Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 29001–29017.
DOI: 10.3934/math.20241407
Received: 21 July 2024
Revised: 24 August 2024
Accepted: 29 August 2024
Published: 14 October 2024

Research article

On θ-hyperbolic sine distance functions and existence results in complete metric spaces

Mohamed Jleli and Bessem Samet*

Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: bsamet@ksu.edu.sa.

Abstract: In this paper, we first introduced the notion of θ-hyperbolic sine distance functions on a metric space and studied their properties. We investigated the existence and uniqueness of fixed points for some classes of single-valued mappings defined on a complete metric space and satisfying contractions involving the θ-hyperbolic sine distance function.

Keywords: metric spaces; hyperbolic functions; θ-hyperbolic sine distance functions; fixed point
Mathematics Subject Classification: 54E50, 54E25, 47H10, 33B10

1. Introduction

A distance function or a metric on a set Q is a mapping $\delta : Q \times Q \to [0, \infty)$ satisfying the axioms:

(i) $\delta(u, v) = 0$ if and only if $u = v$,
(ii) $\delta(u, v) = \delta(v, u)$,
(iii) $\delta(u, v) \leq \delta(u, w) + \delta(w, v)$ (triangle inequality)

for every $u, v, w \in Q$. If δ is a metric on Q, then the pair (Q, δ) is called a metric space. Metric spaces constitute a fundamental notion in various mathematical disciplines such as geometry, topology, and analysis. Moreover, metric spaces are used in many applications such as the design of approximation algorithms, neuroscience, image restoration, and medical image classification (see, e.g., [1–3]). However, in some applications, one of the axioms (i)–(iii) is not appropriate. For instance, in denotational semantic, the metric space topology is not appropriate (see, e.g., [4]). This fact motivated many researchers to propose various concepts generalizing the notion of metric spaces. For instance, Smyth [5] generalized the metric notion by dropping the symmetry condition (ii). Matthews [4] introduced the notion of partial metric spaces, where the distance of an element to itself may not be zero. Mustafa and Sims [6] introduced the notion of G-metric spaces, where $G : Z\times Z\times Z \to [0, \infty)$ is a mapping satisfying certain axioms. The literature includes many other interesting extensions of the notion of metric spaces such as almost metric spaces [9], rectangular metric spaces [8], b-metric spaces [9, 10], suprametric spaces [11], F-metric spaces [12], and hemi metric spaces [13]. We also refer to the monograph [14] that contains several fixed point results in various generalized metric spaces.

Fixed point theory is one of the most important topics in pure and applied mathematics. Indeed, the question of existence of solutions to a nonlinear problem is often reduced to a fixed point problem. For some recent applications of fixed point theory, see, e.g., [15–17]. One of the widely used theorems in the study of existence results for nonlinear problems is the Banach contraction principle [18]. This theorem can be stated as follows: Let (Q, δ) be a complete metric space and $F : Q \to Q$ be a mapping such that

$$\delta(Fu, Fv) \leq \kappa\delta(u, v) \tag{1.1}$$

for every $u, v \in Q$, where $\kappa \in (0, 1)$ is a constant. Then, F admits one and only one fixed point. Moreover, for every $w_0 \in Q$, the Picard sequence $\{F^n w_0\}$ converges to this unique fixed point. A mapping F satisfying (1.1) is called a contraction. The literature includes several generalizations and extensions of Banach's contraction principle. We can classify these generalizations and extensions into two major categories. The first one is concerned with weakening the contractive nature of the mapping such as Kannan's contraction [19], quasi-contractions [20], Meir-Keeler-type contractions [21, 22], contractions involving simulation functions [23, 24], (α, ψ)-contractions [25, 26], (ψ, φ)-weak contractions [27, 28], (CAB) contractions [29], and mappings contracting the perimeter of a triangle [30]. We also refer to [31] for various types of contractions. The second category is concerned with the study of fixed points when the underlying space is equipped with a generalized metric, see, e.g., the above mentioned references related to generalized metric spaces and [32].

Hyperbolic functions arise in many applications of mathematics, physics, chemistry, and engineering. Motivated by this fact and the above cited contributions, we introduce in this paper the notion of θ-hyperbolic sine distance functions associated to a certain metric, where $\theta : [0, \infty) \to [0, \infty)$ satisfies the condition

$$\theta(r) \geq cr^\tau \tag{1.2}$$

for all $r \geq 0$, for some constants $c, \tau > 0$. Namely, given a metric space (Q, δ), the mapping $\delta_\theta : Q \times Q \to [0, \infty)$ defined by $\delta_\theta(u, v) = \theta(\sinh(\delta(u, v)))$ for all $u, v \in Q$, is called the θ-hyperbolic sine distance function associated to the metric δ. In this work, we establish several properties of δ_θ and new fixed point results for mappings $F : Q \to Q$ satisfying contractions involving the θ-hyperbolic sine distance function.

In Section 2, we introduce the concept of θ-hyperbolic sine distance functions and establish several properties. In Section 3, we consider two classes of mappings $F : Q \to Q$, where (Q, δ) is a complete metric space. Namely, we first consider the class of mappings F satisfying

$$\delta_\theta(Fu, Fv) \leq \kappa\delta_\theta(u, v)$$

for all $u, v \in Q$ with $Fu \neq Fv$, where $\kappa \in (0, 1)$ is a constant. We next consider the class of mappings F satisfying

$$\delta_\theta(Fu, Fv) \leq \eta(\delta_\theta(u, v))$$

for all $u, v \in Q$ with $Fu \neq Fv$, where $\eta : [0, \infty) \to [0, \infty)$ is a (c)-comparison function and θ satisfies (1.2) with $\tau = 1$. For both classes, we investigate the existence and uniqueness of fixed points. In Section 4, an application to integral equations is provided.

Throughout this paper, the following notations will be used. By Q, we mean an arbitrary nonempty set. For a mapping $F : Q \to Q$, we denote by $\{F^n\}$ the sequence of mappings $F^n : Q \to Q$ defined by

$$F^0 = I_Q \text{ (the identity mapping)}, \ F^{n+1} = F \circ F^n, \quad n \geq 0.$$

We denote by Fix(F) the set of fixed points of F, that is,

$$\text{Fix}(F) = \{u \in Q : \ Fu = u\}.$$

Similarly, for $\eta : [0, \infty) \to [0, \infty)$, we denote by $\{\eta^n\}$ the sequence of functions $\eta^n : Q \to Q$ defined by

$$\eta^0 = I_{[0,\infty)}, \ \eta^{n+1} = \eta \circ \eta^n, \quad n \geq 0.$$

2. θ-hyperbolic sine distance functions

In this section, we introduce the notion of θ-hyperbolic sine distance functions.

For all $\tau > 0$, we denote by Θ_τ the set of functions $\theta : [0, \infty) \to [0, \infty)$ satisfying the condition

$$\theta(r) \geq cr^\tau, \tag{2.1}$$

for all $r \geq 0$, where $c > 0$ is a constant.

Definition 2.1. Let (Q, δ) be a metric space. For all $\tau > 0$ and $\theta \in \Theta_\tau$, we define the mapping $\delta_\theta : Q \times Q \to [0, \infty)$ by

$$\delta_\theta(u, v) = \theta(\sinh(\delta(u, v))), \quad u, v \in Q,$$

where sinh is the hyperbolic sine function defined by

$$\sinh r = \frac{e^r - e^{-r}}{2}, \quad r \in \mathbb{R}.$$

The mapping δ_θ is called the θ-hyperbolic sine distance function associated to the metric δ.

Some properties of the θ-hyperbolic sine distance function are provided below.

Proposition 2.1. *Let (Q, δ) be a metric space and $\theta \in \Theta_\tau$ for some $\tau > 0$. Then, for all $u, v \in Q$, we have*

(i) $\delta_\theta(u, v) = 0 \implies u = v$.
(ii) *If* $\theta(0) = 0$, *then* $\delta_\theta(u, u) = 0$.
(iii) $\delta_\theta(u, v) = \delta_\theta(v, u)$.

Proof. (i) Let $u, v \in Q$ be such that $\delta_\theta(u, v) = 0$, that is, $\theta(\sinh(\delta(u, v)) = 0$. Then, by (2.1), we have

$$0 = \theta(\sinh(\delta(u, v))) \geq c[\sinh(\delta(u, v))]^\tau,$$

which implies that $\sinh(\delta(u, v)) = 0$, that is, $\delta(u, v) = 0$. Since δ is a metric on Q, it holds that $u = v$, which proves (i).
(ii) If $\theta(0) = 0$, then

$$\begin{aligned} \delta_\theta(u, u) &= \theta(\sinh(\delta(u, u))) \\ &= \theta(\sinh(0)) \\ &= \theta(0) \\ &= 0, \end{aligned}$$

which shows (ii).
(iii) is obvious. □

We point out that a θ-hyperbolic sine distance function is not necessarily a metric, even if $\theta(0) = 0$. The following example shows this fact.

Example 2.1. Let $Q = \mathbb{R}$ and $\delta(u, v) = |u - v|$ for all $u, v \in Q$. Let $\theta(r) = r$ for all $r \geq 0$. Then, $\theta \in \Theta_1$. The θ-hyperbolic sine distance function associated to the metric δ is defined by

$$\delta_\theta(u, v) = \theta(\sinh(|u - v|))$$

for all $u, v \in Q$. On the other hand, for all $n \geq 0$, we have

$$\begin{aligned} \frac{\delta_\theta(0, 2n)}{\delta_\theta(0, n) + \delta_\theta(n, 2n)} &= \frac{\theta(\sinh(2n))}{\theta(\sinh(n)) + \theta(\sinh(n))} \\ &= \frac{\sinh(2n)}{2 \sinh n} \\ &= \frac{e^n + e^{-n}}{2} \to \infty \text{ as } n \to \infty, \end{aligned}$$

which shows that δ_θ does not satisfy the triangle inequality. Consequently, δ_θ is not a metric on Q.

Proposition 2.2. *Let (Q, δ) be a metric space.*

(i) *Let δ_θ be the θ-hyperbolic sine distance function associated to δ, where $\theta \in \Theta_\tau$ for some $\tau > 0$. Then, for all $\iota > 0$, we have*

$$\iota\delta_\theta = \delta_{\theta_\iota},$$

where $\theta_\iota = \iota\theta$.

(ii) *Let $\theta_1, \theta_2 \in \Theta_\tau$ for some $\tau > 0$. Then,*

$$\delta_{\theta_1} + \delta_{\theta_2} = \delta_\theta,$$

where $\theta = \theta_1 + \theta_2$.

Proof. (i) Let $\iota > 0$. Since $\theta \in \Theta_\tau$, then $\theta_\iota \in \Theta_\tau$, and

$$\begin{aligned} \iota\delta_\theta(u, v) &= \iota\theta(\sinh(\delta(u, v))) \\ &= \theta_\iota(\sinh(\delta(u, v))) \\ &= \delta_{\theta_\iota}(u, v) \end{aligned}$$

for all $u, v \in Q$, which proves (i).
(ii) Notice that since $\theta_1, \theta_2 \in \Theta_\tau$, then for all $r \geq 0$, we have

$$\begin{aligned} \theta(r) &= \theta_1(r) + \theta_2(r) \\ &\geq c_1 r^\tau + c_2 r^\tau \\ &= (c_1 + c_2) r^\tau, \end{aligned}$$

where $c_1, c_2 > 0$ are constants, which shows that $\theta \in \Theta_\tau$. Then, for all $u, v \in Q$, we have

$$\begin{aligned}(\delta_{\theta_1} + \delta_{\theta_2})(u, v) &= \delta_{\theta_1}(u, v) + \delta_{\theta_2}(u, v)\\ &= \theta_1(\sinh(\delta(u, v))) + \theta_2(\sinh(\delta(u, v)))\\ &= (\theta_1 + \theta_2)(\sinh(\delta(u, v)))\\ &= \theta(\sinh(\delta(u, v)))\\ &= \delta_\theta(u, v),\end{aligned}$$

which proves (ii). □

Proposition 2.3. *Let $\theta \in \Theta_\tau$ for some $\tau > 0$. Assume that the following conditions hold:*

(i) $\theta(0) = 0$.
(ii) *There exists $r^* > 0$ such that*

$$\theta(\sinh r^*) = r^*.$$

Then, for every nonempty set Q, there exists a metric δ on Q such that the θ-hyperbolic sine distance function associated to δ coincides with δ, i.e., $\delta_\theta = \delta$.

Proof. Let Q be an arbitrary nonempty set. We introduce the mapping $\delta : Q \times Q \to [0, \infty)$ defined by

$$\delta(u, v) = \begin{cases} r^* & \text{if} \quad u \neq v,\\ 0 & \text{if} \quad u = v\end{cases}$$

for all $u, v \in Q$. Since $r^* > 0$, then δ is a metric on Q. Furthermore, using (i) and (ii), we obtain that for all $u, v \in Q$,

$$\begin{aligned}\delta_\theta(u, v) =&\theta(\sinh(\delta(u, v)))\\ =& \begin{cases} \theta(\sinh r^*) & \text{if} \quad u \neq v,\\ \theta(\sinh 0) & \text{if} \quad u = v\end{cases}\\ =& \begin{cases} r^* & \text{if} \quad u \neq v,\\ \theta(0) & \text{if} \quad u = v\end{cases}\\ =& \begin{cases} r^* & \text{if} \quad u \neq v,\\ 0 & \text{if} \quad u = v\end{cases}\\ =& \delta(u, v),\end{aligned}$$

which proves that $\delta_\theta = \delta$. □

Example 2.2. Let us consider the function $\theta : [0, \infty) \to [0, \infty)$ defined by

$$\theta(r) = r^2, \quad r \geq 0.$$

Then, $\theta \in \Theta_2$ and $\theta(0) = 0$. On the other hand, we have

$$\theta\left(\sinh\frac{1}{2}\right) - \frac{1}{2} = \sinh^2\frac{1}{2} - \frac{1}{2} \approx -0.22845968259$$

and

$$\theta(\sinh 1) - 1 = \sinh^2 1 - 1 \approx 0.3810.$$

Since $\theta\left(\sinh \frac{1}{2}\right) - \frac{1}{2} < 0$, $\theta(\sinh 1) - 1 > 0$, and the function $r \mapsto \theta(\sinh r) - r$ is continuous on $[\frac{1}{2}, 1]$, we deduce from the intermediate value theorem that there exists $r^* \in (\frac{1}{2}, 1)$ such that $\theta(\sinh r^*) - r^* = 0$. Then, Proposition 2.3 applies.

3. Fixed point results

In this section, we study the existence and uniqueness of fixed points for some classes of single-valued mappings defined on (Q, δ_θ), where δ_θ is a θ-hyperbolic sine distance function associated to a metric δ.

3.1. The class of θ-hyperbolic contractions

In this subsection, we are concerned with the following class of mappings.

Definition 3.1. Let (Q, δ) be a metric space and $\theta \in \Theta_\tau$ for some $\tau > 0$. A mapping $F : Q \to Q$ is called a θ-hyperbolic contraction on Q, if there exists $\kappa \in (0, 1)$ such that

$$\delta_\theta(Fu, Fv) \leq \kappa \delta_\theta(u, v) \tag{3.1}$$

for all $u, v \in Q$ with $Fu \neq Fv$.

Our first main result is the following fixed point theorem.

Theorem 3.1. *Let (Q, δ) be a complete metric space and $\theta \in \Theta_\tau$ for some $\tau > 0$. Let $F : Q \to Q$ be a mapping such that*

(I) *F is a θ-hyperbolic contraction on Q.*
(II) *For all $u, v \in Q$, if $\lim_{n\to\infty} \delta(F^n u, v) = 0$, then there exists a subsequence $\{F^{n_k} u\}$ of $\{F^n u\}$ such that $\lim_{k\to\infty} \delta(F(F^{n_k} u), Fv) = 0$.*

Then, F possesses one and only one fixed point. Moreover, for all $w_0 \in Q$, the sequence $\{F^n w_0\}$ converges to this unique fixed point.

Proof. We first prove that $\text{Fix}(F) \neq \emptyset$. For an arbitrary $w_0 \in Q$, let $\{w_n\} \subset Q$ be the Picard sequence defined by

$$w_n = F^n w_0, \quad n \geq 0.$$

If $w_m = w_{m+1}$ for some $m \geq 0$, then $w_m \in \text{Fix}(F)$. So, we may assume that $w_n \neq w_{n+1}$ for all $n \geq 0$. Then, using (3.1) with $(u, v) = (w_0, w_1)$ (since $Fw_0 \neq Fw_1$), we get

$$\delta_\theta(Fw_0, Fw_1) \leq \kappa \delta_\theta(w_0, w_1),$$

that is,

$$\delta_\theta(w_1, w_2) \leq \kappa \delta_\theta(w_0, w_1). \tag{3.2}$$

Using again (3.1) with $(u, v) = (w_1, w_2)$, we obtain

$$\delta_\theta(Fw_1, Fw_2) \leq \kappa \delta_\theta(w_1, w_2),$$

that is,

$$\delta_\theta(w_2, w_3) \le \kappa\delta_\theta(w_1, w_2). \tag{3.3}$$

Then, it follows from (3.2) and (3.3) that

$$\delta_\theta(w_2, w_3) \le \kappa^2\delta_\theta(w_0, w_1).$$

Continuing in the same way, we obtain by induction that

$$\delta_\theta(w_n, w_{n+1}) \le \kappa^n\delta_\theta(w_0, w_1), \quad n \ge 0,$$

which is equivalent to

$$\theta(\sinh(\delta(w_n, w_{n+1}))) \le \kappa^n\delta_\theta(w_0, w_1), \quad n \ge 0. \tag{3.4}$$

On the other hand, by (2.1), we have

$$\theta(\sinh(\delta(w_n, w_{n+1}))) \ge c[\sinh(\delta(w_n, w_{n+1}))]^\tau, \quad n \ge 0. \tag{3.5}$$

Furthermore, making use of the elementary inequality

$$\frac{\sinh r}{r} > 1, \quad r > 0,$$

we get

$$c[\sinh(\delta(w_n, w_{n+1}))]^\tau > c\delta^\tau(w_n, w_{n+1}), \quad n \ge 0. \tag{3.6}$$

Thus, in view of (3.4)–(3.6), we obtain

$$\delta(w_n, w_{n+1}) \le \left[\frac{\delta_\theta(w_0, w_1)}{c}\right]^{\frac{1}{\tau}} \left(\kappa^{\frac{1}{\tau}}\right)^n, \quad n \ge 0. \tag{3.7}$$

Then, using (3.7) and the triangle inequality, we obtain that for all $n \ge 0$ and $m \ge 1$,

$$\begin{aligned}
\delta(w_n, w_{n+m}) &\le \delta(w_n, w_{n+1}) + \cdots + \delta(w_{n+m-1}, w_{n+m}) \\
&\le \left[\frac{\delta_\theta(w_0, w_1)}{c}\right]^{\frac{1}{\tau}} \left(\left(\kappa^{\frac{1}{\tau}}\right)^n + \cdots + \left(\kappa^{\frac{1}{\tau}}\right)^{n+m-1}\right) \\
&= \left[\frac{\delta_\theta(w_0, w_1)}{c}\right]^{\frac{1}{\tau}} \frac{1-\left(\kappa^{\frac{1}{\tau}}\right)^m}{1-\kappa^{\frac{1}{\tau}}} \left(\kappa^{\frac{1}{\tau}}\right)^n \\
&\le \left[\frac{\delta_\theta(w_0, w_1)}{c}\right]^{\frac{1}{\tau}} \frac{1}{1-\kappa^{\frac{1}{\tau}}} \left(\kappa^{\frac{1}{\tau}}\right)^n \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Consequently, $\{w_n\}$ is a Cauchy sequence in the complete metric space (Q, δ). Thus, there exists $\bar{w} \in Q$ such that

$$\lim_{n\to\infty} \delta(F^n w_0, \bar{w}) = \lim_{n\to\infty} \delta(w_n, \bar{w}) = 0. \tag{3.8}$$

Hence, by (II), there exists a subsequence $\{F^{n_k} w_0\}$ of $\{F^n w_0\}$ such that

$$\lim_{k\to\infty} \delta(F(F^{n_k} w_0), F\bar{w}) = 0,$$

that is,

$$\lim_{k\to\infty} \delta(F^{n_k+1}w_0, F\bar{w}) = 0. \tag{3.9}$$

Then, in view of (3.8) and (3.9), it holds that $\bar{w} \in \mathrm{Fix}(F)$.

We now show that $\mathrm{Fix}(F) = \{\bar{w}\}$. Indeed, if $\bar{z} \in \mathrm{Fix}(F)$ and $\bar{w} \neq \bar{z}$ (or, equivalently, $F\bar{w} \neq F\bar{z}$), then, using (3.1) with $(u, v) = (\bar{w}, \bar{z})$, we get

$$\delta_\theta(F\bar{w}, F\bar{z}) \leq \kappa\delta_\theta(\bar{w}, \bar{z}),$$

that is,

$$\delta_\theta(\bar{w}, \bar{z}) \leq \kappa\delta_\theta(\bar{w}, \bar{z}).$$

On the other hand, since $\bar{w} \neq \bar{z}$, then by Proposition 2.1 (i), we have $\delta_\theta(\bar{w}, \bar{z}) > 0$. Thus, dividing the above inequality by $\delta_\theta(\bar{w}, \bar{z})$, we reach a contradiction with $\kappa \in (0, 1)$. Consequently, $\bar{w}$ is the unique fixed point of F. This completes the proof of Theorem 3.1. □

Clearly, if F is continuous, then condition (II) of Theorem 3.1 is satisfied. Hence, we deduce from Theorem 3.1 the following result.

Corollary 3.1. *Let (Q, δ) be a complete metric space and $\theta \in \Theta_\tau$ for some $\tau > 0$. Let $F : Q \to Q$ be a mapping such that*

(I) *F is a θ-hyperbolic contraction on Q.*
(II) *F is continuous.*

Then, F possesses one and only one fixed point. Moreover, for all $w_0 \in Q$, the sequence $\{F^n w_0\}$ converges to this unique fixed point.

3.2. The class of (θ, η)-hyperbolic contractions

We recall below the notion of (c)-comparison functions, see, e.g., [31].

Definition 3.2. A (c)-comparison function is a function $\eta : [0, \infty) \to [0, \infty)$ satisfying the properties:

(C_1) η is nondecreasing.
(C_2) For all $t > 0$, we have $\sum_{n\geq 0} \eta^n(t) < \infty$.

Remark 3.1. It can be easily seen that, if η is a (c)-comparison function, then

(i) For all $t > 0$, we have $\lim_{n\to\infty} \eta^n(t) = 0$.
(ii) For all $t > 0$, we have $\eta(t) < t$.
(iii) $\eta(0) = 0$.

Example 3.1. A standard example of (c)-comparison functions, is the function

$$\eta(t) = \ell t, \quad t \geq 0,$$

where $\ell \in (0, 1)$ is a constant.

Example 3.2. Let η be the function defined by

$$\eta(t) = a \arctan t, \quad t \geq 0,$$

where $a \in (0, 1)$ is a constant. Then, η is a (c)-comparison function.

We now consider the following class of mappings.

Definition 3.3. Let (Q, δ) be a metric space, $\theta \in \Theta_1$, and η be a (c)-comparison function. A mapping $F : Q \to Q$ is called a (θ, η)-hyperbolic contraction on Q, if

$$\delta_\theta(Fu, Fv) \leq \eta(\delta_\theta(u, v)) \tag{3.10}$$

for all $u, v \in Q$ with $Fu \neq Fv$.

We have the following fixed point result.

Theorem 3.2. *Let (Q, δ) be a complete metric space, $\theta \in \Theta_1$, and η be a* (c)*-comparison function. Let $F : Q \to Q$ be a mapping such that*

(I) *F is a (θ, η)-hyperbolic contraction on Q.*
(II) *For all $u, v \in Q$, if $\lim_{n\to\infty} \delta(F^n u, v) = 0$, then there exists a subsequence $\{F^{n_k} u\}$ of $\{F^n u\}$ such that $\lim_{k\to\infty} \delta(F(F^{n_k} u), Fv) = 0$.*

Then, F possesses one and only one fixed point. Moreover, for all $w_0 \in Q$, the sequence $\{F^n w_0\}$ converges to this unique fixed point.

Proof. Let us show that $\mathrm{Fix}(F) \neq \emptyset$. For an arbitrary $w_0 \in Q$, let $\{w_n\} \subset Q$ be the Picard sequence defined by

$$w_n = F^n w_0, \quad n \geq 0.$$

Without restriction of the generality, we may assume that $w_n \neq w_{n+1}$ for all $n \geq 0$ (otherwise, there exists $m \geq 0$ such that $w_m \in \mathrm{Fix}(F)$). Then, using (3.10) with $(u, v) = (w_0, w_1)$, we get

$$\delta_\theta(Fw_0, Fw_1) \leq \eta\left(\delta_\theta(w_0, w_1)\right),$$

that is,

$$\delta_\theta(w_1, w_2) \leq \eta\left(\delta_\theta(w_0, w_1)\right).$$

Since η is a nondecreasing function, the above inequality yields

$$\eta\left(\delta_\theta(w_1, w_2)\right) \leq \eta^2\left(\delta_\theta(w_0, w_1)\right). \tag{3.11}$$

Using again (3.10) with $(u, v) = (w_1, w_2)$, we obtain

$$\delta_\theta(Fw_1, Fw_2) \leq \eta\left(\delta_\theta(w_1, w_2)\right),$$

that is,

$$\delta_\theta(w_2, w_3) \leq \eta\left(\delta_\theta(w_1, w_2)\right). \tag{3.12}$$

Then, it follows from (3.11) and (3.12) that

$$\delta_\theta(w_2, w_3) \le \eta^2 \left(\delta_\theta(w_0, w_1)\right).$$

Continuing in the same way, it holds by induction that

$$\delta_\theta(w_n, w_{n+1}) \le \eta^n \left(\delta_\theta(w_0, w_1)\right), \quad n \ge 0,$$

which is equivalent to

$$\theta(\sinh(\delta(w_n, w_{n+1}))) \le \eta^n \left(\delta_\theta(w_0, w_1)\right), \quad n \ge 0. \tag{3.13}$$

On the other hand, since $\theta \in \Theta_1$, by (2.1) (with $\tau = 1$), we have

$$\theta(\sinh(\delta(w_n, w_{n+1}))) \ge c \sinh(\delta(w_n, w_{n+1})), \quad n \ge 0. \tag{3.14}$$

Then, it follows from (3.6) (with $\tau = 1$), (3.13), and (3.14) that

$$\delta(w_n, w_{n+1}) \le \frac{1}{c}\eta^n \left(\delta_\theta(w_0, w_1)\right), \quad n \ge 0. \tag{3.15}$$

Then, by (3.15) and using the triangle inequality, for all $n \ge 0$ and $m \ge 1$, we get

$$\begin{aligned}\delta(w_n, w_{n+m}) &\le \sum_{i=n}^{n+m-1} \delta(w_i, w_{i+1})\\ &\le \frac{1}{c}\sum_{i=n}^{n+m-1} \eta^i \left(\delta_\theta(w_0, w_1)\right)\\ &\le \frac{1}{c}\sum_{i=n}^{\infty} \eta^i \left(\delta_\theta(w_0, w_1)\right).\end{aligned} \tag{3.16}$$

On the other hand, since $\sum_{i\ge 0} \eta^i \left(\delta_\theta(w_0, w_1)\right) < \infty$, it holds that

$$\lim_{n\to\infty} \sum_{i=n}^{\infty} \eta^i \left(\delta_\theta(w_0, w_1)\right) = 0. \tag{3.17}$$

Hence, in view of (3.16) and (3.17), we obtain that $\{w_n\}$ is a Cauchy sequence in the complete metric space (Q, δ). Thus, there exists $\bar{w}$ such that

$$\lim_{n\to\infty} \delta(w_n, \bar{w}) = 0.$$

Proceeding as in the proof of Theorem 3.1, we obtain by (II) that $\bar{w} \in \mathrm{Fix}(F)$.

Suppose now that $\bar{z} \in \mathrm{Fix}(F)$ and $\bar{z} \ne \bar{w}$. Then, using (3.10) with $(u, v) = (\bar{w}, \bar{z})$, we get

$$\delta_\theta(F\bar{w}, F\bar{z}) \le \eta(\delta_\theta(\bar{w}, \bar{z})),$$

that is,

$$\delta_\theta(\bar{w}, \bar{z}) \le \eta(\delta_\theta(\bar{w}, \bar{z})). \tag{3.18}$$

Notice that form Proposition 2.1 (i), one has (since $\bar{z} \neq \bar{w}$) $\delta_\theta(\bar{w}, \bar{z}) > 0$. Then, by Remark 3.1 (ii), we get

$$\eta(\delta_\theta(\bar{w}, \bar{z})) < \delta_\theta(\bar{w}, \bar{z}),$$

which contradicts (3.18). Consequently, $\bar{w}$ is the unique fixed point of F. The proof of Theorem 3.2 is then completed. □

From Theorem 3.2, we deduce the following result.

Corollary 3.2. *Let* (Q, δ) *be a complete metric space,* $\theta \in \Theta_1$*, and* η *be a* (c)*-comparison function. Let* $F : Q \to Q$ *be a mapping such that*

(I) *F is a (θ, η)-hyperbolic contraction on Q.*
(II) *F is continuous.*

Then, F possesses one and only one fixed point. Moreover, for all $w_0 \in Q$, the sequence $\{F^n w_0\}$ converges to this unique fixed point.

Example 3.3. Let $Q = \{q_1, q_2, q_3\}$ and δ be the metric on Q defined by

$$\delta(q_i, q_i) = 0, \ \ \delta(q_i, q_j) = \delta(q_j, q_i), \quad i, j \in \{1, 2, 3\}$$

and

$$\delta(q_1, q_2) = 1, \ \ \delta(q_1, q_3) = 4, \ \ \delta(q_2, q_3) = 5.$$

Notice that δ satisfies the triangle inequality. Indeed, we have

$$\delta(q_1, q_2) = 1 < 4 = \delta(q_1, q_3) < \delta(q_1, q_3) + \delta(q_3, q_2),$$

$$\delta(q_1, q_3) = 4 < 5 = \delta(q_2, q_3) < \delta(q_1, q_2) + \delta(q_2, q_3),$$

and

$$\delta(q_2, q_3) = 5 = \delta(q_2, q_1) + \delta(q_1, q_3).$$

Consequently, (Q, δ) is a metric space.

Consider the mapping $F : Q \to Q$ defined by

$$Fq_1 = q_1, \ \ Fq_2 = q_3, \ \ Fq_3 = q_1.$$

We point out that F is not a contraction in the sense of Banach. Indeed, we have

$$\delta(Fq_1, Fq_2) = \delta(q_1, q_3) = 4 > 1 = \delta(q_1, q_2).$$

We now introduce the mapping $\theta : [0, \infty) \to [0, \infty)$ defined by

$$\theta(r) = \begin{cases} \dfrac{7r}{\sinh 1} & \text{if} \quad 0 \leq r \leq \sinh 1, \\ \dfrac{2r}{\sinh 4} & \text{if} \quad \sinh 1 < r \leq \sinh 4, \\ \dfrac{5r}{\sinh 5} & \text{if} \quad r > \sinh 4. \end{cases}$$

It can be easily seen that

$$\theta(r) \geq \frac{5}{\sinh 5} r, \quad r \geq 0,$$

which shows that $\theta \in \Theta_1$. Furthermore, we have

$$\delta_\theta(Fq_1, Fq_2) = \delta_\theta(q_1, q_3) = \theta(\sinh(\delta(q_1, q_3))) = \theta(\sinh 4) = 2$$

and

$$\delta_\theta(q_1, q_2) = \theta(\sinh(\delta(q_1, q_2))) = \theta(\sinh 1) = 7,$$

which shows that

$$\frac{\delta_\theta(Fq_1, Fq_2)}{\delta_\theta(q_1, q_2)} = \frac{2}{7}. \tag{3.19}$$

We also have

$$\delta_\theta(Fq_2, Fq_3) = \delta_\theta(q_3, q_1) = \theta(\sinh(\delta(q_1, q_3))) = \theta(\sinh 4) = 2$$

and

$$\delta_\theta(q_2, q_3) = \theta(\sinh(\delta(q_2, q_3))) = \theta(\sinh 5) = 5,$$

which shows that

$$\frac{\delta_\theta(Fq_2, Fq_3)}{\delta_\theta(q_2, q_3)} = \frac{2}{5}. \tag{3.20}$$

Thus, from (3.19) and (3.20), we deduce that

$$\delta_\theta(Fq_i, Fq_j) \leq \kappa \delta_\theta(q_i, Fq_j), \quad (i, j) \in \{(1, 2), (2, 1), (2, 3), (3, 2)\},$$

for all $\kappa \in \left[\frac{2}{5}, 1\right)$. This shows that F is a θ-hyperbolic contraction on Q. Then, condition (I) of Theorem 3.1 is satisfied. Notice also that for all $n \geq 2$, we have

$$F^n q_i = q_1, \quad i \in \{1, 2, 3\},$$

which shows that condition (II) of Theorem 3.1 is satisfied. Consequently, Theorem 3.1 applies. On the other hand, we have $\mathrm{Fix}(F) = \{q_1\}$, which confirms Theorem 3.1.

4. An application to integral equations

In this section, making use of Theorem 3.1, we study the existence and uniqueness of solutions to the nonlinear integral equation

$$u(t) = \int_0^1 \xi(t, u(s))\, ds, \quad 0 \leq t \leq 1, \tag{4.1}$$

where $\xi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given function.

The following lemma will be used later (see, e.g., [33]).

Lemma 4.1. *(Jensen's inequality) Let $h : [a, b] \to \mathbb{R}$ be continuous and $f : [c, d] \to \mathbb{R}$ be a convex function such that $h([a, b]) \subset [c, d]$. Then,*

$$f\left(\frac{1}{b-a} \int_a^b h(t)\, dt\right) \leq \frac{1}{b-a} \int_a^b f(h(t))\, dt.$$

We denote by $C([0,1])$ the space of real-valued continuous functions on $[0,1]$. We have the following result.

Theorem 4.1. *Assume that the following conditions hold:*

(i) *The function* ξ *is continuous on* $[0,1]\times\mathbb{R}$.
(ii) *There exists a constant* $\kappa\in(0,1)$ *such that*

$$\sinh\left(|\xi(t,z)-\xi(t,w)|\right)\leq\kappa\sinh\left(|z-w|\right)$$

for all $t\in[0,1]$ *and* $z,w\in\mathbb{R}$.

Then,

(I) *Equation* (4.1) *admits one and only one solution* $u^*\in C([0,1])$.
(II) *For every* $u_0\in C([0,1])$*, the sequence* $\{u_n\}\subset C([0,1])$ *defined by*

$$u_{n+1}(t)=\int_0^1\xi(t,u_n(s))\,ds,\quad 0\leq t\leq 1,$$

converges uniformly to u^*.

Proof. Let $Q=C([0,1])$ and δ be the metric on Q defined by

$$\delta(u,v)=\max_{0\leq t\leq 1}|u(t)-v(t)|,\quad u,v\in Q.$$

It is well known that (Q,δ) is a complete metric space. We introduce the mapping $F:Q\to Q$ defined by

$$(Fu)(t)=\int_0^1\xi(t,u(s))\,ds,\quad 0\leq t\leq 1$$

for all $u\in Q$. Notice that due to (i), we have $FQ\subset Q$. On the other hand, $u\in Q$ is a solution to Eq (4.1) if and only if u is a fixed point of F.

We first show that F is continuous on (Q,δ). Indeed, for all $u,v\in Q$ and $t\in[0,1]$, we have

$$|(Fu)(t)-(Fv)(t)|\leq\int_0^1|\xi(t,u(s))-\xi(t,v(s))|\,ds. \tag{4.2}$$

On the other hand, by (ii), for all $t\in[0,1]$ and $z,w\in\mathbb{R}$, we have (since $\kappa\in(0,1)$)

$$\sinh\left(|\xi(t,z)-\xi(t,w)|\right)\leq\sinh\left(|z-w|\right),$$

which yields

$$\sinh^{-1}\left[\sinh\left(|\xi(t,z)-\xi(t,w)|\right)\right]\leq\sinh^{-1}\left[\sinh\left(|z-w|\right)\right],$$

that is,

$$|\xi(t,z)-\xi(t,w)|\leq|z-w|,\quad t\in[0,1],\ z,w\in\mathbb{R}. \tag{4.3}$$

Here, $\sinh^{-1}$ is the inverse hyperbolic sine function. We recall that $\sinh^{-1}$ is a nondecreasing function on $[0,\infty)$. Next, in view of (4.2) and (4.3), for all $u,v\in Q$ and $t\in[0,1]$, we obtain

$$\begin{aligned}|(Fu)(t)-(Fv)(t)|&\leq\int_0^1|u(s)-v(s)|\,ds\\&\leq\delta(u,v),\end{aligned}$$

which yields

$$\delta(Fu, Fv) \leq \delta(u, v), \quad u, v \in Q.$$

This shows that F is continuous on (Q, δ).

We now show that F is a θ-hyperbolic contraction for some $\theta \in \Theta_\tau$, $\tau > 0$. Indeed, for all $u, v \in Q$ and $t \in [0, 1]$, by (4.2), we have

$$\sinh\left(|(Fu)(t) - (Fv)(t)|\right) \leq \sinh\left[\int_0^1 |\xi(t, u(s)) - \xi(t, v(s))|\, ds\right]. \tag{4.4}$$

On the other hand, since sinh is a convex function on $[0, \infty)$, then by (ii) and Lemma 4.1, we obtain

$$\begin{aligned} \sinh\left[\int_0^1 |\xi(t, u(s)) - \xi(t, v(s))|\, ds\right] &\leq \int_0^1 \sinh\left(|\xi(t, u(s)) - \xi(t, v(s))|\right) ds \\ &\leq \kappa \int_0^1 \sinh\left(|u(s) - v(s)|\right) ds \\ &\leq \kappa \sinh(\delta(u, v)), \end{aligned}$$

which implies by (4.4) that

$$\sinh\left(|(Fu)(t) - (Fv)(t)|\right) \leq \kappa \sinh(\delta(u, v)), \quad t \in [0, 1].$$

Then, taking the maximum over $t \in [0, 1]$ in the above inequality, we obtain

$$\sinh(\delta(Fu, Fv)) \leq \kappa \sinh(\delta(u, v)), \quad u, v \in Q.$$

This shows that F is a θ-hyperbolic contraction with $\theta(t) = t$ for all $t \geq 0$. Finally, applying Theorem 3.1, we obtain (I) and (II). □

5. Conclusions

We introduced the notion of θ-hyperbolic sine distance function associated to a certain metric on Q, where $\theta : [0, \infty) \to [0, \infty)$ is a function that belongs to the set of functions Θ_τ for some $\tau > 0$. As it was shown in Example 2.1, a θ-hyperbolic sine distance function on Q is not necessarily a metric on Q, even if $\theta(0) = 0$. We considered two classes of mappings $F : Q \to Q$ satisfying contractions involving the θ-hyperbolic sine distance function; namely, the class of θ-hyperbolic contractions (see Definition 3.1) and the class of (θ, η)-hyperbolic contractions (see Definition 3.3), where $\eta : [0, \infty) \to [0, \infty)$ is a (c)-comparison function (see Definition 3.2). For each class of mappings, we established the existence and uniqueness of fixed points (see Theorems 3.1 and 3.2).

Inspired from the existing fixed point results from the literature, it would be interesting to investigate other classes of mappings $F : Q \to Q$ satisfying contractions involving θ-hyperbolic sine distance functions. For instance, one can consider contractions of the forms

$$\begin{aligned} \delta_\theta(Fu, Fv) &\leq \kappa\left[\delta_\theta(u, Fu) + \delta_\theta(v, Fv)\right], \\ \delta_\theta(Fu, Fv) &\leq \kappa\left[\delta_\theta(u, Fv) + \delta_\theta(v, Fu)\right], \\ \delta_\theta(Fu, Fv) &\leq \kappa_1 \delta_\theta(u, v) + \kappa_2 \delta(v, Fu), \end{aligned}$$

and many others forms of contractions. The case of multi-valued mappings $F : Q \to \mathcal{P}(Q)$, where $\mathcal{P}(Q)$ denotes the set of nonempty subsets of Q, also deserves to be studied.

It would be also interesting to study the possibility of weakening condition (II) of Theorem 3.1 (or Theorem 3.2) using the concepts introduced in [34].

Author contributions

Both authors contributed equally and significantly in writing this paper.

Acknowledgments

The first author is supported by Researchers Supporting Project number (RSP2024R57), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare that they have no competing interests.

References

1. D. Aronov, Fast algorithm for the metric-space analysis of simultaneous responses of multiple single neurons, *J. Neurosci. Meth.*, **124** (2023), 175–179. https://doi.org/10.1016/S0165-0270(03)00006-2
2. A. L. Gibbs, Convergence in the Wasserstein metric for Markov chain Monte Carlo algorithms with applications to image restoration, *Stoch. Models*, **20** (2004), 473–492. https://doi.org/10.1081/STM-200033117
3. E. Vidal, H. M. Rulot, F. Casacuberta, J. M. Benedi, On the use of a metric-space search algorithm (AESA) for fast DTW-based recognition of isolated words, *IEEE Trans. Acoust. Speech Signal Process.*, **36** (1988), 651–660. https://doi.org/10.1109/29.1575
4. S. G. Matthews, Partial metric spaces, *A. NY. Acad. Sci.*, **728** (1994), 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x
5. M. B. Smyth, Completeness of quasi-uniform and syntopological spaces, *Lond. Math. Soc.*, **49** (1994), 385–400. https://doi.org/10.1112/jlms/49.2.385
6. Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.*, **7** (2006), 289–297.
7. I. A. Bakhtin, Contracting mapping principle in an almost metric space (Russian), *Funkts. Anal.*, **30** (1989), 26–37.
8. A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen*, **57** (2000), 31–37. https://doi.org/10.5486/PMD.2000.2133
9. I. A. Bakhtin, Contracting mapping principle in an almost metric space (Russian), *Funkts. Anal.*, **30** (1989), 26–37.

10. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis*, **1** (1993), 5–11.
11. M. Berzig, First results in suprametric spaces with applications, *Mediterr. J. Math.*, **19** (2022), 1–18. https://doi.org/10.1007/s00009-022-02148-6
12. M. Jleli, B. Samet, On a new generalization of metric spaces, *J. Fixed Point Theory Appl.*, **20** (2018), 128. https://doi.org/10.1007/s11784-018-0606-6
13. V. Ozturk, S. Radenović, Hemi metric spaces and Banach fixed point theorem, *Appl. Gen. Topol.*, **25** (2024), 175–181. https://doi.org/10.4995/agt.2024.19780
14. W. Kirk, N. Shahzad, *Fixed point theory in distance spaces*, Springer International Publishing Switzeralan, 2014. https://doi.org/10.1007/978-3-319-10927-5
15. M. Younis, H. Ahmad, L. Chen, M. Han, Computation and convergence of fixed points in graphical spaces with an application to elastic beam deformations, *J. Geom. Phys.*, **192** (2023), 104955. https://doi.org/10.1016/j.geomphys.2023.104955
16. M. Younis, A. A. N. Abdou, Novel fuzzy contractions and applications to engineering science, *Fractal Fract.*, **8** (2024), 28. https://doi.org/10.3390/fractalfract8010028
17. H. Ahmad, F. U. Din, M. Younis, A fixed point analysis of fractional dynamics of heat transfer in chaotic fluid layers, *J. Comput. Appl. Math.*, **453** (2025), 116144. https://doi.org/10.1016/j.cam.2024.116144
18. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181
19. R. Kannan, Some results on fixed points, *Bull. Calc. Math. Soc.*, **60** (1968), 71–76. https://doi.org/10.2307/2316437
20. L. Ćirić, A generalization of Banach's contraction principle, *P. Am. Math. Soc.*, **45** (1974), 267–273. https://doi.org/10.1090/S0002-9939-1974-0356011-2
21. A. Meir, E. Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.*, **28** (1969), 326–329. https://doi.org/10.1016/0022-247X(69)90031-6
22. J. Caballero, J. Harjani, K. A. Sadarangani, Fixed point theorem for operators of Meir-Keeler type via the degree of nondensifiability and its application in dynamic programming, *J. Fixed Point Theory A.*, **22** (2020), 13. https://doi.org/10.1007/s11784-019-0748-1
23. F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theory for simulation functions, *Filomat*, **29** (2015), 1189–1194. https://doi.org/10.2298/FIL1506189K
24. A. Chanda, L. K. Dey, S. Radenović, Simulation functions: A survey of recent results, *RACSAM Rev. R. Acad. A*, **113** (2019), 2923–2957. https://doi.org/10.1007/s13398-018-0580-2
25. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal.*, **75** (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014
26. I. Kedim, M. Berzig, Fixed point theorems for Maia $\alpha-\psi$ contractive type mappings with applications, *J. Math. Anal. Appl.*, **504** (2021), 125381. https://doi.org/10.1016/j.jmaa.2021.125381
27. O. Popescu, Fixed points for (ψ,φ)-weak contractions, *Appl. Math. Lett.*, **24** (2011), 1–4.

28. H. Aydi, On common fixed point theorems for (ψ, φ)-generalized f-weakly contractive mappings, *Miskolc Math. Notes*, **14** (2013), 19–30. https://doi.org/10.18514/MMN.2013.399

29. A. H. Ansari, M. Berzig, S. Chandok, Some fixed point theorems for (CAB)-contractive mappings and related results, *Math. Morav.*, **19** (2015), 97–112. https://doi.org/10.5937/MatMor1502097A

30. E. Petrov, Fixed point theorem for mappings contracting perimeters of triangles, *J. Fixed Point Theory A.*, **25** (2023), 74. https://doi.org/10.1007/s11784-023-01078-4

31. I. A. Rus, *Generalized contractions and applications*, Cluj University Press, 2001.

32. S. Pakhira, S. M. Hossein, A new fixed point theorem in Gb-metric space and its application to solve a class of nonlinear matrix equations, *J. Comput. Appl. Math.*, **437** (2024), 115474. https://doi.org/10.1016/j.cam.2023.115474

33. D. S. Mitrinović, J. Pečarić, A. M. Fink, *Classical and new inequalities in analysis*, In: Mathematics and its Applications (East European Series), Springer Science and Business Media, 1993.

34. Y. Han, S. Xu, J. Chen, H. Yang, Fixed point theorems for b-generalized contractive mappings with weak continuity conditions, *AIMS Math.*, **9** (2024), 15024–15039. https://doi.org/10.3934/math.2024728