \wedge Y), \end{aligned}$$

where $\tilde{R}^0$ is the Riemannian curvature tensor for $\tilde{D}^0$ on M. They also defined a corresponding δ-invariant δ^U based on this new concept of U sectional curvature for statistical manifolds. It would be of significant interest to reformulate such an optimal inequality by defining a new notion for the (contact) CR δ^U-invariant for (contact) CR-submanifolds in a (respectively, Sasakian statistical manifold) holomorphic statistical manifold.

(3) It would be of significant interest to check whether Proposition 3.3 is valid for any codimension. We have already established its validity for codimension greater than or equal to 2. Thus, the remaining case to consider is when N is a hypersurface, that is, of codimension 1.

(4) From [9, 18], we note that a contact CR-submanifold (N, D, g) in a Sasakian statistical manifold $(M, \tilde{D}, G, \phi, \xi)$ is said to be mixed foliate with respect to $\tilde{D}$ (respectively, $\tilde{D}*$) if N is mixed totally geodesic with respect to $\tilde{D}$ (respectively, $\tilde{D}*$) and $\mathcal{D}$ is completely integrable. Now, let us consider a $(r + 1)$-dimensional generic submanifold (N, D, g) in a $(2s + 1)$-dimensional Sasakian statistical manifold $(M(c), \tilde{D}, G, \phi, \xi)$ of constant ϕ-sectional curvature c, where $\dim(\mathcal{D}) = 2\alpha + 1$ and $\dim(\mathcal{D}^{\perp}) = \beta$. If N satisfies the equality case of (4.5) and $\mathcal{D}$ is integrable, then it follows from Theorem 4.1 that N is mixed foliate with respect to $\tilde{D}$ (respectively, $\tilde{D}*$).

(5) In [10], Furuhata et al. constructed Sasakian statistical structures on the $(2s + 1)$-dimensional unit hypersphere $\mathbb{S}$ in $(2s + 2)$-dimensional Euclidean space $\mathbb{R}$. They showed that $\mathbb{S}$ is a Sasakian statistical manifold of constant statistical sectional curvature 1 and of constant ϕ-sectional curvature $c = 1$ as well by setting $K_X Y = G(X, \xi)G(Y, \xi)\xi$. Building in this, we consider $N = \mathbb{S}^{s_1}(r_1) \times \cdots \times \mathbb{S}^{s_k}(r_k)$ and an immersion $N \to \mathbb{S}^{m+k+1}$ from [23], where $m = s_1 + s_2 + \cdots + s_k$,

$\sum_{i=1}^{k} r_i^2 = 1$. It is straightforward to observe that N is a generic statistical submanifold of a Sasakian statistical manifold $\mathbb{S}^{m+k+1}$, provided that all s_i are odd.

(6) Let $\mathbb{E}^{2s+1}(-3)$ be a Sasakian space form of constant ϕ-sectional curvature -3 with Sasakian structure (G, ϕ, ξ) on $\mathbb{E}^{2s+1}(-3)$ as:

$$G = \begin{pmatrix} \frac{1}{4}(\delta_{ij} + y^i y^j) & 0 & \frac{-1}{4} y^i \\ 0 & \frac{1}{4}\delta_{ij} & 0 \\ \frac{1-}{4} y^j & 0 & \frac{1}{4} \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & \delta^i_j & 0 \\ -\delta^i_j & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix},$$

$$\xi = (0, 0, \cdots, 0, 0, 2), \quad G(\cdot, \xi) = (-y^1, \cdots, -y^s, 0, \cdots, 0, 1),$$

where $(x^1, \cdots, x^n, y^1, \cdots, y^n, z)$ denotes the cartesian coordinates. Now, we can construct a Sasakian statistical structure on $(\mathbb{E}^{2s+1}(-3), G, \phi, \xi)$ by setting $K_X Y = G(X, \xi) G(Y, \xi)\xi$ satisfies Definition 2.2.

Further, we consider $N^{2s} = \mathbb{S}^{2s-1} \times \mathbb{E}^1$. Then N^{2s} is doubly totally contact umbilical submanifold of $\mathbb{E}^{2s+1}(-3)$.

Author contributions

Aliya Naaz Siddiqui: Conceptualization, Methodology, Formal analysis, Writing-Original draft preparation, Writing-Reviewing and Editing. Meraj Ali Khan: Visualization, Investigation, Supervision. Amira Ishan: Project administration, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Acknowledgment

The authors would like to acknowledge the Deanship of Graduate Studies and Scientific Research, Taif University, for funding this work.

Conflict of interest

The authors declare there are no conflicts of interest.

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