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Research article

On mean square of the error term of a multivariable divisor function

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Abstract: Let $\tau(n)$ be the Dirichlet divisor function and $k \geqslant 2$ be a fixed integer. We give an asymptotic formula of the mean square of

$$\Delta_k(x) = \sum_{n_1,\cdots,n_k \leqslant x} \tau(n_1 \cdots n_k) - x^k P_k(\log x).$$

Keywords: divisor function; mean square; Dirichlet series
Mathematics Subject Classification: 11N37

1. Introduction

Let $\tau(n)$ be the Dirichlet divisor function. It is known that for a real number $x \geqslant 2$,

$$\sum_{n \leqslant x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x), \tag{1.1}$$

where γ is Euler's constant. It was first proved by Dirichlet that

$$\Delta(x) \ll \sqrt{x}.$$

Let θ denotes the smallest number such that

$$\Delta(x) \ll x^{\theta+\varepsilon} \tag{1.2}$$

holds for any $\varepsilon > 0$. Many authors worked on making θ smaller, such as Voronoi [1], Corput [2], Kolesnik [3], Huxley [4], etc. Until now the best result, namely $\theta = 131/416$ is due to Huxley [4].

It is conjectured that $\theta = 1/4$, which is supported by the classical mean square result: Suppose T is a large real number, then for any $\varepsilon > 0$, one has

$$\int_1^T \Delta^2(x)dx = \frac{1}{6\pi^2} C_2 T^{3/2} + O\left(T^{\frac{5}{4}+\varepsilon}\right), \tag{1.3}$$

where

$$C_2=\sum_{n=1}^{\infty}\frac{\tau^2(n)}{n^{3/2}}.$$

This result was given by Cramér [5] in 1922. In 1956, Tong [6] showed that the error term in (1.3) can be reduced to $T\log^5 T$. In 1988, Preissmann [7] reduced the error term to $T\log^4 T$. In 2009, Lau and Tsang [8] proved that

$$\int_1^T \Delta^2(x)dx=\frac{1}{6\pi^2}C_2T^{3/2}+O\left(T\log^3 T\log\log T\right).$$

Let $k\geqslant 2$ be a fixed integer. Tóth and Zhai [9] considered the average of divisor function of k variables. They obtain the asymptotic formula

$$\sum_{n_1,\cdots,n_k\leqslant x}\tau(n_1\cdots n_k)=x^kP_k(\log x)+O\left(x^{k-1+\theta+\varepsilon}\right) \tag{1.4}$$

holds for any $\varepsilon>0$, where θ is the exponent in (1.2) and $P_k(t)$ is a polynomial in t of degree k. We denote $\Delta_k(x)$ by

$$\Delta_k(x):=\sum_{n_1,\cdots,n_k\leqslant x}\tau(n_1\cdots n_k)-x^kP_k(\log x). \tag{1.5}$$

We have the following theorem:

Theorem 1.1. *Suppose $T\geqslant 2$ is a large real number, then we have*

$$\int_1^T {\Delta_k}^2(x)dx=\frac{k^2}{4\pi^2}T^{2k-\frac{1}{2}}L_{2k-2}(\log T)+O\left(T^{2k-\frac{3}{5}+\varepsilon}\right), \tag{1.6}$$

where $L_{2k-2}(u)$ is a polynomial in u of degree $2k-2$, and the implied constant about "O" depends on k and ε.

2. Some lemmas

We present some lemmas.

Lemma 2.1. *Suppose $x\geqslant 2$ is large, then for any $1\ll N\ll x$, we have*

$$\Delta(x)=\frac{x^{1/4}}{\sqrt{2}\pi}\sum_{n\leqslant N}\frac{\tau(n)}{n^{3/4}}\cos\left(4\pi\sqrt{nx}-\frac{\pi}{4}\right)+O\left(\frac{x^{1/2+\varepsilon}}{N^{1/2}}\right).$$

Proof. See [10, Chapter 3.2]. □

Lemma 2.2. *Suppose $T\geqslant 2$, $T^{\varepsilon}\ll y\ll T$, and let*

$$\delta_1(x,y)=\frac{x^{1/4}}{\sqrt{2}\pi}\sum_{n\leqslant y}\frac{\tau(n)}{n^{3/4}}\cos\left(4\pi\sqrt{nx}-\frac{\pi}{4}\right),$$

$$\delta_2(x,y)=\Delta(x)-\delta_1(x,y),$$

then we have

$$\int_T^{2T}{\delta_2}^2(x,y)dx\ll\frac{T^{3/2}}{y^{1/2}}\log^3 T+T\log^4 T.$$

Proof. See, for example, the following references: Lau and Tsang [8], Tsang [11], Zhai [12]. □

Lemma 2.3. *Suppose G_0, m_0 are fixed real positive numbers, let G(x) be a monotonic function defined on $[a, b]$ such that*

$$|G(x)| \leqslant G_0$$

and m(x) be a differentiable real function such that

$$|m^{'}(x)| \geqslant m_0$$

on $[a, b]$, $F(\cdot) = \cos(\cdot)$ or $\sin(\cdot)$ or $e(\cdot)$, then

$$\int_a^b G(x)F\left(m(x)\right) dx \ll G_0 {m_0}^{-1}.$$

Proof. See [10, Lemma 2.1]. □

Lemma 2.4. *Let $k \geqslant 2$ be a fixed integer and let $s_1, \cdots, s_k$ be complex numbers. Then for $\Re s_j > 1$, where $j = 1, 2, \cdots, k$, we have*

$$\sum_{n_1,\cdots,n_k=1}^{\infty} \frac{\tau(n_1 \cdots n_k)}{n_1^{s_1} \cdots n_k^{s_k}} = \zeta^2(s_1) \cdots \zeta^2(s_k) F_k(s_1, \cdots, s_k),$$

where

$$F_k(s_1, \cdots, s_k) = \sum_{n_1,\cdots,n_k=1}^{\infty} \frac{f(n_1, \cdots, n_k)}{n_1^{s_1} \cdots n_k^{s_k}}.$$

This series is absolutely convergent provided that $\Re s_j > 0$ and $\Re(s_j + s_l) > 1 (1 \leqslant j, l \leqslant k)$, and $f(n_1, \cdots, n_k)$ is multiplicative and symmetric in all variables.

Moreover, for $r_1, \cdots, r_k \in \{1, 2\}$,

$$\sum_{n_1,\cdots,n_k=1}^{\infty} \frac{f(n_1, \cdots, n_k)\tau_{r_1}(n_1) \cdots \tau_{r_k}(n_k)}{n_1^{s_1} \cdots n_k^{s_k}} \tag{2.1}$$

is absolutely convergent provided that $\Re s_j > 0$ and $\Re(s_j + s_l) > 1 (1 \leqslant j, l \leqslant k)$.

Proof. See Tóth and Zhai [9, Proposition 2.1], and the convergence of (2.1) is a direct corollary. □

Lemma 2.5. *Suppose x, y are large real numbers, $k \geqslant 2$ is a fixed integer, s, w are given real numbers such that $0 < s < 1/2 < w < 1$, f is defined in Lemma 2.4. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ denote the vectors $(m_1, \cdots, m_k)$ and $(m_{k+1}, \cdots, m_{2k})$, D_1 and D_2 denote $\left(\prod_{j=1}^{k-1} m_j\right)$ and $\left(\prod_{j=k+1}^{2k-1} m_j\right)$, respectively. Let*

$$T_{g,k}(x, y; s, w) = \sum_{\substack{m_1,\cdots,m_{2k} \leqslant x \\ n_1,n_2 \leqslant y \\ \frac{m_k}{m_{2k}} = \frac{n_1}{n_2}}} \frac{f(\mathbf{M}_1)f(\mathbf{M}_2)g(\mathbf{M}_1, \mathbf{M}_2)}{D_1 D_2 (m_k m_{2k})^s} \cdot \frac{\tau(n_1)\tau(n_2)}{(n_1 n_2)^w},$$

$$T_{g,k}(s, w) = \sum_{\substack{\mathbf{M}_1,\mathbf{M}_2 \in \mathbb{N}^k \\ n_1,n_2 \in \mathbb{N} \\ \frac{m_k}{m_{2k}} = \frac{n_1}{n_2}}} \frac{f(\mathbf{M}_1)f(\mathbf{M}_2)g(\mathbf{M}_1, \mathbf{M}_2)}{D_1 D_2 (m_k m_{2k})^s} \cdot \frac{\tau(n_1)\tau(n_2)}{(n_1 n_2)^w},$$

where $g(\mathbf{M}_1, \mathbf{M}_2)$ *is any function that satisfies*

$$g(\mathbf{M}_1, \mathbf{M}_2) \ll \left(\prod_{j=1}^{2k} m_j \right)^{\varepsilon},$$

then we have:

(i) $T_{g,k}(s, w)$ *is absolutely convergent.*

(ii) We have

$$T_{g,k}(s, w) - T_{g,k}(x, y; s, w) \ll x^{-2s+\varepsilon} + y^{1-2w} \log^3 y.$$

Proof. (i) Since $m_k/m_{2k} = n_1/n_2$, we can find positive integers t_1, t_2, g_1, g_2 such that

$$\frac{m_k}{m_{2k}} = \frac{n_1}{n_2} = \frac{t_1}{t_2},$$

where $(t_1, t_2) = 1$ and $t_1 g_1 = m_k$, $t_2 g_1 = m_{2k}$, $t_1 g_2 = n_1$, $t_2 g_2 = n_2$. Denote by $\mathbf{M}_1{}'$ and $\mathbf{M}_2{}'$ the vectors $(m_1, \cdots, m_{k-1})$ and $(m_{k+1}, \cdots, m_{2k-1})$, respectively, then we have

$$\begin{aligned}
T_{g,k}(s, w) &\ll \sum_{\substack{\mathbf{M}_1{}', \mathbf{M}_2{}' \in \mathbb{N}^{k-1} \\ t_1, t_2, g_1, g_2 \in \mathbb{N} \\ (t_1, t_2) = 1}} \frac{|f(\mathbf{M}_1{}', t_1 g_1)||f(\mathbf{M}_2{}', t_2 g_1)||g(\mathbf{M}_1{}', t_1 g_1, \mathbf{M}_2{}', t_2 g_1)|\tau(t_1 g_2)\tau(t_2 g_2)}{D_1 D_2 (t_1 g_1)^s (t_2 g_1)^s (t_1 g_2)^w (t_2 g_2)^w} \\
&\leqslant \sum_{\substack{\mathbf{M}_1{}', \mathbf{M}_2{}' \in \mathbb{N}^{k-1} \\ t_1, t_2, g_1, g_2 \in \mathbb{N} \\ (t_1, t_2) = 1}} \frac{|f(\mathbf{M}_1{}', t_1 g_1)||f(\mathbf{M}_2{}', t_2 g_1)||g(\mathbf{M}_1{}', t_1 g_1, \mathbf{M}_2{}', t_2 g_1)|\tau(t_1)\tau(t_2)\tau^2(g_2)}{D_1 D_2 (t_1 g_1)^s (t_2 g_1)^s (t_1 t_2)^w g_2{}^{2w}} \\
&\ll \sum_{\substack{\mathbf{M}_1{}', \mathbf{M}_2{}' \in \mathbb{N}^{k-1} \\ t_1, t_2, g_1 \in \mathbb{N}}} \frac{|f(\mathbf{M}_1{}', t_1 g_1)||f(\mathbf{M}_2{}', t_2 g_1)||g(\mathbf{M}_1{}', t_1 g_1, \mathbf{M}_2{}', t_2 g_1)|\tau(t_1)\tau(t_2)}{D_1 D_2 g_1{}^{2s} (t_1 t_2)^{s+w}} \sum_{g_2=1}^{\infty} \frac{\tau^2(g_2)}{g_2{}^{2w}} \\
&\ll \sum_{\substack{\mathbf{M}_1{}', \mathbf{M}_2{}' \in \mathbb{N}^{k-1} \\ t_1, t_2, g_1 \in \mathbb{N}}} \frac{|f(\mathbf{M}_1{}', t_1 g_1)||f(\mathbf{M}_2{}', t_2 g_1)||g(\mathbf{M}_1{}', t_1 g_1, \mathbf{M}_2{}', t_2 g_1)|\tau(t_1)\tau(t_2)}{D_1 D_2 g_1{}^{2s} (t_1 t_2)^{s+w}} \\
&:= U_1,
\end{aligned}$$

where we use the conclusion: For a given real number c, we have

$$\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^c} < \infty \quad (c > 1), \tag{2.2}$$

moreover, for a large real number U, using partial summation and the conclusion

$$\sum_{n \leqslant U} \tau^2(n) \ll U \log^3 U,$$

we have

$$\sum_{n\leqslant U}\frac{\tau^2(n)}{n^c}\ll\int_2^U\frac{1}{u^c}d\left(\sum_{n\leqslant u}\tau^2(n)\right)$$
$$\ll U^{1-c}\log^3 U. \tag{2.3}$$

Since for any real $\delta>0$,

$$\tau(n)/n^{\delta}\ll 1, \tag{2.4}$$

then

$$\begin{aligned}U_1 &= \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}}}\frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)||g(\mathbf{M}_1,t_1g_1,\mathbf{M}_2,t_2g_1)|}{D_1D_2(t_1g_1)^s(t_2g_1)^s}\cdot\frac{\tau(t_1)\tau(t_2)}{(t_1t_2)^w}\\
&\ll \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}}}\frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)||g(\mathbf{M}_1{}',t_1g_1,\mathbf{M}_2{}',t_2g_1)|}{D_1D_2(t_1g_1)^s(t_2g_1)^s}\\
&\ll \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}}}\frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)|}{(D_1D_2)^{1-\varepsilon}(t_1g_1)^{s-\varepsilon}(t_2g_1)^{s-\varepsilon}}\\
&\ll \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1,g_1{}'\in\mathbb{N}}}\frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1{}')|}{(D_1D_2)^{1-\varepsilon}(t_1g_1)^{s-\varepsilon}(t_2g_1{}')^{s-\varepsilon}}\\
&:= U_2.\end{aligned}$$

Let $t_1g_1=m_k$, $t_2g_1{}'=m_{2k}{}'$, by Lemma 2.4, we have

$$\begin{aligned}U_2 &= \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ m_k,m_{2k}{}'\in\mathbb{N}}}\frac{|f(\mathbf{M}_1{}',m_k)||f(\mathbf{M}_2{}',m_{2k}{}')|\tau(m_k)\tau(m_{2k}{}')}{(D_1D_2)^{1-\varepsilon}m_k{}^{s-\varepsilon}m_{2k}{}'^{s-\varepsilon}}\\
&=\left(\sum_{\substack{\mathbf{M}_1{}'\in\mathbb{N}^{k-1}\\ m_k\in\mathbb{N}}}\frac{|f(\mathbf{M}_1{}',m_k)|\tau(m_k)}{D_1{}^{1-\varepsilon}m_k{}^{s-\varepsilon}}\right)^2\\
&\ll 1.\end{aligned}$$

Thus, we conclude that (i) holds.

(ii) Since $m_1,\cdots,m_{k-1},m_{k+1},\cdots,m_{2k-1}$ are symmetric, we have

$$T_{g,k}(s,w)-T_{g,k}(x,y;s,w)\ll T_1+T_2+T_3, \tag{2.5}$$

where

$$T_1=T_1(x;s,w)=\sum_{\substack{\mathbf{M}_1,\mathbf{M}_2\in\mathbb{N}^k\\ n_1,n_2\in\mathbb{N}\\ \frac{m_k}{m_{2k}}=\frac{n_1}{n_2}\\ m_1>x}}\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)||g(\mathbf{M}_1,\mathbf{M}_2)|}{D_1D_2m_k{}^sm_{2k}{}^s}\cdot\frac{\tau(n_1)\tau(n_2)}{n_1{}^wn_2{}^w},$$

$$T_2 = T_2(x;s,w) = \sum_{\substack{\mathbf{M}_1,\mathbf{M}_2\in\mathbb{N}^k\\ n_1,n_2\in\mathbb{N}\\ \frac{m_k}{m_{2k}}=\frac{n_1}{n_2}\\ m_k>x \text{ or } m_{2k}>x}} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)||g(\mathbf{M}_1,\mathbf{M}_2)|}{D_1D_2{m_k}^s{m_{2k}}^s}\cdot\frac{\tau(n_1)\tau(n_2)}{{n_1}^w{n_2}^w},$$

$$T_3 = T_3(y;s,w) = \sum_{\substack{\mathbf{M}_1,\mathbf{M}_2\in\mathbb{N}^k\\ n_1,n_2\in\mathbb{N}\\ \frac{m_k}{m_{2k}}=\frac{n_1}{n_2}\\ n_1>y \text{ or } n_2>y}} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)||g(\mathbf{M}_1,\mathbf{M}_2)|}{D_1D_2{m_k}^s{m_{2k}}^s}\cdot\frac{\tau(n_1)\tau(n_2)}{{n_1}^w{n_2}^w}.$$

Similar to (i), let $(t_1,t_2)=1$ and $t_1g_1=m_k$, $t_2g_1=m_{2k}$, $t_1g_2=n_1$, $t_2g_2=n_2$, and denote by ${\mathbf{M}_1}'$ and ${\mathbf{M}_2}'$ the vectors $(m_1,\cdots,m_{k-1})$ and $(m_{k+1},\cdots,m_{2k-1})$ respectively, then

$$\begin{aligned}
T_1 &= \sum_{\substack{{\mathbf{M}_1}',{\mathbf{M}_2}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1,g_2\in\mathbb{N}\\ (t_1,t_2)=1\\ m_1>x}} \frac{|f({\mathbf{M}_1}',t_1g_1)||f({\mathbf{M}_2}',t_2g_1)||g({\mathbf{M}_1}',t_1g_1,{\mathbf{M}_2}',t_2g_1)|\tau(t_1g_2)\tau(t_2g_2)}{D_1D_2(t_1g_1)^s(t_2g_1)^s(t_1g_2)^w(t_2g_2)^w}\\
&\leqslant \sum_{\substack{{\mathbf{M}_1}',{\mathbf{M}_2}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1,g_2\in\mathbb{N}\\ (t_1,t_2)=1\\ m_1>x}} \frac{|f({\mathbf{M}_1}',t_1g_1)||f({\mathbf{M}_2}',t_2g_1)||g({\mathbf{M}_1}',t_1g_1,{\mathbf{M}_2}',t_2g_1)|\tau(t_1)\tau(t_2)\tau^2(g_2)}{D_1D_2(t_1g_1)^s(t_2g_1)^s(t_1t_2)^w{g_2}^{2w}}\\
&\ll \sum_{\substack{{\mathbf{M}_1}',{\mathbf{M}_2}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}\\ m_1>x}} \frac{|f({\mathbf{M}_1}',t_1g_1)||f({\mathbf{M}_2}',t_2g_1)||g({\mathbf{M}_1}',t_1g_1,{\mathbf{M}_2}',t_2g_1)|\tau(t_1)\tau(t_2)}{D_1D_2{g_1}^{2s}(t_1t_2)^{s+w}}\sum_{g_2=1}^{\infty}\frac{\tau^2(g_2)}{{g_2}^{2w}}\\
&\ll \sum_{\substack{{\mathbf{M}_1}',{\mathbf{M}_2}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}\\ m_1>x}} \frac{|f({\mathbf{M}_1}',t_1g_1)||f({\mathbf{M}_2}',t_2g_1)||g({\mathbf{M}_1}',t_1g_1,{\mathbf{M}_2}',t_2g_1)|\tau(t_1)\tau(t_2)}{D_1D_2{g_1}^{2s}(t_1t_2)^{s+w}}\\
&:= {T_1}',
\end{aligned} \tag{2.6}$$

where we use (2.2).

Let

$${D_1}' = D_1/m_1,$$

by (2.4), we have

$$\begin{aligned}
{T_1}' &= \sum_{\substack{{\mathbf{M}_1}',{\mathbf{M}_2}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}\\ m_1>x}} \frac{|f({\mathbf{M}_1}',t_1g_1)||f({\mathbf{M}_2}',t_2g_1)||g({\mathbf{M}_1}',t_1g_1,{\mathbf{M}_2}',t_2g_1)|\tau(t_1)\tau(t_2)}{D_1D_2(t_1g_1)^{2s-3\varepsilon}(t_2g_1)^{3\varepsilon}{t_1}^{w-s+3\varepsilon}{t_2}^{s+w-3\varepsilon}}\\
&\ll \sum_{\substack{{\mathbf{M}_1}',{\mathbf{M}_2}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}\\ m_1>x}} \frac{|f({\mathbf{M}_1}',t_1g_1)||f({\mathbf{M}_1}',t_2g_1)||g({\mathbf{M}_1}',t_1g_1,{\mathbf{M}_2}',t_2g_1)|}{D_1D_2(t_1g_1)^{2s-3\varepsilon}(t_2g_1)^{3\varepsilon}}
\end{aligned}$$

$$\ll \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}\\ m_1>x}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)|}{m_1{}^{1-\varepsilon}D_1{}'^{1-\varepsilon}D_2{}^{1-\varepsilon}(t_1g_1)^{2s-4\varepsilon}(t_2g_1)^{2\varepsilon}}$$

$$\ll \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1,g_1{}'\in\mathbb{N}\\ m_1>x}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1{}')|}{m_1{}^{1-\varepsilon}D_1{}'^{1-\varepsilon}D_2{}^{1-\varepsilon}(t_1g_1)^{2s-4\varepsilon}(t_2g_1{}')^{2\varepsilon}}$$

$$:= T_1{}''. \tag{2.7}$$

Let $t_1g_1 = m_k$, $t_2g_1{}' = m_{2k}{}'$, we have

$$T_1{}'' = \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ m_k,m_{2k}{}'\in\mathbb{N}\\ m_1>x}} \frac{|f(\mathbf{M}_1{}',m_k)||f(\mathbf{M}_2{}',m_{2k}{}')|\tau(m_k)\tau(m_{2k}{}')}{m_1{}^{1-2s+5\varepsilon}D_1{}'^{1-\varepsilon}D_2{}^{1-\varepsilon}m_k{}^{2s-4\varepsilon}m_{2k}{}'^{2\varepsilon}} \cdot \frac{1}{m_1{}^{2s-6\varepsilon}}$$

$$\ll x^{-2s+6\varepsilon} \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ m_k,m_{2k}{}'\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}',m_k)||f(\mathbf{M}_2{}',m_{2k}{}')|\tau(m_k)\tau(m_{2k}{}')}{m_1{}^{1-2s+5\varepsilon}D_1{}'^{1-\varepsilon}D_2{}^{1-\varepsilon}m_k{}^{2s-4\varepsilon}m_{2k}{}'^{2\varepsilon}}$$

$$\ll x^{-2s+6\varepsilon} \sum_{\substack{\mathbf{M}_1{}'\in\mathbb{N}^{k-1}\\ m_k\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}',m_k)|\tau(m_k)}{m_1{}^{1-2s+5\varepsilon}D_1{}'^{1-\varepsilon}m_k{}^{2s-4\varepsilon}} \sum_{\substack{\mathbf{M}_1{}'\in\mathbb{N}^{k-1}\\ m_k\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}',m_k)|\tau(m_k)}{D_1{}^{1-\varepsilon}m_k{}^{2\varepsilon}}$$

$$\ll x^{-2s+\varepsilon},$$

the convergence of the two series in the last step can be obtained by Lemma 2.4, and we use the arbitrariness of ε.

For T_2, if $m_k > x$, we replace the condition "$m_1 > x$" by "$t_1g_1 > x$" in (2.6), thus

$$T_2 \ll \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}\\ t_1g_1>x}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)||g(\mathbf{M}_1{}',t_1g_1,\mathbf{M}_2{}',t_2g_1)|\tau(t_1)\tau(t_2)}{D_1D_2g_1{}^{2s}(t_1t_2)^{s+w}} := T_2{}'.$$

Using (2.4) we have

$$T_2{}' = \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}\\ t_1g_1>x}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)||g(\mathbf{M}_1{}',t_1g_1,\mathbf{M}_2{}',t_2g_1)|\tau(t_1)\tau(t_2)}{D_1D_2(t_1g_1)^{2s-3\varepsilon}(t_2g_1)^{3\varepsilon}t_1{}^{w-s+3\varepsilon}t_2{}^{s+w-3\varepsilon}}$$

$$\ll \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}\\ t_1g_1>x}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)||g(\mathbf{M}_1{}',t_1g_1,\mathbf{M}_2{}',t_2g_1)|}{D_1D_2(t_1g_1)^{2s-3\varepsilon}(t_2g_1)^{3\varepsilon}}$$

$$\ll \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}\\ t_1g_1>x}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)|}{(D_1D_2)^{1-\varepsilon}(t_1g_1)^{2s-4\varepsilon}(t_2g_1)^{2\varepsilon}}$$

$$
\begin{aligned}
&\ll \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1,g_1{}'\in\mathbb{N}\\ t_1g_1>x}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1{}')|}{(D_1D_2)^{1-\varepsilon}(t_1g_1)^{2s-4\varepsilon}(t_2g_1{}')^{2\varepsilon}}\\
&:= T_2{}''. \qquad (2.8)
\end{aligned}
$$

Let $t_1g_1 = m_k$, $t_2g_1{}' = m_{2k}{}'$, we have

$$
\begin{aligned}
T_2{}'' &= \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ m_k,m_{2k}{}'\in\mathbb{N}\\ m_k>x}} \frac{|f(\mathbf{M}_1{}',m_k)||f(\mathbf{M}_2{}',m_{2k}{}')|\tau(m_k)\tau(m_{2k}{}')}{(D_1D_2)^{1-\varepsilon}m_k{}^{2s-4\varepsilon}m_{2k}{}'^{2\varepsilon}}\cdot\frac{1}{m_1{}^{2s-6\varepsilon}}\\
&\ll x^{-2s+6\varepsilon}\sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ m_k,m_{2k}{}'\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}',m_k)||f(\mathbf{M}_2{}',m_{2k}{}')|\tau(m_k)\tau(m_{2k}{}')}{(D_1D_2)^{1-\varepsilon}m_k{}^{2s-4\varepsilon}m_{2k}{}'^{2\varepsilon}}\\
&\ll x^{-2s+6\varepsilon}\sum_{\substack{\mathbf{M}_1{}'\in\mathbb{N}^{k-1}\\ m_k\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}',m_k)|\tau(m_k)}{D_1{}^{1-\varepsilon}m_k{}^{2s-4\varepsilon}}\sum_{\substack{\mathbf{M}_1{}'\in\mathbb{N}^{k-1}\\ m_k\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}',m_k)|\tau(m_k)}{D_1{}^{1-\varepsilon}m_k{}^{2\varepsilon}}\\
&\ll x^{-2s+\varepsilon},
\end{aligned}
$$

the convergence of the two series in the last step can be obtained by Lemma 2.4, and we use the arbitrariness of ε. If $m_{2k} > x$, exchange t_1 and t_2, we can also obtain

$$T_2 \ll x^{-2s+\varepsilon}.$$

For T_3, if $n_1 > y$, similar to (i), we obtain $(t_1, t_2) = 1$ and $t_1g_1 = m_k$, $t_2g_1 = m_{2k}$, $t_1g_2 = n_1$, $t_2g_2 = n_2$, then

$$
\begin{aligned}
T_3 &= \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1,g_2\in\mathbb{N}\\ (t_1,t_2)=1\\ t_1g_2>y}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)||g(\mathbf{M}_1{}',t_1g_1,\mathbf{M}_2{}',t_2g_1)|\tau(t_1g_2)\tau(t_2g_2)}{D_1D_2(t_1g_1)^s(t_2g_1)^s(t_1g_2)^w(t_2g_2)^w}\\
&\leqslant \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1,g_2\in\mathbb{N}\\ (t_1,t_2)=1\\ t_1g_2>y}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)||g(\mathbf{M}_1{}',t_1g_1,\mathbf{M}_2{}',t_2g_1)|\tau(t_1)\tau(t_2)\tau^2(g_2)}{D_1D_2(t_1g_1)^s(t_2g_1)^s(t_1t_2)^w g_2{}^{2w}}\\
&\ll \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)||g(\mathbf{M}_1{}',t_1g_1,\mathbf{M}_2{}',t_2g_1)|\tau(t_1)\tau(t_2)}{D_1D_2g_1{}^{2s}(t_1t_2)^{s+w}}\sum_{g_2>\frac{y}{t_1}}\frac{\tau^2(g_2)}{g_2{}^{2w}}\\
&\ll y^{1-2w}\log^3 y\sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}',t_1g_1)||f(\mathbf{M}_2{}',t_2g_1)||g(\mathbf{M}_1{}',t_1g_1,\mathbf{M}_2{}',t_2g_1)|\tau(t_1)\tau(t_2)}{D_1D_2g_1{}^{2s}t_1{}^{1+s-w}t_2{}^{s+w}},
\end{aligned}
$$

where we use (2.3) by taking $U = y/t_1$.

We denote the latter series by Σ, using (2.4) we have

$$\begin{aligned}\Sigma = & \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}', t_1g_1)||f(\mathbf{M}_2{}', t_2g_1)||g(\mathbf{M}_1{}', t_1g_1, \mathbf{M}_2{}', t_2g_1)|\tau(t_1)\tau(t_2)}{D_1D_2(t_1g_1)^s(t_2g_1)^s {t_1}^{1-w}{t_2}^w}\\
\ll & \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}', t_1g_1)||f(\mathbf{M}_2{}', t_2g_1)||g(\mathbf{M}_1{}', t_1g_1, \mathbf{M}_2{}', t_2g_1)|}{D_1D_2(t_1g_1)^s(t_2g_1)^s}\\
\ll & \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}', t_1g_1)||f(\mathbf{M}_2{}', t_2g_1)|}{(D_1D_2)^{1-\varepsilon}(t_1g_1)^{s-\varepsilon}(t_2g_1)^{s-\varepsilon}}\\
\ll & \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ t_1,t_2,g_1,g_1{}'\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}', t_1g_1)||f(\mathbf{M}_2{}', t_2g_1{}')|}{(D_1D_2)^{1-\varepsilon}(t_1g_1)^{s-\varepsilon}(t_2g_1{}')^{s-\varepsilon}}.\end{aligned}$$

Let $t_1g_1 = m_k$, $t_2g_1{}' = m_{2k}{}'$, we obtain

$$\begin{aligned}\Sigma \ll & \sum_{\substack{\mathbf{M}_1{}',\mathbf{M}_2{}'\in\mathbb{N}^{k-1}\\ m_k,m_{2k}{}'\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}', m_k)||f(\mathbf{M}_1{}', m_{2k}{}')|\tau(m_k)\tau(m_{2k}{}')}{(D_1D_2)^{1-\varepsilon}{m_k}^{s-\varepsilon}{m_{2k}{}'}^{s-\varepsilon}}\\
= & \left(\sum_{\substack{\mathbf{M}_1{}'\in\mathbb{N}^{k-1}\\ m_k\in\mathbb{N}}} \frac{|f(\mathbf{M}_1{}', m_k)|\tau(m_k)}{{D_1}^{1-\varepsilon}{m_k}^{s-\varepsilon}}\right)^2\\
\ll & 1\end{aligned}$$

by Lemma 2.4, thus

$$T_3 \ll y^{1-2w}\log^3 y.$$

If $n_2 > y$, exchange t_1 and t_2, similarly we obtain

$$T_3 \ll y^{1-2w}\log^3 y.$$

Above all, we obtain

$$T_1, T_2 \ll x^{-2s+\varepsilon} \quad \text{and} \quad T_3 \ll y^{1-2w}\log^3 y,$$

then (ii) holds from (2.5). □

3. Expression of $\Delta_k(x)$

We shall give a more explicit expression of $\Delta_k(x)$. According to Lemma 2.4,

$$\tau(n_1\cdots n_k) = \sum_{m_1d_1=n_1,\cdots,m_kd_k=n_k} f(m_1,\cdots,m_k)\tau(d_1)\cdots\tau(d_k)$$

holds for any $n_1, \cdots, n_k \in \mathbb{N}$, where f is multiplicative and symmetric in all variables.

Therefore, we deduce by (1.1) that

$$\begin{aligned}
\sum_{n_1,\cdots,n_k\leqslant x}\tau(n_1\cdots n_k) &= \sum_{m_1,\cdots,m_k\leqslant x} f(m_1,\cdots,m_k)\prod_{j=1}^{k}\left(\sum_{d_j\leqslant x/m_j}\tau(d_j)\right)\\
&= \sum_{m_1,\cdots,m_k\leqslant x} f(m_1,\cdots,m_k)\prod_{j=1}^{k}\left(M\left(\frac{x}{m_j}\right)+\Delta\left(\frac{x}{m_j}\right)\right)\\
&= \sum_{m_1,\cdots,m_k\leqslant x} f(m_1,\cdots,m_k)\left(\prod_{j=1}^{k}M\left(\frac{x}{m_j}\right)\right)\\
&\quad + k\sum_{m_1,\cdots,m_k\leqslant x} f(m_1,\cdots,m_k)\left(\prod_{j=1}^{k-1}M\left(\frac{x}{m_j}\right)\right)\Delta\left(\frac{x}{m_k}\right)\\
&\quad + \binom{k}{2}\sum_{m_1,\cdots,m_k\leqslant x} f(m_1,\cdots,m_k)\left(\prod_{j=1}^{k-2}M\left(\frac{x}{m_j}\right)\right)\Delta\left(\frac{x}{m_k}\right)\Delta\left(\frac{x}{m_{k-1}}\right)\\
&\quad + \cdots\\
&\quad + \binom{k}{k-1}\sum_{m_1,\cdots,m_k\leqslant x} f(m_1,\cdots,m_k)M\left(\frac{x}{m_1}\right)\left(\prod_{j=2}^{k}\Delta\left(\frac{x}{m_j}\right)\right)\\
&\quad + \sum_{m_1,\cdots,m_k\leqslant x} f(m_1,\cdots,m_k)\left(\prod_{j=1}^{k}\Delta\left(\frac{x}{m_j}\right)\right)\\
&:= \sum_{m_1,\cdots,m_k\leqslant x} f(m_1,\cdots,m_k)\left(\prod_{j=1}^{k}M\left(\frac{x}{m_j}\right)\right)+\mathbf{E}(x), \qquad (3.1)
\end{aligned}$$

where

$$M(u) = u\log u + (2\gamma-1)u. \qquad (3.2)$$

Here we have

$$\begin{aligned}
\sum_{m_1,\cdots,m_k\leq x} f(m_1,\cdots,m_k)\prod_{j=1}^{k}\left(M\left(\frac{x}{m_j}\right)\right) &= \sum_{m_1,\cdots,m_k=1}^{\infty} f(m_1,\cdots,m_k)\prod_{j=1}^{k}\left(M\left(\frac{x}{m_j}\right)\right)+O(x^{k-1+\varepsilon})\\
&= x^kP_k(\log x)+O\left(x^{k-1+\varepsilon}\right) \qquad (3.3)
\end{aligned}$$

by the proof of Theorem 3.4 in Tóth and Zhai [9]. From (1.2) we have

$$\Delta(\frac{x}{m_j}) \ll (\frac{x}{m_j})^{\theta+\varepsilon}$$

for any $1 \leqslant j \leqslant k$, then

$$\mathbf{E}(x) \quad \ll \quad \sum_{m_1,\cdots,m_k\leqslant x}|f(m_1,\cdots,m_k)|\left(\prod_{j=1}^{k-2}M\left(\frac{x}{m_j}\right)\right)\Delta\left(\frac{x}{m_k}\right)\Delta\left(\frac{x}{m_{k-1}}\right)$$

$$
\begin{aligned}
&\ll x^{k-2+2\theta+\varepsilon} \sum_{m_1,\cdots,m_k\leqslant x} \frac{|f(m_1,\cdots,m_k)|}{m_1\cdots m_{k-2}(m_{k-1}m_k)^{\theta}} \cdot \frac{(m_{k-1}m_k)^{\frac{1}{2}-\theta+\varepsilon}}{(m_{k-1}m_k)^{\frac{1}{2}-\theta+\varepsilon}} \\
&\ll x^{k-2+2\theta+\varepsilon}(x^2)^{\frac{1}{2}-\theta+\varepsilon} \sum_{m_1,\cdots,m_k=1}^{\infty} \frac{|f(m_1,\cdots,m_k)|}{m_1\cdots m_{k-2}(m_{k-1}m_k)^{\frac{1}{2}+\varepsilon}} \\
&\ll x^{k-1+\varepsilon}, \qquad (3.4)
\end{aligned}
$$

where the convergence of latter series can be obtained by Lemma 2.4. Then we conclude that

$$
\Delta_k(x) = \Delta_k{}^*(x) + O\left(x^{k-1+\varepsilon}\right) \qquad (3.5)
$$

follows by (1.4), (3.1), (3.3), and (3.4), where

$$
\Delta_k{}^*(x) = k \sum_{m_1,\cdots,m_k\leqslant x} f(m_1,\cdots,m_k)\left(\prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right)\right)\Delta\left(\frac{x}{m_k}\right).
$$

4. Mean square of $\Delta_k{}^*(x)$

Suppose $T \geqslant 2$, we shall first estimate $\int_T^{2T}(\Delta_k{}^*(x))^2dx$. Since $m_1,\cdots,m_{k-1}$ are symmetric, we can divide $\Delta_k{}^*(x)$ into three parts,

$$
\Delta_k{}^*(x) = M_1 + O(M_2 + M_3),
$$

where the implied constant about "O" depends on k and ε,

$$
\begin{aligned}
M_1 = M_1(x,y) &:= k \sum_{m_1,\cdots,m_k\leqslant y} f(m_1,\cdots,m_k) \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right)\Delta\left(\frac{x}{m_k}\right), \\
M_2 = M_2(x,y) &:= \sum_{\substack{m_1,\cdots,m_k\leqslant x \\ m_1>y}} |f(m_1,\cdots,m_k)| \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right)\left|\Delta\left(\frac{x}{m_k}\right)\right|, \\
M_3 = M_3(x,y) &:= \sum_{\substack{m_1,\cdots,m_k\leqslant x \\ m_k>y}} |f(m_1,\cdots,m_k)| \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right)\left|\Delta\left(\frac{x}{m_k}\right)\right|, \qquad (4.1)
\end{aligned}
$$

where y is a parameter that satisfies $T^{\varepsilon} \ll y \ll T$. So we obtain

$$
\begin{aligned}
\int_T^{2T}(\Delta_k{}^*(x))^2dx = \int_T^{2T} M_1{}^2dx &+ O\left(\int_T^{2T}(M_1M_2 + M_2M_3 + M_1M_3)dx\right) \\
&+ O\left(\int_T^{2T}(M_2{}^2 + M_3{}^2)dx\right). \qquad (4.2)
\end{aligned}
$$

4.1. Mean square of $M_1(x, y)$

First, we deal with $\int_T^{2T} M_1{}^2 dx$. By Lemma 2.1 we obtain

$$M_1(x,y) = k \sum_{m_1,\cdots,m_k \leqslant y} f(m_1,\cdots,m_k) \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right)\left(\delta_1\left(\frac{x}{m_k}, y\right) + \delta_2\left(\frac{x}{m_k}, y\right)\right)$$

$$:= M_{11}(x,y) + M_{12}(x,y),$$

where

$$M_{11}(x,y) = k \sum_{m_1,\cdots,m_k \leqslant y} f(m_1,\cdots,m_k) \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right)\delta_1\left(\frac{x}{m_k}, y\right),$$
$$M_{12}(x,y) = k \sum_{m_1,\cdots,m_k \leqslant y} f(m_1,\cdots,m_k) \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right)\delta_2\left(\frac{x}{m_k}, y\right),$$

and $\delta_1(\cdot, y)$ and $\delta_2(\cdot, y)$ are defined in Lemma 2.2, thus we have

$$\int_T^{2T} M_1{}^2(x,y)dx = \int_T^{2T} M_{11}{}^2(x,y)dx + O\left(\int_T^{2T}\left(M_{11}(x,y)M_{12}(x,y) + M_{12}{}^2(x,y)\right)dx\right). \tag{4.3}$$

Let $\mathbf{M_1}, \mathbf{M_2}$ be defined in Lemma 2.5, using

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta)),$$

we have

$$\begin{aligned}
M_{11}{}^2(x,y) =& k^2 \sum_{m_1,\cdots,m_{2k} \leqslant y} f(\mathbf{M}_1)f(\mathbf{M}_2) \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right) \prod_{j=k+1}^{2k-1} M\left(\frac{x}{m_j}\right) \\
&\times \frac{x^{1/2}}{2\pi^2(m_k m_{2k})^{1/4}} \sum_{n_1,n_2 \leqslant y} \frac{\tau(n_1)\tau(n_2)}{n_1{}^{3/4}n_2{}^{3/4}} \cos\left(4\pi\sqrt{\frac{n_1 x}{m_k}} - \frac{\pi}{4}\right)\cos\left(4\pi\sqrt{\frac{n_2 x}{m_{2k}}} - \frac{\pi}{4}\right) \\
=& k^2 \sum_{m_1,\cdots,m_{2k} \leqslant y} f(\mathbf{M}_1)f(\mathbf{M}_2) \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right) \prod_{j=k+1}^{2k-1} M\left(\frac{x}{m_j}\right) \frac{x^{1/2}}{4\pi^2(m_k m_{2k})^{1/4}} \\
&\times \sum_{n_1,n_2 \leqslant y} \frac{\tau(n_1)\tau(n_2)}{n_1{}^{3/4}n_2{}^{3/4}} \left[\cos\left(4\pi\left(\sqrt{\frac{n_1}{m_k}} - \sqrt{\frac{n_2}{m_{2k}}}\right)\sqrt{x}\right) + \sin\left(4\pi\left(\sqrt{\frac{n_1}{m_k}} + \sqrt{\frac{n_2}{m_{2k}}}\right)\sqrt{x}\right)\right] \\
:=& k^2(S_0(x,y) + S_1(x,y) + S_2(x,y)),
\end{aligned} \tag{4.4}$$

where

$$S_0(x,y) = \frac{x^{1/2}}{4\pi^2} \sum_{m_1,\cdots,m_{2k} \leqslant y} \frac{f(\mathbf{M}_1)f(\mathbf{M}_2)}{(m_k m_{2k})^{1/4}} \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right) \prod_{j=k+1}^{2k-1} M\left(\frac{x}{m_j}\right) \sum_{\substack{n_1,n_2 \leq y \\ n_1 m_{2k} = n_2 m_k}} \frac{\tau(n_1)\tau(n_2)}{n_1{}^{3/4}n_2{}^{3/4}},$$

$$S_1(x,y)=\frac{x^{1/2}}{4\pi^2}\sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{f(\mathbf{M}_1)f(\mathbf{M}_2)}{(m_km_{2k})^{1/4}}\prod_{j=1}^{k-1}M\left(\frac{x}{m_j}\right)\prod_{j=k+1}^{2k-1}M\left(\frac{x}{m_j}\right)$$
$$\times\sum_{\substack{n_1,n_2\leqslant y\\ n_1m_{2k}\neq n_2m_k}}\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\cos\left(4\pi\left(\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}}\right)\sqrt{x}\right),$$
$$S_2(x,y)=\frac{x^{1/2}}{4\pi^2}\sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{f(\mathbf{M}_1)f(\mathbf{M}_2)}{(m_km_{2k})^{1/4}}\prod_{j=1}^{k-1}M\left(\frac{x}{m_j}\right)\prod_{j=k+1}^{2k-1}M\left(\frac{x}{m_j}\right)$$
$$\times\sum_{n_1,n_2\leqslant y}\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\sin\left(4\pi\left(\sqrt{\frac{n_1}{m_k}}+\sqrt{\frac{n_2}{m_{2k}}}\right)\sqrt{x}\right).$$

So, it turns to evaluate

$$\int_0=\int_T^{2T}S_0(x,y)dx\,,\quad \int_1=\int_T^{2T}S_1(x,y)dx\,,\quad \int_2=\int_T^{2T}S_2(x,y)dx.$$

We need to give more explicit expressions for $(\prod_{j=1}^{k-1}M(\frac{x}{m_j}))$ and $(\prod_{j=k+1}^{2k-1}M(\frac{x}{m_j}))$. By the proof of Theorem 3.3 in Tóth and Zhai [9] in the case $f_j(n)=\tau(n),\ a_j=1,\ \delta_j=1\ (1\leqslant j\leqslant k-1)$ we obtain

$$\prod_{j=1}^{k-1}M\left(\frac{x}{m_j}\right)=\frac{x^{k-1}}{m_1\cdots m_{k-1}}\sum_{l_1=0}^{k-1}C_{l_1}(\log m_1,\cdots,\log m_{k-1})(\log x)^{l_1},\tag{4.5}$$

where

$$C_{l_1}(\log m_1,\cdots,\log m_{k-1})=\sum_{j_1,\cdots,j_{k-1}=0,1}c(j_1,\cdots,j_{k-1})(\log m_1)^{j_1}\cdots(\log m_{k-1})^{j_{k-1}},$$

and $c(j_1,\cdots,j_{k-1})$ are constants $(j_1,\cdots,j_{k-1}=0,1)$. Similarly, when $k+1\leqslant j\leqslant 2k-1$, we have

$$\prod_{j=k+1}^{2k-1}M\left(\frac{x}{m_j}\right)=\frac{x^{k-1}}{m_{k+1}\cdots m_{2k-1}}\sum_{l_2=k}^{2k-1}C_{l_2}(\log m_{k+1},\cdots,\log m_{2k-1})(\log x)^{l_2-k},\tag{4.6}$$

where

$$C_{l_2}(\log m_{k+1},\cdots,\log m_{2k-1})=\sum_{j_{k+1},\cdots,j_{2k-1}=0,1}c(j_{k+1},\cdots,j_{2k-1})(\log m_{k+1})^{j_{k+1}}\cdots(\log m_{2k-1})^{j_{2k-1}},$$

and $c(j_{k+1},\cdots,j_{2k-1})$ are constants $(j_{k+1},\cdots,j_{2k-1}=0,1)$.

4.1.1. Evaluation of $\int_0$

We denote $C_{l_1}(\log m_1,\cdots,\log m_{k-1})C_{l_2}(\log m_{k+1},\cdots,\log m_{2k-1})$ by $C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2)$, using (4.5) and (4.6) we have

$$S_0(x,y)=\frac{x^{2k-\frac{3}{2}}}{4\pi^2}\sum_{l_1,l_2=0}^{k-1}(\log x)^{l_1+l_2}\sum_{\substack{m_1,\cdots,m_{2k}\leqslant y\\ n_1,n_2\leqslant y\\ \frac{m_k}{m_{2k}}=\frac{n_1}{n_2}}}\frac{f(\mathbf{M}_1)f(\mathbf{M}_2)C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2)}{D_1D_2(m_km_{2k})^{\frac{1}{4}}}\cdot\frac{\tau(n_1)\tau(n_2)}{(n_1n_2)^{\frac{3}{4}}},\tag{4.7}$$

where D_1, D_2 are defined in Lemma 2.5. Using (2.4) we obtain

$$C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2) \ll \left(\prod_{j=1}^{k-1} m_j \prod_{j=k+1}^{2k-1} m_j\right)^{\varepsilon} \ll \left(\prod_{j=1}^{2k} m_j\right)^{\varepsilon}.$$

Choosing

$$g(\mathbf{M}_1,\mathbf{M}_2) = g_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2) = C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2),$$

$s = 1/4$, $w = 3/4$ in Lemma 2.5, we obtain that $S_0(x,y)$ can be written as

$$\begin{aligned} S_0(x,y) &= \frac{x^{2k-\frac{3}{2}}}{4\pi^2}\sum_{l_1,l_2=0}^{k-1}(\log x)^{l_1+l_2} \times T_{g,k}\left(y,y;\frac{1}{4},\frac{3}{4}\right) \\ &= \frac{x^{2k-\frac{3}{2}}}{4\pi^2}\sum_{l_1,l_2=0}^{k-1}(\log x)^{l_1+l_2}\left(T_{g,k}\left(\frac{1}{4},\frac{3}{4}\right) + O\left(y^{-\frac{1}{2}+\varepsilon}\right)\right). \end{aligned} \tag{4.8}$$

Since $g(\mathbf{M}_1,\mathbf{M}_2)$ is related to l_1, l_2, we denote $T_{g,k}\left(\frac{1}{4},\frac{3}{4}\right)$ in (4.8) by D_{k,l_1,l_2}, we conclude that

$$S_0(x,y) = \frac{x^{2k-\frac{3}{2}}}{4\pi^2}Q_{2k-2}(\log x) + O\left(x^{2k-\frac{3}{2}+\varepsilon}y^{-\frac{1}{2}+\varepsilon}\right), \tag{4.9}$$

where

$$Q_{2k-2}(t) = \sum_{l_1,l_2=0}^{k-1} D_{k,l_1,l_2}\, t^{l_1+l_2}.$$

Then it follows from (4.9) that

$$\int_T^{2T} S_0(x,y)dx = \frac{1}{4\pi^2}\sum_{l_1,l_2=0}^{k-1} D_{k,l_1,l_2}\int_T^{2T} x^{2k-\frac{3}{2}}(\log x)^{l_1+l_2}dx + O\left(T^{2k-\frac{1}{2}+\varepsilon}y^{-\frac{1}{2}}\right). \tag{4.10}$$

4.1.2. Estimates of $\int_1$ and $\int_2$

Let $\mathbf{M_1}, \mathbf{M_2}, D_1, D_2$ be defined in Lemma 2.5, then by (4.5) and (4.6) we have

$$\begin{aligned} \int_2 = \int_T^{2T} \frac{x^{2k-\frac{3}{2}}}{4\pi^2}\sum_{l_1,l_2=0}^{k-1}(\log x)^{l_1+l_2} \sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2)}{D_1D_2} \\ \times \sum_{n_1,n_2\leqslant y}\frac{f(\mathbf{M}_1)f(\mathbf{M}_2)}{{m_k}^{1/4}{m_{2k}}^{1/4}}\cdot\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\sin\left(4\pi\left(\sqrt{\frac{n_1}{m_k}}+\sqrt{\frac{n_2}{m_{2k}}}\right)\sqrt{x}\right)dx. \end{aligned}$$

Let

$$G(x) = x^{2k-\frac{3}{2}}(\log x)^{l_1+l_2}, \quad F(\cdot) = \sin(\cdot), \quad m(x) = 4\pi\left(\sqrt{\frac{n_1}{m_k}}+\sqrt{\frac{n_2}{m_{2k}}}\right)\sqrt{x}, \quad a = T, \ \ b = 2T$$

in Lemma 2.3, change the order of integration and summation, then we have

$$\begin{aligned}\int_2 =& \frac{1}{4\pi^2}\sum_{l_1,l_2=0}^{k-1}\sum_{\substack{m_1,\cdots,m_{2k}\leqslant y\\ n_1,n_2\leqslant y}}\frac{C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2)}{D_1D_2}\cdot\frac{f(\mathbf{M}_1)f(\mathbf{M}_2)}{{m_k}^{1/4}{m_{2k}}^{1/4}}\cdot\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\\ &\times\int_T^{2T}x^{2k-\frac{3}{2}}(\log x)^{l_1+l_2}\sin\left(4\pi\left(\sqrt{\frac{n_1}{m_k}}+\sqrt{\frac{n_2}{m_{2k}}}\right)\sqrt{x}\right)dx\\ \ll&\sum_{\substack{m_1,\cdots,m_{2k}\leqslant y\\ n_1,n_2\leqslant y}}\frac{|C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2)|}{D_1D_2}\cdot\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{{m_k}^{1/4}{m_{2k}}^{1/4}}\cdot\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\\ &\times T^{2k-\frac{3}{2}+\varepsilon}\cdot\frac{T^{\frac{1}{2}}}{\sqrt{\frac{n_1}{m_k}}+\sqrt{\frac{n_2}{m_{2k}}}},\end{aligned}$$

then we use

$$a^2+b^2\geqslant 2ab,\quad C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2)\ll\left(\prod_{j=1}^{2k}m_j\right)^{\varepsilon}$$

to obtain

$$\begin{aligned}\int_2 \ll& T^{2k-1+\varepsilon}\sum_{\substack{m_1,\cdots,m_{2k}\leqslant y\\ n_1,n_2\leqslant y}}\frac{\left(\prod_{j=1}^{2k}m_j\right)^{\varepsilon}}{D_1D_2}\cdot\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{{m_k}^{1/4}{m_{2k}}^{1/4}}\cdot\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\cdot\left(\frac{m_km_{2k}}{n_1n_2}\right)^{\frac{1}{4}}\\ \ll& T^{2k-1+(2k+1)\varepsilon}\sum_{\substack{m_1,\cdots,m_{2k}\leqslant y\\ n_1,n_2\leqslant y}}\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2}\cdot\frac{\tau(n_1)\tau(n_2)}{n_1n_2}\cdot\frac{T^{2\varepsilon}}{{m_k}^{\varepsilon}{m_{2k}}^{\varepsilon}}\\ \ll& T^{2k-1+(2k+3)\varepsilon}\sum_{m_1,\cdots,m_{2k}=1}^{\infty}\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2{m_k}^{\varepsilon}{m_{2k}}^{\varepsilon}}\sum_{n_1,n_2\leqslant y}\frac{\tau(n_1)\tau(n_2)}{n_1n_2}\\ \ll& T^{2k-1+(2k+4)\varepsilon}\left(\sum_{m_1,\cdots,m_{2k}=1}^{\infty}\frac{|f(\mathbf{M}_1)|}{D_1{m_k}^{\varepsilon}}\right)^2,\end{aligned}$$

where by partial summation we obtain

$$\sum_{n_1,n_2\leqslant y}\frac{\tau(n_1)\tau(n_2)}{n_1n_2}\ll\log^4 y\ll\log^4 T\ll T^{\varepsilon}.$$

Since $\varepsilon>0$ is arbitrary, using Lemma 2.4 we conclude that

$$\int_2 \ll T^{2k-1+\varepsilon}. \tag{4.11}$$

Then we turn to estimate $\int_1$, similar to $\int_2$, we have

$$\int_1 = \int_T^{2T}\frac{x^{2k-\frac{3}{2}}}{4\pi^2}\sum_{l_1,l_2=0}^{k-1}(\log x)^{l_1+l_2}\sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2)}{D_1D_2}$$

$$\times \sum_{\substack{n_1,n_2\leqslant y\\ n_1m_{2k}\neq n_2m_k}} \frac{f(\mathbf{M}_1)f(\mathbf{M}_2)}{{m_k}^{1/4}{m_{2k}}^{1/4}}\cdot\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\cos\left(4\pi\left(\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}}\right)\sqrt{x}\right)dx.$$

Let

$$G(x)=x^{2k-\frac{3}{2}}(\log x)^{l_1+l_2},\quad F(\cdot)=\cos(\cdot),\quad m(x)=4\pi(\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}})\sqrt{x},\quad a=T,\ \ b=2T$$

in Lemma 2.3, change the order of integration and summation, then we have

$$\begin{aligned}\int_1 =&\frac{1}{4\pi^2}\sum_{l_1,l_2=0}^{k-1}\sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2)}{D_1D_2}\cdot\frac{f(\mathbf{M}_1)f(\mathbf{M}_2)}{{m_k}^{1/4}{m_{2k}}^{1/4}}\sum_{\substack{n_1,n_2\leqslant y\\ n_1m_{2k}\neq n_2m_k}}\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\\ &\times\int_T^{2T}x^{2k-\frac{3}{2}}(\log x)^{l_1+l_2}\cos\left(4\pi\left(\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}}\right)\sqrt{x}\right)dx\\ \ll&\sum_{l_1,l_2=0}^{k-1}\sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{|C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2)|}{D_1D_2}\cdot\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{{m_k}^{1/4}{m_{2k}}^{1/4}}\sum_{\substack{n_1,n_2\leqslant y_2\\ n_1m_{2k}\neq n_2m_k}}\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\\ &\times T^{2k-\frac{3}{2}+\varepsilon}\cdot\frac{T^{\frac{1}{2}}}{|\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}}|}.\end{aligned}$$

Since

$$C_{l_1,l_2}(\mathbf{M}_1,\mathbf{M}_2)\ll\left(\prod_{j=1}^{2k}m_j\right)^{\varepsilon},$$

we have

$$\begin{aligned}\int_1\ll&\,T^{2k-1+\varepsilon}\sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{\left(\prod_{j=1}^{2k}m_j\right)^{\varepsilon}}{D_1D_2}\cdot\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{{m_k}^{1/4}{m_{2k}}^{1/4}}\sum_{\substack{n_1,n_2\leqslant y\\ n_1m_{2k}\neq n_2m_k}}\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\cdot\frac{1}{|\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}}|}\\ \ll&\,T^{2k-1+(2k+1)\varepsilon}\sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2{m_k}^{1/4}{m_{2k}}^{1/4}}\sum_{\substack{n_1,n_2\leqslant y\\ n_1m_{2k}\neq n_2m_k}}\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\cdot\frac{1}{|\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}}|}\\ :=&\,T^{2k-1+(2k+1)\varepsilon}\sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2{m_k}^{1/4}{m_{2k}}^{1/4}}\left(R_1+R_2\right),\end{aligned}\tag{4.12}$$

where

$$R_1=\sum_{\substack{n_1,n_2\leqslant y,\ n_1m_{2k}\neq n_2m_k\\ \left|\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}}\right|\leq\frac{(n_1n_2)^{1/4}}{10(m_km_{2k})^{1/4}}}}\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\cdot\frac{1}{|\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}}|},$$

$$R_2=\sum_{\substack{n_1,n_2\leqslant y,\ n_1m_{2k}\neq n_2m_k\\ \left|\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}}\right|>\frac{(n_1n_2)^{1/4}}{10(m_km_{2k})^{1/4}}}}\frac{\tau(n_1)\tau(n_2)}{{n_1}^{3/4}{n_2}^{3/4}}\cdot\frac{1}{|\sqrt{\frac{n_1}{m_k}}-\sqrt{\frac{n_2}{m_{2k}}}|}.$$

Then we have

$$R_2 \ll \sum_{n_1,n_2\leqslant y} \frac{\tau(n_1)\tau(n_2)}{n_1{}^{3/4}n_2{}^{3/4}} \cdot \frac{(m_km_{2k})^{1/4}}{(n_1n_2)^{1/4}} \ll (m_km_{2k})^{\frac{1}{4}} \log^4 T, \tag{4.13}$$

where we use partial summation to obtain

$$\sum_{n_1,n_2\leqslant y} \frac{\tau(n_1)\tau(n_2)}{n_1n_2} \ll \log^4 y \ll \log^4 T.$$

And for R_1, by Lagrange's mean value theorem we have

$$\sqrt{\beta_1} - \sqrt{\beta_2} \asymp (\sqrt{\beta_1\beta_2})^{-\frac{1}{2}}|\beta_1 - \beta_2|$$

for any

$$\beta_1 \asymp \beta_2 \in \mathbb{R},$$

thus let

$$\beta_1 = n_1/m_k, \quad \beta_2 = n_2/m_{2k}$$

in R_1, we obtain

$$\begin{aligned} R_1 &\ll \sum_{\substack{n_1,n_2\leqslant y \\ n_1m_{2k}\neq n_2m_k}} \frac{\tau(n_1)\tau(n_2)}{n_1{}^{3/4}n_2{}^{3/4}} \cdot \frac{1}{|\frac{n_1}{m_k} - \frac{n_2}{m_{2k}}|(\frac{m_km_{2k}}{n_1n_2})^{1/4}} \\ &= (m_km_{2k})^{\frac{3}{4}} \sum_{\substack{n_1,n_2\leqslant y \\ n_1m_{2k}\neq n_2m_k}} \frac{\tau(n_1)\tau(n_2)}{n_1^{1/2}n_2^{1/2}|n_1m_{2k} - n_2m_k|}. \end{aligned}$$

For some real numbers N_1, N_2 satisfying $1 \leqslant N_1, N_2 \leqslant y$, one has

$$\begin{aligned} R_1 &\ll (m_km_{2k})^{\frac{3}{4}} \log^2 y \sum_{\substack{N_1<n_1\leqslant 2N_1 \\ N_2<n_2\leqslant 2N_2 \\ n_1m_{2k}\neq n_2m_k}} \frac{\tau(n_1)\tau(n_2)}{(n_1n_2)^{1/2}|n_1m_{2k} - n_2m_k|} \\ &\ll \frac{(m_km_{2k})^{\frac{3}{4}}y^{\varepsilon}}{(N_1N_2)^{1/2}} \sum_{\substack{N_1<n_1\leqslant 2N_1 \\ N_2<n_2\leqslant 2N_2 \\ n_1m_{2k}\neq n_2m_k}} \frac{1}{|n_1m_{2k} - n_2m_k|}, \end{aligned} \tag{4.14}$$

we denote the latter sum by $T(m_k, m_{2k})$. Let

$$|n_1m_{2k} - n_2m_k| = r,$$

we have

$$r \equiv -n_1m_{2k} (\bmod m_k),$$

so we can find a constant c_0 such that

$$1 \leqslant c_0 < m_k, \quad r = m_kt + c_0,$$

where t is an integer such that $0 < t < 2y^2$, thus

$$\begin{aligned} T(m_k, m_{2k}) &= \sum_{N_1 < n_1 \leqslant 2N_1} \sum_{\substack{N_2 < n_2 \leqslant 2N_2 \\ n_1 m_{2k} \neq n_2 m_k}} \frac{1}{|n_1 m_{2k} - n_2 m_k|} \\ &\leqslant \sum_{N_1 < n_1 \leqslant 2N_1} \left(1 + \sum_{1 \leqslant t \leqslant 2y^2} \frac{1}{m_k t + c_0} \right) \\ &\ll N_1 \log y, \end{aligned}$$

similarly, we have

$$T(m_k, m_{2k}) \ll N_2 \log y,$$

thus

$$T(m_k, m_{2k}) \ll (N_1 N_2)^{\frac{1}{2}} \log y.$$

By (4.14) we have

$$R_1 \ll \frac{(m_k m_{2k})^{\frac{3}{4}} y^{\varepsilon}}{(N_1 N_2)^{1/2}} (N_1 N_2)^{\frac{1}{2}} \log y \ll (m_k m_{2k})^{\frac{3}{4}} T^{\varepsilon}. \tag{4.15}$$

Finally by (4.12), (4.13) and (4.15) we obtain

$$R_1 + R_2 \ll T^{\varepsilon} (m_k m_{2k})^{3/4},$$

thus

$$\begin{aligned} \int_1 &\ll T^{2k-1+(2k+1)\varepsilon} \sum_{m_1, \cdots, m_{2k} \leqslant y} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1 D_2 {m_k}^{1/4} {m_{2k}}^{1/4}} \cdot T^{\varepsilon} {m_k}^{3/4} {m_{2k}}^{3/4} \\ &\ll T^{2k-1+(2k+2)\varepsilon} \sum_{m_1, \cdots, m_{2k} \leqslant y} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)| {m_k}^{1/2} {m_{2k}}^{1/2}}{D_1 D_2} \\ &\ll T^{2k-1+(2k+2)\varepsilon} y \sum_{m_1, \cdots, m_{2k} \leqslant y} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1 D_2} \cdot \frac{T^{2\varepsilon}}{{m_k}^{\varepsilon} {m_{2k}}^{\varepsilon}} \\ &\ll T^{2k-1+(2k+4)\varepsilon} y \left(\sum_{m_1, \cdots, m_k = 1}^{\infty} \frac{|f(\mathbf{M}_1)|}{D_1 {m_k}^{\varepsilon}} \right)^2 \\ &\ll T^{2k-1+\varepsilon} y, \end{aligned} \tag{4.16}$$

since $\varepsilon > 0$ is arbitrary, $T^{\varepsilon} \ll y \ll T$, and the convergence of the latter series can be obtained by Lemma 2.4. Above all, by (4.4), (4.10), (4.11), and (4.16) we obtain

$$\begin{aligned} \int_T^{2T} {M_{11}}^2(x, y) dx =& \frac{k^2}{4\pi^2} \sum_{l_1, l_2 = 0}^{k-1} D_{k, l_1, l_2} \int_T^{2T} x^{2k-\frac{3}{2}} (\log x)^{l_1 + l_2} dx \\ &+ O\left(T^{2k-\frac{1}{2}+\varepsilon} y^{-\frac{1}{2}} \right) + O\left(T^{2k-1+\varepsilon} y \right). \end{aligned} \tag{4.17}$$

4.1.3. Other terms in (4.3)

We are going to estimate $\int_T^{2T} M_{12}(x,y)dx$,

$$\begin{aligned}\int_T^{2T} {M_{12}}^2(x,y)dx &= k^2\int_T^{2T}\left(\sum_{m_1,\cdots,m_k\leqslant y} f(\mathbf{M}_1)\prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right)\delta_2\left(\frac{x}{m_k},y\right)\right)^2 dx\\ &\ll \int_T^{2T}\sum_{m_1,\cdots,m_{2k}\leqslant y}|f(\mathbf{M}_1)||f(\mathbf{M}_2)|\left(\prod_{j=1}^{k-1}\frac{x\log x}{m_j}\prod_{j=k+1}^{2k-1}\frac{x\log x}{m_j}\right)\left|\delta_2\left(\frac{x}{m_k},y\right)\right|\left|\delta_2\left(\frac{x}{m_{2k}},y\right)\right|dx\\ &\ll \sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2}\int_T^{2T} x^{2k-2+\varepsilon}\left|\delta_2\left(\frac{x}{m_k},y\right)\right|\left|\delta_2\left(\frac{x}{m_{2k}},y\right)\right|dx\\ &\ll T^{2k-2+\varepsilon}\sum_{m_1,\cdots,m_{2k}\leqslant y}\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2}\left(\int_T^{2T}{\delta_2}^2\left(\frac{x}{m_k},y\right)dx\right)^{\frac12}\left(\int_T^{2T}{\delta_2}^2\left(\frac{x}{m_{2k}},y\right)dx\right)^{\frac12}.\end{aligned}$$

By Lemma 2.2 we have

$$\begin{aligned}\left(\int_T^{2T}{\delta_2}^2\left(\frac{x}{m_k},y\right)dx\right)^{\frac12} &= \left(m_k\int_{\frac{T}{m_k}}^{\frac{2T}{m_k}}{\delta_2}^2(u,y)\,du\right)^{\frac12}\\ &\ll \left(m_k\left(\frac{T^{\frac32}}{{m_k}^{\frac32}y^{\frac12}}\log^3 T+\frac{T}{m_k}\log^4 T\right)\right)^{\frac12}\\ &\ll \frac{T^{\frac34+\varepsilon}}{{m_k}^{\frac14}y^{\frac14}}+T^{\frac12+\varepsilon},\end{aligned}$$

similarly, we have

$$\left(\int_T^{2T}{\delta_2}^2\left(\frac{x}{m_{2k}},y\right)dx\right)^{\frac12}\ll\frac{T^{\frac34+\varepsilon}}{{m_{2k}}^{\frac14}y^{\frac14}}+T^{\frac12+\varepsilon}.$$

Thus,

$$\begin{aligned}\int_T^{2T}{M_{12}}^2(x,y)dx &\ll T^{2k-2+\varepsilon}\sum_{m1,\cdots,m_{2k}\leqslant y}\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2}\left(\frac{T^{\frac34+\varepsilon}}{{m_k}^{\frac14}y^{\frac14}}+T^{\frac12+\varepsilon}\right)\left(\frac{T^{\frac34+\varepsilon}}{{m_{2k}}^{\frac14}y^{\frac14}}+T^{\frac12+\varepsilon}\right)\\ &\ll T^{2k-\frac12+3\varepsilon}y^{-\frac12}\sum_{m1,\cdots,m_{2k}=1}^{\infty}\frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2{m_k}^{\frac14}{m_{2k}}^{\frac14}}\\ &\ll T^{2k-\frac12+3\varepsilon}y^{-\frac12},\end{aligned}\tag{4.18}$$

since $y\ll T$, and the convergence of the latter series is given by Lemma 2.4.

Then by (4.17), (4.18), and Cauchy-Schwarz's inequality, we are able to obtain

$$\int_T^{2T} M_{11}M_{12}dx\ll\left(\int_T^{2T}{M_{11}}^2dx\right)^{\frac12}\left(\int_T^{2T}{M_{12}}^2dx\right)^{\frac12}\ll\frac{T^{2k-\frac12+\varepsilon}}{y^{\frac14}}.\tag{4.19}$$

So from (4.3) and (4.17)–(4.19), we obtain

$$\int_T^{2T} {M_1}^2 dx = \frac{k^2}{4\pi^2} \sum_{l_1,l_2=0}^{k-1} D_{k,l_1,l_2} \int_T^{2T} x^{2k-\frac{3}{2}} (\log x)^{l_1+l_2} dx + O\left(T^{2k-\frac{1}{2}+\varepsilon} y^{-\frac{1}{4}}\right) + O\left(T^{2k-1+\varepsilon} y\right). \tag{4.20}$$

4.2. Error terms in (4.2)

Let $\mathbf{M_1}, \mathbf{M_2}, D_1, D_2$ be defined in Lemma 2.5, then for $\int_T^{2T} {M_3}^2 dx$ we deduce that

$$\int_T^{2T} {M_3}^2(x,y) dx = \int_T^{2T} \left(\sum_{\substack{m_1,\cdots,m_k \leqslant x \\ m_k > y}} |f(m_1,\cdots,m_k)| \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right) \left|\Delta\left(\frac{x}{m_k}\right)\right| \right)^2 dx$$

change order of integration and summation we obtain

$$\begin{aligned}
\int_T^{2T} {M_3}^2(x,y) dx &\leqslant \int_T^{2T} \sum_{\substack{m_1,\cdots,m_{2k} \leqslant 2T \\ m_k, m_{2k} > y}} |f(\mathbf{M}_1) f(\mathbf{M}_2)| \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right) \prod_{j=k+1}^{2k-1} M\left(\frac{x}{m_j}\right) \left|\Delta\left(\frac{x}{m_k}\right)\right| \left|\Delta\left(\frac{x}{m_{2k}}\right)\right| dx \\
&= \sum_{\substack{m_1,\cdots,m_{2k} \leqslant 2T \\ m_k, m_{2k} > y}} |f(\mathbf{M}_1) f(\mathbf{M}_2)| \int_T^{2T} \prod_{j=1}^{k-1} M\left(\frac{x}{m_j}\right) \prod_{j=k+1}^{2k-1} M\left(\frac{x}{m_j}\right) \left|\Delta\left(\frac{x}{m_k}\right)\right| \left|\Delta\left(\frac{x}{m_{2k}}\right)\right| dx \\
&\ll \sum_{\substack{m_1,\cdots,m_{2k} \leqslant 2T \\ m_k, m_{2k} > y}} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1 D_2} \int_T^{2T} x^{2k-2+\varepsilon} \left|\Delta\left(\frac{x}{m_k}\right)\right| \left|\Delta\left(\frac{x}{m_{2k}}\right)\right| dx \\
&\ll T^{2k-2+\varepsilon} \sum_{\substack{m_1,\cdots,m_{2k} \leqslant 2T \\ m_k, m_{2k} > y}} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1 D_2} \int_T^{2T} \left|\Delta\left(\frac{x}{m_k}\right)\right| \left|\Delta\left(\frac{x}{m_{2k}}\right)\right| dx.
\end{aligned}$$

Using Cauchy-Schwarz's inequality and (1.3), we deduce that

$$\begin{aligned}
\int_T^{2T} \left|\Delta\left(\frac{x}{m_k}\right)\right| \left|\Delta\left(\frac{x}{m_{2k}}\right)\right| dx &\ll \left(\int_T^{2T} \Delta^2\left(\frac{x}{m_k}\right) dx\right)^{\frac{1}{2}} \left(\int_T^{2T} \Delta^2\left(\frac{x}{m_{2k}}\right) dx\right)^{\frac{1}{2}} \\
&\ll \left(m_k \int_{\frac{T}{m_k}}^{\frac{2T}{m_k}} \Delta^2(u) du\right)^{\frac{1}{2}} \left(m_{2k} \int_{\frac{T}{m_{2k}}}^{\frac{2T}{m_{2k}}} \Delta^2(u) du\right)^{\frac{1}{2}} \\
&\ll \frac{T^{\frac{3}{2}}}{{m_k}^{\frac{1}{4}} {m_{2k}}^{\frac{1}{4}}},
\end{aligned}$$

thus, by Lemma 2.4, we have

$$\int_T^{2T} {M_3}^2(x,y) dx \ll T^{2k-\frac{1}{2}+\varepsilon} \sum_{\substack{m_1,\cdots,m_{2k} \leqslant 2T \\ m_k, m_{2k} > y}} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1 D_2 {m_k}^{\frac{1}{4}} {m_{2k}}^{\frac{1}{4}}}$$

$$\ll T^{2k-\frac{1}{2}+\varepsilon} \sum_{\substack{m_1,\cdots,m_{2k}\leqslant 2T\\ m_k,m_{2k}>y}} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2{m_k}^{\varepsilon}{m_{2k}}^{\varepsilon}}{m_k}^{-\frac{1}{4}+\varepsilon}{m_{2k}}^{-\frac{1}{4}+\varepsilon}$$

$$\ll T^{2k-\frac{1}{2}+\varepsilon}y^{-\frac{1}{2}+2\varepsilon} \sum_{\substack{m_1,\cdots,m_{2k}\leqslant 2T\\ m_k,m_{2k}>y}} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2{m_k}^{\varepsilon}{m_{2k}}^{\varepsilon}}$$

$$\ll T^{2k-\frac{1}{2}+\varepsilon}y^{-\frac{1}{2}} \sum_{m_1,\cdots,m_{2k}=1}^{\infty} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2{m_k}^{\varepsilon}{m_{2k}}^{\varepsilon}}$$

$$\ll \frac{T^{2k-\frac{1}{2}+\varepsilon}}{y^{\frac{1}{2}}}. \tag{4.21}$$

For $\int_T^{2T} {M_2}^2dx$, let ${D_1}^{'} = D_1/m_1$, ${D_2}^{'} = D_2/m_{k+1}$, similar to $\int_T^{2T} {M_3}^2dx$, we obtain by Lemma 2.4 that

$$\int_T^{2T} {M_2}^2(x,y)dx \ll T^{2k-\frac{1}{2}+\varepsilon} \sum_{\substack{m_1,\cdots,m_{2k}\leqslant 2T\\ m_1,m_{k+1}>y}} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{D_1D_2{m_k}^{\frac{1}{4}}{m_{2k}}^{\frac{1}{4}}}$$

$$\ll T^{2k-\frac{1}{2}+\varepsilon} \sum_{\substack{m_1,\cdots,m_{2k}\leqslant 2T\\ m_1,m_{k+1}>y}} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|{m_1}^{-\frac{1}{4}+\varepsilon}{m_{k+1}}^{-\frac{1}{4}+\varepsilon}}{{D_1}'{D_2}'{m_1}^{\frac{3}{4}+\varepsilon}{m_k}^{\frac{1}{4}}{m_{k+1}}^{\frac{3}{4}+\varepsilon}{m_{2k}}^{\frac{1}{4}}}$$

$$\ll T^{2k-\frac{1}{2}+\varepsilon}y^{-\frac{1}{2}+2\varepsilon} \sum_{\substack{m_1,\cdots,m_{2k}\leqslant 2T\\ m_1,m_{k+1}>y}} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{{D_1}'{D_2}'{m_1}^{\frac{3}{4}+\varepsilon}{m_k}^{\frac{1}{4}}{m_{k+1}}^{\frac{3}{4}+\varepsilon}{m_{2k}}^{\frac{1}{4}}}$$

$$\ll T^{2k-\frac{1}{2}+\varepsilon}y^{-\frac{1}{2}} \sum_{m_1,\cdots,m_{2k}=1}^{\infty} \frac{|f(\mathbf{M}_1)||f(\mathbf{M}_2)|}{{D_1}'{D_2}'{m_1}^{\frac{3}{4}+\varepsilon}{m_k}^{\frac{1}{4}}{m_{k+1}}^{\frac{3}{4}+\varepsilon}{m_{2k}}^{\frac{1}{4}}}$$

$$\ll \frac{T^{2k-\frac{1}{2}+\varepsilon}}{y^{\frac{1}{2}}}. \tag{4.22}$$

By (4.20)–(4.22) and Cauchy-Schwarz's inequality, we are able to estimate the following terms in (4.2)

$$\int_T^{2T} M_1M_2dx \ll \left(\int_T^{2T} {M_1}^2dx\right)^{\frac{1}{2}}\left(\int_T^{2T} {M_2}^2dx\right)^{\frac{1}{2}} \ll \frac{T^{2k-\frac{1}{2}+\varepsilon}}{y^{\frac{1}{4}}},$$

$$\int_T^{2T} M_1M_3dx \ll \left(\int_T^{2T} {M_1}^2dx\right)^{\frac{1}{2}}\left(\int_T^{2T} {M_3}^2dx\right)^{\frac{1}{2}} \ll \frac{T^{2k-\frac{1}{2}+\varepsilon}}{y^{\frac{1}{4}}},$$

$$\int_T^{2T} M_2M_3dx \ll \left(\int_T^{2T} {M_2}^2dx\right)^{\frac{1}{2}}\left(\int_T^{2T} {M_3}^2dx\right)^{\frac{1}{2}} \ll \frac{T^{2k-\frac{1}{2}+\varepsilon}}{y^{\frac{1}{2}}}. \tag{4.23}$$

Above all, taking $y = T^{\frac{2}{5}}$, then

$$\int_T^{2T} \left({\Delta_k}^*(x)\right)^2 dx = \frac{k^2}{4\pi^2}\sum_{l_1,l_2=0}^{k-1} D_{k,l_1,l_2}\int_T^{2T} x^{2k-\frac{3}{2}}(\log x)^{l_1+l_2}dx + O\left(T^{2k-\frac{3}{5}+\varepsilon}\right) \tag{4.24}$$

follows by (4.2) and (4.20)–(4.23).

5. Proof of the theorem

Using (3.5) and the Cauchy-Schwarz's inequality we obtain

$$\begin{aligned}\int_T^{2T} {\Delta_k}^2(x)dx &= \int_T^{2T} ({\Delta_k}^*(x))^2\, dx + O\left(\int_T^{2T} {\Delta_k}^*(x) x^{k-1+\varepsilon} dx\right) + O\left(T^{2k-1+\varepsilon}\right)\\ &= \frac{k^2}{4\pi^2}\sum_{l_1,l_2=0}^{k-1} D_{k,l_1,l_2}\int_T^{2T} x^{2k-\frac{3}{2}}(\log x)^{l_1+l_2} dx + O\left(T^{2k-\frac{3}{5}+\varepsilon}\right)\\ &\quad + O\left(\left(\int_T^{2T} ({\Delta_k}^*(x))^2\, dx\right)^{\frac{1}{2}}\left(\int_T^{2T} x^{2k-2+\varepsilon} dx\right)^{\frac{1}{2}}\right)\\ &= \frac{k^2}{4\pi^2}\sum_{l_1,l_2=0}^{k-1} D_{k,l_1,l_2}\int_T^{2T} x^{2k-\frac{3}{2}}(\log x)^{l_1+l_2} dx + O\left(T^{2k-\frac{3}{5}+\varepsilon}\right).\end{aligned}$$

Then replacing T by $T/2$, $T/2^2$, and so on, and adding up all the results, we obtain

$$\begin{aligned}\int_1^{T} {\Delta_k}^2(x)dx &= \frac{k^2}{4\pi^2}\sum_{l_1,l_2=0}^{k-1} D_{k,l_1,l_2}\int_1^{T} x^{2k-\frac{3}{2}}(\log x)^{l_1+l_2} dx + O\left(T^{2k-\frac{3}{5}+\varepsilon}\right)\\ &= \frac{k^2}{4\pi^2} T^{2k-\frac{1}{2}} L_{2k-2}(\log T) + O\left(T^{2k-\frac{3}{5}+\varepsilon}\right),\end{aligned}$$

where we use integration by part several times to obtain $L_{2k-2}(u)$ is a polynomial in u of degree $2k-2$ denoted by

$$L_{2k-2}(u) = \sum_{l_1,l_2=0}^{k-1} D_{k,l_1,l_2}\sum_{r=0}^{l_1+l_2}\frac{(-1)^r(l_1+l_2)!}{(2k-\frac{1}{2})^{r+1}(l_1+l_2-r)!}u^{l_1+l_2}.$$

To sum up, this finishes the proof of the Theorem.

6. Conclusions

In this paper, we give an asymptotic formula of the mean square of $\Delta_k(x)$, which can be viewed as an analogue of (1.3). We use the convergence of the multivariable Dirichlet series, and it can be used to show the properties of other multivariable arithmetic functions. In 2023, Tóth [13], Heyman and Tóth [14] gave some useful applications of the Dirichlet series.

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Conflict of interest

No potential conflicts of interest were reported by the author.

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