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Research article

Lipschitz estimate for elliptic equations with oscillatory coefficients

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Abstract: Regularity for elliptic equations with oscillatory coefficients was concerned. Problem domains were periodic and consisted of a connected region with normal permeability and a disconnected matrix block subset with high permeability. Coefficients of the elliptic equations depending on the permeability of the domains were highly oscillatory. Let $\epsilon \in (0, 1)$ be the periodic size of domain, $\epsilon\mu \in (0, 1)$ the size ratio of a matrix block to the whole domain, and $\omega^2 \in (1, \infty)$ the permeability ratio of the disconnected matrix block subset to the connected sub-region of the domains. This work presented Lipschitz estimate uniformly in ϵ, μ, ω for the Green's functions and the solutions of the elliptic equations.

Keywords: Lipschitz estimate; permeability; periodic size; Green's function; oscillatory coefficient
Mathematics Subject Classification: 35J05, 35J15, 35J25, 35J70

1. Introduction

Regularity for elliptic equations with oscillatory coefficients is concerned. Problem domains are periodic and consist of a connected region with normal permeability and a disconnected matrix block subset with high permeability. Let $Y \equiv [\frac{-1}{2}, \frac{1}{2})^n$ for $n \geq 3$; $Y_{\mu,m} (= B_{\mu/4}(0))$ be a ball centered at 0 and with radius $\mu/4$ for $\mu \in (0, 1)$; $Y_{\mu,f} \equiv Y \setminus Y_{\mu,m}$; Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$; $\mathcal{I}_\epsilon \equiv \{\mathbf{j} \in \mathbb{Z}^n | \ \epsilon(Y + \mathbf{j}) \subset \Omega\}$ for $\epsilon \in (0, 1)$; $\Omega^\epsilon_{\mu,m} \equiv \cup_{\mathbf{j}\in\mathcal{I}_\epsilon} \epsilon(Y_{\mu,m} + \mathbf{j})$ be a disconnected subset of Ω; $\Omega^\epsilon_{\mu,f}$ $(\equiv \Omega \setminus \Omega^\epsilon_{\mu,m})$ be a connected sub-region of Ω. Here, ϵ is the periodic size of domain; $\epsilon\mu$ is the size ratio of a matrix block to the whole domain. Let $\omega^2 \in (1, \infty)$ denote the permeability ratio of the disconnected matrix block subset to the connected sub-region of Ω. For any $\sigma, \mu, \varpi > 0$, define $\mathbf{E}^\sigma_{\varpi,\mu} \equiv \mathcal{X}_{\Omega^\sigma_{\mu,f}} + \varpi\mathcal{X}_{\Omega^\sigma_{\mu,m}}$ (here, $\mathcal{X}_\mathbf{D}$ is the characteristic function on $\mathbf{D}$). The problem that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{E}^\epsilon_{\omega^2,\mu} \nabla\Phi) = F & \text{in } \Omega, \\ \Phi = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\epsilon, \mu \in (0, 1)$ and $\omega \in (1, \infty)$. Problem (1.1) contains strongly elliptic equations with oscillatory coefficients; it has applications in flows in fractured media, contaminant flow problems, and the stress in composite materials [4,8,11,19,31]. It is clear that a solution of problem (1.1) for each ϵ, μ, ω exists uniquely in $H^1(\Omega)$ if F, g are smooth. However, it is not clear how the regularity of the solutions of problem (1.1) depends on the parameters ϵ, μ, ω. This work presents the Lipschitz estimate uniformly in ϵ, μ, ω for the solutions of problem (1.1).

Regularity for the uniform linear elliptic equations (e.g., $\omega \in [d_1, d_2]$, $d_1 > 0$, and ϵ, μ fixed in (1.1)) with smooth coefficients was investigated extensively in [13, 15, 16, 24]. Regularity for the uniform elliptic equations with non-smooth coefficients can be found in [21,22,27]. Hölder, $W^{1,p}$, and Lipschitz estimates uniformly in ϵ for uniform elliptic equations with Hölder continuous periodic coefficients (e.g., $\omega \in [d_1, d_2]$, $d_1 > 0$, $\epsilon \in (0, 1)$, and μ fixed in (1.1)) and with Dirichlet or Neumann boundary were derived in [2, 3, 20, 23, 32].

For nonuniform linear elliptic equations, regularity results for diffraction problems (e.g., ϵ, μ fixed in (1.1)) are available in [12, 17, 19, 24, 26, 29] and references therein. Hölder and $W^{1,p}$ estimates uniformly in ϵ, μ for the nonuniform elliptic equations (e.g., $\epsilon, \omega \in (0, 1)$ and μ fixed in (1.1)) with Neumann boundary as well as Lipschitz estimate for the non-uniform elliptic equations with Dirichlet boundary were shown in [33, 34]. Elliptic equations with highly oscillatory coefficients (e.g., $(1.1)_1$ with $\epsilon \in (0, 1)$, $\omega \in (0, \infty)$, and $\mu = 1$) were studied in [30], where the Lipschitz estimate uniformly in ϵ, ω for the fundamental solutions and uniform interior Lipschitz estimate for the elliptic solutions were derived.

Quasi-linear elliptic equations with high-contrast coefficients (whose solutions are minimizers of convex functionals with (p, q)-growth) may arise from compressible flows in porous media and non-Newtonian flow through thin fissures [28]. Homogenization for quasi-linear elliptic equations in heterogeneous media (corresponding to the $\epsilon, \omega \in (0, 1)$ case) was studied in [28]. Some $C^{1,\alpha}$ and $W^{2,p}$ estimates of the minimizers of convex functionals with (p, q)-growth in homogeneous media can be found in [5, 6, 10, 18].

The coefficients of problem (1.1) under $\epsilon, \mu \in (0, 1)$ and $\omega \in (1, \infty)$ are globally discontinuous and highly oscillatory as ϵ closes to 0, but the coefficients are locally smooth in the connected sub-region $\Omega^\epsilon_{\mu,f}$ as well as in each cell $\epsilon(Y_{\mu,m} + \mathbf{j})$ of the matrix block subset $\Omega^\epsilon_{\mu,m}$. This work presents the Lipschitz estimate uniformly in ϵ, μ, ω for the Green's functions and the solutions of the strongly elliptic Eq (1.1). We find that the external sources F, g do not generate oscillatory solutions for (1.1) and that the maximum norm of the gradient of elliptic solutions in the discontinuous matrix block subset of the problem domains can be very small. These results are different from those in [33, 34]. In the latter, $\omega \in (0, 1)$; elliptic solutions can be oscillatory; and the maximum norm of the gradient of elliptic solutions in the discontinuous matrix block subset of domains can be very large.

Our results are proved by employing a compactness argument in [2, 3] and ideas from [30]. More precisely, consider the homogenization problems of Eq (1.1) first. Next, find regularity properties satisfied by the solutions of the homogenized equations and show, by an iteration argument, that these properties are also satisfied by the solutions of (1.1) from macroscopic scale to some level of microscopic scale. Then, derive local a priori estimates to explain that the regularity properties are satisfied by the solutions of (1.1) in all scales. In order to get above a priori estimates, the first step is to locally flatten the boundaries of the matrix blocks of the domains and to derive estimates for the elliptic solutions around the boundaries of the matrix blocks. Next is to study the Lipschitz estimates for the diffraction problem of each matrix block. Since the elliptic solutions change rapidly around the boundary of the domains, to obtain the Lipschitz estimate around the boundary, we need estimates uniformly in ϵ, μ, ω for the Green's functions and the corrector functions of the elliptic equations. Lipschitz estimate for (1.1) can be shown by applying partition of unity and these local *a priori* estimates.

The rest of this work is organized as follows: Notations and main results are stated in Section 2. In Section 3, we derive uniform Lipschitz estimates for diffraction problems and a uniform estimate for strongly elliptic equations. Section 4 is to consider the Hölder estimate for elliptic equations with oscillatory coefficients. Interior Lipschitz estimate for the elliptic equations is obtained in Section 5. Section 6 is to study the boundary Lipschitz estimates for elliptic solutions. Main results (which are Lipschitz estimates for the Green's functions and the solutions of the elliptic equations) are proved in Section 7. Section 8 shows an estimate for a diffraction problem (i.e., proof of Lemma 3.5).

2. Notations and main results

$C^{k,\alpha}$, L^p, $W^{k,p}$, H^k, and $L^{p,\lambda}$ are the Hölder space, Lebesgue space, Sobolev space, Hilbert space, and Morrey space, respectively [13]. C_0^∞ contains C^∞ functions with compact support; $H^1_{loc}(\mathbb{R}^n)$ contains local H^1 functions; $H^1_\#(\mathbb{R}^n) \subset H^1_{loc}(\mathbb{R}^n)$ contains periodic functions with period Y. $[\zeta]_{C^\alpha(\mathbf{D})} \equiv \sup\limits_{x,y\in\mathbf{D}} \frac{|\zeta(x)-\zeta(y)|}{|x-y|^\alpha}$ is the C^α semi-norm of ζ in $\mathbf{D}$ and $supp(\zeta)$ is the support of ζ. For any set $\mathbf{D}$ and $r > 0$, $\mathbf{D}/r = \frac{1}{r}\mathbf{D} \equiv \{x|\ rx \in \mathbf{D}\}$; $|\mathbf{D}|$ is the volume of $\mathbf{D}$; $\mathbf{D}_1 \Subset \mathbf{D}_2$ means that $\mathbf{D}_1$ is a compact subset of $\mathbf{D}_2$;

$$(\zeta)_{\mathbf{D}} \equiv \fint_{\mathbf{D}} \zeta \, dx \equiv \frac{1}{|\mathbf{D}|} \int_{\mathbf{D}} \zeta \, dx.$$

$\beta^x_r \equiv dist(x, \partial\Omega/r)$ denotes the distance from x to the boundary $\partial\Omega/r$; if $r = 1$, set $\beta^x \equiv \beta^x_r$. Suppose U, V are two vectors, and $\langle U, V\rangle$ denotes the inner product of U and V. For any $r > 0$, $B_r(x)$ is a ball centered at x with radius r, $\mathbf{E}^{\epsilon,r}_{\omega,\mu}(x) \equiv \mathbf{E}^{\epsilon}_{\omega,\mu}(rx)$, and $\vec{\mathbf{n}}^{,r}(x) \equiv \vec{\mathbf{n}}(rx)$ is the unit outward normal vector on $\partial\Omega/r$. For any $\sigma, \mu, \varpi > 0$, define $O^\sigma_{\mu,m} \equiv \cup_{\mathbf{j}\in\mathbb{Z}^n}\sigma(Y_{\mu,m}+\mathbf{j})$, $O^\sigma_{\mu,f} \equiv \mathbb{R}^n \backslash O^\sigma_{\mu,f}$, and $\mathbf{K}^\sigma_{\varpi,\mu}(x) \equiv \mathcal{X}_{O^\sigma_{\mu,f}} + \varpi\mathcal{X}_{O^\sigma_{\mu,m}}$. Set $\mathbf{K}_{\varpi,\mu}(x) \equiv \mathbf{K}^\sigma_{\varpi,\mu}(x)$ if $\sigma = 1$. Let us make the following statements:

A1. $\omega \in (1, \infty)$, $\mu \in (0, 1)$, $n \geq 3$, $\omega^2\mu^n \leq 1 \leq \omega^2\mu$,
A2. $\epsilon \in (0, 1)$, $\alpha \in (0, 1)$,
A3. Ω is a $C^{1,\alpha}$ connected domain.

If $x \in \Omega$, a Green's function $\Gamma^\epsilon_{\omega,\mu}(x, \cdot)$ for the elliptic operator $-\nabla \cdot (\mathbf{E}^\epsilon_{\omega^2,\mu}\nabla)$ in Ω is the solution of

$$\begin{cases} -\nabla_y \cdot (\mathbf{E}^\epsilon_{\omega^2,\mu}\nabla_y\Gamma^\epsilon_{\omega,\mu}(x, \cdot)) = \delta(x, \cdot) & \text{in } \Omega, \\ \Gamma^\epsilon_{\omega,\mu}(x, \cdot) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\delta(x, \cdot)$ is the Dirac delta function with pole at x. Under A1–A3, a Green's function $\Gamma^\epsilon_{\omega,\mu}(x, \cdot) \in W^{1,1}(\Omega)$ exists uniquely and $\Gamma^\epsilon_{\omega,\mu}(x, y) = \Gamma^\epsilon_{\omega,\mu}(y, x)$ for $x, y \in \Omega$ [25]. Moreover, we prove the following.

Theorem 2.1. *Under A1–A3, there is a constant c independent of ϵ, μ, ω such that any Green's function for problem (2.1) satisfies*

$$\begin{cases} \left|\Gamma^\epsilon_{\omega,\mu}(x, y)\right| \leq c|x - y|^{2-n} \\ \left|\nabla_y\Gamma^\epsilon_{\omega,\mu}(x, y)\right| + \left|\nabla_x\Gamma^\epsilon_{\omega,\mu}(x, y)\right| \leq c|x - y|^{1-n} \end{cases} \quad \textit{for } x, y \in \Omega.$$

Lipschitz estimate uniformly in ϵ is the best possible estimate for the solutions of uniform elliptic equations with periodic coefficients [3, 20, 32]. Here, we show

Theorem 2.2. *Under A1–A3 and $g \in C^{1,\alpha}(\partial\Omega)$, there is a constant c independent of ϵ, μ, ω such that*

(1) if $F \in L^p(\Omega)$ for $p > n$, then

$$\|\nabla\Phi\|_{L^\infty(\Omega)} \le c\left(\|F\|_{L^p(\Omega)} + \|g\|_{C^{1,\alpha}(\partial\Omega)}\right), \tag{2.2}$$

(2) if $F \in L^p(\Omega)$ for $p > n$ and $F \in L^{2,n+2\lambda-2}(\Omega)$ for $\lambda \in (0,1)$, then

$$\|\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla\Phi\|_{L^\infty(\Omega)} \le c\left(\|F\|_{L^p(\Omega)} + \|g\|_{C^{1,\alpha}(\partial\Omega)} + |\epsilon\mu|^{\lambda}\sup_{\mathbf{j}\in\mathcal{I}_\epsilon}\|F\|_{L^{2,n+2\lambda-2}(\epsilon(Y+\mathbf{j}))}\right). \tag{2.3}$$

Theorem 2.2 implies that the external sources F, g do not generate oscillatory solutions for problem (1.1) and that the maximum norm of the gradient of elliptic solutions in the discontinuous matrix block subset of the problem domains can be very small. Theorems 2.1 and 2.2 are proved in §7.

3. Diffraction problems and strongly elliptic equations

This section derives uniform Lipschitz estimates for diffraction problems and a uniform estimate for strongly elliptic equations.

3.1. Diffraction problems

Set $\varpi > 0$, $z = (z_1, \cdots, z_{n-1}, z_n) = (z', z_n) \in \mathbb{R}^n$, and

$$\begin{cases} \mathbf{T}_\varpi(x) \equiv \mathcal{X}_{\{(z',z_n)|z_n\ge 0\}}(x) + \varpi\mathcal{X}_{\{(z',z_n)|z_n<0\}}(x), \\ \mathbb{K}_\varpi(x) \equiv \mathbb{A}\mathcal{X}_{\{(z',z_n)|z_n\ge 0\}}(x) + \varpi\mathbb{A}\mathcal{X}_{\{(z',z_n)|z_n<0\}}(x). \end{cases}$$

By [35, Lemma 4.9], and the Poincaré inequality, we see

Lemma 3.1. *If $\varpi, \lambda \in (0,1)$, $\mathbf{e} \in [0,2]$, and $0 < \mathbf{m}_1 < \mathbb{A} \in C^{1,0}(B_2(0))$, there is a constant c independent of $\varpi, \mathbf{e}$ such that any solution of*

$$-\nabla\cdot(\mathbb{K}_{\varpi^2}\nabla\Psi) = G \qquad in\ B_2(0)$$

satisfies

$$\|\mathbf{T}_{\varpi^{\mathbf{e}}}\nabla\Psi\|_{L^\infty(B_1(0))} \le c\left(\min\{\|\mathbf{T}_{\varpi^{\mathbf{e}}}\Psi\|_{L^2(B_2(0))},\ \|\mathbf{T}_{\varpi^{\mathbf{e}}}\nabla\Psi\|_{L^2(B_2(0))}\} + \|\mathbf{T}_{\varpi^{\mathbf{e}-2}}G\|_{L^{2,n+2\lambda-2}(B_2(0))}\right).$$

By change of variables, Lemma 3.1 implies the following result.

Lemma 3.2. *If $\omega \in (1,\infty)$, $\lambda \in (0,1)$, $\mathbf{e} \in [0,2]$, and $0 < \mathbf{m}_1 < \mathbb{A} \in C^{1,0}(B_2(0))$, there is a constant c independent of $\omega, \mathbf{e}$ such that any solution of*

$$-\nabla\cdot(\mathbb{K}_{\omega^2}\nabla\Psi) = G \qquad in\ B_2(0)$$

satisfies

$$\|\mathbf{T}_{\omega^{\mathbf{e}}}\nabla\Psi\|_{L^\infty(B_1(0))} \le c\left(\min\{\|\mathbf{T}_{\omega^{\mathbf{e}}}\Psi\|_{L^2(B_2(0))},\ \|\mathbf{T}_{\omega^{\mathbf{e}}}\nabla\Psi\|_{L^2(B_2(0))}\} + \|\mathbf{T}_{\omega^{\mathbf{e}-2}}G\|_{L^{2,n+2\lambda-2}(B_2(0))}\right).$$

Define $H^1(\mathbf{D})/\mathbb{R} \equiv \{\zeta \in H^1(\mathbf{D}) |\ (\zeta)_{\mathbf{D}} = 0\}$ for any set $\mathbf{D}$. See §2 for $(\zeta)_{\mathbf{D}}$. We have the following extension result.

Lemma 3.3. *If $\mu \in (0,1)$, there is a mapping $\Pi_\mu : H^1(Y_{\mu,m})/\mathbb{R} \to H_0^1(2Y_{\mu,m})$ such that*

$$\begin{cases} \Pi_\mu \phi = \phi & \text{in } Y_{\mu,m} \\ \|\Pi_\mu \phi\|_{H^1(2Y_{\mu,m})} \leq c \|\nabla \phi\|_{L^2(Y_{\mu,m})} \end{cases} \quad \text{for any } \phi \in H^1(Y_{\mu,m})/\mathbb{R},$$

where c is a constant independent of μ.

Proof. Let c be a constant independent of μ. By [15, Theorem 7.25] and the Poincaré inequality, there is a linear mapping $\Pi : H^1(\frac{1}{\mu} Y_{\mu,m})/\mathbb{R} \to H_0^1(\frac{2}{\mu} Y_{\mu,m})$ such that

$$\begin{cases} \Pi\zeta = \zeta & \text{in } \frac{1}{\mu} Y_{\mu,m} \\ \|\Pi\zeta\|_{H^1(\frac{2}{\mu} Y_{\mu,m})} \leq c \|\nabla \zeta\|_{L^2(\frac{1}{\mu} Y_{\mu,m})} \end{cases} \quad \text{for any } \zeta \in H^1(\tfrac{1}{\mu} Y_{\mu,m})/\mathbb{R}.$$

Define a mapping $\Pi_\mu : H^1(Y_{\mu,m})/\mathbb{R} \to H_0^1(2Y_{\mu,m})$ as follows: Set $\zeta(x) \equiv \phi(\mu x)$ for $\phi \in H^1(Y_{\mu,m})/\mathbb{R}$ and $x \in \frac{1}{\mu} Y_{\mu,m}$, and set $\Pi_\mu \phi(y) \equiv \Pi\zeta(\frac{1}{\mu} y)$ for $\Pi\zeta \in H^1(\frac{2}{\mu} Y_{\mu,m})$ and $y \in 2Y_{\mu,m}$. Then,

$$\begin{cases} \Pi_\mu \phi = \phi & \text{in } Y_{\mu,m}, \\ \|\Pi_\mu \phi\|_{H^1(2Y_{\mu,m})} \leq c\mu^{\frac{n}{2}-1} \|\Pi\zeta\|_{H^1(\frac{2}{\mu} Y_{\mu,m})} \leq c\mu^{\frac{n}{2}-1} \|\nabla\zeta\|_{L^2(\frac{1}{\mu} Y_{\mu,m})} \leq c\|\nabla\phi\|_{L^2(Y_{\mu,m})}. \end{cases}$$

So, we prove the lemma. □

Next is a local L^2-gradient estimate for elliptic solutions inside the set Y. The idea is from [30].

Lemma 3.4. *If $\omega \in (1,\infty)$ and $\mu \in (0,1)$, any solution $\Psi \in H^1(Y)$ of*

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\mu} \nabla \Psi) = G \qquad \text{in } Y \tag{3.1}$$

satisfies

$$\|\omega^2 \nabla \Psi\|_{L^2(Y_{\mu,m})} \leq c \left(\|\nabla \Psi\|_{L^2(2Y_{\mu,m} \setminus Y_{\mu,m})} + \|G\|_{H^{-1}(2Y_{\mu,m})} \right),$$

where c is a constant independent of μ, ω. See §2 for $\mathbf{K}_{\omega^2,\mu}$.

Proof. Let c be a constant independent of μ, ω. Take a constant h so that the average is $(\Psi - h)_{Y_{\mu,m}} = 0$. By Lemma 3.3, there is a $\zeta \in H_0^1(2Y_{\mu,m})$ satisfying

$$\begin{cases} \zeta = \Psi - h & \text{in } Y_{\mu,m}, \\ \|\zeta\|_{H^1(2Y_{\mu,m})} \leq c\|\nabla\Psi\|_{L^2(Y_{\mu,m})}. \end{cases} \tag{3.2}$$

Test (3.1) against ζ to get

$$\int_{2Y_{\mu,m}} \mathbf{K}_{\omega^2,\mu} \nabla\Psi \nabla\zeta \, dx = \int_{2Y_{\mu,m}} G\zeta \, dx.$$

Then, we have

$$\omega^2 \|\nabla\Psi\|^2_{L^2(Y_{\mu,m})} \leq c \left(\|\nabla\Psi\|_{L^2(2Y_{\mu,m} \setminus Y_{\mu,m})} + \|G\|_{H^{-1}(2Y_{\mu,m})} \right) \|\nabla\zeta\|_{L^2(2Y_{\mu,m})}.$$

Together with (3.2), we prove the lemma. □

Lemma 3.5. *If $\omega \in (1,\infty)$, $\mu \in (0,1)$, $\omega^2\mu \geq 1$, and $p > n$, any solution of*

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\mu}\nabla\Psi) = G \qquad \text{in } Y \tag{3.3}$$

satisfies $\|\nabla\Psi\|_{L^\infty(B_{2/5}(0))} \leq c(\|\Psi\|_{L^2(Y\setminus B_{1/4}(0))} + \|\mathbf{K}_{1/\omega^2,\mu}G\|_{L^p(Y)})$, where c is a constant independent of μ,ω. See §2 for $\mathbf{K}_{1/\omega^2,\mu}$.

Proof of Lemma 3.5 is given in §8.

Lemma 3.6. *If $\omega \in (1,\infty)$, $\mu,\lambda \in (0,1)$, $\omega^2\mu \geq 1$, and $p > n$, any solution of*

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\mu}\nabla\Psi) = G \qquad \text{in } Y$$

satisfies

$$\|\mathbf{K}_{\omega^2,\mu}\nabla\Psi\|_{L^\infty(2Y_{\mu,m})} \leq c\left(\|\Psi\|_{L^2(Y\setminus B_{1/4}(0))} + \|\mathbf{K}_{1/\omega^2,\mu}G\|_{L^p(Y)} + \mu^\lambda\|G\|_{L^{2,n+2\lambda-2}(8Y_{\mu,m})}\right),$$

where c is a constant independent of μ,ω.

Proof. Let c be independent of μ,ω. If $\zeta(x) = \frac{1}{\mu}\Psi(\mu x)$ and $\phi(x) = \mu G(\mu x)$, then

$$-\nabla \cdot (\mathbf{K}^{1/\mu}_{\omega^2,\mu}\nabla\zeta) = \phi \qquad \text{in } \tfrac{1}{\mu}Y.$$

By partition of unity, the argument in [1, pages 3964 and 3965], Lemma 3.2 with $\mathbf{e} = 2$, and the Poincaré inequality,

$$\begin{aligned}\|\mathbf{K}^{1/\mu}_{\omega^2,\mu}\nabla\zeta\|_{L^\infty(\frac{2}{\mu}Y_{\mu,m})} &\leq c(\|\mathbf{K}^{1/\mu}_{\omega^2,\mu}\zeta\|_{L^2(\frac{4}{\mu}Y_{\mu,m})} + \|\phi\|_{L^{2,n+2\lambda-2}(\frac{4}{\mu}Y_{\mu,m})})\\ &\leq c(\|\mathbf{K}^{1/\mu}_{\omega^2,\mu}\nabla\zeta\|_{L^2(\frac{4}{\mu}Y_{\mu,m})} + \|\phi\|_{L^{2,n+2\lambda-2}(\frac{4}{\mu}Y_{\mu,m})}).\end{aligned} \tag{3.4}$$

(3.4) can be written as, by Lemmas 3.4 and 3.5,

$$\begin{aligned}\|\mathbf{K}_{\omega^2,\mu}\nabla\Psi\|_{L^\infty(2Y_{\mu,m})} &\leq c\left(\mu^{\frac{-n}{2}}\|\mathbf{K}_{\omega^2,\mu}\nabla\Psi\|_{L^2(4Y_{\mu,m})} + \mu^\lambda\|G\|_{L^{2,n+2\lambda-2}(4Y_{\mu,m})}\right)\\ &\leq c\left(\mu^{\frac{-n}{2}}\|\nabla\Psi\|_{L^2(8Y_{\mu,m}\setminus Y_{\mu,m})} + \mu^{\frac{-n}{2}}\|G\|_{H^{-1}(8Y_{\mu,m})} + \mu^\lambda\|G\|_{L^{2,n+2\lambda-2}(4Y_{\mu,m})}\right)\\ &\leq c\left(\|\nabla\Psi\|_{L^\infty(8Y_{\mu,m}\setminus Y_{\mu,m})} + \mu^{1-\frac{n}{2}}\|G\|_{L^2(8Y_{\mu,m})} + \mu^\lambda\|G\|_{L^{2,n+2\lambda-2}(4Y_{\mu,m})}\right)\\ &\leq c\left(\|\Psi\|_{L^2(Y\setminus B_{1/4}(0))} + \|\mathbf{K}_{1/\omega^2,\mu}G\|_{L^p(Y)} + \mu^\lambda\|G\|_{L^{2,n+2\lambda-2}(8Y_{\mu,m})}\right).\end{aligned} \tag{3.5}$$

Lemma 3.6 follows from (3.5). □

3.2. Strongly elliptic equations

By Lemma 3.2 in [19, page 88], we have the following.

Lemma 3.7. *Suppose $\epsilon,\mu,r,\frac{\epsilon}{r} \in (0,1)$. There is a constant c (independent of ϵ,μ,r) and a linear continuous mapping $\widetilde{\Pi}_{\epsilon/r} : H^1(\frac{\epsilon}{r}(2Y_{\mu,m}\setminus Y_{\mu,m})) \to H^1(\frac{2\epsilon}{r}Y_{\mu,m})$ such that if $\phi \in H^1(\frac{\epsilon}{r}(2Y_{\mu,m}\setminus Y_{\mu,m}))$, then*

$$\begin{cases}\widetilde{\Pi}_{\epsilon/r}\phi = \phi \quad \text{in } \frac{\epsilon}{r}(2Y_{\mu,m}\setminus Y_{\mu,m}),\\ \|\widetilde{\Pi}_{\epsilon/r}\phi\|_{L^2(\frac{2\epsilon}{r}Y_{\mu,m})} \leq c\|\phi\|_{L^2(\frac{\epsilon}{r}(2Y_{\mu,m}\setminus Y_{\mu,m}))},\\ \|\nabla\widetilde{\Pi}_{\epsilon/r}\phi\|_{L^2(\frac{2\epsilon}{r}Y_{\mu,m})} \leq c\|\nabla\phi\|_{L^2(\frac{\epsilon}{r}(2Y_{\mu,m}\setminus Y_{\mu,m}))}.\end{cases}$$

Lemma 3.8. *Suppose A1–A3, $\frac{\epsilon}{r}, r, R \in (0,1)$, and $\frac{\epsilon}{r} < R$. There is a constant c independent of $\epsilon, \mu, \omega, r, R$ such that*

(1) if $0 \in \partial\Omega/r$, $\Psi_b \in C^1(B_{2R}(0) \cap \partial\Omega/r)$, and

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu} \nabla \Psi) = 0 & \text{in } B_{2R}(0) \cap \Omega/r, \\ \Psi = \Psi_b & \text{on } B_{2R}(0) \cap \partial\Omega/r, \end{cases} \tag{3.6}$$

then

$$\|\mathbf{E}^{\epsilon,r}_{\omega^2,\mu} \nabla \Psi\|_{L^2(B_R(0)\cap\Omega/r)} \le c\left(R^{-1}\|\Psi\|_{L^2(B_{2R}(0)\cap\Omega/r)} + \|\Psi_b\|_{C^1(B_{2R}(0)\cap\partial\Omega/r)}\right), \tag{3.7}$$

(2) if $B_{2R}(0) \Subset \Omega/r$ and $-\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu} \nabla \Psi) = 0$ in $B_{2R}(0)$, then

$$\|\mathbf{E}^{\epsilon,r}_{\omega^2,\mu} \nabla \Psi\|_{L^2(B_R(0))} \le cR^{-1}\|\Psi\|_{L^2(B_{2R}(0))}. \tag{3.8}$$

Proof. Let c be a constant independent of $\epsilon, \mu, \omega, r, R$.
Step I. Let $\eta \in C_0^\infty(B_{\frac{3}{2}R}(0))$ denote a bell-shaped function with $\eta \in [0,1]$, $\eta = 1$ in $B_R(0)$ and $\|\nabla \eta\|_{L^\infty(B_{\frac{3}{2}R}(0)} \le \frac{c}{R}$. By Lemma 3.4,

$$\|\mathbf{E}^{\epsilon,r}_{\omega^2,\mu} \nabla \Psi\|_{L^2(B_{R/2}(0)\cap\Omega/r)} \le c\|\nabla \Psi\|_{L^2(B_R(0)\cap\Omega/r)}. \tag{3.9}$$

If $\Psi_b \in C^1(B_{2R}(0) \cap \partial\Omega/r)$, there is a function $\zeta \in C^1(B_{2R}(0) \cap \Omega/r)$ such that $\Psi_b = \zeta$ on $B_{\frac{3}{2}R}(0) \cap \partial\Omega/r$ and $\|\zeta\|_{C^1(B_{\frac{3}{2}R}(0)\cap\Omega/r)} \le \|\Psi_b\|_{C^1(B_{2R}(0)\cap\partial\Omega/r)}$ by [15, Lemma 6.38]. Test (3.6) against $(\Psi - \zeta)\eta^2$ to see, by A1,

$$\begin{aligned} \|\mathbf{E}^{\epsilon,r}_{\omega,\mu} \nabla \Psi\|^2_{L^2(B_R(0)\cap\Omega/r)} &\le c\|\mathbf{E}^{\epsilon,r}_{\omega,\mu}(\Psi - \zeta)\nabla\eta\|^2_{L^2(B_{\frac{3}{2}R}(0)\cap\Omega/r)} + c\|\mathbf{E}^{\epsilon,r}_{\omega,\mu}\nabla\zeta\|^2_{L^2(B_{\frac{3}{2}R}(0)\cap\Omega/r)} \\ &\le c\left(R^{-2}\|\mathbf{E}^{\epsilon,r}_{\omega,\mu}\Psi\|^2_{L^2(B_{2R}(0)\cap\Omega/r)} + \|\Psi_b\|^2_{C^1(B_{2R}(0)\cap\partial\Omega/r)}\right). \end{aligned} \tag{3.10}$$

Step II. If $\frac{\epsilon}{r}(Y_{\mu,m} + \mathbf{j}) \subset B_{2R}(0) \cap \Omega/r$, we see, by Lemmas 3.7 and 3.5 and A1,

$$\begin{aligned} \|\omega\Psi\|_{L^2(\frac{\epsilon}{r}(Y_{\mu,m}+\mathbf{j}))} &\le \omega\left(\|\widetilde{\Pi}_{\epsilon/r}\Psi\|_{L^2(\frac{\epsilon}{r}(Y_{\mu,m}+\mathbf{j}))} + \|\Psi - \widetilde{\Pi}_{\epsilon/r}\Psi\|_{L^2(\frac{\epsilon}{r}(Y_{\mu,m}+\mathbf{j}))}\right) \\ &\le c\,\omega\left(\|\Psi\|_{L^2(\frac{\epsilon}{r}(2Y_{\mu,m}\setminus Y_{\mu,m}+\mathbf{j}))} + \frac{\epsilon\mu}{r}\|\nabla(\Psi - \widetilde{\Pi}_{\epsilon/r}\Psi)\|_{L^2(\frac{\epsilon}{r}(Y_{\mu,m}+\mathbf{j}))}\right) \\ &\le c\,\omega\left(\left|\frac{\epsilon\mu}{r}\right|^{\frac{n}{2}}\|\Psi\|_{L^\infty(\frac{\epsilon}{r}(2Y_{\mu,m}\setminus Y_{\mu,m}+\mathbf{j}))} + \left|\frac{\epsilon\mu}{r}\right|^{1+\frac{n}{2}}\|\nabla\Psi\|_{L^\infty(2\frac{\epsilon}{r}(Y_{\mu,m}+\mathbf{j}))}\right) \\ &\le c\omega\mu^{\frac{n}{2}}\|\Psi\|_{L^2(\frac{\epsilon}{r}(Y_{\mu,f}+\mathbf{j}))} \le c\|\Psi\|_{L^2(\frac{\epsilon}{r}(Y_{\mu,f}+\mathbf{j}))}. \end{aligned} \tag{3.11}$$

See §1 for $Y_{\mu,f}$. In (3.11), we use, by Lemma 3.5,

$$\left|\frac{\epsilon}{r}\right|^{\frac{n}{2}}\|\Psi\|_{L^\infty(\frac{\epsilon}{r}(2Y_{\mu,m}\setminus Y_{\mu,m}))} + \left|\frac{\epsilon}{r}\right|^{1+\frac{n}{2}}\|\nabla\Psi\|_{L^\infty(\frac{\epsilon}{r}(2Y_{\mu,m}\setminus Y_{\mu,m}))} \le c\|\Psi\|_{L^2(\frac{\epsilon}{r}Y_{\mu,f})}.$$

(3.7) follows from (3.9)–(3.11). (3.8) follows by (3.9), (3.11), and a modification of the argument for (3.10). □

From now on, A1–A3 are always assumed.

4. Hölder estimate

This section includes three subsections. The first subsection (see §§4.1) is to consider a homogenization problem for periodic elliptic equations. Then, we derive the Hölder estimate for the elliptic equations in the interior region (see §§4.2) and around the boundary (see §§4.3). If $i \in \{1, 2, \cdots, n\}$, find $\mathbb{X}_{\omega,\mu,i} \in H^1_\#(\mathbb{R}^n)$ by solving

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\mu}(\nabla \mathbb{X}_{\omega,\mu,i} + \vec{e}_i)) = 0 \quad \text{in } Y, \\ (\mathbb{X}_{\omega,\mu,i})_Y = 0, \end{cases} \tag{4.1}$$

where $\vec{e}_i$ is the unit vector in the i-th coordinate direction. See §2 for $\mathbf{K}_{\omega^2,\mu}$, $H^1_\#(\mathbb{R}^n)$, and $(\mathbb{X}_{\omega,\mu,i})_Y$. By the energy method and A1, $\|\mathbf{K}_{\omega,\mu} \nabla \mathbb{X}_{\omega,\mu,i}\|_{L^2(Y)} \leq c$, where c is a constant independent of μ, ω. If $\zeta(x) = \mathbb{X}_{\omega,\mu,i}(x) + \langle x, \vec{e}_i \rangle$ for $i = 1, \cdots, n$, then $\nabla \cdot (\mathbf{K}_{\omega^2,\mu} \nabla \zeta) = 0$ in Y. By Lemmas 3.5 and 3.6, the Poincaré inequality, and $(4.1)_2$,

$$\begin{cases} \|\mathbf{K}_{\omega^2,\mu}(\nabla \mathbb{X}_{\omega,\mu,i} + \vec{e}_i)\|_{L^\infty(Y)} = \|\mathbf{K}_{\omega^2,\mu} \nabla \zeta\|_{L^\infty(Y)} \leq c, \\ \|\mathbb{X}_{\omega,\mu,i}\|_{L^\infty(Y)} \leq c, \end{cases} \tag{4.2}$$

where c is independent of i, μ, ω. Define a $n \times n$ matrix $\mathcal{K}_{\omega,\mu}$, whose (i, j)-entry is

$$\mathcal{K}_{\omega,\mu,i,j} \equiv \int_Y \mathbf{K}_{\omega^2,\mu} \left(\delta_{i,j} + \partial_j \mathbb{X}_{\omega,\mu,i}(x) \right) dx \quad \text{where} \quad \delta_{i,j} \equiv \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4.3}$$

By remark in [19, pages 43 and 44] and (4.2), $\mathcal{K}_{\omega,\mu}$ is a continuous symmetric positive definite matrix of μ, ω and satisfies

$$\mathbf{m}_3 I \leq \mathcal{K}_{\omega,\mu} \leq \mathbf{m}_4 I, \tag{4.4}$$

where I is the identity matrix and $\mathbf{m}_3, \mathbf{m}_4$ are positive constants independent of μ, ω. For any $\sigma > 0$ and $i = 1, \cdots, n$, define

$$\mathbb{X}^\sigma_{\omega,\mu,i}(x) \equiv \sigma \mathbb{X}_{\omega,\mu,i}(x/\sigma), \qquad \mathbb{X}^\sigma_{\omega,\mu}(x) \equiv \left(\mathbb{X}^\sigma_{\omega,\mu,1}(x), \cdots, \mathbb{X}^\sigma_{\omega,\mu,n}(x) \right). \tag{4.5}$$

After translation and rotation, we can move any point $z \in \partial\Omega$ to 0. By A3, there is a number $\gamma_* \in (0, 2)$ and a $C^{1,\alpha}$ function $\Upsilon : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\begin{cases} \Upsilon(0) = 0 = \nabla \Upsilon(0), \\ \|\nabla \Upsilon\|_{L^\infty(\mathbb{R}^{n-1})} \leq \mathbf{m}_5, \\ \mathbf{m}_6\, x_n \leq \beta^x \leq \mathbf{m}_7\, x_n \quad \text{for any } x = (0, x_n) \in B_{\gamma_*}(0) \cap \Omega, \\ B_{\gamma_*}(0) \cap \Omega/r = B_{\gamma_*}(0) \cap \{(x', x_n) \in \mathbb{R}^n |\, r x_n > \Upsilon(r x')\} \quad \text{if } r \in (0, 1]. \end{cases} \tag{4.6}$$

If $r = 0$, we define $B_{\gamma_*}(0) \cap \Omega/r \equiv B_{\gamma_*}(0) \cap \{(x', x_n) \in \mathbb{R}^n |\, x_n > 0\}$. See §2 for β^x. Here, $\gamma_*, \mathbf{m}_5, \mathbf{m}_6, \mathbf{m}_7$ are positive numbers independent of $z\, (\in \partial\Omega), x\, (\in B_{\gamma_*}(0) \cap \Omega)$. By [15, Lemma 6.38] and its remark,

Lemma 4.1. *Under A3 and (4.6), there is an operator* $\widehat{\Pi} : C^1(B_{2R}(0) \cap \partial\Omega/r) \to C^1(B_{2R}(0) \cap \Omega/r)$ *for any* $R, r \in (0, \frac{\gamma_*}{2})$ *such that if* $\psi \in C^1(B_{2R}(0) \cap \partial\Omega/r)$, *then* $\widehat{\Pi}(\psi) = \psi$ *on* $B_{3R/2}(0) \cap \partial\Omega/r$ *and* $\|\widehat{\Pi}(\psi)\|_{C^1(B_{3R/2}(0) \cap \Omega/r)} \leq c \|\psi\|_{C^1(B_{2R}(0) \cap \partial\Omega/r)}$, *where c is a constant independent of R, r. Note space C^1 above can be replaced by space C^α for $\alpha \in (0, 1)$.*

4.1. Homogenization

Lemma 4.2. *Suppose* $0 \in \partial\Omega$ *and a sequence* $\{\epsilon, \mu_\epsilon, \omega_\epsilon, r_\epsilon, \psi_\epsilon, \psi_{b,\epsilon}, \mathcal{K}_{\omega_\epsilon,\mu_\epsilon}\}$ *satisfies*

$$\begin{cases} -\nabla\cdot(\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon}\nabla\psi_\epsilon) = 0 & \text{in } B_1(0)\cap\Omega/r_\epsilon, \\ \psi_\epsilon = \psi_{b,\epsilon} & \text{on } B_1(0)\cap\partial\Omega/r_\epsilon, \\ \psi_{b,\epsilon}(0) = 0, \\ \|\psi_\epsilon\|_{L^2(B_1(0)\cap\Omega/r_\epsilon)},\ \|\nabla\psi_{b,\epsilon}\|_{C(B_1(0)\cap\partial\Omega/r_\epsilon)} \le c, \end{cases} \tag{4.7}$$

and

$$\epsilon,\ \epsilon/r_\epsilon \to 0, \quad r_\epsilon \in (0,1) \to r \in [0,1], \quad \mathcal{K}_{\omega_\epsilon,\mu_\epsilon} \to \mathcal{K}_*. \tag{4.8}$$

Then,

(S1) $\|\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon}\nabla\psi_\epsilon\|_{L^2(B_{4/5}(0)\cap\Omega/r_\epsilon)}$ *are bounded independent of* $\epsilon, \mu_\epsilon, \omega_\epsilon, r_\epsilon$,
(S2) a subsequence of $\{\psi_\epsilon, \psi_{b,\epsilon}\}$ *(same notation for subsequence) satisfies*

$$\begin{cases} \psi_\epsilon \to \psi & \text{in } L^2(B_{4/5}(0)\cap\Omega/r) \text{ strongly} \\ \widehat{\Pi}(\psi_{b,\epsilon}) \to \psi_b & \text{in } C(B_{4/5}(0)\cap\Omega/r) \text{ strongly} \\ \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon}\nabla\psi_\epsilon \to \zeta & \text{in } L^2(B_{4/5}(0)\cap\Omega/r) \text{ weakly} \end{cases} \quad \text{as } \frac{\epsilon}{r_\epsilon} \to 0,$$

(S3) $\begin{cases} -\nabla\cdot\zeta = 0 & \text{in } B_{4/5}(0)\cap\Omega/r, \\ \psi = \psi_b & \text{on } B_{4/5}(0)\cap\partial\Omega/r, \end{cases}$
(S4) $\zeta = \mathcal{K}_*\nabla\psi$.

Here, $\mathcal{K}_{\omega_\epsilon,\mu_\epsilon}$ *(see (4.3)) and* $\mathcal{K}_*$ *are symmetric positive definite matrices; convergence of* $\mathcal{K}_{\omega_\epsilon,\mu_\epsilon}$ *in (4.8) is from (4.4); see Lemma 4.1 for* $\widehat{\Pi}$.

If $B_1(0) \Subset \Omega/r$, *(S1)–(S4) are also true. In this case,* $\psi_{b,\epsilon}$ *and* ψ_b *in (4.7) and (S2) should be neglected.*

Proof of Lemma 4.2 is given in §8.

4.2. Interior region

Assume $B_1(0) \Subset \Omega$.

Lemma 4.3. *For any* $\alpha \in (0,1)$, *there are* $\theta_1, \theta_2 \in (0,1)$ *(depending on* α*) with* $\theta_1 < \theta_2^2$ *and* $\epsilon_0 \in (0,1)$ *(depending on* $\alpha, \theta_1, \theta_2$*) so that if* $\nu \in (0,\epsilon_0)$ *and* $\theta \in [\theta_1,\theta_2]$, *any solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{E}^{\nu}_{\omega^2,\mu}\nabla\psi) = 0 & \text{in } B_1(0), \\ \|\psi\|_{L^2(B_1(0))} \le 1 \end{cases} \tag{4.9}$$

satisfies

$$\left(\fint_{B_\theta(0)} \left|\psi - (\psi)_{B_\theta(0)}\right|^2 dz\right)^{1/2} \le \theta^\alpha. \tag{4.10}$$

See §2 for $(\psi)_{B_\theta(0)}$.

Proof. Consider the elliptic equation

$$-\nabla\cdot(\mathcal{K}_*\nabla\psi)=0\qquad\text{in } B_{4/5}(0),\tag{4.11}$$

where $\mathcal{K}_*$ is a constant symmetric positive definite matrix satisfying (4.4). Any solution ψ of (4.11) satisfies

$$\|\psi\|_{C^1(B_{1/2}(0))}\le c\|\psi\|_{L^2(B_{4/5}(0))},$$

where c is a constant. For any $\alpha'\in(\alpha,1)$, we have, by Theorem 1.2 in [13, page 70],

$$\fint_{B_\theta(0)}\left|\psi-(\psi)_{B_\theta(0)}\right|^2dz\le\theta^{2\alpha'}\fint_{B_{4/5}(0)}|\psi|^2dz,\tag{4.12}$$

if θ (depending on α) is sufficiently small. Let us find $\theta_1,\theta_2\in(0,1)$ such that $\theta_1<\theta_2^2$ and (4.12) holds for any $\theta\in[\theta_1,\theta_2]$.

We claim (4.10). If not, there is a sequence $\{\nu,\mu_\nu,\omega_\nu,\theta_\nu,\psi_\nu,\mathcal{K}_{\omega_\nu,\mu_\nu}\}$ satisfying (4.9) and

$$\begin{cases}\nu\to0,\quad\theta_\nu\to\theta,\quad\mathcal{K}_{\omega_\nu,\mu_\nu}\to\mathcal{K}_*,\\ \fint_{B_{\theta_\nu}(0)}\left|\psi_\nu-(\psi_\nu)_{B_{\theta_\nu}(0)}\right|^2dz>\theta_\nu^{2\alpha}.\end{cases}\tag{4.13}$$

See (4.3) for $\mathcal{K}_{\omega_\nu,\mu_\nu}$. Convergence of $\mathcal{K}_{\omega_\nu,\mu_\nu}$ in $(4.13)_1$ is from (4.4). By Lemma 4.2, there is a subsequence (same notation for subsequence) such that

$$\begin{cases}\psi_\nu\to\psi & \text{in } L^2(B_{4/5}(0))\text{ strongly}\\ \mathbf{E}^\nu_{\omega_\nu^2,\mu_\nu}\nabla\psi_\nu\to\mathcal{K}_*\nabla\psi & \text{in } L^2(B_{4/5}(0))\text{ weakly}\end{cases}\qquad\text{as }\nu\to0,\tag{4.14}$$

where $\mathcal{K}_*$ is a constant symmetric positive definite matrix satisfying (4.4). Also, the limit function ψ in (4.14) satisfies (4.11). Equations (4.12)–(4.14) imply

$$\theta^{2\alpha}\;=\;\lim_{\nu\to0}\theta_\nu^{2\alpha}\le\lim_{\nu\to0}\fint_{B_{\theta_\nu}(0)}\left|\psi_\nu-(\psi_\nu)_{B_{\theta_\nu}(0)}\right|^2dz=\fint_{B_\theta(0)}\left|\psi-(\psi)_{B_\theta(0)}\right|^2dz\le\theta^{2\alpha'}\fint_{B_{4/5}(0)}|\psi|^2dz.$$

We get a contradiction if θ_2 is sufficiently small. So, Lemma 4.3 is proved. □

Lemma 4.4. *For any $\alpha\in(0,1)$, there are $\theta_1,\theta_2\in(0,1)$ (depending on α) with $\theta_1<\theta_2^2$ and $\epsilon_0\in(0,1)$ (depending on α,θ_1,θ_2) so that if $\epsilon\in(0,\epsilon_0)$, $\theta\in[\theta_1,\theta_2]$, and $k\in\mathbb{N}$ with $\frac{\epsilon}{\epsilon_0}<\theta^k$, any solution of*

$$-\nabla\cdot(\mathbf{E}^\epsilon_{\omega^2,\mu}\nabla U)=0\qquad\textit{in } B_1(0)\tag{4.15}$$

satisfies

$$\left(\fint_{B_{\theta^k}(0)}\left|U-(U)_{B_{\theta^k}(0)}\right|^2dz\right)^{1/2}\le\theta^{k\alpha}\|U\|_{L^2(B_1(0))}.\tag{4.16}$$

Proof. This lemma is proved by induction for k. Let $J\equiv\|U\|_{L^2(B_1(0))}$. For $k=1$, if $\psi\equiv\frac{U}{J}$, then ψ satisfies (4.9) with $\nu=\epsilon$. (4.16) follows from Lemma 4.3. If (4.16) holds for some $k\in\mathbb{N}$ with $\frac{\epsilon}{\epsilon_0}<\theta^k$, define

$$\psi(z)\equiv J^{-1}\theta^{-k\alpha}(U(\theta^kz)-(U)_{B_{\theta^k}(0)})\qquad\text{in } B_1(0).$$

Then, ψ satisfies (4.9) with $\nu = \epsilon/\theta^k$. By Lemma 4.3,

$$\fint_{B_\theta(0)} \left|\psi - (\psi)_{B_\theta(0)}\right|^2 dz \quad \le \quad \theta^{2\alpha}. \tag{4.17}$$

Rewrite the left-hand side of (4.17) in terms of U in $B_{\theta^{k+1}}(0)$ to see

$$\fint_{B_\theta(0)} \left|\psi - (\psi)_{B_\theta(0)}\right|^2 dz = \fint_{B_{\theta^{k+1}}(0)} \frac{\left|U - (U)_{B_{\theta^{k+1}}(0)}\right|^2}{J^2\ \theta^{2k\alpha}} dz. \tag{4.18}$$

(4.17) and (4.18) imply (4.16) for $k+1$. This proves Lemma 4.4. □

Lemma 4.5. *For any $\alpha \in (0,1)$, there is a number $\epsilon_\dagger \in (0,1)$ (depending on α) such that if $\epsilon \in (0, \epsilon_\dagger)$, any solution of (4.15) satisfies*

$$[U]_{C^\alpha(B_{1/2}(0))} \le c\|U\|_{L^2(B_1(0))}, \tag{4.19}$$

where c is a constant independent of ϵ, μ, ω.

Proof. Let $\theta_1, \theta_2, \epsilon_0$ be the same as Lemma 4.4; define $J \equiv \|U\|_{L^2(B_1(0))}$ and $\epsilon_\dagger \equiv \frac{\epsilon_0\theta_2}{2}$; let $\epsilon < \epsilon_\dagger$. For any $h \in (\frac{\epsilon}{\epsilon_0}, \theta_2]$, there are $\theta \in [\theta_1, \theta_2]$ and $k \in \mathbb{N}$ such that $h = \theta^k$. Lemma 4.4 implies, for any $h \in (\frac{\epsilon}{\epsilon_0}, \theta_2]$,

$$\fint_{B_h(0)} \left|U - (U)_{B_h(0)}\right|^2 dz \quad \le \quad h^{2\alpha} J^2. \tag{4.20}$$

Since $\epsilon < \epsilon_\dagger \equiv \frac{\epsilon_0\theta_2}{2}$, take $h = \frac{2\epsilon}{\epsilon_0} \in (\frac{\epsilon}{\epsilon_0}, \theta_2]$. Define

$$\psi(z) \equiv J^{-1}\epsilon^{-\alpha}\left(U(\epsilon z) - (U)_{B_{2\epsilon/\epsilon_0}(0)}\right) \qquad \text{in } B_{2/\epsilon_0}(0).$$

By (4.20), ψ satisfies $(4.9)_1$ with $\nu = 1$ and $\|\psi\|_{L^2(B_{2/\epsilon_0}(0))} \le c$. By [15, Theorems 8.17 and 8.22], and Lemma 3.5,

$$[\psi]_{C^\alpha(B_{1/\epsilon_0}(0))} \le c, \tag{4.21}$$

where c is constant independent of ϵ, μ, ω. By Theorem 1.2 in [13, page 70], inequality (4.21) implies (4.20) true for $h \le \frac{\epsilon}{\epsilon_0}$. In other words, (4.20) holds for $h \le \theta_2$, i.e.,

$$\fint_{B_h(0)} \left|U - (U)_{B_h(0)}\right|^2 dz \le ch^{2\alpha}J^2 \qquad \text{for } h \le \theta_2, \tag{4.22}$$

where c is a constant independent of ϵ, μ, ω. For any $z \in B_{1/2}(0)$, we repeat the above argument to see that (4.22) is also true with 0 replaced by any $z \in B_{1/2}(0)$. Theorem 1.2 in [13, page 70] implies $[U]_{C^\alpha(B_{1/2}(0))} \le cJ$, where c is independent of ϵ, μ, ω. □

By Lemma 4.5, [15, Theorems 8.17 and 8.22], and Lemma 3.5, we conclude the following.

Corollary 4.6. *There is a constant c independent of ϵ, μ, ω such that any solution of (4.15) satisfies (4.19).*

4.3. Boundary region

Assume $0 \in \partial\Omega$.

Lemma 4.7. *For any $\alpha \in (0,1)$, there are $\breve{\theta}_1, \breve{\theta}_2 \in (0,1)$ (depending on $\alpha, \partial\Omega$) with $\breve{\theta}_1 < \breve{\theta}_2^2$ and $\breve{\epsilon}_0 \in (0,1)$ (depending on $\alpha, \breve{\theta}_1, \breve{\theta}_2, \partial\Omega$) so that if $\epsilon, \frac{\epsilon}{r} \in (0, \breve{\epsilon}_0)$, $r \in (0,1)$, and $\theta \in [\breve{\theta}_1, \breve{\theta}_2]$, any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu} \nabla \psi) = 0 & \text{in } B_1(0) \cap \Omega/r, \\ \psi = \psi_b & \text{on } B_1(0) \cap \partial\Omega/r, \\ \|\psi\|_{L^2(B_1(0)\cap\Omega/r)}, \|\nabla \psi_b\|_{C(B_1(0)\cap\partial\Omega/r)} \le 1, \\ \psi(0) = 0 \end{cases} \tag{4.23}$$

satisfies

$$\left(\fint_{B_\theta(0)\cap\Omega/r} |\psi|^2 dz \right)^{1/2} \le \theta^\alpha. \tag{4.24}$$

Proof. Consider the elliptic equation

$$\begin{cases} -\nabla \cdot (\mathcal{K}_* \nabla \psi) = 0 & \text{in } B_{4/5}(0) \cap \Omega/r, \\ \psi = \psi_b & \text{on } B_{4/5}(0) \cap \partial\Omega/r, \\ \psi(0) = 0, \end{cases} \tag{4.25}$$

where $r \in [0,1]$ and $\mathcal{K}_*$ is a constant symmetric positive definite matrix satisfying (4.4). See (4.6) for the definition of $B_{4/5}(0) \cap \Omega/r$ for $r \in [0,1]$. Any solution ψ of (4.25) satisfies, by [15, Theorems 8.25 and 8.29],

$$\|\psi\|_{C^{\alpha'}(B_{1/2}(0)\cap\Omega/r)} \le c \left(\|\psi\|_{L^2(B_{4/5}(0)\cap\Omega/r)} + \|\nabla \psi_b\|_{C(B_{4/5}(0)\cap\partial\Omega/r)} \right),$$

where $\alpha' \in (\alpha, 1)$ and c is a constant depending on $\partial\Omega$. By Theorem 1.2 in [13, page 70],

$$\fint_{B_\theta(0)\cap\Omega/r} \left|\psi\right|^2 dz \le \theta^{2\alpha'} \left(\|\psi\|^2_{L^2(B_{4/5}(0)\cap\Omega/r)} + \|\nabla \psi_b\|^2_{C(B_{4/5}(0)\cap\partial\Omega/r)} \right), \tag{4.26}$$

for sufficiently small θ (depending on $\alpha, \partial\Omega$). Let us find $\breve{\theta}_1, \breve{\theta}_2 \in (0,1)$ such that $\breve{\theta}_1 < \breve{\theta}_2^2$ and (4.26) holds for any $\theta \in [\breve{\theta}_1, \breve{\theta}_2]$.

We claim (4.24). If not, there is a sequence $\{\epsilon, \mu_\epsilon, \omega_\epsilon, r_\epsilon, \theta_\epsilon, \psi_\epsilon, \psi_{b,\epsilon}, \mathcal{K}_{\omega_\epsilon,\mu_\epsilon}\}$ satisfying (4.23) and

$$\begin{cases} \epsilon, \frac{\epsilon}{r_\epsilon} \to 0, \quad r_\epsilon \to r, \quad \theta_\epsilon \to \theta, \quad \mathcal{K}_{\omega_\epsilon,\mu_\epsilon} \to \mathcal{K}_*, \\ \fint_{B_{\theta_\epsilon}(0)\cap\Omega/r_\epsilon} |\psi_\epsilon|^2 \, dz > \theta_\epsilon^{2\alpha}. \end{cases} \tag{4.27}$$

See (4.3) for $\mathcal{K}_{\omega_\epsilon,\mu_\epsilon}$. Convergence of $\mathcal{K}_{\omega_\epsilon,\mu_\epsilon}$ in $(4.27)_1$ is from (4.4). By Lemma 4.2, there is a subsequence (same notation for subsequence) such that

$$\begin{cases} \psi_\epsilon \to \psi & \text{in } L^2(B_{4/5}(0) \cap \Omega/r) \text{ strongly} \\ \widehat{\Pi}(\psi_{b,\epsilon}) \to \psi_b & \text{in } C(B_{4/5}(0) \cap \Omega/r) \text{ strongly} \\ \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla \psi_\epsilon \to \mathcal{K}_* \nabla \psi & \text{in } L^2(B_{4/5}(0) \cap \Omega/r) \text{ weakly} \end{cases} \quad \text{as } \frac{\epsilon}{r_\epsilon} \to 0, \tag{4.28}$$

where $\mathcal{K}_*$ is a constant symmetric positive definite matrix satisfying (4.4). The limit function ψ in (4.28) satisfies (4.25). Equations (4.26)–(4.28) imply

$$\begin{aligned}\theta^{2\alpha} &= \lim_{\epsilon\to 0}\theta_\epsilon^{2\alpha} \le \lim_{\epsilon\to 0}\fint_{B_{\theta_\epsilon}(0)\cap\Omega/r_\epsilon} |\psi_\epsilon|^2\, dz = \fint_{B_\theta(0)\cap\Omega/r} |\psi|^2\, dz \\ &\le \theta^{2\alpha'}\left(\|\psi\|^2_{L^2(B_{4/5}(0)\cap\Omega/r)} + \|\nabla\psi_b\|^2_{C(B_{4/5}(0)\cap\partial\Omega/r)}\right).\end{aligned}$$

We have a contradiction if θ_2 is sufficiently small. So, Lemma 4.7 is proved. □

Lemma 4.8. *For any $\alpha \in (0,1)$, there are $\breve\theta_1,\breve\theta_2 \in (0,1)$ (depending on $\alpha,\partial\Omega$) with $\breve\theta_1 < \breve\theta_2^2$ and $\breve\epsilon_0 \in (0,1)$ (depending on $\alpha,\breve\theta_1,\breve\theta_2,\partial\Omega$) such that if $\epsilon \in (0,\breve\epsilon_0)$, $\theta \in [\breve\theta_1,\breve\theta_2]$, and $k \in \mathbb{N}$ with $\frac{\epsilon}{\breve\epsilon_0} < \theta^k$, any solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{E}^\epsilon_{\omega^2,\mu}\nabla U) = 0 & \text{in } B_1(0)\cap\Omega, \\ U = U_b & \text{on } B_1(0)\cap\partial\Omega, \\ U(0) = 0 \end{cases} \tag{4.29}$$

satisfies

$$\left(\fint_{B_{\theta^k}(0)\cap\Omega} \left|U\right|^2 dz\right)^{1/2} \le \theta^{k\alpha}\left(\|U\|_{L^2(B_1(0)\cap\Omega)} + \|\nabla U_b\|_{C(B_1(0)\cap\partial\Omega)}\right). \tag{4.30}$$

Proof. This is proved by induction. Set $\breve J \equiv \|U\|_{L^2(B_1(0)\cap\Omega)} + \|\nabla U_b\|_{C(B_1(0)\cap\partial\Omega)}$. For $k = 1$, if $\psi \equiv \frac{U}{\breve J}$ and $\psi_b \equiv \frac{U_b}{\breve J}$, then ψ,ψ_b satisfy (4.23) with $r = 1$. So, (4.30) follows from Lemma 4.7. If (4.30) holds for some $k \in \mathbb{N}$ with $\frac{\epsilon}{\breve\epsilon_0} < \theta^k$, define

$$\begin{cases} \psi(z) \equiv \breve J^{-1}\theta^{-k\alpha}U(\theta^k z) & \text{in } B_1(0)\cap\Omega/\theta^k, \\ \psi_b(z) \equiv \breve J^{-1}\theta^{-k\alpha}U_b(\theta^k z) & \text{on } B_1(0)\cap\partial\Omega/\theta^k. \end{cases}$$

Then, ψ and ψ_b satisfy (4.23) with $r = \theta^k$. By Lemma 4.7,

$$\fint_{B_\theta(0)\cap\Omega/\theta^k} |\psi|^2 dz \;\le\; \theta^{2\alpha}. \tag{4.31}$$

Rewrite the left-hand side of (4.31) in terms of U in $B_{\theta^{k+1}}(0)\cap\Omega$ to see

$$\fint_{B_\theta(0)\cap\Omega/\theta^k} |\psi|^2 dz = \fint_{B_{\theta^{k+1}}(0)\cap\Omega} \frac{\left|U\right|^2}{\breve J^2\,\theta^{2k\alpha}} dz. \tag{4.32}$$

(4.31) and (4.32) imply (4.30) for $k+1$. This proves Lemma 4.8. □

Lemma 4.9. *For any $\alpha \in (0,1)$, there is a number $\breve\epsilon_\dagger \in (0,1)$ (depending on $\alpha,\partial\Omega$) such that if $\epsilon \in (0,\breve\epsilon_\dagger)$, any solution of (4.29) satisfies*

$$[U]_{C^\alpha(B_{1/2}(0)\cap\Omega)} \le c\big(\|U\|_{L^2(B_1(0)\cap\Omega)} + \|\nabla U_b\|_{C(B_1(0)\cap\partial\Omega)}\big), \tag{4.33}$$

where c is a constant independent of ϵ,μ,ω.

Proof. $\breve{\theta}_1, \breve{\theta}_2, \breve{\epsilon}_0, \breve{J}$ are the same as Lemma 4.8; $\breve{\epsilon}_\dagger \equiv \min\{\frac{\breve{\epsilon}_0\breve{\theta}_2}{3}, \epsilon_\dagger\}$. See Lemma 4.5 for $\epsilon_\dagger$. There is a constant c independent of ϵ, μ, ω so that, by $(4.29)_3$,

$$\|U_b\|_{C^1(B_1(0)\cap\partial\Omega)} \le c\|\nabla U_b\|_{C(B_1(0)\cap\partial\Omega)}. \tag{4.34}$$

For any $x \in B_{\breve{\theta}_2/3}(0) \cap \Omega$, define $\beta^x \equiv |x - x_*| = \min_{y\in\partial\Omega} |x - y|$, where $x_* \in \partial\Omega$, and define

$$\begin{cases} \zeta(z) = U(z) - U(x_*) & \text{in } B_1(0) \cap \Omega, \\ \zeta_b(z) = U_b(z) - U(x_*) & \text{on } B_1(0) \cap \partial\Omega. \end{cases} \tag{4.35}$$

Then, ζ and ζ_b satisfy

$$\begin{cases} -\nabla \cdot (\mathbf{E}^\epsilon_{\omega^2,\mu} \nabla\zeta) = 0 & \text{in } B_{1/2}(x_*) \cap \Omega, \\ \zeta = \zeta_b & \text{on } B_{1/2}(x_*) \cap \partial\Omega, \\ \zeta(x_*) = 0. \end{cases} \tag{4.36}$$

Let c be a constant independent of $\epsilon, \mu, \omega, x, x_*$. Since $\breve{\theta}_1 < \breve{\theta}_2^2$, for any $h \in [\frac{\epsilon}{\breve{\epsilon}_0}, \breve{\theta}_2]$, there are $\theta \in [\breve{\theta}_1, \breve{\theta}_2]$ and $k \in \mathbb{N}$ such that $h = \theta^k$. By Lemma 4.8, any solution of (4.36) satisfies

$$⨍_{B_h(x_*)\cap\Omega} |\zeta|^2 dy \le h^{2\alpha} \breve{J}_*^2 \qquad \text{for } h \in [\tfrac{\epsilon}{\breve{\epsilon}_0}, \breve{\theta}_2], \tag{4.37}$$

where $\breve{J}_* \equiv \|\zeta\|_{L^2(B_{1/2}(x_*)\cap\Omega)} + \|\nabla\zeta_b\|_{C(B_{1/2}(x_*)\cap\partial\Omega)}$. Next, consider case (1) $\beta^x > \frac{2\epsilon}{3\breve{\epsilon}_0}$ and case (2) $\beta^x \le \frac{2\epsilon}{3\breve{\epsilon}_0}$ separately.

Case 1. For $\beta^x > \frac{2\epsilon}{3\breve{\epsilon}_0}$. If $\rho \in [\frac{\beta^x}{2}, \frac{\breve{\theta}_2}{3}]$, then $B_\rho(x) \subset B_{3\rho}(x_*)$. (4.37) with $h = 3\rho$ implies

$$⨍_{B_\rho(x)\cap\Omega} \left|\zeta - (\zeta)_{B_\rho(x)\cap\Omega}\right|^2 dy \le c ⨍_{B_{3\rho}(x_*)\cap\Omega} |\zeta|^2\, dy \le c\rho^{2\alpha} \breve{J}_*^2. \tag{4.38}$$

Define

$$\begin{cases} \mathcal{A}(y) \equiv \mathbf{E}^\epsilon_{\omega^2,\mu}(\beta^x y + x) \\ \psi(y) \equiv |\breve{J}_*|^{-1} |\beta^x|^{-\alpha} \left(\zeta(\beta^x y + x) - (\zeta)_{B_{\beta^x}(x)}\right) \end{cases} \quad \text{in } B_1(0).$$

By (4.38) with $\rho = \beta^x$, we get $\|\psi\|_{L^2(B_1(0))} \le c$. Also, ψ satisfies

$$-\nabla \cdot (\mathcal{A}\nabla\psi) = 0 \qquad \text{in } B_1(0). \tag{4.39}$$

Apply Corollary 4.6 to (4.39) to obtain $[\psi]_{C^\alpha(B_{1/2}(0))} \le c$, which implies

$$⨍_{B_\rho(x)} \left|\zeta - (\zeta)_{B_\rho(x)}\right|^2 dy \le c\rho^{2\alpha} \breve{J}_*^2 \quad \text{for } \rho < \tfrac{\beta^x}{2}. \tag{4.40}$$

Case 2. For $\beta^x \le \frac{2\epsilon}{3\breve{\epsilon}_0}$. If $\rho \in [\frac{\epsilon}{3\breve{\epsilon}_0}, \frac{\breve{\theta}_2}{3}]$, then $B_\rho(x) \subset B_{3\rho}(x_*)$. (4.37) with $h = 3\rho$ implies

$$⨍_{B_\rho(x)\cap\Omega} \left|\zeta - (\zeta)_{B_\rho(x)\cap\Omega}\right|^2 dy \le c ⨍_{B_{3\rho}(x_*)\cap\Omega} |\zeta|^2\, dy \le c\rho^{2\alpha} \breve{J}_*^2. \tag{4.41}$$

Again, we define

$$\begin{cases} \mathcal{A}(y) \equiv \mathbf{E}^{\epsilon}_{\omega^2,\mu}(\epsilon y + x), \\ \psi(y) \equiv |\breve{J}_*|^{-1}\epsilon^{-\alpha}\left(\zeta(\epsilon y + x) - (\zeta)_{B_{\epsilon/\breve{\epsilon}_0}(x)\cap\Omega}\right) & \text{in } B_{1/\breve{\epsilon}_0}(0) \cap (\Omega - \{x\})/\epsilon, \\ \psi_b(y) \equiv |\breve{J}_*|^{-1}\epsilon^{-\alpha}\left(\zeta_b(\epsilon y + x) - (\zeta)_{B_{\epsilon/\breve{\epsilon}_0}(x)\cap\Omega}\right) & \text{on } B_{1/\breve{\epsilon}_0}(0) \cap (\partial\Omega - \{x\})/\epsilon. \end{cases}$$

Then, ψ satisfies

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla\psi) = 0 & \text{in } B_{1/\breve{\epsilon}_0}(0) \cap (\Omega - \{x\})/\epsilon, \\ \psi = \psi_b & \text{on } B_{1/\breve{\epsilon}_0}(0) \cap (\partial\Omega - \{x\})/\epsilon. \end{cases}$$

By (4.41) for $\rho = \frac{\epsilon}{\breve{\epsilon}_0}$, $\|\psi\|_{L^2(B_{1/\breve{\epsilon}_0}(0)\cap(\Omega-\{x\})/\epsilon)} + \|\nabla\psi_b\|_{C(B_{1/\breve{\epsilon}_0}(0)\cap(\partial\Omega-\{x\})/\epsilon)} \le c$. By [15, Theorems 8.25 and 8.29] and Lemma 3.5,

$$[\psi]_{C^\alpha(B_{1/2\breve{\epsilon}_0}(0)\cap(\Omega-\{x\})/\epsilon)} \le c. \tag{4.42}$$

(4.42) implies that (4.41) holds for $\rho \le \frac{\epsilon}{2\breve{\epsilon}_0}$. (4.34), (4.35), (4.38), and (4.40)–(4.42) imply that, if $x \in B_{\breve{\theta}_2/3}(0) \cap \Omega$ and $\rho < \breve{\theta}_2/3$,

$$⨍_{B_\rho(x)\cap\Omega} \left|U - (U)_{B_\rho(x)\cap\Omega}\right|^2 dy = ⨍_{B_\rho(x)\cap\Omega} \left|\zeta - (\zeta)_{B_\rho(x)\cap\Omega}\right|^2 dy \le c\rho^{2\alpha}\breve{J}_*^2 \le c\rho^{2\alpha}\breve{J}^2. \tag{4.43}$$

The Hölder estimate of U follows from (4.43) and Theorem 1.2 in [13, page 70]. □

By [15, Theorems 8.25 and 8.29], Lemma 3.5, and Lemma 4.9, we conclude the following.

Corollary 4.10. *There is a constant c independent of ϵ, μ, ω so that any solution of (4.29) satisfies (4.33).*

Lemma 4.11. *If $z \in \Omega$, $q \in (0, 2]$, and $0 < h < R < 3$, any solution of (4.29) satisfies*

$$\sup_{B_h(z)\cap\Omega} |U| \le c\left((R-h)^{\frac{-n}{q}}\|U\|_{L^q(B_R(z)\cap\Omega)} + \|\nabla U_b\|_{C(B_R(z)\cap\partial\Omega)}\right),$$

where c is a constant independent of $\epsilon, \mu, \omega, h, R, z$.

Proof. We trace the argument in [14, pages 80–82]. Let c denote a constant independent of $\epsilon, \mu, \omega, h, R$. First, consider $z \in \partial\Omega$ case. By translation, we move z to the origin and assume (4.6). For any $\theta, \tau \in (0, 1)$, find $x \in B_{\tau R}(0) \cap \Omega$ so that $|U|^2(x) > \sup_{B_{\tau R}(0)\cap\Omega} |U|^2 - \theta$. By Corollaries 4.6 and 4.10,

$$\begin{aligned} \sup_{B_{\tau R}(0)\cap\Omega} |U|^2 &< \theta + |U(x)|^2 \le \theta + 2\left|(U)_{B_{\frac{1}{2}(1-\tau)R}(x)\cap\Omega}\right|^2 + 2\left|U(x) - (U)_{B_{\frac{1}{2}(1-\tau)R}(x)\cap\Omega}\right|^2 \\ &\le \theta + 2\left|(U)_{B_{\frac{1}{2}(1-\tau)R}(x)\cap\Omega}\right|^2 + c\left|(1-\tau)R\right|^{2\alpha}[U]^2_{C^\alpha(B_{\frac{1}{2}(1-\tau)R}(x))} \\ &\le \theta + c\left|(U)_{B_{\frac{1}{2}(1-\tau)R}(x)\cap\Omega}\right|^2 + c(1-\tau)^{2\alpha}\left((U^2)_{B_R(0)\cap\Omega} + \|\nabla U_b\|^2_{C(B_R(0)\cap\partial\Omega)}\right) \\ &\le \theta + \frac{c}{(1-\tau)^n}(U^2)_{B_R(0)\cap\Omega} + c\|\nabla U_b\|^2_{C(B_R(0)\cap\partial\Omega)}, \end{aligned}$$

where c is independent of θ. Let $\theta \to 0$ and define $h \equiv \tau R$ to obtain

$$\sup_{B_h(0)\cap\Omega} |U| \le \frac{c}{(R-h)^{\frac{n}{2}}}\|U\|_{L^2(B_R(0)\cap\Omega)} + c\|\nabla U_b\|_{C(B_R(0)\cap\partial\Omega)}. \tag{4.44}$$

So, we obtain Lemma 4.11 for $0 \in \partial\Omega$ and the $q = 2$ case.

To show Lemma 4.11 for $0 \in \partial\Omega$ and the $q \in (0, 2)$ case, we apply Young's inequality to (4.44) to see

$$\begin{aligned} \sup_{B_h(0)\cap\Omega} |U| &\le \frac{c}{(R-h)^{\frac{n}{2}}} \|U\|^{q/2}_{L^q(B_R(0)\cap\Omega)} \sup_{B_R(0)\cap\Omega} |U|^{\frac{2-q}{2}} + c\,\|\nabla U_b\|_{C(B_R(0)\cap\partial\Omega)} \\ &\le \frac{1}{2}\sup_{B_R(0)\cap\Omega} |U| + \frac{c}{(R-h)^{\frac{n}{q}}}\|U\|_{L^q(B_R(0)\cap\Omega)} + c\|\nabla U_b\|_{C(B_R(0)\cap\partial\Omega)}. \end{aligned} \tag{4.45}$$

Lemma 4.11 for $0 \in \partial\Omega$ and the $q < 2$ case follows from (4.45) and Lemma 3.1 on [13, page 161].

Next, consider the $B_R(z) \cap \partial\Omega = \emptyset$ case. Lemma 4.11 is proved by repeating the above process without U_b. So, we skip the proof. □

By Corollaries 4.6 and 4.10 and Lemma 4.11, we obtain the following.

Corollary 4.12. *If $q \in (0, 2]$, there is a constant c independent of ϵ, μ, ω so that any solution of (4.29) satisfies*

$$\|U\|_{C^\alpha(B_{1/2}(0)\cap\Omega)} \le c\big(\|U\|_{L^q(B_1(0)\cap\Omega)} + \|\nabla U_b\|_{C(B_1(0)\cap\partial\Omega)}\big).$$

5. Interior Lipschitz estimate

The section is to study the interior Lipschitz estimates for strongly elliptic equations. Assume $B_1(0) \Subset \Omega$.

Lemma 5.1. *For any $\alpha \in (0, 1)$, there are constants $\theta, \epsilon_0 \in (0, 1)$ depending on α such that if $\nu \in (0, \epsilon_0)$, any solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{E}^\nu_{\omega^2,\mu}\nabla\psi) = 0 & \text{in } B_1(0), \\ \|\psi\|_{L^\infty(B_1(0))} \le 1 \end{cases} \tag{5.1}$$

satisfies

$$\sup_{z\in B_\theta(0)} \left|\psi(z) - \psi(0) - (z + \mathbb{X}^\nu_{\omega,\mu}(z))\mathbf{b}_{\omega,\mu,\nu}\right| \le \theta^{1+\alpha}, \tag{5.2}$$

where $\mathbf{b}_{\omega,\mu,\nu} \equiv \mathcal{K}^{-1}_{\omega,\mu}(\mathbf{E}^\nu_{\omega^2,\mu}\nabla\psi)_{B_\theta(0)}$ and $\mathcal{K}^{-1}_{\omega,\mu}$ is the inverse matrix of $\mathcal{K}_{\omega,\mu}$. See §2 for $(\mathbf{E}^\nu_{\omega^2,\mu}\nabla\psi)_{B_\theta(0)}$ and (4.3) for $\mathcal{K}_{\omega_\nu,\mu_\nu}$.

Proof. Consider $-\nabla\cdot(\mathcal{K}_*\nabla\psi) = 0$ in $B_{4/5}(0)$, where $\mathcal{K}_*$ is a constant symmetric positive definite matrix satisfying (4.4). By [15, Corollary 6.3], there is a $\theta \in (0, 1)$ such that, for any $\alpha' \in (\alpha, 1)$,

$$\sup_{B_\theta(0)} \left|\psi(z) - \psi(0) - z(\nabla\psi)_{B_\theta(0)}\right| \le \theta^{1+\alpha'}\|\psi\|_{L^\infty(B_{4/5}(0))}. \tag{5.3}$$

We claim (5.2). If not, there is a sequence $\{\nu, \mu_\nu, \omega_\nu, \psi_\nu, \mathcal{K}_{\omega_\nu,\mu_\nu}\}$ satisfying (5.1) and

$$\begin{cases} \nu \to 0, \quad \mathcal{K}_{\omega_\nu,\mu_\nu} \to \mathcal{K}_*, \\ \sup_{z\in B_\theta(0)} \left|\psi_\nu(z) - \psi_\nu(0) - (z + \mathbb{X}^\nu_{\omega_\nu,\mu_\nu}(z))\mathbf{b}_{\omega_\nu,\mu_\nu,\nu}\right| > \theta^{1+\alpha}. \end{cases} \tag{5.4}$$

By Lemma 4.2, $(5.1)_2$, and Corollary 4.6, there is a subsequence (same notation for subsequence) of $\{\psi_\nu\}$ such that, as $\nu \to 0$,

$$\begin{cases} \psi_\nu \to \psi & \text{in } C(B_{4/5}(0)), \\ \|\psi\|_{L^\infty(B_1(0))} \le 1, \\ \mathbf{E}^\nu_{\omega_\nu^2,\mu_\nu} \nabla\psi_\nu \to \mathcal{K}_* \nabla\psi & \text{in } L^2(B_{4/5}(0)) \text{ weakly}, \\ -\nabla \cdot (\mathcal{K}_* \nabla\psi) = 0 & \text{in } B_{4/5}(0), \end{cases} \tag{5.5}$$

where $\mathcal{K}_*$ is a constant symmetric positive definite matrix satisfying (4.4). By $(5.4)_1$ and $(5.5)_3$, we see

$$\lim_{\nu\to 0} \mathbf{b}_{\omega_\nu,\mu_\nu,\nu} = \lim_{\nu\to 0} \mathcal{K}^{-1}_{\omega_\nu,\mu_\nu} \fint_{B_\theta(0)} \mathbf{E}^\nu_{\omega_\nu^2,\mu_\nu} \nabla\psi_\nu \, dx = (\nabla\psi)_{B_\theta(0)}. \tag{5.6}$$

By $(5.4)_2$, (5.5), (4.2), (4.5), (5.6), and (5.3),

$$\begin{aligned} \theta^{1+\alpha} &\le \lim_{\nu\to 0} \sup_{z\in B_\theta(0)} \left|\psi_\nu(z) - \psi_\nu(0) - (z + \mathbb{X}^\nu_{\omega_\nu,\mu_\nu}(z))\mathbf{b}_{\omega_\nu,\mu_\nu,\nu}\right| \\ &= \sup_{z\in B_\theta(0)} \left|\psi(z) - \psi(0) - z(\nabla\psi)_{B_\theta(0)}\right| \le \theta^{1+\alpha'} \|\psi\|_{L^\infty(B_{4/5}(0))}. \end{aligned}$$

We get a contradiction. So, (5.2) is true. □

Lemma 5.2. *For any $\alpha \in (0,1)$, there are constants $\theta, \epsilon_0 \in (0,1)$ depending on α such that if $\epsilon \in (0,\epsilon_0)$, $k \in \mathbb{N}$ with $\frac{\epsilon}{\epsilon_0} < \theta^k$, and*

$$-\nabla \cdot (\mathbf{E}^\epsilon_{\omega^2,\mu} \nabla U) = 0 \quad \text{in } B_1(0), \tag{5.7}$$

there are constants $\mathbf{a}^{\epsilon,k}_{\omega,\mu}$, $\mathbf{b}^{\epsilon,k}_{\omega,\mu}$ satisfying

$$\begin{cases} |\mathbf{a}^{\epsilon,k}_{\omega,\mu}| + |\mathbf{b}^{\epsilon,k}_{\omega,\mu}| \le cJ, \\ \sup_{z\in B_{\theta^k}(0)} \left|U(z) - U(0) - \epsilon\mathbf{a}^{\epsilon,k}_{\omega,\mu} - \left(z + \mathbb{X}^\epsilon_{\omega,\mu}(z)\right)\mathbf{b}^{\epsilon,k}_{\omega,\mu}\right| \le \theta^{k(1+\alpha)} J, \end{cases} \tag{5.8}$$

where $J \equiv \|U\|_{L^\infty(B_1(0))}$ and c is a constant independent of ϵ, μ, ω.

Proof. For $k = 1$, (5.8) is from Lemma 5.1 with $\nu = \epsilon$ and $\psi = \frac{U}{J}$. In this case, $\mathbf{a}^{\epsilon,1}_{\omega,\mu} = 0$, $\mathbf{b}^{\epsilon,1}_{\omega,\mu} = \mathcal{K}^{-1}_{\omega,\mu}(\mathbf{E}^\epsilon_{\omega^2,\mu}\nabla U)_{B_\theta(0)}$. By Lemma 3.8 and (4.4), $|\mathbf{b}^{\epsilon,1}_{\omega,\mu}| \le cJ$, where c is a constant independent of ϵ, μ, ω. If (5.8) holds for some $k \in \mathbb{N}$ with $\frac{\epsilon}{\epsilon_0} < \theta^k$, define

$$\psi(z) \equiv \frac{U(\theta^k z) - U(0) - \epsilon\mathbf{a}^{\epsilon,k}_{\omega,\mu} - \left(\theta^k z + \mathbb{X}^\epsilon_{\omega,\mu}(\theta^k z)\right)\mathbf{b}^{\epsilon,k}_{\omega,\mu}}{\theta^{k(1+\alpha)} J} \quad \text{in } B_1(0).$$

By induction and (4.1), ψ satisfies (5.1) with $\nu = \epsilon/\theta^k$. Apply Lemma 5.1 to obtain

$$\sup_{z\in B_\theta(0)} \left|\psi(z) - \psi(0) - (z + \mathbb{X}^{\epsilon/\theta^k}_{\omega,\mu}(z))\mathbf{b}_{\omega,\mu,\epsilon/\theta^k}\right| \le \theta^{1+\alpha}, \tag{5.9}$$

where $\mathbf{b}_{\omega,\mu,\epsilon/\theta^k} \equiv \mathcal{K}^{-1}_{\omega,\mu}(\mathbf{E}^{\epsilon/\theta^k}_{\omega^2,\mu}\nabla\psi)_{B_\theta(0)}$. Define

$$\mathbf{a}^{\epsilon,k+1}_{\omega,\mu} \equiv -\mathbb{X}^1_{\omega,\mu}(0)\mathbf{b}^{\epsilon,k}_{\omega,\mu} \qquad \text{and} \qquad \mathbf{b}^{\epsilon,k+1}_{\omega,\mu} \equiv \mathbf{b}^{\epsilon,k}_{\omega,\mu} + J\theta^{k\alpha}\mathbf{b}_{\omega,\mu,\epsilon/\theta^k}. \tag{5.10}$$

By Lemma 3.8, (4.4), and $\|\psi\|_{L^\infty(B_1(0))} \le 1$, we see $|\mathbf{b}_{\omega,\mu,\epsilon/\theta^k}|$ is a constant independent of ϵ, μ, ω, k. By (5.10) and (4.2), we obtain $(5.8)_1$. Rewrite (5.9) in terms of U in $B_{\theta^{k+1}}(0)$ and apply (5.10) to obtain $(5.8)_2$. □

Lemma 5.3. *There is a number $\epsilon_0 \in (0,1)$ such that if $\epsilon \in (0,\epsilon_0)$, any solution of (5.7) satisfies*

$$||\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla U||_{L^\infty(B_{1/2}(0))} \le c||U||_{L^\infty(B_1(0))}, \tag{5.11}$$

where c is a constant independent of ϵ, μ, ω.

Proof. Let $\alpha, \theta, \epsilon_0, J$ be the same as Lemma 5.2 and c be a constant independent of ϵ, μ, ω. Suppose $k \in \mathbb{N}$ satisfies $\theta^{k+1} \le \frac{\epsilon}{\epsilon_0} < \theta^k$. By Lemma 5.2,

$$\sup_{z\in B_{\epsilon/\epsilon_0}(0)} \left| U(z) - U(0) - \epsilon \mathbf{a}^{\epsilon,k}_{\omega,\mu} - \left(z + \mathbb{X}^{\epsilon}_{\omega,\mu}(z)\right) \mathbf{b}^{\epsilon,k}_{\omega,\mu} \right| \le c \left|\frac{\epsilon}{\epsilon_0}\right|^{1+\alpha} J. \tag{5.12}$$

Define

$$\psi(z) \equiv \frac{U(\epsilon z) - U(0) - \epsilon \mathbf{a}^{\epsilon,k}_{\omega,\mu} - (\epsilon z + \epsilon \mathbb{X}^{1}_{\omega,\mu}(z))\mathbf{b}^{\epsilon,k}_{\omega,\mu}}{\epsilon^{1+\alpha} J} \quad \text{in } B_{1/\epsilon_0(0)}.$$

(4.1) and (5.12) imply that ψ satisfies $(5.1)_1$ with $\nu = 1$ and $||\psi||_{L^\infty(B_{1/\epsilon_0(0)})} \le c$. [15, Corollary 6.3] and Lemma 3.6 imply

$$||\mathbf{E}^{\epsilon,\epsilon}_{\omega^2,\mu}\nabla \psi||_{L^\infty(B_{1/2\epsilon_0}(0))} \le c. \tag{5.13}$$

(5.13), $(5.8)_1$, and (4.2) imply (5.11). □

Remark 5.4. By [15, Corollary 6.3], Lemmas 3.6 and 5.3, we conclude any solution of (5.7) satisfies (5.11).

6. Boundary Lipschitz estimate for elliptic solutions

Assume (4.6). Let $\vec{d} = (d_1, \cdots, d_n)$, $d_i \in [\frac{\gamma_*}{2}, \gamma_*]$, and $\mathcal{R}_{\vec{d}} \equiv \Pi_{i=1}^n[-d_i, d_i]$. See (4.6) for γ_*. Find a bell-shaped function $\widehat{\eta} \in C_0^\infty(\mathcal{R}_{\vec{d}})$ satisfying $\widehat{\eta} \in [0,1]$ and $\widehat{\eta} = 1$ in $\Pi_{i=1}^n[-\frac{\gamma_*}{4}, \frac{\gamma_*}{4}]$. For any $r, \frac{\epsilon}{r} \in (0,1)$, find $\mathbb{W}^{\epsilon,r}_{\omega,\mu,n} \in H^1(\mathcal{R}_{\vec{d}} \cap \Omega/r)$ satisfying

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}(\nabla \mathbb{W}^{\epsilon,r}_{\omega,\mu,n} + \vec{e}_n)) = 0 & \text{in } \mathcal{R}_{\vec{d}} \cap \Omega/r, \\ \mathbb{W}^{\epsilon,r}_{\omega,\mu,n} = (1 - \widehat{\eta}\,)\, \mathbb{X}^{\epsilon/r}_{\omega,\mu,n} & \text{on } \partial(\mathcal{R}_{\vec{d}} \cap \Omega/r), \end{cases} \tag{6.1}$$

where $\vec{e}_n$ is the unit vector in the n-th coordinate direction. See (4.5) for $\mathbb{X}^{\epsilon/r}_{\omega,\mu,n}$ and §2 for $\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}$. Adjust the constant vector $\vec{d}$ of $\mathcal{R}_{\vec{d}}$ so that if $\frac{\epsilon}{r}(Y_{\mu,m} + \mathbf{j}) \subset \Omega^\epsilon_{\mu,m}/r$ for any $\mathbf{j} \in \mathbb{Z}^n$, then $|\mathcal{R}_{\vec{d}} \cap \frac{\epsilon}{r}(Y + \mathbf{j})|$ is either 0 or $|\frac{\epsilon}{r}|^n$. Define

$$\begin{cases} \mathcal{D}^{\cdot r} \equiv \mathcal{R}_{\vec{d}} \cap \Omega/r, \\ \mathcal{D}^{\epsilon,r}_* \equiv \bigcup_{\mathbf{j}\in\mathbb{Z}^n;\ \frac{\epsilon}{r}(Y_{\mu,m}+\mathbf{j})\subset\mathcal{R}_{\vec{d}}\cap\Omega^\epsilon_{\mu,m}/r} \frac{\epsilon}{r}(Y + \mathbf{j}). \end{cases} \tag{6.2}$$

From $(6.2)_1$ and the definition of $\Omega^\epsilon_{\mu,m}$ in §1, we see

$$\begin{cases} \widehat{\eta} = 0 & \text{on } \partial\mathcal{D}^{\cdot r} \cap \partial\mathcal{D}^{\epsilon,r}_*, \\ \mathcal{D}^{\cdot r} \setminus \mathcal{D}^{\epsilon,r}_* \subset \{x \in \Omega^\epsilon_{\mu,f}/r \,\big|\, \beta^x_r = dist(x, \partial\Omega/r) \le 2\frac{\epsilon}{r}\}. \end{cases} \tag{6.3}$$

Let $\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,\cdot)$ be a Green's function of $-\nabla\cdot(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla)$ in $\mathcal{D}^{\cdot r}$, that is, a solution of

$$\begin{cases}-\nabla_y\cdot(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla_y\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,\cdot))=\delta(x,\cdot) & \text{in } \mathcal{D}^{\cdot r},\\ \mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,\cdot)=0 & \text{on } \partial\mathcal{D}^{\cdot r},\end{cases} \tag{6.4}$$

where $\delta(x,\cdot)$ is the Dirac delta function with pole at x. So, $\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,\cdot)\in W^{1,1}(\mathcal{D}^{\cdot r})$ exists uniquely [25]. Below is a local L^∞ estimate.

Lemma 6.1. *If $r,\frac{\epsilon}{r}\in(0,1)$, $x\in\mathcal{D}^{\cdot r}$, and $t>0$, any solution of*

$$\begin{cases}-\nabla\cdot(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\varphi)=0 & \textit{in } B_t(x)\cap\mathcal{D}^{\cdot r},\\ \varphi=0 & \textit{on } B_t(x)\cap\partial\mathcal{D}^{\cdot r}\end{cases} \tag{6.5}$$

satisfies

$$|\varphi|(x)\le c\left|\fint_{B_t(x)\cap\mathcal{D}^{\cdot r}}|\varphi(z)|^2\,dz\right|^{1/2}, \tag{6.6}$$

for some constant c independent of $\epsilon,\mu,\omega,r,x,t$.

Proof. To start, we assume $x=0\in\mathcal{D}^{\cdot r}$ and define $\zeta(z)=\varphi(tz)$. Then, (6.5) implies

$$\begin{cases}-\nabla\cdot(\mathbf{E}^{\epsilon,rt}_{\omega^2,\mu}\nabla\zeta)=0 & \text{in } B_1(0)\cap\mathcal{D}^{\cdot r}/t,\\ \zeta=0 & \text{on } B_1(0)\cap\partial\mathcal{D}^{\cdot r}/t.\end{cases}$$

Note $\frac{\epsilon}{rt}\le 1$ or $\frac{\epsilon}{rt}>1$. If $\frac{\epsilon}{rt}\le 1$ (resp., $\frac{\epsilon}{rt}>1$), then Lemma 4.11 (resp., [15, Theorems 8.25 and 8.29] and Lemmas 3.2 and 3.5) implies

$$\|\zeta\|_{L^\infty(B_{1/4}(0)\cap\mathcal{D}^{\cdot r}/t)}\le c\|\zeta\|_{L^2(B_1(0)\cap\mathcal{D}^{\cdot r}/t)}, \tag{6.7}$$

where c is a constant independent of ϵ,μ,ω,r,t. By (6.7),

$$|\varphi(0)|=|\zeta(0)|\le c\left|\int_{B_1(0)\cap\mathcal{D}^{\cdot r}/t}|\zeta(z)|^2dz\right|^{\frac12}\le c\left|\fint_{B_t(0)\cap\mathcal{D}^{\cdot r}}|\varphi(y)|^2dy\right|^{\frac12}.$$

So, (6.6) holds for the $x=0$ case. If $x\neq 0$, (6.6) is proved by shifting x to the origin of the coordinate system and repeating the above argument. □

Lemma 6.2. *If $r,\frac{\epsilon}{r},\alpha\in(0,1)$, there is a constant c independent of $\epsilon,\mu,\omega,r,\alpha$ such that any Green's function of (6.4) satisfies, for $x,y\in\mathcal{D}^{\cdot r}$,*

$$\begin{cases}\left|\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)\right|\le c|x-y|^{2-n},\\ |\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)|\le c|\beta^x_r|^\alpha|x-y|^{2-n-\alpha},\\ |\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)|\le c|\beta^x_r|^\alpha|\beta^y_r|^\alpha|x-y|^{2-n-2\alpha},\end{cases} \tag{6.8}$$

and

$$\begin{cases}\left|\nabla_y\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)\right|\le c|x-y|^{1-n} & \textit{for } |x-y|\le 4\frac{\epsilon}{r},\\ |\nabla_y\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)|\le c\dfrac{r}{\epsilon}|\beta^x_r|^\alpha|\beta^y_r|^\alpha|x-y|^{2-n-2\alpha} & \textit{for } |x-y|\ge 4\frac{\epsilon}{r}.\end{cases} \tag{6.9}$$

See §2 for $\beta^x_r\equiv dist(x,\partial\Omega/r)$.

Proof. Let c be a constant independent of $\epsilon, \mu, \omega, r, \alpha$ and set $\rho \equiv |x-y|$.

Step I. For $(6.8)_1$. Take $\xi \in C_0^\infty(B_{\rho/3}(y) \cap \mathcal{D}^{,r})$ and find $\varphi \in H_0^1(\mathcal{D}^{,r})$ satisfying

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu} \nabla\varphi) = \xi & \text{in } \mathcal{D}^{,r}, \\ \varphi = 0 & \text{on } \partial\mathcal{D}^{,r}. \end{cases}$$

Note φ is solvable uniquely in $H_0^1(\mathcal{D}^{,r})$. By the Sobolev embedding theorem [15],

$$\|\nabla\varphi\|_{L^2(\mathcal{D}^{,r})} \le c\|\xi\|_{L^{\frac{2n}{n+2}}(\mathcal{D}^{,r})} \le c\,\rho\|\xi\|_{L^2(B_{\rho/3}(y)\cap\mathcal{D}^{,r})}. \tag{6.10}$$

By [25] and Lemma 6.1,

$$\begin{cases} \varphi(x) = \displaystyle\int_{B_{\rho/3}(y)\cap\mathcal{D}^{,r}} \mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,z)\xi(z)\,dz, \\ |\varphi(x)| \le c\left|\displaystyle⨍_{B_{\rho/3}(x)\cap\mathcal{D}^{,r}} |\varphi|^2\,dz\right|^{\frac{1}{2}} \le c\left|\displaystyle⨍_{B_{\rho/3}(x)\cap\mathcal{D}^{,r}} |\varphi|^{\frac{2n}{n-2}}\,dz\right|^{\frac{n-2}{2n}}. \end{cases} \tag{6.11}$$

(6.10) and (6.11) imply

$$\begin{aligned} \left|\int_{B_{\rho/3}(y)\cap\mathcal{D}^{,r}} \mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,z)\xi(z)dz\right| &\le c\left|⨍_{B_{\rho/3}(x)\cap\mathcal{D}^{,r}} |\varphi|^{\frac{2n}{n-2}}dz\right|^{\frac{n-2}{2n}} \\ &\le c\rho^{\frac{2-n}{2}}\|\nabla\varphi\|_{L^2(\mathcal{D}^{,r})} \le c\rho^{\frac{4-n}{2}}\|\xi\|_{L^2(B_{\rho/3}(y)\cap\mathcal{D}^{,r})}. \end{aligned} \tag{6.12}$$

(6.12) and Lemma 6.1 imply

$$\left|\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)\right| \le c\left|⨍_{B_{\rho/3}(y)\cap\mathcal{D}^{,r}} |\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,z)|^2dz\right|^{\frac{1}{2}} \le c\rho^{2-n}.$$

So, $(6.8)_1$ is proved.

Step II. For $(6.8)_{2,3}$. Recall $\rho \equiv |x-y|$. By $(6.8)_1$, it is enough to show $(6.8)_2$ for case $\beta_r^x < \frac{\rho}{6}$. By $(6.8)_1$,

$$|\mathbf{G}^{\epsilon,r}_{\omega,\mu}(\tilde{x},y)| \le c|x-y|^{2-n} \qquad \text{for any } \tilde{x} \in B_{\rho/3}(x)\cap\mathcal{D}^{,r}.$$

Applying [15, Theorems 8.25 and 8.29], Lemmas 3.2 and 3.5, Corollary 4.12, and $(4.6)_3$ to $\mathbf{G}^{\epsilon,r}_{\omega,\mu}(\cdot,y)$ in $B_{\rho/3}(x)\cap\mathcal{D}^{,r}$, we obtain

$$|\mathbf{G}^{\epsilon,r}_{\omega,\mu}(\tilde{x},y)| \le c|\beta_r^{\tilde{x}}|^\alpha|x-y|^{2-n-\alpha} \qquad \text{for any } \tilde{x} \in B_{\rho/6}(x)\cap\mathcal{D}^{,r}.$$

$(6.8)_2$ follows by setting $\tilde{x} = x$. $(6.8)_3$ is obtained by $(6.8)_2$ and an argument similar to that for $(6.8)_2$. So, we skip its proof.

Step III. For (6.9). If $\rho = |x-y| \le \frac{4\epsilon}{r}$, then [15, Theorem 4.16], Lemmas 3.2 and 3.5, and $(6.8)_1$ imply

$$\|\nabla_y\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,\cdot)\|_{L^\infty(B_{\rho/2}(y)\cap\mathcal{D}^{,r})} \le c\rho^{-1}\|\mathbf{G}(x,\cdot)\|_{L^\infty(B_{3\rho/4}(y)\cap\mathcal{D}^{,r})} \le c|x-y|^{1-n}.$$

So, $(6.9)_1$ holds. If $|x-y| \ge \frac{4\epsilon}{r}$, set $t = \frac{\epsilon}{r}$. By [15, Theorem 4.16] and Lemmas 3.2 and 3.5,

$$\|\nabla_y\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,\cdot)\|_{L^\infty(B_{t/2}(y)\cap\mathcal{D}^{,r})} \le c\,t^{-1}\|\mathbf{G}(x,\cdot)\|_{L^\infty(B_{3t/4}(y)\cap\mathcal{D}^{,r})}. \tag{6.13}$$

$(6.9)_2$ follows from (6.13) and $(6.8)_3$. □

Recall problem (2.1). Trace the argument of Lemmas 6.1 and 6.2 to see

Corollary 6.3. *There is a constant c independent of ϵ, μ, ω such that any Green's function of problem (2.1) satisfies*

$$\left|\Gamma^{\epsilon}_{\omega,\mu}(x,y)\right| \leq c|x-y|^{2-n} \qquad \textit{for } x, y \in \Omega.$$

Lemma 6.4. *Solution of (6.1) exists uniquely in $H^1(\mathcal{D}^{,r})$. For any $r, \frac{\epsilon}{r} \in (0,1)$, there is a constant c independent of ϵ, μ, ω, r such that*

$$\|\mathbb{W}^{\epsilon,r}_{\omega,\mu,n}\|_{L^\infty(\mathcal{D}^{,r})} \leq c\frac{\epsilon}{r}.$$

Proof. Let c denote a constant independent of ϵ, μ, ω, r.

Step I. Unique existence of a solution of (6.1) in $H^1(\mathcal{D}^{,r})$ is from the Lax-Milgram theorem [15] and (4.2). If $\mathbb{Y}^{\epsilon,r}_{\omega,\mu,n} \equiv \mathbb{W}^{\epsilon,r}_{\omega,\mu,n} - \mathbb{X}^{\epsilon/r}_{\omega,\mu,n}$ in $\mathcal{D}^{\epsilon,r}_*$ (see (6.2)), then

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu} \nabla \mathbb{Y}^{\epsilon,r}_{\omega,\mu,n}) = 0 & \text{in } \mathcal{D}^{\epsilon,r}_*, \\ \mathbb{Y}^{\epsilon,r}_{\omega,\mu,n} = \mathbb{W}^{\epsilon,r}_{\omega,\mu,n} - \mathbb{X}^{\epsilon/r}_{\omega,\mu,n} & \text{on } \partial\mathcal{D}^{\epsilon,r}_*. \end{cases}$$

By the maximal principle [15], (4.2), and $(6.3)_1$,

$$\sup_{\mathcal{D}^{\epsilon,r}_*} |\mathbb{W}^{\epsilon,r}_{\omega,\mu,n}| \leq \frac{c\,\epsilon}{r} + \sup_{\partial\mathcal{D}^{\epsilon,r}_* \setminus \partial\mathcal{D}^{,r}} |\mathbb{W}^{\epsilon,r}_{\omega,\mu,n}|. \tag{6.14}$$

Next, we show

$$|\mathbb{W}^{\epsilon,r}_{\omega,\mu,n}(x)| \leq c\frac{\epsilon}{r} \qquad \text{for } x \in \mathcal{D}^{,r} \setminus \mathcal{D}^{\epsilon,r}_*. \tag{6.15}$$

If (6.15) is true, (6.14) and (6.15) imply Lemma 6.4.

Step II. We claim (6.15). The solution of (6.1) can be written as $\mathbb{W}^{\epsilon,r}_{\omega,\mu,n} = \mathbb{X}^{\epsilon/r}_{\omega,\mu,n} + \mathbb{U}_1 + \mathbb{U}_2$, where $\mathbb{U}_1$ is the solution of

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu} \nabla \mathbb{U}_1) = 0 & \text{in } \mathcal{D}^{,r}, \\ \mathbb{U}_1 = -\widehat{\eta}\, \mathbb{X}^{\epsilon/r}_{\omega,\mu,n} & \text{on } \partial\mathcal{D}^{,r}, \end{cases}$$

and $\mathbb{U}_2$ is the solution of

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu} (\nabla \mathbb{U}_2 + \nabla \mathbb{X}^{\epsilon/r}_{\omega,\mu,n} + \vec{e}_n)) = 0 & \text{in } \mathcal{D}^{,r}, \\ \mathbb{U}_2 = 0 & \text{on } \partial\mathcal{D}^{,r}. \end{cases}$$

By (4.2), (4.5), and the maximal principle [15],

$$\|\mathbb{X}^{\epsilon/r}_{\omega,\mu,n}\|_{L^\infty(\mathcal{D}^{,r})} + \|\mathbb{U}_1\|_{L^\infty(\mathcal{D}^{,r})} \leq c\frac{\epsilon}{r}. \tag{6.16}$$

Set $\widehat{\mathcal{D}^{,r}_{\ell}} \equiv \{x \in \mathcal{D}^{,r} |\ dist(x, \partial\Omega/r) \leq \ell\}$. Find $\breve{\eta} \in C^\infty(\mathcal{D}^{,r})$ so that $\breve{\eta} \in [0,1]$, $\breve{\eta} = 1$ in $\widehat{\mathcal{D}^{,r}_{2\epsilon/r}}$, $\|\nabla\breve{\eta}\|_{L^\infty(\mathcal{D}^{,r})} \leq c\frac{r}{\epsilon}$, and $supp(\breve{\eta}) \subset \widehat{\mathcal{D}^{,r}_{3\epsilon/r}}$. By (6.2) and (6.3), $\mathcal{D}^{,r} \setminus \mathcal{D}^{\epsilon,r}_* \subset \widehat{\mathcal{D}^{,r}_{2\epsilon/r}}$ and $\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}(x) = \mathbf{K}^{\epsilon/r}_{\omega^2,\mu}(x)$ for $x \in \mathcal{D}^{\epsilon,r}_*$. For any $x \in \mathcal{D}^{,r} \setminus \mathcal{D}^{\epsilon,r}_*$, by Green's formula, (6.8), (6.3), and (6.4),

$$\mathbb{U}_2(x) \quad = \quad -\int_{\mathcal{D}^{,r}} \nabla_y \mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y) \mathbf{E}^{\epsilon,r}_{\omega^2,\mu} \left(\nabla \mathbb{X}^{\epsilon/r}_{\omega,\mu,n}(y) + \vec{e}_n\right) dy$$

$$
\begin{aligned}
&= -\int_{\mathcal{D}^r} \nabla_y \mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)(1-\breve{\eta}(y))\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\left(\nabla \mathbb{X}^{\epsilon/r}_{\omega,\mu,n}(y)+\vec{e}_n\right)\,dy \\
&\quad -\int_{\mathcal{D}^r} \nabla_y \mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)\breve{\eta}(y)\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\left(\nabla \mathbb{X}^{\epsilon/r}_{\omega,\mu,n}(y)+\vec{e}_n\right)\,dy \\
&= -\int_{\widehat{\mathcal{D}^r_{3\epsilon/r}}} \mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)\nabla\breve{\eta}(y)\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\left(\nabla \mathbb{X}^{\epsilon/r}_{\omega,\mu,n}(y)+\vec{e}_n\right)\,dy \\
&\quad -\int_{\widehat{\mathcal{D}^r_{3\epsilon/r}}} \nabla_y \mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)\breve{\eta}(y)\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\left(\nabla \mathbb{X}^{\epsilon/r}_{\omega,\mu,n}(y)+\vec{e}_n\right)\,dy.
\end{aligned}
$$

If $x \in \mathcal{D}^r \setminus \mathcal{D}^{\epsilon,r}_*$ and $y \in \widehat{\mathcal{D}^r_{3\epsilon/r}}$, then $\beta^x_r, \beta^y_r \le c\frac{\epsilon}{r}$. By $(4.2)_1$, $(6.8)_{1,3}$, and $\alpha \in (\frac{1}{2},1)$,

$$
\begin{aligned}
&\Big|\int_{\widehat{\mathcal{D}^r_{3\epsilon/r}}} \mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)\nabla\breve{\eta}(y)\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\left(\nabla \mathbb{X}^{\epsilon/r}_{\omega,\mu,n}(y)+\vec{e}_n\right)\,dy\Big| \\
&\le c\frac{r}{\epsilon}\left(\int_{\widehat{\mathcal{D}^r_{3\epsilon/r}}\cap B_{4\epsilon/r}(x)} + \int_{\widehat{\mathcal{D}^r_{3\epsilon/r}}\setminus B_{4\epsilon/r}(x)}\right)\left|\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)\right|\,dy \\
&\le c\frac{r}{\epsilon}\left(\int_{B_{4\epsilon/r}(x)}|x-y|^{2-n}dy + \left|\frac{\epsilon}{r}\right|^{2\alpha}\int_{\widehat{\mathcal{D}^r_{3\epsilon/r}}\setminus B_{4\epsilon/r}(x)}|x-y|^{2-n-2\alpha}dy\right) \le c\frac{\epsilon}{r}.
\end{aligned}
$$

Similarly, $(4.2)_1$, (6.9), and $\alpha \in (\frac{1}{2},1)$ imply

$$
\begin{aligned}
&\Big|\int_{\widehat{\mathcal{D}^r_{3\epsilon/r}}} \nabla_y \mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)\breve{\eta}(y)\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\left(\nabla \mathbb{X}^{\epsilon/r}_{\omega,\mu,n}(y)+\vec{e}_n\right)\,dy\Big| \\
&\le c\left(\int_{\widehat{\mathcal{D}^r_{3\epsilon/r}}\cap B_{4\epsilon/r}(x)} + \int_{\widehat{\mathcal{D}^r_{3\epsilon/r}}\setminus B_{4\epsilon/r}(x)}\right)\left|\nabla_y\mathbf{G}^{\epsilon,r}_{\omega,\mu}(x,y)\right|\,dy \\
&\le c\left(\int_{B_{4\epsilon/r}(x)}|x-y|^{1-n}dy + \left|\frac{\epsilon}{r}\right|^{2\alpha-1}\int_{\widehat{\mathcal{D}^r_{3\epsilon/r}}\setminus B_{4\epsilon/r}(x)}|x-y|^{2-n-2\alpha}dy\right) \le c\frac{\epsilon}{r}.
\end{aligned}
$$

So, $\|\mathbb{U}_2\|_{L^\infty(\mathcal{D}^r\setminus\mathcal{D}^{\epsilon,r}_*)} \le c\frac{\epsilon}{r}$. Together with (6.16), we obtain (6.15). So, Lemma 6.4 is proved. □

Lemma 6.5. *Let θ, ϵ_0 be the same as in Lemma 5.1 and $0 \in \partial\Omega/r$. There are constants $\widetilde{\theta}, \widetilde{\epsilon}_0 \in (0,1)$ with $\widetilde{\theta} < \theta$, $\widetilde{\epsilon}_0 < \epsilon_0$ such that if $\frac{\epsilon}{r} < \widetilde{\epsilon}_0$, $r \in (0,1)$, and*

$$
\begin{cases}
-\nabla\cdot(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\psi) = 0 & \text{in } B_1(0)\cap\Omega/r,\\
\psi = \psi_b & \text{on } B_1(0)\cap\partial\Omega/r,
\end{cases}
\tag{6.17}
$$

and if

$$
\begin{cases}
\psi_b(0) = \partial_T\psi_b(0) = 0,\\
\|\psi\|_{L^\infty(B_1(0)\cap\Omega/r)}, [\nabla\psi_b]_{C^\alpha(B_1(0)\cap\partial\Omega/r)} \le 1,
\end{cases}
\tag{6.18}
$$

then

$$
\sup_{B_{\widetilde{\theta}}(0)\cap\Omega/r}\left|\psi(x) - \left(x_n + \mathbb{W}^{\epsilon,r}_{\omega,\mu,n}(x)\right)\mathbf{d}_{\omega,\mu,\epsilon,r}\right| \le \widetilde{\theta}^{1+\tau}.
$$

Here, $x = (x_1,\cdots,x_{n-1},x_n) = (x',x_n)$; $\alpha \in (0,1)$; $\tau = \frac{\alpha}{2}$; $\partial_T\psi_b(0)$ is the tangential derivative of ψ_b at 0; $\mathbf{d}_{\omega,\mu,\epsilon,r}$ is the n-th component of $\mathcal{K}^{-1}_{\omega,\mu}(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\psi)_{B_{\widetilde{\theta}}(0)\cap\Omega/r}$; $\mathcal{K}^{-1}_{\omega,\mu}$ is the inverse matrix of $\mathcal{K}_{\omega,\mu}$ (see (4.3)).

Proof. The proof is similar to that of Lemma 5.1. Let $r \in [0,1]$ and (ψ, ψ_b) satisfy

$$\begin{cases} -\nabla \cdot (\mathcal{K}_* \nabla \psi) = 0 & \text{in } B_{4/5}(0) \cap \Omega/r, \\ \psi = \psi_b & \text{on } B_{4/5}(0) \cap \partial\Omega/r, \\ \psi_b \in C^{1,\alpha}(B_1(0) \cap \partial\Omega/r), & \psi_b(0) = \partial_T \psi_b(0) = 0, \end{cases}$$

where $\mathcal{K}_*$ is a constant symmetric positive definite matrix satisfying (4.4). See (4.6) for the definition of $B_{4/5}(0) \cap \Omega/r$ for $r \in [0,1]$. By [15, Theorem 4.16], there are $\widetilde{\theta} \in (0, \frac{2}{3})$ and τ' satisfying $\tau < \tau' < \alpha$ such that

$$\sup_{B_{\widetilde{\theta}}(0)\cap\Omega/r} \left| \psi - x_n (\partial_n \psi)_{B_\theta(0)\cap\Omega/r} \right| \le \theta^{1+\tau'} \left(\|\psi\|_{L^\infty(B_{4/5}(0)\cap\Omega/r)} + [\nabla \psi_b]_{C^\alpha(B_{4/5}(0)\cap\partial\Omega/r)} \right). \tag{6.19}$$

Fix a small $\widetilde{\theta} \in (0,1)$ so that (6.19) holds. Lemma 6.5 follows by a contradiction argument (see Lemma 5.1), Lemma 4.2, Corollary 4.12 (for uniform convergence of solutions), and Lemma 6.4 (for $\lim_{\epsilon/r\to 0} \|\mathbb{W}^{\epsilon,r}_{\omega,\mu,n}\|_{L^\infty(B_{4/5}(0)\cap\Omega/r)} = 0$). □

Lemma 6.6. *$\widetilde{\theta}, \widetilde{\epsilon}_0, \tau, \alpha$ are the same as Lemma 6.5. If $\epsilon < \widetilde{\epsilon}_0$, $k \in \mathbb{N}$ with $\frac{\epsilon}{\widetilde{\epsilon}_0} < \widetilde{\theta}^k$, and*

$$\begin{cases} -\nabla \cdot (\mathbf{E}^\epsilon_{\omega^2,\mu} \nabla U) = 0 & \text{in } B_1(0) \cap \Omega, \\ U = U_b & \text{on } B_1(0) \cap \partial\Omega, \\ U_b(0) = \partial_T U_b(0) = 0, \end{cases} \tag{6.20}$$

there are constants $\mathbf{d}^{\epsilon,k-1}_{\omega,\mu}$ satisfying

$$\begin{cases} |\mathbf{d}^{\epsilon,k-1}_{\omega,\mu}| \le c\widetilde{J}, \\ \sup\limits_{B_{\widetilde{\theta}^k}(0)\cap\Omega} \left| U - \sum_{j=0}^{k-1} \widetilde{\theta}^{\tau j} \left(x_n + \widetilde{\theta}^j \mathbb{W}^{\epsilon,\widetilde{\theta}^j}_{\omega,\mu,n}(\widetilde{\theta}^{-j}x) \right) \mathbf{d}^{\epsilon,j}_{\omega,\mu} \right| \le \widetilde{\theta}^{k(1+\tau)} \widetilde{J}, \end{cases} \tag{6.21}$$

where $\widetilde{J} \equiv \|U\|_{L^\infty(B_1(0)\cap\Omega)} + [\nabla U_b]_{C^\alpha(B_1(0)\cap\partial\Omega)}$ and c is independent of ϵ, μ, ω.

Proof. This is proved by induction. If $k = 1$, set $\psi \equiv U/\widetilde{J}, \psi_b \equiv U_b/\widetilde{J}$. Then, ψ and ψ_b satisfy (6.17) and (6.18). So, (6.21) holds by Lemma 6.5 with $r = 1$. Here, $\mathbf{d}^{\epsilon,0}_{\omega,\mu}$ is the n-th component of $\mathcal{K}^{-1}_{\omega,\mu}(\mathbf{E}^\epsilon_{\omega^2,\mu}\nabla U)_{B_{\widetilde{\theta}}(0)\cap\Omega}$. By Lemma 3.8 and (4.4), $|\mathbf{d}^{\epsilon,0}_{\omega,\mu}| \le c\widetilde{J}$, where c is a constant independent of ϵ, μ, ω. Suppose (6.21) holds for some k with $\frac{\epsilon}{\widetilde{\epsilon}_0} < \widetilde{\theta}^k$, and we define, in $B_1(0) \cap \Omega/\widetilde{\theta}^k$,

$$\begin{cases} \psi(x) \equiv \widetilde{J}^{-1}\widetilde{\theta}^{-k(1+\tau)} \left(U(\widetilde{\theta}^k x) - \sum_{j=0}^{k-1} \widetilde{\theta}^{\tau j} \left(\widetilde{\theta}^k x_n + \widetilde{\theta}^j \mathbb{W}^{\epsilon,\widetilde{\theta}^j}_{\omega,\mu,n}(\widetilde{\theta}^{k-j}x) \right) \mathbf{d}^{\epsilon,j}_{\omega,\mu} \right), \\ \psi_b(x) \equiv \widetilde{J}^{-1}\widetilde{\theta}^{-k(1+\tau)} \left(U_b(\widetilde{\theta}^k x) - \sum_{j=0}^{k-1} \widetilde{\theta}^{\tau j} \widetilde{\theta}^k \, x_n \, \mathbf{d}^{\epsilon,j}_{\omega,\mu} \right). \end{cases}$$

By induction and (6.1), ψ and ψ_b satisfy (6.17) and (6.18) with $r = \widetilde{\theta}^k$. Lemma 6.5 implies

$$\sup_{B_{\widetilde{\theta}}(0)\cap\Omega/\widetilde{\theta}^k} \left| \psi(x) - \left(x_n + \mathbb{W}^{\epsilon,\widetilde{\theta}^k}_{\omega,\mu,n}(x) \right) \mathbf{d}_{\omega,\mu,\epsilon,\widetilde{\theta}^k} \right| \le \widetilde{\theta}^{1+\tau}, \tag{6.22}$$

where $\mathbf{d}_{\omega,\mu,\epsilon,\widetilde{\theta}^k}$ is the n-th component of $\mathcal{K}^{-1}_{\omega,\mu}(\mathbf{E}^{\epsilon,\widetilde{\theta}^k}_{\omega^2,\mu}\nabla\psi)_{B_{\widetilde{\theta}}(0)\cap\Omega/\widetilde{\theta}^k}$. By Lemma 3.8 and (4.4), $|\mathbf{d}_{\omega,\mu,\epsilon,\widetilde{\theta}^k}|$ is a constant independent of $\epsilon,\mu,\omega,\widetilde{\theta}^k$. Rewrite (6.22) in terms of U in $B_{\widetilde{\theta}^{k+1}}(0)$ to obtain

$$\sup_{B_{\widetilde{\theta}^{k+1}}(0)\cap\Omega}\left|U(x)-\sum_{j=0}^{k-1}\widetilde{\theta}^{\tau j}\left(x_n+\widetilde{\theta}^j\mathbb{W}^{\epsilon,\widetilde{\theta}^j}_{\omega,\mu,n}(\widetilde{\theta}^{-j}x)\right)\mathbf{d}^{\epsilon,j}_{\omega,\mu}\ -\widetilde{\theta}^{k\tau}\widetilde{J}\left(x_n+\widetilde{\theta}^k\mathbb{W}^{\epsilon,\widetilde{\theta}^k}_{\omega,\mu,n}(\widetilde{\theta}^{-k}x)\right)\mathbf{d}_{\omega,\mu,\epsilon,\widetilde{\theta}^k}\right|\le\widetilde{\theta}^{(k+1)(1+\tau)}\widetilde{J}.$$

If $\mathbf{d}^{\epsilon,k}_{\omega,\mu}\equiv\widetilde{J}\mathbf{d}_{\omega,\mu,\epsilon,\widetilde{\theta}^k}$, then (6.21) holds for $k+1$. □

Lemma 6.7. *Let $0\in\partial\Omega$ and $\widetilde{\epsilon}_0,\alpha$ be the same as Lemma 6.6. Suppose $\epsilon\in(0,\widetilde{\epsilon}_0)$, and any solution of (6.20) satisfies*

$$\|\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla U\|_{L^\infty(B_{1/2}(0)\cap\Omega)}\le c\left(\|U\|_{L^\infty(B_1(0)\cap\Omega)}+[\nabla U_b]_{C^\alpha(B_1(0)\cap\partial\Omega)}\right),\tag{6.23}$$

where c is a constant independent of ϵ,μ,ω.

Proof. By (4.6), there is a local coordinate $x=(x',x_n)$ so that

$$B_1(0)\cap\Omega=\left\{(x',x_n)\in\Omega|\ \ |x'|^2+|x_n|^2<1,\quad x_n>\Upsilon(x')\right\}.$$

To obtain (6.23), it suffices to show, for any $(0,x_n)\in B_{1/2}(0)\cap\Omega$,

$$\left|\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla U(0,x_n)\right|\le c\left(\|U\|_{L^\infty(B_1(0)\cap\Omega)}+[\nabla U_b]_{C^\alpha(B_1(0)\cap\partial\Omega)}\right).\tag{6.24}$$

This is because one can derive (6.23) for any $x\in B_1(0)\cap\Omega$ by moving the origin along the boundary $\partial\Omega$ and repeating the same argument as that for (6.24).

Let $\widetilde{\theta},\widetilde{J},\tau$ be the same as Lemma 6.6; c is a constant independent of ϵ,μ,ω; let k satisfy $\widetilde{\theta}^{k+1}\le\frac{\epsilon}{\widetilde{\epsilon}_0}<\widetilde{\theta}^k$. If $x\equiv(0,x_n)\in B_{1/2}(0)\cap\Omega$, either (1) $\frac{1}{2}\widetilde{\theta}^\ell\le x_n<\frac{1}{2}\widetilde{\theta}^{\ell-1}$ and $1\le\ell\le k$ or (2) $0<x_n<\frac{1}{2}\widetilde{\theta}^k$.

Case 1. For $\frac{1}{2}\widetilde{\theta}^\ell\le x_n<\frac{1}{2}\widetilde{\theta}^{\ell-1}$ and $1\le\ell\le k$. By Lemma 6.6,

$$\sup_{B_{\widetilde{\theta}^{\ell-1}}(0)\cap\Omega}\left|U(y)-\sum_{j=0}^{\ell-2}\widetilde{\theta}^{\tau j}\left(y_n+\widetilde{\theta}^j\mathbb{W}^{\epsilon,\widetilde{\theta}^j}_{\omega,\mu,n}(\widetilde{\theta}^{-j}y)\right)\mathbf{d}^{\epsilon,j}_{\omega,\mu}\right|\le c\,\widetilde{\theta}^{\ell(1+\tau)}\widetilde{J}.\tag{6.25}$$

Hence, by Lemma 6.4, $(4.6)_3$, and (6.25),

$$\sup_{B_{\widetilde{\theta}^{\ell-1}}(0)\cap\Omega}|U|\le c\widetilde{J}\left(\widetilde{\theta}^{\ell(1+\tau)}+(\beta^x+\epsilon)\sum_{j=0}^{\ell-2}\widetilde{\theta}^{\tau j}\right)\le c\beta^x\widetilde{J}.\tag{6.26}$$

See §2 for β^x. Note $\epsilon\le\widetilde{\epsilon}_0\widetilde{\theta}^k\le2\widetilde{\epsilon}_0\frac{1}{2}\widetilde{\theta}^\ell\le2\widetilde{\epsilon}_0x_n\le c\widetilde{\epsilon}_0\beta^x$. By (6.26),

$$\sup_{B_{\beta^x/2}(x)}|U|\le c\beta^x\widetilde{J}.\tag{6.27}$$

Define

$$\begin{cases}\mathcal{A}(y)\equiv\mathbf{E}^{\epsilon}_{\omega^2,\mu}(\beta^xy+x)\\ \psi(y)\equiv|\beta^x|^{-1}\widetilde{J}^{-1}U(\beta^xy+x)\end{cases}\quad\text{in }B_{1/2}(0).$$

By (6.27), $\|\psi\|_{L^\infty(B_{1/2}(0))} \le c$. Also ψ satisfies $-\nabla\cdot(\mathcal{A}\nabla\psi) = 0$ in $B_{1/2}(0)$. By Remark 5.4, $\|\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla U\|_{L^\infty(B_{\beta x/4}(x))} \le c\widetilde{J}$. This proves (6.24) for Case (1).

Case 2. For $0 < x_n < \frac{1}{2}\widetilde{\theta}^k$. By Lemma 6.6,

$$\sup_{B_{\widetilde{\theta}^k}(0)\cap\Omega}\left|U(y) - \sum_{j=0}^{k-1}\widetilde{\theta}^{\tau j}\left(y_n + \widetilde{\theta}^j\mathbb{W}^{\epsilon,\widetilde{\theta}^j}_{\omega,\mu,n}(\widetilde{\theta}^{-j}y)\right)\mathbf{d}^{\epsilon,j}_{\omega,\mu}\right| \le c\widetilde{J}\widetilde{\theta}^{k(1+\tau)}.$$

By Lemma 6.4 and $(6.21)_1$,

$$\sup_{B_{\widetilde{\theta}^k}(0)\cap\Omega}|U(y)| \le c\widetilde{J}\epsilon. \tag{6.28}$$

Define

$$\begin{cases}\psi(x) \equiv \widetilde{J}^{-1}\epsilon^{-1}U(\epsilon x) & \text{in } B_1(0)\cap\Omega/\epsilon,\\ \psi_b(x) \equiv \widetilde{J}^{-1}\epsilon^{-1}U_b(\epsilon x) & \text{on } B_1(0)\cap\partial\Omega/\epsilon.\end{cases}$$

By (6.28),

$$\begin{cases}\psi_b(0) = \partial_T\psi_b(0) = 0,\\ \|\psi\|_{L^\infty(B_1(0)\cap\Omega/\epsilon)} + [\nabla\psi_b]_{C^\alpha(B_1(0)\cap\partial\Omega/\epsilon)} \le c.\end{cases}$$

By (6.1), ψ and ψ_b satisfy

$$\begin{cases}-\nabla\cdot(\mathbf{E}^{\epsilon,\epsilon}_{\omega^2,\mu}\nabla\psi) = 0 & \text{in } B_1(0)\cap\Omega/\epsilon,\\ \psi = \psi_b & \text{on } B_1(0)\cap\partial\Omega/\epsilon.\end{cases}$$

By the definition of $\mathbf{E}^{\epsilon,\epsilon}_{\omega^2,\mu}$ (see §2), [15, Theorem 4.16], and Lemma 3.6,

$$\|\mathbf{E}^{\epsilon,\epsilon}_{\omega^2,\mu}\nabla\psi\|_{L^\infty(B_{1/2}(0)\cap\Omega/\epsilon)} \le c.$$

This proves (6.24) for Case (2). □

Remark 6.8. By [15, Theorem 4.16], Lemma 3.6, and Lemma 6.7, any solution of (6.20) satisfies (6.23).

7. Proofs of Theorems 2.1 and 2.2

We now prove Theorems 2.1 and 2.2.

Lemma 7.1. *There is a constant c independent of ϵ,μ,ω such that any Green's function for problem (2.1) satisfies*

$$|\nabla_x\Gamma^{\epsilon}_{\omega,\mu}(x,y)| + |\nabla_y\Gamma^{\epsilon}_{\omega,\mu}(x,y)| \le c|x-y|^{1-n} \qquad \text{for } x,y\in\Omega. \tag{7.1}$$

Proof. If $x,y\in\Omega$ and $x\ne y$, set $\rho\equiv|x-y|$. Consider

$$\begin{cases}-\nabla\cdot(\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla\Gamma^{\epsilon}_{\omega,\mu}(x,\cdot)) = 0 & \text{in } B_{\rho/2}(y)\cap\Omega,\\ \Gamma^{\epsilon}_{\omega,\mu}(x,\cdot) = 0 & \text{on } B_{\rho/2}(y)\cap\partial\Omega.\end{cases}$$

By [15, Theorem 4.16], Lemmas 3.2 and 3.5, Remarks 5.4 and 6.8, and Corollary 6.3,

$$\|\nabla\Gamma^{\epsilon}_{\omega,\mu}(x,\cdot)\|_{L^\infty(B_{\rho/4}(y)\cap\Omega)} \leq \frac{c}{\rho}\|\Gamma^{\epsilon}_{\omega,\mu}(x,\cdot)\|_{L^\infty(B_{\rho/2}(y)\cap\Omega)} \leq c\rho^{1-n}. \tag{7.2}$$

Since $\Gamma^{\epsilon}_{\omega,\mu}(x,y) = \Gamma^{\epsilon}_{\omega,\mu}(y,x)$ for any $x, y \in \Omega$ [25], a similar argument as (7.2) gives

$$\|\nabla\Gamma^{\epsilon}_{\omega,\mu}(\cdot,y)\|_{L^\infty(B_{\rho/4}(x)\cap\Omega)} \leq c\rho^{1-n}.$$

So, (7.1) is proved. □

Theorem 2.1 follows from Corollary 6.3 and Lemma 7.1.

Next, we prove Theorem 2.2. Suppose $g \in C^{1,\alpha}(\partial\Omega)$ for $\alpha \in (0,1)$, $F \in L^p(\Omega)$ for $p > n$, and Φ is a solution of problem (1.1). Define

$$\psi(x) = \int_\Omega \Gamma^{\epsilon}_{\omega,\mu}(x,y)F(y)\,dy.$$

By [25] and Lemma 7.1,

$$\begin{cases} -\nabla\cdot(\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla\psi) = F & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \\ \|\nabla\psi\|_{L^\infty(\Omega)} \leq c\|F\|_{L^p(\Omega)}. \end{cases}$$

Suppose $U \equiv \Phi - \psi$, then

$$\begin{cases} -\nabla\cdot(\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla U) = 0 & \text{in } \Omega, \\ U = g & \text{on } \partial\Omega. \end{cases}$$

By partition of unity, Remarks 5.4 and 6.8, and Lemma 4.11,

$$\|\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla U\|_{L^\infty(\Omega)} \leq c(\|U\|_{L^\infty(\Omega)} + \|g\|_{C^{1,\alpha}(\partial\Omega)}) \leq c\|g\|_{C^{1,\alpha}(\partial\Omega)}.$$

Therefore,

$$\|\nabla\Phi\|_{L^\infty(\Omega)} \leq c\big(\|F\|_{L^p(\Omega)} + \|g\|_{C^{1,\alpha}(\partial\Omega)}\big). \tag{7.3}$$

So, (2.2) follows from (7.3).

Next, consider problem (1.1) in each cell $\epsilon(Y + \mathbf{j})$ for $\mathbf{j} \in \mathcal{I}_\epsilon$ (see §1). For convenience, let $\mathbf{j} = 0$, that is, consider

$$-\nabla\cdot(\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla\Phi) = F \qquad \text{in } \epsilon Y. \tag{7.4}$$

Let us define $\zeta(x) \equiv \frac{1}{\epsilon}\Phi(\epsilon x)$ and $\phi(x) \equiv \epsilon F(\epsilon x)$. Then,

$$\begin{cases} -\nabla\cdot(\mathbf{E}^{\epsilon,\epsilon}_{\omega^2,\mu}\nabla\zeta) = \phi & \text{in } Y, \\ \|\mathbf{E}^{\epsilon,\epsilon}_{\omega^2,\mu}\nabla\zeta\|_{L^\infty(\frac{1}{2}Y)} = \|\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla\Phi\|_{L^\infty(\frac{\epsilon}{2}Y)}, & \\ \|\phi\|_{L^p(Y)} = \epsilon^{1-\frac{n}{p}}\|F\|_{L^p(\epsilon Y)} & \text{for any } p > n, \\ \|\phi\|_{L^{2,n+2\lambda-2}(Y)} = \epsilon^\lambda\|F\|_{L^{2,n+2\lambda-2}(\epsilon Y)} & \text{for any } \lambda \in (0,1). \end{cases} \tag{7.5}$$

By Lemma 3.6 and (7.5),

$$\|\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla\Phi\|_{L^\infty(\frac{\epsilon}{2}Y)} \;=\; \|\mathbf{E}^{\epsilon,\epsilon}_{\omega^2,\mu}\nabla\zeta\|_{L^\infty(\frac{1}{2}Y)} \leq c\left(\|\nabla\zeta\|_{L^\infty(Y)} + \|\phi\|_{L^p(Y)} + \mu^\lambda\|\phi\|_{L^{2,n+2\lambda-2}(Y)}\right)$$

$$= \quad c\left(\|\nabla\Phi\|_{L^\infty(\epsilon Y)} + \epsilon^{1-\frac{n}{p}}\|F\|_{L^p(\epsilon Y)} + |\epsilon\mu|^\lambda\|F\|_{L^{2,n+2\lambda-2}(\epsilon Y)}\right). \tag{7.6}$$

(7.3) and (7.6) imply

$$\|\mathbf{E}^\epsilon_{\omega^2,\mu}\nabla\Phi\|_{L^\infty(\Omega)} \leq c\left(\|F\|_{L^p(\Omega)} + \|g\|_{C^{1,\alpha}(\partial\Omega)} + |\epsilon\mu|^\lambda \sup_{\mathbf{j}\in I_\epsilon}\|F\|_{L^{2,n+2\lambda-2}(\epsilon(Y+\mathbf{j}))}\right).$$

So, (2.3) is proved.

8. Conclusions

This work studies the regularity of Dirichlet problem for strongly elliptic equations with oscillatory coefficients in periodic heterogeneous media. Diffusion coefficients of the elliptic equations are highly oscillatory functions. This work shows that the elliptic solutions are not oscillatory solutions and the Lipschitz norms of the elliptic solutions are bounded above uniformly in periodic size and the magnitude of the permeability of the media. These results are useful in the analysis of the regularity of the solutions in porous medium problems. The regularity of Neumann problem for strongly elliptic equations with oscillatory coefficients will be considered later.

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Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this manuscript.

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Appendix I

Proof of Lemma 3.5.

Let $\Gamma(z-y)$ denote the fundamental solution of the Laplace equation in $\mathbb{R}^n$ [15] and $\mathbf{D}$ be a bounded smooth domain. Define a single-layer and a double-layer potential as, for any smooth function ζ on the boundary $\partial\mathbf{D}$,

$$\begin{cases} \mathscr{S}_{\partial\mathbf{D}}(\zeta)(z) \equiv \displaystyle\int_{\partial\mathbf{D}} \Gamma(z-y)\,\zeta(y)\,dS \\ \mathscr{D}_{\partial\mathbf{D}}(\zeta)(z) \equiv \displaystyle\int_{\partial\mathbf{D}} \nabla_y\Gamma(z-y)\vec{\mathbf{n}}_y\,\zeta(y)\,dS \end{cases} \quad \text{for } z \in \partial\mathbf{D},$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to $\partial\mathbf{D}$. By [12, pages 148–151], [7, page 226], and a similar proof as [34, Lemma 3.2], we see

Lemma 8.1. *For any* $\alpha \in (0,1)$*, the linear operators*

$$\begin{cases} \mathcal{S}_{\partial\mathbf{D}} : C^{\alpha}(\partial\mathbf{D}) \to C^{1,\alpha}(\partial\mathbf{D}), \\ \mathcal{D}_{\partial\mathbf{D}} : C^{\alpha}(\partial\mathbf{D}) \to C^{1,\alpha}(\partial\mathbf{D}) \end{cases}$$

are bounded; $I - \mathbf{m}_2\mathcal{D}_{\partial\mathbf{D}}$ *for* $\mathbf{m}_2 \in [-2,2]$ *are invertible in* $C^{1,\alpha}(\partial\mathbf{D})$*; and*

$$\|\zeta\|_{C^{1,\alpha}(\partial\mathbf{D})} \le c\|(I - \mathbf{m}_2\mathcal{D}_{\partial\mathbf{D}})(\zeta)\|_{C^{1,\alpha}(\partial\mathbf{D})},$$

where I *is the identity operator and* c *is a constant independent of* $\mathbf{m}_2$.

If $\vec{\mathbf{n}}$ is the unit vector outward normal to $\partial\frac{1}{\mu}Y_{\mu,m}$, we define, for any $y \in \partial\frac{1}{\mu}Y_{\mu,m}$ and any function ζ on $\frac{1}{\mu}Y$,

$$\begin{cases} \zeta_{\pm}(y) \equiv \lim_{t\to0^+} \zeta(y \pm t\vec{\mathbf{n}}), & \lfloor\zeta\rfloor(y) \equiv \zeta_+(y) - \zeta_-(y), \\ \partial^{\pm}_{\mathbf{n}}\zeta \equiv \nabla\zeta_{\pm} \cdot \vec{\mathbf{n}}, & \lfloor\partial_{\mathbf{n}}\zeta\rfloor(y) \equiv \partial^{+}_{\mathbf{n}}\zeta(y) - \partial^{-}_{\mathbf{n}}\zeta(y). \end{cases} \tag{8.1}$$

Proof of Lemma 3.5. Set $\widehat{J} \equiv \|\Psi\|_{L^2(Y\setminus B_{1/4}(0))} + \|\mathbf{K}_{1/\omega^2,\mu}G\|_{L^p(Y)}$ and let c be a constant independent of μ, ω. By [15, Theorem 9.11], any solution Ψ of (3.3) satisfies

$$\|\Psi\|_{C^{1,\alpha}(B_{9/20}(0)\setminus B_{7/20}(0))} \le c\widehat{J} \qquad \text{for } \alpha \in (0,1). \tag{8.2}$$

Next, we find $\zeta \in C^{1,\alpha}(B_{2/5}(0))$ by solving

$$\begin{cases} -\Delta\zeta = \mathbf{K}_{1/\omega^2,\mu}G & \text{in } B_{2/5}(0), \\ \zeta = \Psi & \text{on } \partial B_{2/5}(0). \end{cases}$$

By [15, Theorem 9.13],

$$\|\zeta\|_{C^{1,\alpha}(B_{2/5}(0))} \le c\widehat{J}. \tag{8.3}$$

Define $\phi \equiv \Psi - \zeta$ in $B_{2/5}(0)$ and $\widehat{\phi}(y) \equiv \phi(\mu y)$, $\widehat{\zeta}(y) \equiv \zeta(\mu y)$ in $B_{2/5\mu}(0)$. Then

$$\begin{cases} -\Delta\widehat{\phi} = 0 & \text{in } B_{\frac{2}{5\mu}}(0) \setminus \partial\frac{1}{\mu}Y_{\mu,m}, \\ \lfloor\widehat{\phi}\rfloor = 0 & \text{on } \partial\frac{1}{\mu}Y_{\mu,m}, \\ \lfloor\mathbf{K}^{1/\mu}_{\omega^2,\mu}\nabla\widehat{\phi}\rfloor \cdot \vec{\mathbf{n}} = E & \text{on } \partial\frac{1}{\mu}Y_{\mu,m}, \\ \widehat{\phi} = 0 & \text{on } \partial B_{\frac{2}{5\mu}}(0), \end{cases} \tag{8.4}$$

where $\vec{\mathbf{n}}$ is the unit vector normal to $\partial\frac{1}{\mu}Y_{\mu,m}$ and $E = -\lfloor\mathbf{K}^{1/\mu}_{\omega^2,\mu}\nabla\widehat{\zeta}\rfloor \cdot \vec{\mathbf{n}}$. See §2 for $\mathbf{K}^{1/\mu}_{\omega^2,\mu}$ and (8.1) for $\lfloor\widehat{\phi}\rfloor$, $\lfloor\mathbf{K}^{1/\mu}_{\omega^2,\mu}\nabla\widehat{\phi}\rfloor$. Note, by (8.2), (8.3), and the definition of single-layer potential,

$$\begin{cases} \|E\|_{C^{\alpha}(\partial\frac{1}{\mu}Y_{\mu,m})} \le c\omega^2\mu\widehat{J}, \\ \left\|\mathcal{S}_{\partial B_{2/5\mu}(0)}(\partial_{\mathbf{n}}\widehat{\phi}|_{\partial B_{2/5\mu}(0)})\right\|_{C^{1,\alpha}(\partial\frac{1}{\mu}Y_{\mu,m})} \le c\widehat{J}, \end{cases} \tag{8.5}$$

where $\partial_{\mathbf{n}}\widehat{\phi}|_{\partial B_{2/5\mu}(0)}$ is the normal derivative of $\widehat{\phi}$ on $\partial B_{2/5\mu}(0)$. On the set $\partial\frac{1}{\mu}Y_{\mu,m}$, by Green's formula, (8.4), and [12, pages 148–151],

$$\begin{cases}\frac{\widehat{\phi}}{2}+\mathscr{D}_{\partial\frac{1}{\mu}Y_{\mu,m}}(\widehat{\phi})=\mathscr{S}_{\partial\frac{1}{\mu}Y_{\mu,m}}(\nabla\widehat{\phi}_{-}\cdot\vec{\mathbf{n}}|_{\partial\frac{1}{\mu}Y_{\mu,m}}),\\ \frac{\widehat{\phi}}{2}-\mathscr{D}_{\partial\frac{1}{\mu}Y_{\mu,m}}(\widehat{\phi})=-\mathscr{S}_{\partial\frac{1}{\mu}Y_{\mu,m}}(\nabla\widehat{\phi}_{+}\cdot\vec{\mathbf{n}}|_{\partial\frac{1}{\mu}Y_{\mu,m}})+\mathscr{S}_{\partial B_{2/5\mu}(0)}(\partial_{\mathbf{n}}^{-}\widehat{\phi}|_{\partial B_{2/5\mu}(0)}).\end{cases}$$

Therefore,

$$\left(I-\frac{2(1-\omega^2)}{1+\omega^2}\mathscr{D}_{\partial\frac{1}{\mu}Y_{\mu,m}}\right)\widehat{\phi}=\frac{2}{1+\omega^2}\left(\mathscr{S}_{\partial\frac{1}{\mu}Y_{\mu,m}}(-E)+\mathscr{S}_{\partial B_{2/5\mu}(0)}(\partial_{\mathbf{n}}^{-}\widehat{\phi}|_{\partial B_{2/5\mu}(0)})\right),$$

where I is the identity matrix. Apply (8.5) and Lemma 8.1 to see

$$\|\widehat{\phi}\|_{C^{1,\alpha}(\partial\frac{1}{\mu}Y_{\mu,m})}\leq\frac{c}{\omega^2}\left(\|E\|_{C^{\alpha}(\partial\frac{1}{\mu}Y_{\mu,m})}+\left\|\mathscr{S}_{\partial B_{2/5\mu}(0)}(\partial_{\mathbf{n}}\widehat{\phi}|_{\partial B_{2/5\mu}(0)})\right\|_{C^{1,\alpha}(\partial\frac{1}{\mu}Y_{\mu,m})}\right)\leq c\widehat{J}(\mu+\omega^{-2}). \tag{8.6}$$

By the maximal principle, (8.4), and (8.6),

$$\|\widehat{\phi}\|_{W^{1,\infty}(\frac{1}{\mu}Y_{\mu,m})}+\|\widehat{\phi}\|_{W^{1,\infty}(B_{2/5\mu}(0)\setminus\frac{1}{\mu}Y_{\mu,m})}\leq c\widehat{J}(\mu+\omega^{-2}). \tag{8.7}$$

By assumptions, (8.7), and the definition of $\widehat{\phi}$, we see $\|\nabla\phi\|_{L^\infty(B_{2/5}(0))}\leq c\widehat{J}$, which implies Lemma 3.5.

Appendix II

Proof of Lemma 4.2.

Lemma 8.2. *If* $\epsilon,\mu_\epsilon\in(0,1)$, $\omega_\epsilon\in(1,\infty)$, $\xi_\epsilon(x)=\mathbf{K}^{\epsilon}_{\omega_\epsilon^2,\mu_\epsilon}(x)\nabla(x+\mathbb{X}^{\epsilon}_{\omega_\epsilon,\mu_\epsilon}(x))$, *and* $\mathcal{K}_*=\lim_{\epsilon\to0}\mathcal{K}_{\omega_\epsilon,\mu_\epsilon}$, *then* ξ_ϵ *converges to* $\mathcal{K}_*$ *weakly in* $L^2(\Omega)$.

Proof. By [9, Proposition 1.46], we only need to show

$$\begin{cases}\|\xi_\epsilon\|_{L^2(\Omega)}\leq c\ (independent\ of\ \epsilon),\\ \int_{\mathbf{D}\cap\Omega}\xi_\epsilon(x)\,dx\xrightarrow[\epsilon\to0]{}\int_{\mathbf{D}\cap\Omega}\mathcal{K}_*\,dx=|\mathbf{D}\cap\Omega|\,\mathcal{K}_*,\end{cases} \tag{8.8}$$

for any rectangle $\mathbf{D}$. (4.2) implies $(8.8)_1$. By (2.9) in [9, page 35] and (4.2),

$$\int_{\mathbf{D}\cap\Omega}\xi_\epsilon(x)\,dx=\sum_{\mathbf{j}\in\mathfrak{A}(\epsilon)}\epsilon^n\mathcal{K}_{\omega_\epsilon,\mu_\epsilon}+\sum_{\mathbf{j}\in\mathfrak{A}'(\epsilon)}\int_{\epsilon(Y+\mathbf{j})\cap\Omega}\xi_\epsilon(x)\,dx\xrightarrow[\epsilon\to0]{}|\mathbf{D}\cap\Omega|\,\mathcal{K}_*,$$

where $\mathfrak{A}(\epsilon)\equiv\{\mathbf{j}\in\mathbb{Z}^n|\ \epsilon(Y+\mathbf{j})\subset\mathbf{D}\cap\Omega\}$ and $\mathfrak{A}'(\epsilon)\equiv\{\mathbf{j}\in\mathbb{Z}^n|\ \epsilon(Y+\mathbf{j})\setminus(\mathbf{D}\cap\Omega)\neq\emptyset\}$. Hence $(8.8)_2$ is true. So, we prove Lemma 8.2. □

Now, we give the proof of Lemma 4.2. (4.6)–(4.8) are assumed. The proof contains three steps.

Step I. (S1) is from (1) of Lemma 3.8. (S2) is from (S1), Lemma 4.1, (4.7), and the Sobolev embedding theorem [15]. If $\phi\in C_0^\infty(B_{4/5}(0)\cap\Omega/r)$, then $supp(\phi)\cap\partial\Omega/r_\epsilon=\emptyset$ when $\frac{\epsilon}{r_\epsilon}$ is small. Test $(4.7)_1$ against ϕ to see

$$\int_{B_1(0)\cap\Omega/r_\epsilon}\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon}\nabla\psi_\epsilon\nabla\phi\,dx=0. \tag{8.9}$$

As $\epsilon \to 0$, then (S2), (8.9), $(4.7)_2$, and Lemma 4.1 imply (S3).

Step II. Let $\mathbf{D}_\epsilon \equiv B_{4/5}(0) \cap \Omega/r_\epsilon$, $\mathbf{D} \equiv B_{4/5}(0) \cap \Omega/r$, and $\phi \in C_0^\infty(\mathbf{D})$. To show (S4), we consider the identity

$$\int_{\mathbf{D}_\epsilon} \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla\psi_\epsilon\, \phi\nabla(x_i + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i})\, dx = \int_{\mathbf{D}_\epsilon} \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_i + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i})\, \phi\nabla\psi_\epsilon\, dx. \tag{8.10}$$

See (4.5) for $\mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i}$. If $\frac{\epsilon}{r_\epsilon} < dist(supp(\phi), \partial\Omega/r)$, the left-hand side of (8.10) satisfies, by $(4.7)_1$, (4.2), (4.5), (S2), and (S3),

$$\begin{aligned}\int_{\mathbf{D}_\epsilon} \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla\psi_\epsilon\, \phi\nabla(x_i + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i})\, dx &= -\int_{\mathbf{D}_\epsilon} \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla\psi_\epsilon\, (x_i + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i})\nabla\phi\, dx \\ &\xrightarrow[\frac{\epsilon}{r_\epsilon}\to 0]{} -\int_{\mathbf{D}} \zeta\, x_i \nabla\phi\, dx = \int_{\mathbf{D}} \zeta\, \vec{e}_i\, \phi\, dx,\end{aligned} \tag{8.11}$$

where $\vec{e}_i$ is the unit vector in the i-th coordinate direction in $\mathbb{R}^n$. Consider the right-hand side of (8.10). If $\frac{\epsilon}{r_\epsilon} < dist(supp(\phi), \partial\Omega/r)$, then $\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} = \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon}$ in $supp(\phi)$. By (4.1) and Green's formula,

$$\begin{aligned}\int_{\mathbf{D}_\epsilon} \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_i + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i})\, \phi\nabla\psi_\epsilon\, dx &= -\int_{\mathbf{D}_\epsilon} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_i + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i})\psi_\epsilon\nabla\phi\, dx \\ = -\int_{\mathbf{D}_\epsilon} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_i + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i})\psi\nabla\phi\, dx &- \int_{\mathbf{D}_\epsilon} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_i + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i})(\psi_\epsilon - \psi)\nabla\phi\, dx.\end{aligned}$$

Lemma 8.2, (4.3), and (4.8) imply

$$\lim_{\frac{\epsilon}{r_\epsilon}\to 0} -\int_{\mathbf{D}_\epsilon} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_i + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i})\psi\nabla\phi\, dx = -\int_{\mathbf{D}} \mathcal{K}_*\, \vec{e}_i\, \psi\nabla\phi\, dx = \int_{\mathbf{D}} \vec{e}_i\, \phi\, \mathcal{K}_*\nabla\psi\, dx. \tag{8.12}$$

(4.2) and (S2) imply

$$\left|\int_{\mathbf{D}_\epsilon} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_i + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,i})(\psi_\epsilon - \psi)\nabla\phi\, dx\right| \xrightarrow[\frac{\epsilon}{r_\epsilon}\to 0]{} 0. \tag{8.13}$$

(8.10)–(8.13) imply $\int_{\mathbf{D}} \zeta\, \vec{e}_i\phi\, dx = \int_{\mathbf{D}} \mathcal{K}_*\nabla\psi\, \vec{e}_i\phi\, dx$. Since ϕ and i are arbitrary, this proves $\zeta = \mathcal{K}_*\nabla\psi$. So, we prove (S4).

Step III. To show (S1)–(S4) for $B_1(0) \Subset \Omega/r$ case, we simply repeat the procedure in Steps I and II and neglect $\psi_{b,\epsilon}, \psi_b$. Details are omitted.

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