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Research article

A fixed point theorem for non-negative functions

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Abstract: In this paper, we are concerned with the study of the existence and uniqueness of fixed points for the class of functions $f : C \to C$ satisfying the inequality

$$\ell\left(\alpha f(t) + (1-\alpha)f(s)\right) \le \sigma\ell(\alpha t + (1-\alpha)s)$$

for every $t, s \in C$ with $f(t) \neq f(s)$, where C is a closed subset of $[0, \infty)$, $\alpha, \sigma \in (0, 1)$ are constants, and $\ell : [0, \infty) \to [0, \infty)$ is a function satisfying the condition $\inf_{t>0} \frac{\ell(t)}{t^\rho} > 0$ for some constant $\rho > 0$. Namely, under a weak continuity condition imposed on f, we show that f possesses a unique fixed point, and for every $t_0 \in C$, the Picard sequence defined by $t_{n+1} = f(t_n)$, $n \ge 0$, converges to this fixed point. Next, we study the special cases when C is a closed interval and ℓ is a convex or concave function. Namely, making use of the Hermite-Hadamard inequalities, we obtain several new fixed point theorems. To the best of our knowledge, the considered class of functions was never previously investigated in the literature.

Keywords: fixed points; non-negative functions; convex functions; concave functions; Hermite-Hadamard inequalities
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1. Introduction

The theory of fixed points constitutes one of the important topics in pure and applied mathematics. Indeed, the most results related to the existence of solutions for nonlinear problems arising in physics and engineering are based on the use of certain fixed point theorems. For some contributions related to the applications of the theory of fixed points, we refer to the series of papers [1–3].

One of the most important fixed point results is the Banach fixed point theorem [4], which states that; if (M, d) is a complete metric space and $f : M \to M$ is a mapping satisfying the inequality

$$d(f(u), f(v)) \leq kd(u, v) \tag{1.1}$$

for every $u, v \in M$, where $k \in (0, 1)$ is a constant, then f possesses a unique fixed point, and for every $u_0 \in M$, the Picard sequence $u_{n+1} = f(u_n)$ converges to this fixed point. A mapping f satisfying (1.1) is called a contraction on M. The literature includes several generalizations and extensions of Banach's fixed point theorem. Some of them are concerned with the study of fixed points for mappings satisfying various kinds of contractions. For instance, Boyd and Wong [5] considered nonlinear contractions involving a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying certain conditions. Reich [6,7] studied a contraction of the form

$$d(f(u), f(v)) \leq k_1 d(f(u), f(v)) + k_2 d(f(u), u) + k_3 d(f(v), v),$$

where $k_1, k_2, k_3 \geq 0$ and $k_1 + k_2 + k_3 < 1$. Ćirić [8] proposed a contraction of the form

$$d(f(u), f(v)) \leq k \max\{d(f(u), f(v)), d(f(u), u), d(f(v), v), d(f(u), v), d(f(v), u)\},$$

where $k \in (0, 1)$. Further contributions for other kinds of contractions can be found in [9–11]. Other fixed point results were obtained when the underlying set is equipped with a generalized metric such as b-metric spaces [12], rectangular metric spaces [13], G-metric spaces [14], partial metric spaces [15], JS-metric spaces [16], supra-metric spaces [17], Hemi metric spaces [18], and fractional metric spaces [19].

Another category of fixed point results is concerned with the study of fixed points for mappings satisfying functional inequalities. For instance, in the monograph [20], Guo et al. considered the class of mappings $f : \mathrm{Int}(P) \to \mathrm{Int}(P)$ satisfying the inequality

$$f(tx) \geq t^{\xi} f(x), \quad 0 < t < 1, \ x \in \mathrm{Int}(P),$$

where $\xi \in (0, 1)$ is a constant. Here, P is a normal solid cone of a Banach space E and $\mathrm{Int}(P)$ is the interior of P. Namely, it was proved that, if f is an increasing operator (with respect to the partial order induced by the cone P), then f possesses a unique fixed point. Moreover, for any $x_0 \in \mathrm{Int}(P)$, the sequence $x_{n+1} = f(x_n)$, $n \geq 0$, converges to this fixed point. Other results that belong to the same category can be found in [21–23]. In the mentioned contributions, the mapping f is always supposed to be monotone or mixed monotone with respect to the partial order induced by the cone.

The present contribution belongs to the category of fixed point results for mappings satisfying functional inequalities. Our main idea is motivated by the following example. Let us consider the function $f : [0, 1] \to [0, 1]$ defined by

$$f(t) = \begin{cases} \dfrac{t}{2} & \text{if} \quad 0 \leq t < 1, \\ \dfrac{1}{4} & \text{if} \quad t = 1. \end{cases}$$

Let d be the standard metric on $[0,1]$, that is,

$$d(u,v)=|u-v|,\quad u,v\in[0,1].$$

It is clear that 0 is the unique fixed point of f. However, there is no $k\in[0,1)$ such that f satisfies (1.1) for every $u,v\in[0,1]$. This can be easily seen by remarking that f is not continuous at 1 (with respect to the metric d). Then, the Banach fixed theorem is not applicable in this case. On the other hand, the function f satisfies an interesting property. Indeed, if $\alpha\in(0,1)$, then for every $t,s\in[0,1]$, we have

$$\alpha f(t)+(1-\alpha)f(s)\le\frac{1}{2}\left[\alpha t+(1-\alpha)s\right].\tag{1.2}$$

So, a natural question arises: Is it possible to obtain suitable conditions under which a function f satisfying inequalities of type (1.2) possesses a unique fixed point? The aim of this work is to investigate this question. Namely, we are concerned with the class of real-valued functions $f:C\to C$ satisfying the inequality

$$\ell\left(\alpha f(t)+(1-\alpha)f(s)\right)\le\sigma\ell(\alpha t+(1-\alpha)s)$$

for every $t,s\in C$ with $f(t)\neq f(s)$. Here, $C\subset[0,\infty)$, $\alpha,\sigma\in(0,1)$ are constants, and

$$\ell:[0,\infty)\to[0,\infty)$$

is a function satisfying $\inf_{t>0}\frac{\ell(t)}{t^\rho}>0$ for some constant $\rho>0$. Notice that in the special case $\ell(t)=t$, the above inequality reduces to (1.2) with $\sigma=\frac{1}{2}$. First, we establish a fixed point theorem for the above class of functions. Next, we discuss the particular cases when C is an interval and ℓ is a convex or concave function. Namely, making use of Hermite-Hadamard inequalities, we deduce several new fixed point theorems. We also provide an example where our approach can be used while the Banach fixed point theorem is not applicable.

We point out that unlike the most contributions related to the study of fixed points for mappings satisfying functional inequalities, in this paper, no monotony condition is imposed on the function f.

Our main result is stated and proved in Section 2. Some particular cases of our main result are studied in Section 3.

We end this section by fixing some notations that will be used throughout this paper. Let $f:C\to C$. We denote by $\mathrm{Fix}(f)$ the set of fixed points of f, that is,

$$\mathrm{Fix}(f)=\{t\in C: f(t)=t\}.$$

For $t\in C$, we denote by $\{f^n(t)\}$ the sequence in C defined by

$$f^0(t)=t,\quad f^{n+1}(t)=f(f^n(t)),\quad n\ge 0.$$

2. Main result

For $\rho>0$, let us denote by L_ρ the set of functions $\ell:[0,\infty)\to[0,\infty)$ satisfying the condition

$$c_\rho:=\inf_{t>0}\frac{\ell(t)}{t^\rho}>0.\tag{2.1}$$

For a nonempty subset $C \subset [0,\infty)$, $\rho > 0$, $\ell \in L_\rho$, and $\alpha, \sigma \in (0,1)$, let $\mathbb{F}_C(\rho, \ell, \sigma, \alpha)$ be the set of functions $f : C \to C$ satisfying the inequality

$$\ell\left(\alpha f(t) + (1-\alpha) f(s)\right) \leq \sigma \ell(\alpha t + (1-\alpha)s) \tag{2.2}$$

for every $t, s \in C$ with $f(t) \neq f(s)$.

Our main result is the following fixed point theorem.

Theorem 2.1. *Let $C \subset [0,\infty)$, $C \neq \emptyset$, and $f : C \to C$. Assume that the following conditions hold:*

(i) *C is a closed subset;*
(ii) *$f \in \mathbb{F}_C(\rho, \ell, \sigma, \alpha)$ for some $\rho > 0$, $\ell \in L_\rho$, and $\alpha, \sigma \in (0,1)$;*
(iii) *For every $t, s \in C$, if $\lim_{n\to\infty} f^n(t) = s$, then $\{f^n(t)\}$ admits a subsequence $\{f^{n_q}(t)\}$ such that $\lim_{q\to\infty} f(f^{n_q}(t)) = f(s)$.*

Then, f possesses a unique fixed point. Moreover, for every $t_0 \in C$, the sequence $\{f^n(t_0)\}$ converges to this fixed point.

Proof. The first step of the proof is to show that $\mathrm{Fix}(f) \neq \emptyset$. Indeed, for an arbitrary $t_0 \in C$, let $\{t_n\} \subset C$ be the sequence defined by

$$t_{n+1} = f(t_n) \geq 0, \quad n \geq 0,$$

that is,

$$t_n = f^n(t_0), \quad n \geq 0.$$

If for some $k \geq 0$, we have $t_k = t_{k+1}$, then $t_k \in \mathrm{Fix}(f)$ and $\mathrm{Fix}(f) \neq \emptyset$. So, we may assume that

$$t_n \neq t_{n+1}, \quad n \geq 0,$$

which implies that

$$f(t_n) \neq f(t_{n+1}), \quad n \geq 0.$$

Then, by (ii), taking $(t, s) = (t_0, t_1)$ in (2.2), we obtain

$$\ell\left(\alpha f(t_0) + (1-\alpha) f(t_1)\right) \leq \sigma \ell(\alpha t_0 + (1-\alpha) t_1),$$

that is,

$$\ell\left(\alpha t_1 + (1-\alpha) t_2\right) \leq \sigma \ell(\alpha t_0 + (1-\alpha) t_1). \tag{2.3}$$

Similarly, (2.2) with $(t, s) = (t_1, t_2)$ yields

$$\ell\left(\alpha f(t_1) + (1-\alpha) f(t_2)\right) \leq \sigma \ell(\alpha t_1 + (1-\alpha) t_2),$$

that is,

$$\ell\left(\alpha t_2 + (1-\alpha) t_3\right) \leq \sigma \ell(\alpha t_1 + (1-\alpha) t_2). \tag{2.4}$$

Then, from (2.3) and (2.4), we deduce that

$$\ell\left(\alpha t_2 + (1-\alpha) t_3\right) \leq \sigma^2 \ell(\alpha t_0 + (1-\alpha) t_1).$$

Repeating the same argument as above, we obtain by induction that

$$\ell\left(\alpha t_n + (1-\alpha)t_{n+1}\right) \le \sigma^n \ell(\alpha t_0 + (1-\alpha)t_1), \quad n \ge 0. \tag{2.5}$$

On the other hand, by (2.1), for all $n \ge 0$, we have

$$\ell\left(\alpha t_n + (1-\alpha)t_{n+1}\right) \ge c_\rho \left(\alpha t_n + (1-\alpha)t_{n+1}\right)^\rho,$$

which implies by (2.5) that

$$(\alpha t_n + (1-\alpha)t_{n+1})^\rho \le \frac{1}{c_\rho}\sigma^n \ell(\alpha t_0 + (1-\alpha)t_1), \quad n \ge 0. \tag{2.6}$$

We now discuss two cases.
Case 1: If $t_n > t_{n+1}$ for some $n \ge 0$.
In this case, we obtain

$$\begin{aligned} \alpha t_n + (1-\alpha)t_{n+1} &= \alpha(t_n - t_{n+1}) + t_{n+1} \\ &\ge \alpha(t_n - t_{n+1}), \end{aligned}$$

which implies that

$$(\alpha t_n + (1-\alpha)t_{n+1})^\rho \ge \alpha^\rho (t_n - t_{n+1})^\rho.$$

Case 2: If $t_n < t_{n+1}$ for some $n \ge 0$.
In this case, we obtain

$$\begin{aligned} \alpha t_n + (1-\alpha)t_{n+1} &= (1-\alpha)(t_{n+1} - t_n) + t_n \\ &\ge (1-\alpha)(t_{n+1} - t_n), \end{aligned}$$

which implies that

$$(\alpha t_n + (1-\alpha)t_{n+1})^\rho \ge (1-\alpha)^\rho (t_{n+1} - t_n)^\rho.$$

Consequently, in both cases, we have

$$(\alpha t_n + (1-\alpha)t_{n+1})^\rho \ge \tau_\alpha^\rho |t_{n+1} - t_n|^\rho, \quad n \ge 0,$$

where

$$\tau_\alpha = \min\{\alpha, 1-\alpha\}.$$

Thus, by (2.6), we obtain

$$\tau_\alpha^\rho |t_{n+1} - t_n|^\rho \le \frac{1}{c_\rho}\sigma^n \ell(\alpha t_0 + (1-\alpha)t_1), \quad n \ge 0,$$

that is,

$$|t_{n+1} - t_n| \le \left[\frac{1}{\tau_\alpha^\rho c_\rho}\ell(\alpha t_0 + (1-\alpha)t_1)\right]^{\frac{1}{\rho}} \sigma_\rho^n, \quad n \ge 0, \tag{2.7}$$

where $\sigma_\rho = \sigma^{\frac{1}{\rho}} \in (0,1)$. Hence, for all $n \geq 0$ and $m \geq 1$, using (2.7), we obtain

$$\begin{aligned}
|t_n - t_{n+m}| &\leq |t_n - t_{n+1}| + \cdots + |t_{n+m-1} - t_{n+m}| \\
&\leq \left(\frac{1}{\tau_\alpha^\rho c_\rho}\ell(\alpha t_0 + (1-\alpha)t_1)\right)^{\frac{1}{\rho}} \left(\sigma_\rho^n + \cdots + \sigma_\rho^{n+m-1}\right) \\
&= \left(\frac{1}{\tau_\alpha^\rho c_\rho}\ell(\alpha t_0 + (1-\alpha)t_1)\right)^{\frac{1}{\rho}} \sigma_\rho^n \frac{1-\sigma_\rho^m}{1-\sigma_\rho} \\
&\leq \frac{1}{1-\sigma_\rho}\left(\frac{1}{\tau_\alpha^\rho c_\rho}\ell(\alpha t_0 + (1-\alpha)t_1)\right)^{\frac{1}{\rho}} \sigma_\rho^n.
\end{aligned}$$

Since $\sigma_\rho^n \to 0$ as $n \to \infty$, it holds that $\{t_n\}$ is a Cauchy sequence. Thus, there exists $t^* \geq 0$ such that

$$\lim_{n\to\infty} f^n(t_0) = \lim_{n\to\infty} t_n = t^*. \tag{2.8}$$

Furthermore, since C is closed and $\{t_n\} \subset C$, then

$$t^* \in C. \tag{2.9}$$

Next, by (iii), (2.8), and (2.9), we deduce that $\{f^n(t_0)\}$ admits a subsequence $\{f^{n_q}(t_0)\}$ such that

$$\lim_{q\to\infty} f^{n_q+1}(t_0) = \lim_{q\to\infty} f(f^{n_q}(t_0)) = f(t^*). \tag{2.10}$$

In view of (2.8) and (2.10), we obtain $t^* = f(t^*)$, that is, $t^* \in \mathrm{Fix}(f)$.

The second step is to show that t^* is the unique fixed point of f. We use the contradiction assuming that there exists $s^* \in \mathrm{Fix}(f)$ such that $t^* \neq s^*$ (or, equivalently, $f(t^*) \neq f(s^*)$). Then, using (2.2) with $(t,s) = (t^*, s^*)$, we obtain

$$\ell\left(\alpha f(t^*) + (1-\alpha)f(s^*)\right) \leq \sigma\ell(\alpha t^* + (1-\alpha)s^*),$$

that is,

$$\ell\left(\alpha t^* + (1-\alpha)s^*\right) \leq \sigma\ell(\alpha t^* + (1-\alpha)s^*). \tag{2.11}$$

On the other hand, if

$$\alpha t^* + (1-\alpha)s^* = 0,$$

then $t^* = s^* = 0$, which is impossible, since $t^* \neq s^*$. Then,

$$\alpha t^* + (1-\alpha)s^* > 0,$$

which implies by (2.1) that

$$\ell\left(\alpha t^* + (1-\alpha)s^*\right) > 0.$$

Hence, dividing (2.11) by $\ell\left(\alpha t^* + (1-\alpha)s^*\right)$, we reach a contradiction with $\sigma < 1$. Then, t^* is the unique fixed point of f. The proof of Theorem 2.1 is completed. □

3. Some special cases

It can be easily seen that, if f is continuous on C, then condition (iii) of Theorem 2.1 is satisfied. Then, from Theorem 2.1, we deduce the following result.

Corollary 3.1. *Let $C \subset [0,\infty)$, $C \neq \emptyset$, and $f : C \to C$. Assume that the following conditions hold:*

(i) *C is a closed subset;*
(ii) *$f \in \mathbb{F}_C(\rho, \ell, \sigma, \alpha)$ for some $\rho > 0$, $\ell \in L_\rho$, and $\alpha, \sigma \in (0,1)$;*
(iii) *f is continuous on C.*

Then, f possesses a unique fixed point. Moreover, for every $t_0 \in C$, the sequence $\{f^n(t_0)\}$ converges to this fixed point.

Next, we consider the case when $C \subset [0,\infty)$ is a closed interval and ℓ is a convex or concave function on C. Just before, we recall the following lemma (see, e.g., [24]).

Lemma 3.1. *(Hermite-Hadamard inequalities) Let $C \subset [0,\infty)$ be an interval.*

(ι) *If $\ell : C \to \mathbb{R}$ is a convex function, then for all $a, b \in C$ with $a < b$, we have*

$$\ell\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \ell(z)\,dz.$$

($\iota\iota$) *If $\ell : C \to \mathbb{R}$ is a concave function, then for all $a, b \in C$ with $a < b$, we have*

$$\frac{1}{b-a}\int_a^b \ell(z)\,dz \leq \ell\left(\frac{a+b}{2}\right).$$

Corollary 3.2. *Let $C \subset [0,\infty)$, $C \neq \emptyset$, and $f : C \to C$. Assume that the following conditions hold:*

(i) *C is a closed interval;*
(ii) *There exist $\rho > 0$ and $\ell \in L_\rho$ such that ℓ is convex on C;*
(iii) *There exist $\alpha, \sigma \in (0,1)$ such that the inequality*

$$\alpha\ell(f(t)) + (1-\alpha)\ell(f(s)) \leq \sigma\ell(\alpha t + (1-\alpha)s) \tag{3.1}$$

holds for every $t, s \in C$ with $f(t) \neq f(s)$;
(iv) *f is continuous on C.*

Then, f possesses a unique fixed point. Moreover, for every $t_0 \in C$, the sequence $\{f^n(t_0)\}$ converges to this fixed point.

Proof. By the convexity of ℓ on C, for every $t, s \in C$ with $f(t) \neq f(s)$, we obtain

$$\ell\left(\alpha f(t) + (1-\alpha)f(s)\right) \leq \alpha\ell(f(t)) + (1-\alpha)\ell(f(s)),$$

which implies by (3.1) that

$$\ell\left(\alpha f(t) + (1-\alpha)f(s)\right) \leq \sigma\ell(\alpha t + (1-\alpha)s).$$

Consequently, $f \in \mathbb{F}_C(\rho, \ell, \sigma, \alpha)$. Then, applying Corollary 3.1, we obtain the desired result. □

Corollary 3.3. *Let $C \subset [0,\infty)$, $C \neq \emptyset$, and $f : C \to C$. Assume that the following conditions hold:*

(i) *C is a closed interval;*
(ii) *There exist $\rho > 0$ and $\ell \in L_\rho$ such that ℓ is convex on C;*
(iii) *There exist $\alpha, \sigma \in (0,1)$ such that the inequality*

$$\frac{1}{|f(t)-f(s)|}\int_{\min\{f(t),f(s)\}}^{\max\{f(t),f(s)\}} \ell(z)\,dz \leq \sigma \ell\left(\frac{t+s}{2}\right) \tag{3.2}$$

holds for every $t, s \in C$ with $f(t) \neq f(s)$;
(iv) *f is continuous on C.*

Then, f possesses a unique fixed point. Moreover, for every $t_0 \in C$, the sequence $\{f^n(t_0)\}$ converges to this fixed point.

Proof. Since ℓ is convex on C, by Lemma 3.1 (ι), for every $t, s \in C$ with $f(t) \neq f(s)$, we have

$$\frac{1}{|f(t)-f(s)|}\int_{\min\{f(t),f(s)\}}^{\max\{f(t),f(s)\}} \ell(z)\,dz \geq \ell\left(\frac{1}{2}f(t)+\frac{1}{2}f(s)\right),$$

which implies by (3.2) that

$$\ell\left(\frac{1}{2}f(t)+\frac{1}{2}f(s)\right) \leq \sigma\ell\left(\frac{1}{2}t+\frac{1}{2}s\right).$$

This shows that f satisfies (2.2) with $\alpha = \frac{1}{2}$, that is, $f \in \mathbb{F}_C(\rho, \ell, \sigma, \frac{1}{2})$. Then, applying Corollary 3.1, we obtain the desired result. □

Corollary 3.4. *Let $C \subset [0,\infty)$, $C \neq \emptyset$, and $f : C \to C$. Assume that the following conditions hold:*

(i) *C is a closed interval;*
(ii) *There exist $\rho > 0$ and $\ell \in L_\rho$ such that ℓ is concave on C;*
(iii) *There exist $\alpha, \sigma \in (0,1)$ such that the inequality*

$$\ell\left(\alpha f(t) + (1-\alpha) f(s)\right) \leq \sigma\left[\alpha\ell(t) + (1-\alpha)\ell(s)\right] \tag{3.3}$$

holds for every $t, s \in C$ with $f(t) \neq f(s)$;
(iv) *f is continuous on C.*

Then, f possesses a unique fixed point. Moreover, for every $t_0 \in C$, the sequence $\{f^n(t_0)\}$ converges to this fixed point.

Proof. By the concavity of ℓ on C, for every $t, s \in C$ with $f(t) \neq f(s)$, we obtain

$$\ell(\alpha t + (1-\alpha)s) \geq \alpha\ell(t) + (1-\alpha)\ell(s),$$

which implies by (3.3) that

$$\ell\left(\alpha f(t) + (1-\alpha) f(s)\right) \leq \sigma\ell(\alpha t + (1-\alpha)s).$$

This shows that $f \in \mathbb{F}_C(\rho, \ell, \sigma, \alpha)$. Then, applying Corollary 3.1, we obtain the desired result. □

Corollary 3.5. *Let $C \subset [0,\infty)$, $C \neq \emptyset$, and $f : C \to C$. Assume that the following conditions hold:*

(i) *C is a closed interval;*
(ii) *There exist $\rho > 0$ and $\ell \in L_\rho$ such that ℓ is concave on C;*
(iii) *There exist $\alpha, \sigma \in (0,1)$ such that the inequality*

$$\ell\left(\frac{f(t)+f(s)}{2}\right) \leq \frac{\sigma}{|t-s|}\int_{\min\{t,s\}}^{\max\{t,s\}} \ell(z)\,dz \tag{3.4}$$

holds for every $t, s \in C$ with $f(t) \neq f(s)$;
(iv) *f is continuous on C.*

Then, f possesses a unique fixed point. Moreover, for every $t_0 \in C$, the sequence $\{f^n(t_0)\}$ converges to this fixed point.

Proof. Since ℓ is concave on C, by Lemma 3.1 (u), for every $t, s \in C$ with $f(t) \neq f(s)$, we have

$$\frac{1}{|t-s|}\int_{\min\{t,s\}}^{\max\{t,s\}} \ell(z)\,dz \leq \ell\left(\frac{t+s}{2}\right),$$

which implies by (3.4) that

$$\ell\left(\frac{1}{2}f(t)+\frac{1}{2}f(s)\right) \leq \sigma\ell\left(\frac{1}{2}t+\frac{1}{2}s\right).$$

This shows that $f \in \mathbb{F}_C(\rho, \ell, \sigma, \frac{1}{2})$. Then, applying Corollary 3.1, we obtain the desired result. □

We now give an example to illustrate Theorem 2.1.

Example 3.1. Let $C = \{0, 1, 3, 5\}$ and $f : C \to C$ be the function defined by

$$f(0) = 0,\ \ f(1) = 5,\ \ f(5) = 3,\ \ f(3) = 0.$$

Observe that

$$\frac{|f(0)-f(1)|}{|0-1|} = |f(0)-f(1)| = 5 > 1,$$

which shows that there is no $k \in (0,1)$ such that

$$|f(t)-f(s)| \leq k|t-s|$$

for every $t, s \in C$. Then, the Banach fixed point theorem is not applicable in this example.

We now introduce the function $\ell : [0,\infty) \to [0,\infty)$ defined by

$$\ell(t) = \begin{cases} 8t & \text{if} \quad 0 \leq t \leq \dfrac{1}{2}, \\ \dfrac{2}{3}t & \text{if} \quad \dfrac{1}{2} < t \leq \dfrac{3}{2}, \\ 3t & \text{if} \quad \dfrac{3}{2} < t \leq 2, \\ \dfrac{4}{5}t & \text{if} \quad 2 < t \leq \dfrac{5}{2}, \\ 3t & \text{if} \quad \dfrac{5}{2} < t \leq 3, \\ \dfrac{7}{4}t & \text{if} \quad t > 3. \end{cases}$$

It can be easily seen that

$$\ell(t) \geq \frac{2}{3}t, \quad t \geq 0,$$

which shows that $\ell \in L_1$. On the other hand, for all $t, s \in C$, we have

$$f(t) \neq f(s) \Longleftrightarrow (t, s) \in U \cup U',$$

where

$$U = \{(0,1), (0,5), (1,3), (1,5), (3,5)\}$$

and

$$U' = \{(t, s) : (s, t) \in U\}.$$

Let $(t, s) \in U$. If $(t, s) = (0, 1)$, then

$$\frac{\ell\left(\frac{f(t)+f(s)}{2}\right)}{\ell\left(\frac{t+s}{2}\right)} = \frac{\ell\left(\frac{5}{2}\right)}{\ell\left(\frac{1}{2}\right)} = \frac{1}{2}.$$

If $(t, s) = (0, 5)$, then

$$\frac{\ell\left(\frac{f(t)+f(s)}{2}\right)}{\ell\left(\frac{t+s}{2}\right)} = \frac{\ell\left(\frac{3}{2}\right)}{\ell\left(\frac{5}{2}\right)} = \frac{1}{2}.$$

If $(t, s) = (1, 3)$, then

$$\frac{\ell\left(\frac{f(t)+f(s)}{2}\right)}{\ell\left(\frac{t+s}{2}\right)} = \frac{\ell\left(\frac{5}{2}\right)}{\ell(2)} = \frac{1}{3}.$$

If $(t, s) = (1, 5)$, then

$$\frac{\ell\left(\frac{f(t)+f(s)}{2}\right)}{\ell\left(\frac{t+s}{2}\right)} = \frac{\ell(4)}{\ell(3)} = \frac{7}{9}.$$

If $(t, s) = (3, 5)$, then

$$\frac{\ell\left(\frac{f(t)+f(s)}{2}\right)}{\ell\left(\frac{t+s}{2}\right)} = \frac{\ell\left(\frac{3}{2}\right)}{\ell(4)} = \frac{1}{7}.$$

The above calculations show that for all $(t, s) \in U$, we have

$$\ell\left(\frac{f(t) + f(s)}{2}\right) \leq \frac{7}{9}\ell\left(\frac{t+s}{2}\right).$$

Notice that by symmetry, the above inequality holds also for all $(t, s) \in U'$. Consequently, f satisfies (2.2) for every $t, s \in C$ with $f(t) \neq f(s)$, where $\alpha = \frac{1}{2}$, $\sigma = \frac{7}{9}$, and ℓ is the mapping defined above, that is, $f \in \mathbb{F}_C\left(1, \ell, \frac{7}{9}, \frac{1}{2}\right)$. Furthermore, for all $n \geq 3$, we have

$$f^n(t) = 0, \quad t \in C.$$

This shows that condition (iii) of Theorem 2.1 is satisfied. Thus, Theorem 2.1 applies. Notice that $\mathrm{Fix}(f) = \{0\}$, which confirms Theorem 2.1.

4. Conclusions

We introduced the new class of functions $f : C \to C$ satisfying the functional inequality

$$\ell\left(\alpha f(t) + (1-\alpha) f(s)\right) \leq \sigma \ell(\alpha t + (1-\alpha)s)$$

for every $t, s \in C$ with $f(t) \neq f(s)$, where C is a closed subset of $[0, \infty)$, $\alpha, \sigma \in (0, 1)$ are constants, and $\ell : [0, \infty) \to [0, \infty)$ is a function satisfying the condition $\inf_{t>0} \frac{\ell(t)}{t^\rho} > 0$ for some constant $\rho > 0$. We proved that, if f is continuous (or, more generally, f satisfies condition (iii) of Theorem 2.1), then f possesses a unique fixed point. Moreover, for any $t_0 \in C$, the Picard sequence $\{f^n(t_0)\}$ converges to this fixed point. Next, making use of the Hermite-Hadamard inequalities, we deduced from our main result new fixed point theorems in the special cases when f is a convex or concave function.

The proposed approach needs to be more developed. For instance, the following issues deserve to be studied:

I. The study of fixed points for mappings $f : C \to C$ satisfying new functional inequalities of type (2.2). For instance, one can study the possibility of extending the obtained results to functions f satisfying the inequality

$$\ell(\alpha f(t) + (1-\alpha) f(s)) \leq \sigma_1 \ell(\alpha t + (1-\alpha)s) + \sigma_2 \ell\left(\alpha t + (1-\alpha) f(t)\right) + \sigma_3 \ell\left(\alpha s + (1-\alpha) f(s)\right)$$

for every $t, s \in C$ with $f(t) \neq f(s)$, where $\sigma_1 + \sigma_2 + \sigma_3 > 0$.

II. The extension of the obtained results from $\mathbb{R}$ to a Banach space E partially ordered by a cone P. Indeed, one can study the class of mappings $f : C \to C$ satisfying the functional inequality

$$\ell(\alpha f(u) + (1-\alpha) f(v)) \preceq \sigma \ell(\alpha u + (1-\alpha)v)$$

for every $u, v \in C$ with $f(u) \neq f(v)$, where $C \subset P$, $\sigma \in (0, 1)$, $\ell : P \to P$, and $\preceq$ is the partial order induced by the cone P.

III. It would be interesting to study the possible applications of the obtained results.

Author contributions

Hassen Aydi: Conceptualization, methodology, investigation, formal analysis, writing review and editing; Bessem Samet: Conceptualization, methodology, validation, investigation, writing original draft preparation; Manuel De La Sen: Methodology, validation, formal analysis, investigation, writing review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interest.

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