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Research article

Fractional stochastic heat equation with mixed operator and driven by fractional-type noise

Mounir Zili[1], **Eya Zougar**[2,*] **and Mohamed Rhaima**[3]

[1] Department of Mathematics, University of Monastir, Faculty of Sciences, Avenue de l'environnement, 5019 Monastir, Tunisia

[2] Department of Mathematics, University of Polytechnique Hauts-de-France, CERAMATHS, Campus Mont Houy, 59313,Valenciennes, CEDEX 9-France

[3] Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: Eya.Zougar@uphf.fr.

Abstract: We investigated a novel stochastic fractional partial differential equation (FPDE) characterized by a mixed operator that integrated the standard Laplacian, the fractional Laplacian, and the gradient operator. The equation was driven by a random noise, which admitted a covariance measure structure with respect to the time variable and behaved as a Wiener process in space. Our analysis included establishing the existence of a solution in the general case and deriving an explicit form for its covariance function. Additionally, we delved into a specific case where the noise was modeled as a generalized fractional Brownian motion (gfBm) in time, with a particular emphasis on examining the regularity of the solution's sample paths.

Keywords: stochastic fractional partial differential equations; fractional Brownian and sub-Brownian motions; mild solution; sample paths; heat equation
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1. Introduction

In recent decades, fractional partial differential equations (FPDEs) have garnered significant attention across various fields such as mathematics [5, 34], physics [14], engineering [20], chemistry [26], fluid mechanics [1], nuclear reactor dynamics [15], chaotic dynamical systems [19], mechanics of materials [8], biology [18], hydrology [7], finance [21], and social sciences [9]. In this paper, we focus on the following specific FPDE for a fixed $d \geq 1$:

$$\frac{\partial u(t,x)}{\partial t} = \mathcal{L}^{a,b} u(t,x), \tag{1.1}$$

with $(t,x) \in [0,T] \times \mathbb{R}^d, \forall T > 0$. Here, $\mathcal{L}^{a,b}$ is the mixed fractional operator given as:

$$\mathcal{L}^{a,b} = \Delta + a^\alpha \, \Delta^{\alpha/2} + b\,.\nabla, \tag{1.2}$$

where $\alpha \in (1,2]$, $a \in (0,M]$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \in \mathbb{R}^d$, $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\mathbb{R}^d$, $\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_d} \end{pmatrix}$ is the gradient on $\mathbb{R}^d$, $b.\nabla = \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}$, and $\Delta^{\alpha/2}$ is the operator defined by:

$$\Delta^{\alpha/2} u(x) = \mathcal{A}(d,\alpha) \lim_{\delta \to 0} \int_{\{z \in \mathbb{R}^d; \|z-x\| > \delta\}} \frac{u(z) - u(x)}{\|z - x\|^{d+\alpha}} \, dz, \quad \forall u \in C_c^2(\mathbb{R}^d), \tag{1.3}$$

with $\mathcal{A}(d,\alpha) = \alpha \, 2^{\alpha-1} \, \pi^{-d/2} \frac{\Gamma\left(\frac{d+\alpha}{2}\right)}{\Gamma\left(1 - \frac{\alpha}{2}\right)}$. Here, Γ is denoting the Gamma function and $C_c^2(\mathbb{R}^d)$ is the space of twice continuously differentiable functions on $\mathbb{R}^d$ with compact support.

Operator (1.2) was introduced in [5] in a more general case, where b is a function in a certain Kato class, and it can be seen as the infinitesimal generator of some diffusion processes related to anomalous diffusion (see [5, 34] and references therein).

In the present paper, we introduce a stochastic counterpart of Eq (1.1), defined by

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} &= \mathcal{L}^{a,b} u(t,x) + \dot{W}(t,x), \quad t > 0, \; x \in \mathbb{R}^d, \\ u(0,x) &= 0, \end{cases} \tag{1.4}$$

where $\dot{W}$ denotes the formal derivative of a centered Gaussian field W, which behaves as a Wiener process with respect to the space variable, and as a process that admits a covariance measure structure, in the sense of [13], with respect to the time variable. In particular, for fixed $x \in \mathbb{R}^d$, $W(.,x)$ extends many interesting Gaussian fractional processes such as: mixed fractional Brownian motion (mfBm) (see e.g., [28]), mixed subfractional Brownian motion (msfBm) (see, e.g., [6, 16]), generalized fractional Brownian motion (gfBm) [29, 31, 32], and so on. Investigation of Eq (1.4) represents a novel mathematical problem that has not been explored before. In [34], Zili and Zougar have investigated equation of the form (1.4) with a different type of random force, specifically space-time white noise $\{W_t\}_{t \geq 0}$, that is, a centered Gaussian process with covariance function $\mathbb{E} W_t W_s = t \wedge s$. In this paper, we deal with a distinct and more general stochastic problem that involves a different framework and additional complexity.

Equation (1.4) illustrates heat propagation in inhomogeneous media, influenced by anomalous diffusion and subject to stochastic perturbations. Recent studies have uncovered atypical behaviors in diffusion processes within nonhomogeneous media. These anomalous diffusion phenomena are best described by fractional-order models, as classical integer-order models fail to capture their unique characteristics (see, for example, [12]). This can be considered as a main motivation for this work.

Stochastic FPDEs of type (1.4) have been widely studied in the literature with specific operators that are special cases of the general operator considered in this work. For instance, the case where the operator is limited to the fractional Laplacian $\Delta^{\alpha/2}$ has been thoroughly investigated in works like [2, 11], considering various types of additive Gaussian noises. In [25,27], the authors examined equations resembling (1.4), particularly when $a = 1$ and $b = 0$, resulting in the operator $\mathcal{L}^{1,0} = \Delta + \Delta^{\alpha/2}$. They explored these equations under the influence of diverse additive drifts and various types of fractional noises.

Therefore, in addition to the generalization of the stochastic FPDE introduced in [34], Eq (1.4) represents a further extension of the fractional models investigated in [10, 11, 25, 27, 34], and this can be regarded as another important motivation for the investigation of such an equation's solution. It's worth noting that in numerous other papers (e.g., [17, 23, 24, 30, 33], and the references therein), researchers have explored different types of fractional stochastic PDEs. In these cases, the term "fractional" typically pertains to the additive noise rather than to the fractional Laplacian operator, as is the case in our study.

This research represents a pioneering study of the solution for the novel, unexplored FPDE (1.4). Our primary contribution lies in laying the foundational groundwork, providing a solid basis for future investigations in this area. We first provide a sufficient condition for the existence of the solution, and we give an explicit expression of its covariance function. Then, we focus our attention on the interesting specific case, where $W(., x)$ behaves as a gfBm, in the sense introduced by M. Zili in [29, 31, 32]. The gfBm is an extension of both fBm and sfBm, defined as a linear combination of two independent fBm and sfBm. So, it is about a process which depends on three parameters: the Hurst index and the coefficients of the linear combination. This should allow researchers to construct more adequate models, permitting, for example, to control the level of correlation between the increments of the studied phenomena, and, consequently, to overcome the deficiency of fBm and sfBm models due to their dependence on one single constant, which is the Hurst parameter. More information about the gfBm and the motivations of its introduction can be found in [31].

The results of this paper are obtained by introducing the canonical Hilbert space associated to the Gaussian noise W by applying many integration techniques, calculation, and analysis tools, and especially by suitably exploiting the two-sided estimates of the fundamental solution $G_{a,b}$ of the operator $\mathcal{L}^{a,b}$, already established in the case where $d \geq 1$, by Zili and Zougar in [34], and moreover by the explicit expression of its Fourier transform.

The paper is organized as follows: In next section, we give some examples of applications of the fractional PDE. In Section 3, we first introduce the random noise that drives our stochastic FPDE and, in particular, the processes admitting a covariance measure structure. We give some interesting examples, and we explain the mode of Wiener integration with respect to our noise that we will use. Then, we specify our meaning of solution and give some characteristics of the Green fundamental function $G_{a,b}$ of Eq (1.1) that will play a main role in the whole of this paper. After that, we give a sufficient condition for the existence of the mild solution of the stochastic FPDE (1.4), and we present an explicit expression of its covariance function. In Section 4, we focus on the interesting particular case when the process is a white-space Gaussian field, behaving as a gfBm in time. We especially analyze the regularity of the sample paths of the solution with respect to the time variable. Finally, Section 5 offers a discussion of the results, while Section 6 presents the conclusion of the paper.

2. Some application of the deterministic FPDE

As mentioned in the introduction, Eq (1.1) serves as a good model across various domains. It has become increasingly important in modeling complex systems where classical integer-order models fail to capture the full dynamics. By incorporating fractional derivatives, these equations provide a more comprehensive framework for handling anomalous diffusion, nonlocal processes, and long-range interactions. Below, we explore two examples of applications of FPDEs: one in fluid dynamics and the other in financial mathematics.

2.1. Application 1: Fluid dynamics

FPDEs are vital in fluid dynamics, especially for systems involving turbulence, anomalous diffusion, and non-Newtonian fluids. While traditional PDEs using the classical Laplacian operator Δ effectively model normal diffusion, many real-world systems, such as turbulent or porous media, involve more complex transport processes. In turbulent flows, like those in atmospheric dynamics or ocean currents, particles exhibit super-diffusive behavior, where their mean squared displacement grows faster than under normal diffusion. This can be modeled using the fractional Laplacian, which captures nonlocal transport mechanisms like Lévy flights, where particles make large, unpredictable jumps. The drift term $b.\nabla$ models advection, representing directional transport driven by external forces or velocity fields, such as wind or ocean currents. In oceanography, for example, this describes how water is carried by currents, while diffusion terms capture smaller-scale turbulence and mixing. FPDEs are also effective for modeling non-Newtonian fluids, like polymers or biological fluids, where the stress-strain relationship is nonlinear, and memory effects are important.

Example 2.1. *In the study of pollutant dispersion in turbulent ocean currents, the drift term accounts for the overall flow of water carrying the pollutant, while the standard Laplacian Δ represents local diffusion. The fractional Laplacian $\Delta^{\alpha/2}$ captures the nonlocal effects, such as sudden shifts in concentration caused by turbulence. Together, these terms provide a full description of how pollutants spread in complex fluid environments. For further details, readers are advised to consult [4].*

2.2. Application 2: Finance mathematics

FPDEs are also extensively used in financial mathematics, particularly for modeling asset prices and option pricing in markets with jumps and volatility clustering. In these models, FPDEs capture the stochastic processes governing financial instruments. The classical Laplacian Δ corresponds to Brownian motion, which assumes continuous price changes, as seen in models like the Black-Scholes equation. However, real-world markets often exhibit large price jumps and heavy-tailed distributions, which the standard diffusion operator cannot account for. So, to address this, the fractional Laplacian is used to model jumps and heavy-tailed returns. This operator captures the behavior of Lévy processes, which describe sudden, unpredictable price changes, such as those caused by market shocks or economic news. It is particularly useful for pricing exotic options or derivatives sensitive to these large, infrequent movements. The drift term represents the expected return or trend of an asset, influenced by predictable market factors like economic growth or interest rates. Combined with the fractional Laplacian, this term models both the overall direction and the random, jump-like fluctuations of asset prices.

Example 2.2. *In option pricing under jump-diffusion models, the drift term accounts for the average return on an asset, while the fractional Laplacian captures the sudden jumps in asset prices caused by market shocks. This approach is particularly relevant for pricing credit derivatives or insurance contracts against market crashes, as it allows for a more accurate assessment of risk and pricing, considering both gradual changes in asset prices and the possibility of rare but significant market events. For more information, readers are encouraged to see [21].*

3. Stochastic FPDE driven by a noise with covariance measure structure

Let us describe the random noise driving the stochastic FPDE (1.4), subject to this study.

3.1. Description of the random noise W

To start, we define the processes with covariance measure structure.

3.1.1. Processes with a covariance measure structure:

Consider a zero mean square integrable process $(X_t)_{t\in[0,T]}$ with covariance function $R_X(t,r) = \mathbf{E}[X_t X_r]$, for $(t,r) \in [0,T]^2$. The covariance R_X defines naturally a finite additive measure $\mu_{R_X} := \mu$ on the algebra $\mathcal{R}$ of finite disjoint rectangles included in the set $[0,T]^2$ by:

$$\mu(J) = \Delta_J R_X$$

where $\Delta_J R_X$ denotes the rectangular increment of R_X over the rectangle $J = [a_1, b_1) \times [a_2, b_2)$ given by:

$$\Delta_J R_X = R_X(b_1, b_2) - R_X(a_1, b_2) - R_X(a_2, b_1) + R_X(a_1, a_2).$$

The process X is said to have a covariance measure structure if μ can be extended to a signed sigma finite measure on $\mathcal{B}([0,T]^2)$. Some important characteristics of such processes can be found in [13]. In particular, we have:

Lemma 3.1. *Any zero mean square integrable process $(X_t)_{t\in[0,T]}$ with covariance function R such that*

$$\frac{\partial^2 R_X}{\partial r \partial t} \text{ is integrable on } [0,T]^2,$$

has a covariance measure structure. Furthermore, the measure μ generated by R_X admits a density with respect to the Lebesgue measure on $[0,T]^2$ given by $\dfrac{\partial^2 R_X}{\partial r \partial t}$.

Let us give a few examples of processes with covariance measure structure.

Example 3.1. *Let us denote by $M^H = \{M_t^H(\theta,\nu); t \geq 0\} = \{M_t^H; t \geq 0\}$ the mixed-fBm of parameters θ, ν, and H such that $H \in (0,1)$, $(\theta,\nu) \in \mathbb{R}^2 \setminus \{(0,0)\}$; that is, the centered Gaussian process, starting from zero, with covariance*

$$R_M^{H,\theta,\nu}(t,r) := \mathbb{C}ov\Big(M_t^H(\theta,\nu), M_r^H(\theta,\nu)\Big) = \theta^2 (t \wedge r) + \frac{\nu^2}{2}\left(t^{2H} + r^{2H} - |t-r|^{2H}\right), \tag{3.1}$$

where $t \wedge r = \frac{1}{2}(t + r - |t - r|)$. *Some specific examples of this process include:* $M^H(0,1) = B^H$, *which represents an fBm and* $M^H(1,0) = B$, *which corresponds to standard Bm. So, the mfBm is clearly an extension of the fBm and of the Wiener process. We refer to [28] for further information on this process.*

If $(\theta, \nu, H) \in \mathbb{R} \times \mathbb{R}^\star \times \left(\frac{1}{2}, 1\right)$ *or* $(\theta, \nu) \in \mathbb{R}^\star \times \{0\}$, $M^H(\theta, \nu)$ *admits a covariance measure structure on* $[0,T]^2$ *which has a density given by:*

$$\frac{\partial^2 R_M^{H,\theta,\nu}}{\partial t \partial r}(t,r) = \theta^2 \delta_0(t-r) + \nu^2 \sigma_H |t-r|^{2H-2},$$

where $\sigma_H = H(2H-1)$, δ_0 *is the Dirac measure and* $\mathbb{R}^\star$ *is the set* $\mathbb{R} \setminus \{0\}$.

Example 3.2. *Consider* θ *and* ν *two real constants such that* $(\theta, \nu) \neq (0,0)$ *and* $H \in (0,1)$. *A gfBm of parameters* θ, ν, *and* H *is a process* $Z^H = \{Z_t^H(\theta,\nu); t \geq 0\} = \{Z_t^H; t \geq 0\}$, *defined on the probability space* $(\Omega, F, \mathbb{P})$ *by:*

$$\forall t \in \mathbb{R}_+ \quad Z_t^H = Z_t^H(\theta,\nu) = \theta B_t^H + \nu B_{-t}^H \tag{3.2}$$

where $(B_t^H)_{t\in\mathbb{R}}$ *is a two-sided fBm of parameter H. We offer several examples of this process:* $Z^H(1,0)$ *represents an fBm, while* $Z^H\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ *denotes the sfBm. So, the gfBm is in the same time, a generalization of the fBm, of the sfBm, and of course of the standard Brownian motion. For more information, the reader can read [29, 31, 32].*

The gfBm $(Z_t^H(\theta,\nu))_{t\in\mathbb{R}_+}$ *is a centered Gaussian process with covariance function*

$$\begin{aligned} R_Z^{H,\theta,\nu}(t,r) &= \mathbb{C}ov\Big(Z_t^H(\theta,\nu), Z_r^H(\theta,\nu)\Big) \\ &= \frac{1}{2}(\theta+\nu)^2\left(t^{2H} + r^{2H}\right) - \nu\theta\,(t+r)^{2H} - \frac{\theta^2+\nu^2}{2}|t-r|^{2H} \end{aligned} \tag{3.3}$$

for every $t, r \in (0, +\infty)$. *Then, when* $H \in (\frac{1}{2}, 1)$, $Z^H(\theta,\nu)$ *admits a covariance measure structure on* $[0,T]^2$ *which has a density given by:*

$$\frac{\partial^2 R_Z^{H,\theta,\nu}}{\partial t \partial r}(t,r) := \sigma_H\left[(\theta^2+\nu^2)\,|t-r|^{2H-2} - 2\nu\theta\,(t+r)^{2H-2}\right]. \tag{3.4}$$

3.1.2. The random noise with covariance measure structure:

In this section, we introduce the random noise that drives the parabolic Eq (1.4). On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a zero-mean Gaussian field $W = \{W(t,A); t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance:

$$\mathbb{C}ov\,(W(t,A)W(r,B)) = \lambda^d(A \cap B)\,R_W(t,r) \tag{3.5}$$

where λ^d is the Lebesgue measure, and R_W is the covariance of a stochastic process that generates a covariance measure μ.

To the Gaussian field W, we can associate a Hilbert space that will be called the canonical Hilbert space of W and will be denoted by $\mathcal{H}$. Consider $\mathcal{E}$ the set of linear combinations of elementary functions $\mathbf{1}_{[0,t]} \times A$, $(t,A) \in [0,T] \times \mathcal{B}_b(\mathbb{R}^d)$, and let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product

$$< \mathbf{1}_{[0,t]} \times A, \mathbf{1}_{[0,r]} \times B >_{\mathcal{H}} := \mathbb{C}ov\,(W(t,A)W(s,B)).$$

We have the following expression of the scalar product in $\mathcal{H}$:

$$< \varphi, \psi >_{\mathcal{H}} = \int_0^T \int_0^T \mu(dr, dw) \int_{\mathbb{R}^d} \varphi(r, z)\, \psi(w, z)\, dz \tag{3.6}$$

for any $\varphi, \psi \in \mathcal{H}$ such that

$$\int_0^T \int_0^T |\mu|(dr, dw) \int_{\mathbb{R}^d} |\varphi(r, z)|\, |\psi(w, z)|\, dz < \infty, \tag{3.7}$$

where $|\mu|$ denotes the total variation measure associated to μ.

Following [24], by a routine extension of the construction done in [13], it is possible to define Wiener integrals with respect to the process W whose covariance is given by (3.5). This Wiener integral will act as an isometry between the Hilbert space $\mathcal{H}$ and $L^2(\Omega)$ in the sense that:

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \varphi(r, z) W(dr, dz) \int_0^T \int_{\mathbb{R}^d} \psi(r, z) W(dr, dz)\right] = \int_0^T \int_0^T \mu(dr, dw) \int_{\mathbb{R}^d} \varphi(r, z)\, \psi(w, z)\, dz. \tag{3.8}$$

3.2. The stochastic FPDE driven by a noise with covariance measure structure

In this part, we will analyze the existence of the solution to Eq (1.4) driven by a random noise characterized by (3.5). The notion of the solution to Eq (1.4) is defined in the mild sense. We call a mild solution to (1.4) the stochastic process

$$u^{a,b}(t, x) = \int_0^T \int_{\mathbb{R}^d} G_{a,b}(t - r, x, z)\, \mathbf{1}_{(0,t)}(r)\, W(dr, dz), \qquad \forall t \geq 0, x \in \mathbb{R}^d, \tag{3.9}$$

where W is the Gaussian noise with covariance given by (3.5), $G_{a,b}$ denotes the fundamental solution for the operator $\mathcal{L}^{a,b}$, and the integral in (3.9) is a Wiener integral with respect to the Gaussian noise W.

Let us first recall some useful properties of $G_{a,b}$.

3.2.1. Main properties of the fundamental solution $G_{a,b}$:

In [5], the authors established existence and uniqueness of a fundamental solution $G_{a,b}$ of the operator $\mathcal{L}^{a,b}$, and they provide some characterizations and estimates, some of which we quote in the following lemma.

Lemma 3.2. *Let $d \geq 1$. There exist two positive constants c_1, c_2 such that, for all $t > 0, x, z, b \in \mathbb{R}^d$, and $a \in [0, M]$, we have*

$$c_1^{-1}\, p_{c_2}^a(t, x, z) \leq G_{a,b}(t, x, z) \leq c_1\, p_{1/c_2}^a(t, x, z). \tag{3.10}$$

with $p_c^a(t, x, z) = t^{-d/2} \exp\left(-\frac{c\|x - z\|^2}{t}\right) + t^{-d/2} \wedge \frac{a^\alpha t}{\|x - z\|^{d+\alpha}}, \forall c > 0.$

The following useful characteristic of $G_{a,b}$ was proved in [34].

Lemma 3.3. *For every $t > 0$ and $x \in \mathbb{R}^d$, the Fourier transform of $G_{a,b}(t, x, .)$, denoted by $\mathcal{F}\left(G_{a,b}(t, x, .)\right)$, is given by*

$$\mathcal{F}\left(G_{a,b}(t, x, .)\right)(\xi) = \exp\left(-i(x - tb).\xi\right) \exp\left(-tA_\alpha^a(\xi)\right) \tag{3.11}$$

for every $\xi \in \mathbb{R}^d$, with $A_\alpha^a(\xi) = \|\xi\|^2 + a^\alpha\, \|\xi\|^\alpha$.

In the rest of the paper, we will denote the following function: $a_{t,r} = t - r, \quad \forall (t, r) \in [0, T]^2$.

3.2.2. Existence of the solution for the stochastic FPDE defined in (1.4):

It is well-known that the mild solution to (1.4) exists when the Wiener integral in (3.9) is well-defined, and this happens when the integrand $G_{a,b}$ belongs to $\mathcal{H} = L^2([0,T]\times\mathbb{R}^d)$.

Let us now give a sufficient condition for the existence of the mild solution defined in (3.9).

Theorem 1. *We assume that* $\int_0^t \int_0^t (2t-r-w)^{-d/\alpha}\, |\mu|(dr,dw)$ *is finite, for every* $t \in [0,T]$, *then the mild solution given in (3.9) is well-defined. Moreover, if* $\sup_{t\in[0,T]} \int_0^t \int_0^t (2t-r-w)^{-d/\alpha}\, |\mu|(dr,dw) < \infty$, *then*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}\left[|u^{a,b}(t,x)|^2\right] < \infty.$$

Proof. Consider $t \in [0,T]$. By the Wiener isometry characteristic (3.8), we have

$$\mathbb{E}[|u^{a,b}(t,x)|^2] = \int_0^t \int_0^t \mu(dr,dw) \int_{\mathbb{R}^d} G_{a,b}(a_{t,r},x,z)\, G_{a,b}(a_{t,w},x,z)\, dz. \tag{3.12}$$

By applying the Plancherel theorem, we obtain

$$\begin{aligned}
\mathbb{E}[|u^{a,b}(t,x)|^2] &\le (2\pi)^{-d} \int_0^t \int_0^t |\mu|(dr,dw) \int_{\mathbb{R}^d} \left|\mathcal{F}G_{a,b}(a_{t,r},x,.)(\xi)\right| \overline{\left|\mathcal{F}G_{a,b}(a_{t,w},x,.)(\xi)\right|}\, d\xi \\
&= (2\pi)^{-d} \int_0^t \int_0^t |\mu|(dr,dw) \\
&\times \int_{\mathbb{R}^d} \left|\exp\left(-i(x-a_{t,r}b).\xi\right)\right| e^{-a_{t,r}A^a_\alpha(\xi)} \left|\exp\left(i(x-a_{t,w}b).\xi\right)\right| e^{-a_{t,w}A^a_\alpha(\xi)} d\xi \\
&= (2\pi)^{-d} \int_0^t \int_0^t |\mu|(dr,dw) \int_{\mathbb{R}^d} \left|e^{ib.\xi(w-r)}\right| e^{-(2t-r-w)A^a_\alpha(\xi)} d\xi.
\end{aligned}$$

Since $\alpha/2 \in (1/2,1)$, the function $x \longmapsto x^{\alpha/2}$ is concave on $[0,+\infty)$. Therefore, using the Jensen inequality, we get:

$$A^a_\alpha(\xi) \ge a^\alpha \|\xi\|^\alpha = a^\alpha \left(\sum_{j=1}^d |\xi_j|^2\right)^{\alpha/2} \ge a^\alpha d^{\frac{\alpha}{2}-1} \sum_{j=1}^d |\xi_j|^\alpha \tag{3.13}$$

for every $\xi \in \mathbb{R}^d$. Hence,

$$\mathbb{E}[|u^{a,b}(t,x)|^2] \le (2\pi)^{-d} \int_0^t \int_0^t |\mu|(dr,dw) \left(\int_{\mathbb{R}} e^{-D_\alpha(2t-r-w)|\xi_1|^\alpha} d\xi_1\right)^d$$

with $D_\alpha = a^\alpha\, d^{\alpha/2-1}$. By the change variable $z_1 = (2t-r-w)^{1/\alpha}\xi_1$, we get

$$\int_{\mathbb{R}} e^{-D_\alpha(2t-r-w)|\xi_1|^\alpha} d\xi_1 = (2t-r-w)^{-1/\alpha} \int_{\mathbb{R}} e^{-D_\alpha|z_1|^\alpha} dz_1 = \gamma_\alpha\, (2t-r-w)^{-1/\alpha},$$

with

$$\gamma^d_\alpha = \left(\int_{\mathbb{R}} e^{-D_\alpha|z_1|^\alpha} dz_1\right)^d. \tag{3.14}$$

Consequently,

$$\mathbb{E}[|u^{a,b}(t,x)|^2] \leq (2\pi)^{-d}\,\gamma_\alpha^d \int_0^t \int_0^t (2t-r-w)^{-d/\alpha}\,|\mu|(dr,dw) < +\infty, \tag{3.15}$$

which achieves the proof of Theorem 1. □

Remark 3.1. *In the particular case where the noise W in Eq (1.4) is defined by a Wiener process, that is,* $R_W(t,s) = t \wedge s$, *W defines a covariance measure* μ *given by* $\mu(du,dv) = \delta_0(u-v)\,du\,dv$, *where* δ_0 *is the Dirac measure. This case was analyzed by Zili and Zougar in [34]. They established that a mild solution exists precisely when the integral* $\int_0^t (t-r)^{-d/\alpha} dr$ *is finite, which is equivalent to the condition* $d = 1$.

3.2.3. Covariance function of the solution:

In the following theorem, we give an explicit expression of the covariance function of the mild solution for Eq (1.4).

Theorem 2. *For every* $t, s \in [0,T]$, $x, b \in \mathbb{R}^d$, *and* $a \in (0,M]$, *we have:*

$$\mathbb{E}[u^{a,b}(t,x)u^{a,b}(s,x)] = (2\pi)^{-d} \int_0^t \int_0^s \mu(dr,dw) \int_{\mathbb{R}^d} e^{ib.\xi(a_{t,r}-a_{s,w})} e^{-(a_{t,r}+a_{s,w})A_\alpha^a(\xi)} d\xi. \tag{3.16}$$

Proof. Again, by the Wiener isometry characteristic (3.8) and the Plancheral formula, we get

$$\begin{aligned}
& \mathbb{E}[u^{a,b}(t,x)u^{a,b}(s,x)] \\
=\ & \int_0^t \int_0^s \mu(dr,dw) \int_{\mathbb{R}^d} G_{a,b}(a_{t,r},x,z)\, G_{a,b}(a_{s,w},x,z)\, dz \\
=\ & (2\pi)^{-d} \int_0^t \int_0^s \mu(dr,dw) \int_{\mathbb{R}^d} \mathcal{F}G_{a,b}(a_{t,r},x,.)(\xi)\, \overline{\mathcal{F}G_{a,b}(a_{s,w},x,.)(\xi)}\, d\xi \\
=\ & (2\pi)^{-d} \int_0^t \int_0^s \mu(dr,dw) \\
\times\ & \int_{\mathbb{R}^d} \exp\left(-i(x-a_{t,r}b).\xi\right) e^{-a_{t,r}A_\alpha^a(\xi)} \exp\left(i(x-a_{s,w}b).\xi\right)\, e^{-a_{s,w}A_\alpha^a(\xi)} d\xi \\
=\ & (2\pi)^{-d} \int_0^t \int_0^s \mu(dr,dw) \int_{\mathbb{R}^d} e^{ib.\xi(t-r-s+w)} e^{-(t+s-r-w)A_\alpha^a(\xi)} d\xi.
\end{aligned}$$

Then, the proof is established. □

An immediate consequence of the previous theorem.

Corollary 3.1. *For every* $t \in [0,T]$, $x, b \in \mathbb{R}^d$, *and* $a \in (0,M]$, *we have:*

$$\mathbb{E}[|u^{a,b}(t,x)|^2] = (2\pi)^{-d} \int_0^t \int_0^t \mu(dr,dw) \int_{\mathbb{R}^d} e^{-ib.\xi(r-w)} e^{-(2t-r-w)A_\alpha^a(\xi)} d\xi. \tag{3.17}$$

Remark 3.2. *(1) In the case where the noise W in Eq (1.4) is defined by a Wiener process, the covariance and variance expressions, respectively, given by Eqs (3.16) and (3.17) become*

$$\begin{aligned} \mathbb{E}[u^{a,b}(t,x)u^{a,b}(s,x)] &= (2\pi)^{-1}\int_0^{t\wedge s}\int_{\mathbb{R}} e^{ib.\xi(t-s)}e^{-(t+s-2r)A_\alpha^a(\xi)}d\xi\, dr \\ \mathbb{E}[|u^{a,b}(t,x)|^2] &= (2\pi)^{-1}\int_0^{t}\int_{\mathbb{R}} e^{-2(t-r)A_\alpha^a(\xi)}d\xi\, dr \end{aligned} \tag{3.18}$$

for any $x, y \in \mathbb{R}$ and $t, s \in [0, T]$, leading to exactly the same expressions obtained in [34].

(2) In the particular case where $a = 0$ and $b = 0$, Eq (1.4) coincides with the standard stochastic heat equation, which was studied in many references (see, for example, [24]). In fact, the expressions for covariance and variance can be deduced from Theorem 2 and Corollary 3.1.

4. Case of the generalized-fBm Z^H

In this section, we will focus on the particular case where the noise is the gfBm $Z^H(\theta,\nu)$ with respect to the time variable (see Example 3.2), in the particular case when $H > \frac{1}{2}$. Consider $R = R_Z^{H,\theta,\nu}$ the covariance function given in (3.3) and denote

$$\alpha_H = 2H(2H-1), \quad c_1^H(\theta,\nu) = \alpha_H\frac{\theta^2+\nu^2}{2} \quad \text{and} \quad c_2^H(\theta,\nu) = -\alpha_H\theta\nu,$$

for any $(\theta,\nu) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Throughout, Cte denotes a generic positive constant, and, in what follows, for any $\alpha \in (1, 2]$, we denote

$$\lambda_d^{H,\alpha} = 2H - \frac{d}{\alpha}. \tag{4.1}$$

4.1. Existence of the solution and some elementary properties

We first justify the existence of the solution defined by (3.9).

Corollary 4.1. *Suppose that the noise is the gfBm Z^H with respect to the time variable. If $\lambda_d^{H,\alpha} > 0$, then the mild solution defined in (3.9) exists and, for every $T > 0$, we have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}\left[|u^{a,b}(t,x)|^2\right] < \infty. \tag{4.2}$$

Moreover, $\forall t, s \in [0, T]$, $x, b \in \mathbb{R}^d$, and $a \in (0, M]$,

$$\mathbb{E}[u^{a,b}(t,x)u^{a,b}(s,x)] = (2\pi)^{-d}\int_0^t\int_0^s dr\, dw\, h^{H,\theta,\nu}(r,w)\int_{\mathbb{R}^d} e^{ib.\xi(a_{t,r}-a_{s,w})}e^{-(a_{t,r}+a_{s,w})A_\alpha^a(\xi)}d\xi, \tag{4.3}$$

with $h^{H,\theta,\nu}(r,w) = c_1^H(\theta,\nu)|r-w|^{2H-2} + c_2^H(\theta,\nu)(r+w)^{2H-2}$.

Proof. From (3.4), we have clearly that

$$\frac{\partial^2 R_Z^{H,\theta,\nu}}{\partial r\partial w}(r,w) = c_1^H(\theta,\nu)|r-w|^{2H-2} + c_2^H(\theta,\nu)(r+w)^{2H-2} := h^{H,\theta,\nu}(r,w),$$

for any $(r, w) \in [0, T] \times \mathbb{R}^d$. We first note that $h^{H,\theta,\nu}(r, w) \geq 0$. Indeed, the constant $c_1^H(\theta, \nu)$ is clearly positive. Moreover, on the one hand, if $\theta\nu \leq 0$ then, since $H > \dfrac{1}{2}$, the constant $c_2^H(\theta, \nu)$ is nonnegative, and as consequence, $h^{H,\theta,\nu}$ is positive. Also, if $\theta\nu \geq 0$, then by writing the covariance function of $Z^H(\theta, \nu)$ in the following form

$$h^{H,\theta,\nu}(r, w) = \alpha_H \left\{ \frac{(\theta - \nu)^2}{2} |r - w|^{2H-2} + \theta\nu \left[|r - w|^{2H-2} - (r + w)^{2H-2} \right] \right\},$$

we clearly see that $h^{H,\theta,\nu}$ is positive too, because, for $H > 1/2$, we have $|r-w|^{2H-2} \geq (r+w)^{2H-2}$. Hence, the covariance measure generated by $R^{H,\theta,\nu}$ is positive, and by applying Theorem 1, we deduce that, if $\int_0^t \int_0^t (2t - r - w)^{-d/\alpha} \mu(dr, dw)$ is finite for any $t \in [0, T]$, then the mild solution defined in (3.9) is well-defined. Therefore, there exists a positive constant depending on H, θ, ν, such that

$$\int_0^t \int_0^t (2t - r - w)^{-d/\alpha} h^{H,\theta,\nu}(r, w)\, dr\, dw \leq c(H, \theta, \nu) \int_0^t \int_0^t (2t - r - w)^{-d/\alpha} |r - w|^{2H-2}\, dr\, dw.$$

Then, by the change of variables $\tilde{r} = t - r$ and $\tilde{w} = t - w$, we get

$$\begin{aligned}
&\int_0^t \int_0^t (2t - r - w)^{-d/\alpha} h^{H,\theta,\nu}(r, w) dr\, dw \\
\leq\;& c(H, \theta, \nu) \int_0^t \int_0^t (r + w)^{-d/\alpha} (r - w)^{2H-2} dr\, dw \\
=\;& c(H, \theta, \nu) \left[\int_0^t \int_0^r (r + w)^{-d/\alpha} (r - w)^{2H-2} drdw + \int_0^t \int_r^t (r + w)^{-d/\alpha} (w - r)^{2H-2} drdw \right] \\
=\;& c(H, \theta, \nu) \left[\int_0^t \int_0^r (r + w)^{-d/\alpha} (r - w)^{2H-2} drdw + \int_0^t \int_0^w (r + w)^{-d/\alpha} (w - r)^{2H-2} drdw \right] \\
=\;& 2\, c(H, \theta, \nu) \int_0^t \int_0^r (r + w)^{-d/\alpha} (r - w)^{2H-2} dv\, du \\
=\;& 2\, c(H, \theta, \nu) \int_0^t r^{2H-2-\frac{d}{\alpha}} \int_0^r \left(1 - \frac{w}{r}\right)^{2H-2} \left(1 + \frac{w}{r}\right)^{-d/\alpha} dw\, dr \\
=\;& 2\, c(H, \theta, \nu)\, C_1 \int_0^t r^{2H-1-d/\alpha} dr,
\end{aligned}$$

where in the last line we used the change of variables $\tilde{w} = \frac{w}{r}$ and the constant C_1 is given by $C_1 = \int_0^1 (1 - \tilde{w})^{2H-2} (1 + \tilde{w})^{-d/\alpha} d\tilde{w}$, which is finite because $2H - 2 > -1$.

Since $d < 2\alpha H$, we have $\int_0^t r^{2H-1-d/\alpha} dr < \infty$, for every $0 \leq t \leq T$. Therefore, using Inequality (3.15), we get

$$\begin{aligned}
\sup_{(t,x) \in I \times \mathbb{R}^d} \mathbb{E}\left[|u^{a,b}(t, x)|^2 \right] \;&\leq\; Cte \sup_{t \in [0,T]} \int_0^t \int_0^t (2t - r - w)^{-d/\alpha} h^{H,\theta,\nu}(r, w)\, dr\, dw \\
&\leq\; Cte \sup_{t \in [0,T]} \int_0^t r^{2H-1-d/\alpha} dr < \infty.
\end{aligned}$$

All this with Theorems 1 and 2 allow us to finish the proof of Corollary 4.1. □

Remark 4.1. *For any $\alpha \in (1,2]$ and $H \in (\frac{1}{2},1)$, the sufficient condition, $\lambda_d^{H,\alpha} > 0$, for the existence of the mild solution defined in (3.9), is equivalent to*

$$d = 1 \text{ or } (d = 2 \text{ and } \alpha H > 1) \text{ or } (d = 3 \text{ and } \alpha H > 3/2).$$

4.2. Temporal regularity of the sample paths

In this section, we will focus on the study of the regularity of the trajectories of the solution to the stochastic FPDE (1.4) with respect to the time variable. In what follows, we suppose that $H \in (\frac{1}{2},1)$ and $\alpha \in (1,2]$ such that $\lambda_d^{H,\alpha} > 0$.

Theorem 3. *Let $u^{a,b}$ be the mild solution to Equation (1.4). There exists a positive constant Cte such that,*

$$\mathbb{E}[|u^{a,b}(t,x) - u^{a,b}(s,x)|^2] \leq Cte\ |t-s|^{\lambda_d^{H,\alpha}} \tag{4.4}$$

for any $(t,s) \in [0,T]^2$ and $x \in \mathbb{R}^d$.

In order to prove Theorem 3, we need the following technical lemmas.

4.2.1. Technical lemmas

Lemma 4.1. *For every $\lambda \geq 0$, $\beta \in (1,2]$, $H \in (\frac{1}{2},1)$, and $d \in \{1,2,3\}$ such that $\lambda_d^{H,\beta} > 0$, the improper double integral is*

$$\int_0^{+\infty}\int_0^{+\infty} |r-w|^{2H-2} g^{d,\beta,\lambda}(r,w)\,dr\,dw < \infty,$$

with $g^{d,\beta,\lambda}(r,w) := (2(1+\lambda)+r+w)^{-\frac{d}{\beta}} - 2(1+\lambda+r+w)^{-\frac{d}{\beta}} + (r+w)^{-\frac{d}{\beta}}$.

Proof. Denoting $J(\beta,d,\lambda)$ as the above double integral, we have

$$\begin{aligned} J(\beta,d,\lambda) &= \int_0^{+\infty}\int_0^{r} |r-w|^{2H-2} g^{d,\beta,\lambda}(r,w)\,dr\,dw + \int_0^{+\infty}\int_r^{+\infty} |r-w|^{2H-2} g^{d,\beta,\lambda}(r,w)\,dr\,dw \\ &= 2\int_0^{+\infty}\int_0^{r} |r-w|^{2H-2} g^{d,\beta,\lambda}(r,w)\,dr\,dw. \end{aligned}$$

For any $r,w \in (0,\infty)$, we have $2(1+\lambda)+r+w \geq 1+\lambda+r+w \geq r+w$. So, by using the fact that the function $x \in \mathbb{R}_+ \mapsto x^{-d/\beta}$ is decreasing, we get

$$\begin{aligned} |r-w|^{2H-2} g^{d,\beta,\lambda}(r,w) &= |r-w|^{2H-2}\left[(2(1+\lambda)+r+w)^{-\frac{d}{\beta}} - 2(1+\lambda+r+w)^{-\frac{d}{\beta}} + (r+w)^{-\frac{d}{\beta}}\right] \\ &\leq 4|r-w|^{2H-2}(r+w)^{-\frac{d}{\beta}}. \end{aligned}$$

Hence, by the change of variables $\tilde{w} = \frac{w}{r}$, we get

$$\begin{aligned} \int_0^1\int_0^r |r-w|^{2H-2} g^{d,\beta,\lambda}(r,w)\,dr\,dw &\leq 4\int_0^1\int_0^r (r-w)^{2H-2}(r+w)^{-\frac{d}{\beta}}\,dr\,dw \\ &= 4\int_0^1 r^{2H-1-\frac{d}{\beta}}\,dr \int_0^1 (1-\tilde{w})^{2H-2}(1+\tilde{w})^{-\frac{d}{\beta}}\,d\tilde{w}. \end{aligned} \tag{4.5}$$

Since $H > 1/2$, the integral $\int_0^1 (1-\tilde{w})^{2H-2}(1+\tilde{w})^{-\frac{d}{\beta}}\, d\tilde{w}$ is finite, and since $2\beta H > d$, the integral $\int_0^1 r^{2H-1-\frac{d}{\beta}}\, dr$ is also finite. Therefore, we obtain $\int_0^1 \int_0^r |r-w|^{2H-2} g^{d,\beta,\lambda}(r,w)\, dw\, dr < \infty$.

Now, when $u+v$ is close to infinity,

$$\begin{aligned} & |r-w|^{2H-2}\left[(2(1+\lambda)+r+w)^{-\frac{d}{\beta}} - 2(1+\lambda+r+w)^{-\frac{d}{\beta}} + (r+w)^{-\frac{d}{\beta}}\right] \\ = \; & |r-w|^{2H-2}(r+w)^{-\frac{d}{\beta}}\left[\left(1+\frac{2(1+\lambda)}{r+w}\right)^{-\frac{d}{\beta}} - 2\left(1+\frac{(1+\lambda)}{r+w}\right)^{-\frac{d}{\beta}} + 1\right] \\ \cong \; & \frac{d}{\beta}\left(\frac{d}{\beta}+1\right)(1+\lambda)^2\, |r-w|^{2H-2}(r+w)^{-\frac{d}{\beta}-2}. \end{aligned}$$

Therefore, for every $(u,v) \in [0,+\infty)^2$ such that $r \geq 1$ and $0 \leq w \leq r$, we have

$$|r-w|^{2H-2} g^{d,\beta,\lambda}(r,w) \leq \; Cte\, |r-w|^{2H-2}(r+w)^{-\frac{d}{\beta}-2}.$$

Hence, by the change of variables $\tilde{w} = \frac{w}{r}$, we get

$$\begin{aligned} \int_1^{+\infty}\int_0^r |r-w|^{2H-2} g^{d,\beta,\lambda}(r,w) & \leq \; Cte \int_1^{+\infty}\int_0^r |r-w|^{2H-2}(r+w)^{-2-\frac{d}{\beta}} dr dw \\ & \leq \; Cte \int_1^{\infty} r^{2H-3-\frac{d}{\beta}} dr \int_0^1 |1-\tilde{w}|^{2H-2}(1+\tilde{w})^{-2-\frac{d}{\beta}} d\tilde{w}. \end{aligned}$$

Both integrals appearing in the last line are finite because $H > \frac{1}{2}$ and $2H - \frac{d}{\beta} < 2$. As a consequence,

$$\int_1^{+\infty}\int_0^r |r-w|^{2H-2} g^{d,\beta,\lambda}(r,w)\, dw\, dr < \infty,$$

which implies that $J(\beta,d,\lambda)$ is finite. □

The following useful lemma was obtained by Balan and Tudor in [3].

Lemma 4.2. *We have*

$$\int_0^s \int_0^s |r-w|^{2H-2}\, \exp\left(-\frac{(r+w)z}{2}\right) dr\, dw \leq c'_H \left(s^{2H}+1\right)\left(\frac{1}{1+z}\right)^{2H} \tag{4.6}$$

for any $s \in [0,T]$ and $z \geq 0$ where c'_H denotes a positive constant depending only on H.

4.2.2. Proof of Theorem 3:

Consider $t, s \in [0,T]$ and $x \in \mathbb{R}^d$. Without loss of generality, we assume that $s \leq t$. By the Wiener isometry characteristic (3.8), we have

$$\begin{aligned} \mathbb{E}[|u^{a,b}(t,x) - u^{a,b}(s,x)|^2] \; = \; & \int_0^T du \int_0^T dv\, h^{H,\theta,\nu}(u,v) \int_{\mathbb{R}} dz\, \left[G_{a,b}(a_{t,u},x,z)\mathbf{1}^{(u)}_{(0,t)} - G_{a,b}(a_{s,u},x,z)\mathbf{1}^{(u)}_{(0,s)}\right] \\ & \times\left[G_{a,b}(a_{t,v},x,z)\mathbf{1}^{(v)}_{(0,t)} - G_{a,b}(a_{s,v},x,z)\mathbf{1}^{(v)}_{(0,s)}\right], \end{aligned} \tag{4.7}$$

where we recall that $h^{H,\theta,\nu}(u,v)=c_1^H(\theta,\nu)|u-v|^{2H-2}+c_2^H(\theta,\nu)(u+v)^{2H-2}$. Therefore,

$$\begin{aligned}
&\mathbb{E}[|u^{a,b}(t,x)-u^{a,b}(s,x)|^2]\\
\leq\ & \int_s^t\int_s^t dv\,du\,h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d}G_{a,b}(a_{t,u},x,z)\,G_{a,b}(a_{t,v},x,z)\,dz\\
+\ & 2\int_s^t\int_0^s dv\,du\,h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d}G_{a,b}(a_{t,u},x,z)\,[G_{a,b}(a_{t,v},x,z)-G_{a,b}(a_{s,v},x,z)]\,dz\\
+\ & \int_0^s\int_0^s dv\,du\,h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d}[G_{a,b}(a_{t,u},x,z)-G_{a,b}(a_{s,u},x,z)]\,[G_{a,b}(a_{t,v},x,y)-G_{a,b}(a_{s,v},x,z)]\,dz\\
=\ & A_1^{a,b}(t,s,x)+A_2^{a,b}(t,s,x)+A_3^{a,b}(t,s,x).
\end{aligned}$$

Let us start by the first term. By applying the Plancherel theorem, we obtain

$$\begin{aligned}
|A_1^{a,b}(t,s,x)| &= (2\pi)^{-d}\left|\int_s^t\int_s^t dv\,du\,h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d}\mathcal{F}G_{a,b}(a_{t,u},x,z)\,\overline{\mathcal{F}G_{a,b}(a_{t,v},x,z)}\,dz\right|\\
&= (2\pi)^{-d}\left|\int_s^t\int_s^t dv\,du\,h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d}e^{ib.\xi(v-u)}e^{-(2t-u-v)A_\alpha^a(\xi)}d\xi\right|\\
&\leq (2\pi)^{-d}\int_s^t\int_s^t dv\,du\,\left|h^{H,\theta,\nu}(u,v)\right|\int_{\mathbb{R}^d}\left|e^{ib.\xi(v-u)}\right|\,e^{-(2t-u-v)A_\alpha^a(\xi)}d\xi.
\end{aligned}$$

For every $H\in\left(\frac{1}{2},1\right)$, $u\in(0,t)$, and $v\in(0,s)$, we have $|u-v|^{2H-2}\geq(u+v)^{2H-2}$, which implies that

$$|h^{H,\theta,\nu}(u,v)|\leq c^H(\theta,\nu)\,|a_{u,v}|^{2H-2}, \tag{4.8}$$

with $c^H(\theta,\nu)=\alpha_H\dfrac{(|\theta|+|\nu|)^2}{2}$. This with Inequality (3.13) implies that

$$\begin{aligned}
|A_1^{a,b}(t,s,x)| &\leq (2\pi)^{-d}\,c^H(\theta,\nu)\int_s^t\int_s^t dv\,du\,|a_{u,v}|^{2H-2}\int_{\mathbb{R}^d}e^{-a^\alpha(2t-u-v)\|\xi\|^\alpha}d\xi\\
&\leq (2\pi)^{-d}\,c^H(\theta,\nu)\int_s^t\int_s^t dv\,du\,|a_{u,v}|^{2H-2}\left(\int_{\mathbb{R}}e^{-D_\alpha(2t-u-v)|\xi_1|^\alpha}d\xi_1\right)^d
\end{aligned}$$

with $D_\alpha=a^\alpha d^{\frac{\alpha}{2}-1}$. By the change of variables $z_1=(2t-u-v)^{1/\alpha}\,\xi_1$, we get

$$\int_{\mathbb{R}}e^{-D_\alpha(2t-u-v)|\xi_1|^\alpha}d\xi_1=(2t-u-v)^{-1/\alpha}\int_{\mathbb{R}}e^{-D_\alpha|z_1|^\alpha}dz_1=\gamma_\alpha\,(2t-u-v)^{-1/\alpha},$$

with γ_α is defined in (3.14), which is clearly finite. This, with the change of variables $U=a_{t,u}$ and $V=a_{t,v}$, then, $\tilde{u}=\dfrac{u}{a_{t,S}}$ and $\tilde{v}=\dfrac{v}{a_{t,S}}$, allowing us to get:

$$\begin{aligned}
|A_1^{a,b}(t,s,x)| &\leq (2\pi)^{-d}\gamma_\alpha^d\,c^H(\theta,\nu)\int_s^t\int_s^t|a_{u,v}|^{2H-2}\,(2t-u-v)^{-d/\alpha}\,dv\,du\\
&= (2\pi)^{-d}\gamma_\alpha^d\,c^H(\theta,\nu)\int_0^{t-s}\int_0^{t-s}|a_{u,v}|^{2H-2}\,(u+v)^{-d/\alpha}\,dv\,du\\
&= (2\pi)^{-d}\gamma_\alpha^d\,c^H(\theta,\nu)\,(t-s)^{2H-\frac{d}{\alpha}}\int_0^1\int_0^1|a_{u,v}|^{2H-2}\,(u+v)^{-d/\alpha}\,dv\,du\\
&= C_\alpha^d(H)\,(t-s)^{2H-\frac{d}{\alpha}},
\end{aligned}$$

with $C_\alpha^d(H) = (2\pi)^{-d}\gamma_\alpha^d\, c^H(\theta,\nu)\, \int_0^1\int_0^1 |a_{u,v}|^{2H-2}\,(u+v)^{-d/\alpha}\,dv\,du$, which is finite because $2\alpha H > d$.

Now, let us consider the second term. By applying the Plancherel theorem and from Lemma 3.3, we obtain

$$
\begin{aligned}
&|A_2^{a,b}(t,s,x)|\\
=\ & 2\left|\int_s^t\int_0^s h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d} G_{a,b}(a_{t,u},x,y)\,[G_{a,b}(a_{t,v},x,z)-G_{a,b}(a_{s,v},x,z)]\,dz\,dv\,du\right|\\
=\ & 2^{1-d}\pi^{-d}\left|\int_s^t\int_0^s dv\,du\,h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d}\overline{\mathcal{F}G_{a,b}(a_{t,u},x,z)}\,[\mathcal{F}G_{a,b}(a_{t,v},x,z)-\mathcal{F}G_{a,b}(a_{s,v},x,z)]\,dz\right|\\
=\ & 2^{1-d}\pi^{-d}\left|\int_s^t\int_0^s dv\,du\,h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d} d\xi\exp\left(i(x-a_{t,u}b).\xi\right)e^{-a_{t,u}A_\alpha^a(\xi)}\right.\\
&\left.\times\Big[\exp\left(-i(x-a_{t,v}b).\xi\right)e^{-a_{t,v}A_\alpha^a(\xi)}-\exp\left(-i(x-a_{s,v}b).\xi\right)\,e^{-a_{s,v}A_\alpha^a(\xi)}\Big]\right|
\end{aligned}
\tag{4.9}
$$

for any $s,t\in[0,T]$ and $x\in\mathbb{R}^d$. With a simple simplification, we get

$$
\begin{aligned}
|A_2^{a,b}(t,s,x)| =\ & 2^{1-d}\pi^{-d}\left|\int_s^t\int_0^s h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d} e^{ix.\xi}e^{-i(t-u)b.\xi}\,e^{-(t-u)A_\alpha^a(\xi)}\,e^{-ix.\xi}\,e^{-ivb.\xi}\right.\\
&\left.\times\,\Big[\exp\left(itb.\xi\right)e^{-(t-v)A_\alpha^a(\xi)}-\exp\left(isb.\xi\right)\,e^{-(s-v)A_\alpha^a(\xi)}\Big]\,d\xi\,dv\,du\right|\\
=\ & 2^{1-d}\pi^{-d}\left|\int_s^t\int_0^s dv\,du\,h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d} d\xi\,e^{-(t-u)A_\alpha^a(\xi)}\,e^{-i(v-u)b.\xi}\right.\\
&\left.\times\Big[e^{-(t-v)A_\alpha^a(\xi)}-e^{-i(t-s)b.\xi}\,e^{-(s-v)A_\alpha^a(\xi)}\Big]\right|\\
=\ & 2^{1-d}\pi^{-d}\left|\int_s^t\int_0^s dv\,du\,h^{H,\theta,\nu}(u,v)\int_{\mathbb{R}^d} d\xi\,e^{-(t+s-u-v)A_\alpha^a(\xi)}\,e^{-i(v-u)b.\xi}\right.\\
&\left.\times\Big[e^{-(t-s)A_\alpha^a(\xi)}-e^{-i(t-s)b.\xi}\Big]\right|\\
\le\ & 2^{1-d}\pi^{-d}\int_s^t\int_0^s dv\,du\,|h^{H,\theta,\nu}(u,v)|\int_{\mathbb{R}^d} d\xi\,e^{-(t+s-u-v)A_\alpha^a(\xi)}\left|e^{-i(v-u)b.\xi}\right|\\
&\times\left|e^{-(t-s)A_\alpha^a(\xi)}-e^{-i(t-s)b.\xi}\right|\\
\le\ & 2^{1-d}\pi^{-d}\int_s^t\int_0^s dv\,du\,|h^{H,\theta,\nu}(u,v)|\int_{\mathbb{R}^d} d\xi\,e^{-(t+s-u-v)A_\alpha^a(\xi)}\left|e^{-(t-s)A_\alpha^a(\xi)}-e^{-i(t-s)b.\xi}\right|.
\end{aligned}
\tag{4.10}
$$

We discuss here three cases:

First case: if $d=3$: It is clear that for any $\alpha\in(1,2]$, $1-\frac{3}{\alpha}<0$. From (4.10), we have

$$
|A_2^{a,b}(t,s,x)| \le 2^{2-d}\pi^{-d}\int_s^t\int_0^s |h^{H,\theta,\nu}(u,v)|\left(\int_{\mathbb{R}^3} e^{-(t+s-u-v)A_\alpha^a(\xi)}\,d\xi\right)dv\,du.
$$

Following the same technique employed above, we get

$$
|A_2^{a,b}(t,s,x)| \le 2^{2-d}\pi^{-d}\int_s^t\int_0^s |h^{H,\theta,\nu}(u,v)|\left(\int_{\mathbb{R}} e^{-(t+s-u-v)|\xi_1|^2}\,d\xi_1\right)^3\,dv\,du.
$$

By (4.8) and by the change of variables $z_1 = (t+s-u-v)^{1/2}\xi_1$, we get

$$|A_2^{a,b}(t,s,x)| \leq 2^{2-d}\pi^{3/2-d}\, c^H(\theta,\nu)\int_s^t\int_0^s (u-v)^{2H-2}\,(t+s-u-v)^{-3/2}\,dv\,du.$$

Now, since $u\in(s,t)$ and $v\in(0,s)$, we have $(u-v)^{2H-2}\leq(u-s)^{2H-2}$. Therefore, by the change of variables $\tilde{u}=u-s$, and $\tilde{u}=\dfrac{u}{a_{t,s}}$, we get

$$\begin{aligned}
|A_2^{a,b}(t,s,x)| &\leq Cte\int_s^t (u-s)^{2H-2}\,\left[(t+s-u)^{-1/2}-(t-u)^{-1/2}\right]du\\
&\leq Cte\int_s^t (u-s)^{2H-2}\,(t-u)^{-1/2}du\\
&= Cte\,(t-s)^{2H-\frac{3}{2}}\int_0^1 u^{2H-2}\,(1-u)^{-1/2}\,du\\
&= Cte\,(t-s)^{2H-\frac{3}{2}},
\end{aligned}$$

where the last line is due to the fact that the integral $\int_0^1 u^{2H-2}\,(1-u)^{-1/2}\,du$ is finite since $2-2H<1$.

Second case: if $d=2$: It is clear that for any $\alpha\in(1,2]$, $1-\frac{2}{\alpha}<0$. From (4.10), we have

$$\begin{aligned}
|A_2^{a,b}(t,s,x)| &\leq 2^{2-d}\pi^{-d}\int_s^t\int_0^s |h^{H,\theta,\nu}(u,v)|\int_{\mathbb{R}^2} e^{-(t+s-u-v)A_\alpha^a(\xi)}\,d\xi\,dv\,du.\\
&\leq 2^{2-d}\pi^{-d}\int_s^t\int_0^s |h^{H,\theta,\nu}(u,v)|\left(\int_{\mathbb{R}} e^{-D_\alpha(t+s-u-v)|\xi_1|^\alpha}\,d\xi_1\right)^2\,dv\,du.
\end{aligned}$$

By (4.8) and by the change of variables $z_1=(t+s-u-v)^{1/\alpha}\xi_1$, we get

$$|A_2^{a,b}(t,s,x)| \leq Cte\int_s^t\int_0^s (u-v)^{2H-2}\,(t+s-u-v)^{-2/\alpha}\,dv\,du.$$

Now, since $u\in(s,t)$ and $v\in(0,s)$, we have $(u-v)^{2H-2}\leq(u-s)^{2H-2}$. Therefore, by the change of variables $\tilde{u}=u-s$, and $\tilde{u}=\dfrac{u}{a_{t,s}}$, we get

$$\begin{aligned}
|A_2^{a,b}(t,s,x)| &\leq Cte\int_s^t\int_0^s (u-s)^{2H-2}\,(t+s-u-v)^{-2/\alpha}\,dv\,du\\
&= Cte\int_s^t (u-s)^{2H-2}\,\left[(t-u)^{-\frac{2}{\alpha}+1}-(t+s-u)^{-\frac{2}{\alpha}+1}\right]du\\
&\leq Cte\int_s^t (u-s)^{2H-2}\,(t-u)^{-\frac{2}{\alpha}+1}du\\
&= Cte\,(t-s)^{2H-\frac{2}{\alpha}}\int_0^1 u^{2H-2}\,(1-u)^{-\frac{2}{\alpha}+1}\,du\\
&= Cte\,(t-s)^{2H-\frac{2}{\alpha}},
\end{aligned}$$

where in the last line we used the fact that the integral $\int_0^1 u^{2H-2}\ (1-u)^{-2/\alpha+1}\, du$ is finite because $2H-1>0$ and $2-\dfrac{2}{\alpha}>0$.

Third case: if $d=1$: Starting from (4.10) and using (4.8), we get

$$
\begin{aligned}
|A_2^{a,b}(t,s,x)| &\leq \pi^{-1}\, c^H(\theta,\nu)\int_s^t\int_0^s (u-v)^{2H-2}\int_{\mathbb{R}} e^{-(t+s-u-v)A_\alpha^a(\xi)}\left|e^{-(t-s)A_\alpha^a(\xi)}-e^{-i(t-s)b.\xi}\right| d\xi\, dv\, du\\
&\leq Cte\left\{A_{2,1}^{a,b}(t,s,x)+A_{2,2}^{a,b}(t,s,x)\right\}, \quad \text{with}\\
A_{2,1}^{a,b}(t,s,x) &= \int_s^t\int_0^s (u-v)^{2H-2}\left(\int_{\mathbb{R}} e^{-(t+s-u-v)A_\alpha^a(\xi)}\left|1-e^{-(t-s)A_\alpha^a(\xi)}\right| d\xi\right) dv\, du\\
A_{2,2}^{a,b}(t,s,x) &= \int_s^t\int_0^s (u-v)^{2H-2}\left(\int_{\mathbb{R}} e^{-(t+s-u-v)A_\alpha^a(\xi)}\left|1-e^{-i(t-s)b\xi}\right| d\xi\right) dv\, du.
\end{aligned}
\tag{4.11}
$$

Concerning the first term $A_{2,1}$, from the mean theorem, we have for any $\xi\in\mathbb{R}$ and $(t,s)\in I^2$

$$\left|1-e^{-(t-s)A_\alpha^a(\xi)}\right| \leq |t-s|\ A_\alpha^a(\xi).$$

Moreover, since $u\in(s,t)$ and $v\in(0,s)$, we have $(u-v)^{2H-2}\leq(u-s)^{2H-2}$. Therefore,

$$
\begin{aligned}
A_{2,1}^{a,b}(t,s,x) &= |t-s|\int_s^t\int_0^s (u-v)^{2H-2}\int_{\mathbb{R}} e^{-(t+s-u-v)A_\alpha^a(\xi)}\ A_\alpha^a(\xi)\ d\xi\, dv\, du\\
&\leq |t-s|\int_s^t\int_0^s (u-s)^{2H-2}\int_{\mathbb{R}} e^{-(t+s-u-v)A_\alpha^a(\xi)}\ A_\alpha^a(\xi)\ d\xi\, dv\, du\\
&= |t-s|\int_{\mathbb{R}} A_\alpha^a(\xi)\int_s^t (u-s)^{2H-2}e^{-(t-u)A_\alpha^a(\xi)}\int_0^s e^{-(s-v)A_\alpha^a(\xi)}\, dv\ d\xi\, du\\
&= |t-s|\int_{\mathbb{R}} A_\alpha^a(\xi)\int_s^t (u-s)^{2H-2}e^{-(t-u)A_\alpha^a(\xi)}\left(\frac{1-e^{-sA_\alpha^a(\xi)}}{A_\alpha^a(\xi)}\right) d\xi\, du\\
&\leq |t-s|\int_s^t (u-s)^{2H-2}\int_{\mathbb{R}} e^{-(t-u)A_\alpha^a(\xi)}\ d\xi\, du\\
&\leq |t-s|\int_s^t (u-s)^{2H-2}\int_{\mathbb{R}} e^{-a^\alpha(t-u)|\xi|^\alpha}\ d\xi\, du.
\end{aligned}
$$

By the change of variables $z=(t-u)^{1/\alpha}\xi$, $v=u-s$, and $w=\dfrac{v}{a_{t,s}}$, we get

$$
\begin{aligned}
A_{2,1}^{a,b}(t,s,x) &\leq |t-s|\int_{\mathbb{R}} e^{-a^\alpha|z|^\alpha}dz\int_s^t (u-s)^{2H-2}\,(t-u)^{-1/\alpha}\, du\\
&= |t-s|^{2H-1/\alpha}\ \int_{\mathbb{R}} e^{-a^\alpha|z|^\alpha}dz\ \int_0^1 w^{2H-2}\,(1-w)^{-1/\alpha}\, dw\\
&= Cte\ |t-s|^{2H-\frac{1}{\alpha}},
\end{aligned}
$$

where the last line is due to the fact that both integrals $\int_{\mathbb{R}} e^{-a^\alpha|z|^\alpha}dz$ and $\int_0^1 w^{2H-2}\,(1-w)^{-1/\alpha}\, dw$ are finite.

Now, concerning the second term $A_{2,2}$, we have

$$\left|1 - e^{-ia_{t,s}b\xi}\right| = \left|2i\, e^{-i\frac{a_{t,s}b\xi}{2}}\right| \left|\frac{e^{i\frac{a_{t,s}b\xi}{2}} - e^{-i\frac{a_{t,s}b\xi}{2}}}{2i}\right| \leq 2\left|\sin\left(\frac{a_{t,s}b\,\xi}{2}\right)\right| \leq \left|a_{t,s}\right|\ |b|\,|\xi|$$

for any $\xi \in \mathbb{R}$. Therefore,

$$\begin{aligned} A^{a,b}_{2,2}(t,s,x) &\leq |b|\,|t-s| \int_s^t \int_0^s (u-s)^{2H-2} \int_{\mathbb{R}} e^{-(t+s-u-v)A^a_\alpha(\xi)}\ |\xi|\ d\xi\, dv\, du \\ &= |b|\,|t-s| \int_{\mathbb{R}} |\xi| \int_s^t (u-s)^{2H-2} e^{-(t-u)a^\alpha|\xi|^\alpha} \left(\int_0^s e^{-(s-v)a^\alpha|\xi|^\alpha}\, dv\right) d\xi\, du \\ &= |b|\,|t-s| \int_{\mathbb{R}} |\xi| \int_s^t (u-s)^{2H-2} e^{-(t-u)a^\alpha|\xi|^\alpha} \left(\frac{1 - e^{-a^\alpha s|\xi|^\alpha}}{a^\alpha|\xi|^\alpha}\right) d\xi\, du \\ &\leq a^{-\alpha}\, |b|\,|t-s| \int_s^t (u-s)^{2H-2} \int_{\mathbb{R}} e^{-(t-u)a^\alpha|\xi|^\alpha}\ \frac{1}{|\xi|^{\alpha-1}}\ d\xi\, du. \end{aligned}$$

By the change of variables $z = (t-u)^{1/\alpha}\,\xi$, $v = u - s$, and $w = \frac{v}{t-s}$, we get

$$\begin{aligned} A^{a,b}_{2,1}(t,s,x) &\leq a^{-\alpha}\, |b|\,|t-s| \int_{\mathbb{R}} \frac{e^{-a^\alpha|z|^\alpha}}{|z|^{\alpha-1}}\ dz \int_s^t (u-s)^{2H-2}\,(t-u)^{1-\frac{2}{\alpha}}\, du \\ &= Cte|t-s|^{2H-\frac{2}{\alpha}+1} \int_0^1 w^{2H-2}\,(1-w)^{1-\frac{2}{\alpha}}\, dw \\ &\leq Cte\ |t-s|^{2H-\frac{1}{\alpha}} \end{aligned}$$

where the second line is due to the fact that the integral $\int_{\mathbb{R}} \frac{e^{-a^\alpha|z|^\alpha}}{|z|^{\alpha-1}}\ dz$ is finite, and in the last line we used that $\int_0^1 w^{2H-2}\,(1-w)^{1-\frac{2}{\alpha}}\, dw < \infty$, because $1 - \frac{2}{\alpha} > -1$ and $2H - 2 > -1$.

Now, let us consider the third term $A^{a,b}_3$. By applying the Plancherel theorem, using again (4.8), and with some simple computations, we get:

$$\begin{aligned} &|A^{a,b}_3(t,s,x)| \leq c^H(\theta,\nu) \int_0^s \int_0^s dv\, du\, |u-v|^{2H-2} \\ &\times \int_{\mathbb{R}^d} \left|G_{a,b}(a_{t,u},x,z) - G_{a,b}(a_{s,u},x,z)\right| \left|G_{a,b}(a_{t,v},x,z) - G_{a,b}(a_{t,v},x,z)\right| dz \\ = \ &(2\pi)^{-d} c^H(\theta,\nu) \int_0^s \int_0^s dv\, du\, |u-v|^{2H-2} \\ &\times \int_{\mathbb{R}^d} \overline{\left|\mathcal{F}G_{a,b}(a_{t,u},x,z) - \mathcal{F}G_{a,b}(a_{s,u},x,z)\right|} \left|\mathcal{F}G_{a,b}(a_{t,v},x,z) - \mathcal{F}G_{a,b}(a_{s,v},x,z)\right| dz \\ = \ &Cte \int_0^s \int_0^s dv\, du\, |u-v|^{2H-2} \int_{\mathbb{R}^d} \left|\exp\left(i(x - a_{t,u}b).\xi\right) e^{-a_{t,u}A^a_\alpha(\xi)} - \exp\left(i(x - a_{s,u}b).\xi\right)\ e^{-a_{s,u}A^a_\alpha(\xi)}\right| \\ &\left|\exp\left(-i(x - a_{t,v}b).\xi\right) e^{-a_{s,v}A^a_\alpha(\xi)} - \exp\left(-i(x - a_{s,v}b).\xi\right)\ e^{-a_{s,v}A^a_\alpha(\xi)}\right| d\xi \\ = \ &Cte \int_0^s \int_0^s dv\, du\, |a_{u,v}|^{2H-2} \int_{\mathbb{R}^d} e^{-(2s-u-v)A^a_\alpha(\xi)} \left|\exp\left(-(t-s)[A^a_\alpha(\xi) - ib.\xi]\right) - 1\right|^2 d\xi. \end{aligned}$$

Note that for any $\xi \in \mathbb{R}^d$, $s, t \in [0, T]$, we have,

$$\left|e^{-(t-s)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha-ib.\xi\right)} - 1\right|^2 = \left|e^{-x+iy} - 1\right|^2$$

with $x = (t-s)\left(\|\xi\|^2 + a^\alpha\|\xi\|^\alpha\right)$ and $y = (t-s)b.\xi$, and that

$$\left|e^{-x+iy} - 1\right|^2 = \left|e^{\frac{-x+iy}{2}}\right|^2 \left|e^{\frac{-x+iy}{2}} - e^{\frac{x-iy}{2}}\right|^2 = 2\,e^{-x}(\cosh x - \cos y)$$

for all real numbers x and y. Therefore,

$$\begin{aligned} |A_3^{a,b}(t,s,x)| &\leq Cte \int_{\mathbb{R}^d}\int_0^s\int_0^s dv\,du\,|u-v|^{2H-2} e^{-(2s-u-v)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)} e^{-(t-s)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)} \\ &\quad\times\left[\cosh\left((t-s)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)\right) - \cos\left((t-s)b.\xi\right)\right] d\xi \\ &= Cte\left\{A_{3,1}^{a,b}(t,s,x) + A_{3,2}^{a,b}(t,s,x)\right\}, \end{aligned}$$

$$\begin{aligned} \text{with } A_{3,1}^{a,b}(t,s,x) &= \int_{\mathbb{R}^d}\int_0^s\int_0^s dv\,du\,|u-v|^{2H-2} e^{-(2s-u-v)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)} e^{-(t-s)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)} \\ &\quad\times\left[1 - \cos\left((t-s)b.\xi\right)\right] d\xi \\ \text{and } A_{3,2}^{a,b}(t,s,x) &= \int_{\mathbb{R}^d}\int_0^s\int_0^s dv\,du\,|u-v|^{2H-2} e^{-(2s-u-v)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)} e^{-(t-s)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)} \\ &\quad\times\left[\cosh\left((t-s)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)\right) - 1\right] d\xi. \end{aligned}$$

Concerning the term $A_{3,1}^{a,b}$, we first bound it in the case where $d = 1$. From the expression of $A_{3,1}^{a,b}$, we easily get that

$$A_{3,1}^{a,b}(t,s,x) \leq \int_{\mathbb{R}} |1 - \cos((t-s)b\xi)| \int_0^s\int_0^s dv\,du\,|u-v|^{2H-2} e^{-(2s-u-v)|\xi|^2} d\xi.$$

We note that, for any $\xi \in \mathbb{R}$, we have

$$|1 - \cos((t-s)b\xi)| \leq |1 - \cos((t-s)b\xi)|^H \times 2^{1-H}.$$

Moreover, $|1-\cos((t-s)b\xi)|^H = \left|2\sin^2\left(\dfrac{(t-s)b\xi}{2}\right)\right|^H \leq 2^{H-2H}|t-s|^{2H}|b|^{2H}|\xi|^{2H}$. It follows that

$$|1 - \cos((t-s)b\xi)| \leq |\xi|^{2H} \times |t-s|^{2H}|b|^{2H}\,2^{1-2H},$$

and, consequently,

$$\begin{aligned} A_{3,1}^{a,b}(t,s,x) &\leq Cte\,|t-s|^{2H}\int_{\mathbb{R}}|\xi|^{2H}\int_0^s\int_0^s dv\,du\,|u-v|^{2H-2}e^{-(2s-u-v)|\xi|^2}d\xi \\ &\leq Cte\,T^{\frac{d}{\alpha}}\,|t-s|^{2H-\frac{d}{\alpha}}\int_{\mathbb{R}}|\xi|^{2H}\int_0^s\int_0^s dv\,du\,|u-v|^{2H-2}e^{-(2s-u-v)|\xi|^2}d\xi \\ &= Cte\,|t-s|^{2H-\frac{d}{\alpha}}\left[J_1 + J_2\right], \end{aligned}$$

with

$$\begin{aligned} J_1 &= \int_{|\xi|\le 1} |\xi|^{2H} \int_0^s \int_0^s dv\, du\, |u-v|^{2H-2}\, e^{-(2s-u-v)|\xi|^2} d\xi \\ J_2 &= \int_{1\le|\xi|} |\xi|^{2H} \int_0^s \int_0^s dv\, du\, |u-v|^{2H-2}\, e^{-(2s-u-v)|\xi|^2} d\xi. \end{aligned}$$

On the one hand, by the fact that $|a_{u,v}|^{2H-2}\, e^{-(2s-u-v)|\xi|^2} \le s^{2H-2}$ for every $u, v \in (0, s)$ and $\xi \in \mathbb{R}$, we can write that, for every $s \in [0, T]$ and $\xi \in \mathbb{R}$,

$$J_1 \quad \le \quad T^{2H} \int_{|\xi|\le 1} |\xi|^{2H} d\xi < \infty.$$

On the other hand, by Lemma 4.2 and the Assumption $2H > 1$, we get

$$J_2 \le Cte \int_{1\le|\xi|} \frac{|\xi|^{2H}}{(1+|\xi|^2)^{2H}}\, d\xi \le Cte \int_{1\le|\xi|} \frac{1}{|\xi|^{2H}}\, d\xi < \infty.$$

All this implies that

$$A^{a,b}_{3,1}(t, s, x) \le Cte\ |t-s|^{2H-\frac{d}{\alpha}}.$$

Now, let us bound $A^{a,b}_{3,1}$ in the case where $d \in \{2, 3\}$. By the change of variables $\tilde{u} = 2a_{s,u}$ and $\tilde{v} = 2a_{s,v}$, we get

$$\begin{aligned} & A^{a,b}_{3,1}(t, s, x) \\ \le\ & \int_{\mathbb{R}^d} e^{-(t-s)A^a_\alpha(\xi)}\, |1-\cos((t-s)b.\xi)| \int_0^s \int_0^s |u-v|^{2H-2} e^{-(2s-u-v)A^a_\alpha(\xi)} dv\, du\, d\xi \\ =\ & 2^{-2H} \int_{\mathbb{R}^d} e^{-(t-s)A^a_\alpha(\xi)}\, |1-\cos((t-s)b.\xi)| \int_0^{2s} \int_0^{2s} |\tilde{u}-\tilde{v}|^{2H-2} e^{-\frac{(\tilde{u}+\tilde{v})}{2}A^a_\alpha(\xi)} d\tilde{u}\, d\tilde{v}\, d\xi. \end{aligned} \tag{4.12}$$

Using again Lemma 4.2, we get

$$\int_0^{2s} \int_0^{2s} |u-v|^{2H-2}\ \exp\left(-\frac{(u+v)A^a_\alpha(\xi)}{2}\right) dv\, du \le Cte\ \left((2s)^{2H}+1\right)\left(\frac{1}{1+A^a_\alpha(\xi)}\right)^{2H} \tag{4.13}$$

for any $s \in [0, T]$ and $\xi \in \mathbb{R}^d$.

Moreover, for every fixed $a, x \in \mathbb{R}^*_+$, $x \mapsto \dfrac{e^{-ax}}{(1+x)^{2H}}$ is decreasing. All this with the change of variables $z = (t-s)\xi$, allows us to get

$$\begin{aligned} A^{a,b}_{3,1}(t, s, x) &\le Cte\left(2^{2H}T^{2H}+1\right) \int_{\mathbb{R}^d} \frac{e^{-(t-s)A^a_\alpha(\xi)}}{\left[1+A^a_\alpha(\xi)\right]^{2H}}\ [1-\cos((t-s)b.\xi)]\, d\xi \\ &\le Cte \int_{\mathbb{R}^d} \frac{e^{-a^\alpha(t-s)\|\xi\|^\alpha}}{\left[1+a^\alpha\, \|\xi\|^\alpha\right]^{2H}}\ [1-\cos((t-s)b.\xi)]\, d\xi \\ &\le Cte \int_{\mathbb{R}^d} \frac{1-\cos((t-s)b.\xi)}{\left[1+a^\alpha\, \|\xi\|^\alpha\right]^{2H}}\, d\xi, \\ &= Cte\ |t-s|^{2\alpha H-d} \int_{\mathbb{R}^d} \left(\frac{1}{|t-s|^\alpha + a^\alpha\, \|z\|^\alpha}\right)^{2H}\ [1-\cos(b.z)]\, dz. \end{aligned} \tag{4.14}$$

Let us discuss two cases: $|t-s| \geq 1$ and $|t-s| < 1$.

First case: When $|t-s| \geq 1$,

$$\begin{aligned} A_{3,1}^{a,b}(t,s,x) &\leq Cte\ |t-s|^{2\alpha H-d} \int_{\mathbb{R}^d} \left(\frac{1}{1+a^\alpha \|z\|^\alpha}\right)^{2H} |1-\cos(b.z)|\, dz \\ &\leq Cte\ T^{2\alpha H-2H+\frac{d}{\alpha}-d} |t-s|^{2H-\frac{d}{\alpha}} \int_{\mathbb{R}^d} \left(\frac{1}{1+a^\alpha \|z\|^\alpha}\right)^{2H} dz, \end{aligned}$$

where in the last line we used that $2\alpha H - 2H + \frac{d}{\alpha} - d > 0$, due to the assumptions $\alpha > 1$ and $2H > \frac{d}{\alpha}$.

We note that the integral $\int_{\mathbb{R}^d} \left(\frac{1}{1+a^\alpha \|z\|^\alpha}\right)^{2H} dz$ is finite, because $2H\alpha > d$. All this implies the existence of a nonnegative constant Cte such that

$$A_{3,1}^{a,b}(t,s,x) \leq Cte\, |t-s|^{2H-\frac{d}{\alpha}}. \tag{4.15}$$

Second case: Suppose that $|t-s| < 1$: We first note that, since $\alpha \in (1,2]$, $H \in (\frac{1}{2}, 1)$, $2\alpha H > d$, and since we are in the case where $d \in \{2,3\}$, we necessarily have $\alpha H \in (1,2)$.

Denoting $D_1 = \left\{z \in \mathbb{R}^d; \|z\| \geq 1\right\}$, and $D_2 = \left\{z \in \mathbb{R}^d; \|z\| \leq 1\right\}$ from (4.14), we get

$$\begin{aligned} A_{3,1}^{a,b}(t,s,x) &\leq Cte|t-s|^{2\alpha H-d} \int_{\mathbb{R}^d} \frac{1-\cos(b.z)}{a^{2\alpha H} \|z\|^{2\alpha H}}\, dz \\ &\leq Cte\ |t-s|^{2\alpha H-d} \left[\int_{D_1} \frac{2}{\|z\|^{\alpha 2H}}\ dz + \int_{D_2} \frac{|1-\cos(b.z)|}{\|z\|^{\alpha 2H}}\ dz\right] \\ &\leq Cte\ |t-s|^{2H-\frac{d}{\alpha}} \left[2I_1^H + I_2^H\right], \end{aligned}$$

with $I_1^H = \int_{D_1} \frac{1}{\|z\|^{\alpha 2H}}\ dz$ and $I_2^H = \int_{D_2} \frac{|1-\cos(b.z)|}{\|z\|^{\alpha 2H}}\ dz$.

Since $2\alpha H > d$, the integral I_1^H is clearly finite. As regards I_2^H, for any $z \in \mathbb{R}^d$, we have

$$|1-\cos(b.z)| = |1-\cos(b.z)|^{\frac{\alpha H}{2}} \times |1-\cos(b.z)|^{1-\frac{\alpha H}{2}}$$

and, as consequence, $|1-\cos(b.z)| \leq |1-\cos(b.z)|^{\frac{\alpha H}{2}} \times 2^{1-\frac{\alpha H}{2}}$. Moreover,

$$|1-\cos(b.z)|^{\frac{\alpha H}{2}} = \left|2\sin^2\left(\frac{b.z}{2}\right)\right|^{\frac{\alpha H}{2}} \leq 2^{-\frac{\alpha H}{2}} \|b\|^{\alpha H} \|z\|^{\alpha H}.$$

It follows that

$$|1-\cos(b.z)| \leq 2^{1-\alpha H} \|b\|^{\alpha H} \|z\|^{\alpha H}$$

and, consequently, since $\alpha H < 2$, we have

$$I_2^H \leq Cte\ \int_{D_2} \frac{1}{\|z\|^{\alpha H}}\ dz < \infty.$$

Therefore, Inequality (4.15) is satisfied for every $(t,s) \in [0,T]^2$.

Let us consider the term $A_{3,2}^{a,b}$. We discuss two cases:

First case: if $|t-s| \geq 1$: $\forall \xi \in \mathbb{R}^d$, we have $e^{-(t-s)A_\alpha^a(\xi)}\left[\cosh\left((t-s)A_\alpha^a(\xi)\right)-1\right] \leq 2$. Then, by using again Lemma 4.2 and by applying the change of variables $z=(t-s)^{1/\alpha}\xi$, we get:

$$
\begin{aligned}
A_{3,2}^{a,b}(t,s,x) \leq Cte \int_{\mathbb{R}^d} \frac{1}{\left[1+a^\alpha\|\xi\|^\alpha\right]^{2H}}\, d\xi &= Cte(t-s)^{2H-\frac{d}{\alpha}} \int_{\mathbb{R}^d} \frac{1}{\left[(t-s)+a^\alpha\|z\|^\alpha\right]^{2H}}\, dz \\
&\leq Cte(t-s)^{2H-\frac{d}{\alpha}} \int_{\mathbb{R}^d} \frac{dz}{\left[1+a^\alpha\|z\|^\alpha\right]^{2H}}.
\end{aligned}
$$

Since the last integral is finite because $2\alpha H > d$, we deduce that

$$
A_{3,2}^{a,b}(t,s,x) \leq Cte\ (t-s)^{2H-\frac{d}{\alpha}}.
$$

Second case: if $|t-s| \leq 1$: Denoting $E_1 = \left\{\xi \in \mathbb{R}^d; \|\xi\| \geq (a_{t,s})^{-1/2}\right\}$ and $E_2 = \left\{\xi \in \mathbb{R}^d; \|\xi\| \leq (a_{t,s})^{-1/2}\right\}$, we can write

$$
A_{3,2}^{a,b}(t,s,x) = C_1^{a,b}(t,s,x) + C_2^{a,b}(t,s,x), \quad \text{with} \tag{4.16}
$$

$$
\begin{aligned}
C_1^{a,b}(t,s,x) &= \int_{E_1}\int_0^s\int_0^s dv\, du\, |u-v|^{2H-2} e^{-(2s-u-v)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)} e^{-(t-s)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)} \\
&\quad \times \left[\cosh\left((t-s)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)\right)-1\right] d\xi \\
C_2^{a,b}(t,s,x) &= \int_{E_2}\int_0^s\int_0^s dv\, du\, |u-v|^{2H-2} e^{-(2s-u-v)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)} e^{-(t-s)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)} \\
&\quad \times \left[\cosh\left((t-s)\left(\|\xi\|^2+a^\alpha\|\xi\|^\alpha\right)\right)-1\right] d\xi.
\end{aligned}
$$

We note that if $\xi \in E_1$, then, $\xi \in D_1$. Moreover, the function $x \mapsto e^{-x}\left[\cosh(x)-1\right]$ is increasing on $\mathbb{R}_+$, and $A_\alpha^a(\xi) \leq (1+a^\alpha)\|\xi\|^2$ for every $\xi \in D_1$. Therefore,

$$
e^{-(t-s)A_\alpha^a(\xi)}\left[\cosh\left((t-s)A_\alpha^a(\xi)\right)-1\right] \leq e^{-(t-s)(1+a^\alpha)\|\xi\|^2}\left[\cosh\left((t-s)\|\xi\|^2(1+a^\alpha)\right)-1\right],
$$

for any $\xi \in D_1$, and, consequently, using Lemma 4.2, then by the change variable $z=(t-s)^{1/2}\xi$, we get:

$$
\begin{aligned}
C_1^{a,b}(t,s,x) &\leq \int_{E_1} e^{-(t-s)(1+a^\alpha)\|\xi\|^2}\left[\cosh\left((t-s)(1+a^\alpha)\|\xi\|^2\right)-1\right] \\
&\quad \times \int_0^s\int_0^s |u-v|^{2H-2} e^{-(2s-u-v)A_\alpha^a(\xi)} dv\, du\, d\xi \\
&\leq Cte \int_{E_1} e^{-(t-s)(1+a^\alpha)\|\xi\|^2} \frac{\left[\cosh\left((t-s)\|\xi\|^2(1+a^\alpha)\right)-1\right]}{\left[1+A_\alpha^a(\xi)\right]^{2H}}\, d\xi \\
&\leq Cte \int_{\|\xi\|\geq(t-s)^{-1/2}} e^{-(1+a^\alpha)(t-s)\|\xi\|^2} \frac{\left[\cosh\left((t-s)\|\xi\|^2(1+a^\alpha)\right)-1\right]}{\|\xi\|^{4H}}\, d\xi \\
&= Cte\ |t-s|^{2H-\frac{d}{2}} \int_{\|z\|\geq 1} e^{-(1+a^\alpha)\|z\|^2} \frac{\left[\cosh\left(\|z\|^2(1+a^\alpha)\right)-1\right]}{\|z\|^{4H}}\, dz \\
&\leq Cte\ |t-s|^{2H-\frac{d}{\alpha}} \int_{\|z\|\geq 1} \frac{1}{\|z\|^{4H}}\, dz \\
&= Cte\ |t-s|^{2H-\frac{d}{\alpha}},
\end{aligned}
$$

where in the last line we used that the integral $\int_{\|z\|\geq 1} \frac{dz}{\|z\|^{4H}}$ is finite since $4H > 2\alpha H > d$.

Now, concerning $C_2^{a,b}$, it can be written as: $C_2^{a,b}(t,s,x) = C_{2,1}^{a,b}(t,s,x) + C_{2,2}^{a,b}(t,s,x)$, with

$$\begin{aligned} C_{2,1}^{a,b}(t,s,x) &= \int_{\|\xi\|\leq 1} \left[\cosh\left((t-s)A_\alpha^a(\xi)\right) - 1\right] \int_0^s \int_0^s |u-v|^{2H-2} e^{-(2s-u-v)A_\alpha^a(\xi)} e^{-(t-s)A_\alpha^a(\xi)} dv\, du\, d\xi, \\ C_{2,2}^{a,b}(t,s,x) &= \int_{1\leq\|\xi\|\leq (t-s)^{-1/2}} \left[\cosh\left((t-s)A_\alpha^a(\xi)\right) - 1\right] \\ &\times \int_0^s \int_0^s |u-v|^{2H-2} e^{-(2s-u-v)A_\alpha^a(\xi)} e^{-(t-s)A_\alpha^a(\xi)} dv\, du\, d\xi. \end{aligned}$$

Using again the fact that the function $x \mapsto e^{-x}[\cosh(x) - 1]$ is increasing on $\mathbb{R}_+$; and that $A_\alpha^a(\xi) \leq (1+a^\alpha)\|\xi\|^\alpha$ for any $\xi \in D_2$, we obtain

$$\begin{aligned} C_{2,1}^{a,b}(t,s,x) &\leq \int_{D_2} \int_0^s \int_0^s |u-v|^{2H-2} e^{-(2s-u-v)A_\alpha^a(\xi)} e^{-(1+a^\alpha)(t-s)\|\xi\|^\alpha} \\ &\quad \times \left[\cosh\left((t-s)(1+a^\alpha)\|\xi\|^\alpha\right) - 1\right] dv\, du\, d\xi \\ &\leq \int_{D_2} \int_0^s \int_0^s |u-v|^{2H-2} e^{-a^\alpha(2s-u-v)\|\xi\|^\alpha} e^{-(1+a^\alpha)(t-s)\|\xi\|^\alpha} \\ &\quad \times \left[\cosh\left((t-s)(1+a^\alpha)\|\xi\|^\alpha\right) - 1\right] dv\, du\, d\xi \end{aligned}$$

We note that for any $\xi \in D_2$ and $u, v \in (0,s) \subset [0,T]$, we have

$$e^{-(2s-u-v)a^\alpha\|\xi\|^\alpha} = e^{-(2s-u-v)(1+a^\alpha)\|\xi\|^\alpha} e^{(2s-u-v)\|\xi\|^\alpha} \leq e^{-(2s-u-v)(1+a^\alpha)\|\xi\|^\alpha} e^{2T}.$$

Moreover, for any $a > 0$, $u, v \in (0,s)$ and $(t,s) \in [0,T]^2$, we have:

$$e^{-a(t-s)} e^{-a(2s-u-v)} [\cosh(a(t-s)) - 1] = \frac{1}{2}[e^{-a(2t-u-v)} - 2e^{-a(t+s-u-v)} + e^{-a(2s-u-v)}]. \tag{4.17}$$

Therefore, making the change of variables $U = \frac{a_{s,u}}{a_{t,s}}$, $V = \frac{a_{s,v}}{a_{t,s}}$, and $z = (t-s)^{1/\alpha}\xi$, we get

$$\begin{aligned} & C_{2,1}^{a,b}(t,s,x) \\ \leq\ & Cte \int_{D_2} \int_0^s \int_0^s |u-v|^{2H-2} \left[e^{-(2t-u-v)(1+a^\alpha)\|\xi\|^\alpha} - 2e^{-(t+s-u-v)(1+a^\alpha)\|\xi\|^\alpha} + e^{-(2s-u-v)(1+a^\alpha)\|\xi\|^\alpha}\right] d\xi\, dv\, du \\ \leq\ & Cte \int_{\mathbb{R}^d} \int_0^s \int_0^s |u-v|^{2H-2} \left[e^{-(2t-u-v)(1+a^\alpha)\|\xi\|^\alpha} - 2e^{-(t+s-u-v)(1+a^\alpha)\|\xi\|^\alpha} + e^{-(2s-u-v)(1+a^\alpha)\|\xi\|^\alpha}\right] d\xi\, dv\, du \\ \leq\ & Cte \int_{\mathbb{R}^d} \int_0^s \int_0^s dv\, du\, |u-v|^{2H-2} \left[e^{-(t-s)(2+\frac{s-u}{t-s}+\frac{s-v}{t-s})(1+a^\alpha)\|\xi\|^\alpha}\right. \\ & \left. -2e^{-(t-s)(1+\frac{s-u}{t-s}+\frac{s-v}{t-s})(1+a^\alpha)\|\xi\|^\alpha} + e^{-(t-s)(\frac{s-u}{t-s}+\frac{s-v}{t-s})(1+a^\alpha)\|\xi\|^\alpha}\right] d\xi \\ =\ & Cte\, |t-s|^{2H} \int_{\mathbb{R}^d} \int_0^{\frac{s}{a_{t,s}}} \int_0^{\frac{s}{a_{t,s}}} dV\, dU |U-V|^{2H-2} \\ \times\ & \left[e^{-a_{t,s}(2+U+V)(1+a^\alpha)\|\xi\|^\alpha} - 2e^{-a_{t,s}(1+U+V)(1+a^\alpha)\|\xi\|^\alpha} + e^{-a_{t,s}(U+V)(1+a^\alpha)\|\xi\|^\alpha}\right] d\xi \\ \leq\ & Cte|t-s|^{2H-\frac{d}{\alpha}} \int_{\mathbb{R}^d} \int_0^{+\infty} \int_0^{+\infty} |u-v|^{2H-2} \\ \times\ & \left[e^{-(2+u+v)(1+a^\alpha)\|z\|^\alpha} - 2e^{-(1+u+v)(1+a^\alpha)\|z\|^\alpha} + e^{-(u+v)(1+a^\alpha)\|z\|^\alpha}\right] dz\, dv\, du. \end{aligned}$$

Now, making the changes of variables $\xi = (2+u+v)^{1/\alpha} z,\ \xi = (1+u+v)^{1/\alpha} z$, and $\xi = (u+v)^{1/\alpha} z$, we obtain

$$C^{a,b}_{2,1}(t,s,x) \leq Cte\, I(\alpha,d) \int_{\mathbb{R}^d} e^{-(1+a^\alpha)\|\xi\|^\alpha} d\xi\, |t-s|^{2H-\frac{d}{\alpha}},$$

with

$$I(\alpha,d) = \int_0^{+\infty}\int_0^{+\infty} |u-v|^{2H-2}\left[(2+u+v)^{-\frac{d}{\alpha}} - 2(1+u+v)^{-\frac{d}{\alpha}} + (u+v)^{-\frac{d}{\alpha}}\right] dv\, du.$$

The above Gaussian integral $\int_{\mathbb{R}^d} e^{-(1+a^\alpha)\|\xi\|^\alpha} d\xi$ is clearly finite. Applying Lemma 4.1 with $\lambda = 0$ and $\beta = \alpha$, we get that $I(\alpha,d)$ is also finite. As a consequence,

$$C^{a,b}_{2,1}(t,s,x) \leq Cte\, |t-s|^{2H-\frac{d}{\alpha}}. \tag{4.18}$$

Now, let us investigate $C^{a,b}_{2,2}$. The function $x \mapsto e^{-x}[\cosh(x)-1]$ is increasing on $\mathbb{R}_+$ and for any ξ, such that $\|\xi\| \geq 1$, we have $A^a_\alpha(\xi) \leq (1+a^\alpha)\|\xi\|^2$. Proceeding as above, we get:

$$\begin{aligned}
& C^{a,b}_{2,2}(t,s,x) \\
\leq & \int_{1\leq\|\xi\|\leq(t-s)^{-1/2}} e^{-(1+a^\alpha)(t-s)\|\xi\|^2}\left[\cosh\left((t-s)(1+a^\alpha)\|\xi\|^2\right)-1\right] \\
& \times \int_0^s\int_0^s |u-v|^{2H-2} e^{-(2s-u-v)A^a_\alpha(\xi)} dv\, du\, d\xi \\
\leq & \int_{1\leq\|\xi\|\leq(t-s)^{-1/2}} e^{-(1+a^\alpha)(t-s)\|\xi\|^2}\left[\cosh\left((t-s)(1+a^\alpha)\|\xi\|^2\right)-1\right] \\
& \times \int_0^s\int_0^s e^{-(2s-u-v)(1+a^\alpha)\|\xi\|^2}\, e^{a^\alpha(2s-u-v)\|\xi\|^2}\, |u-v|^{2H-2} dv\, du\, d\xi.
\end{aligned}$$

Now, using (4.17) and denoting $E = \{\xi \in \mathbb{R}^d; 1 \leq \|\xi\| \leq (t-s)^{-1/2}\}$, we get;

$$\begin{aligned}
& C^{a,b}_{2,2}(t,s,x) \\
\leq & \int_E d\xi \int_0^s\int_0^s dv\, du\, e^{a^\alpha(2s-u-v)\|\xi\|^2}\, |u-v|^{2H-2} \\
\times & \left[e^{-(2t-u-v)(1+a^\alpha)\|\xi\|^2} - 2e^{-(t+s-u-v)(1+a^\alpha)\|\xi\|^2} + e^{-(2s-u-v)(1+a^\alpha)\|\xi\|^2}\right] \\
= & \int_E d\xi \int_0^s\int_0^s dv\, du\, |u-v|^{2H-2} \times \left[e^{a^\alpha(2s-u-v)\|\xi\|^2}\, e^{-(2t-u-v)(1+a^\alpha)\|\xi\|^2}\right. \\
- & \left. 2e^{a^\alpha(2s-u-v)\|\xi\|^2} e^{-(t+s-u-v)(1+a^\alpha)\|\xi\|^2} + e^{a^\alpha(2s-u-v)\|\xi\|^2} e^{-(2s-u-v)(1+a^\alpha)\|\xi\|^2}\right] \\
= & \int_E\int_0^s\int_0^s |u-v|^{2H-2} \times \left[e^{-2a^\alpha(t-s)\|\xi\|^2}\, e^{-(2t-u-v)\|\xi\|^2}\right. \\
- & \left. 2e^{-a^\alpha(t-s)\|\xi\|^2}\, e^{-(t+s-u-v)\|\xi\|^2} + e^{-(2s-u-v)\|\xi\|^2}\right] d\xi\, dv\, du.
\end{aligned}$$

By the change variable $U = \dfrac{s-u}{a_{t,s}}$, $V = \dfrac{s-v}{a_{t,s}}$, and $z = (t-s)^{1/2}\xi$, we obtain

$$\begin{aligned} C_{2,2}^{a,b}(t,s,x) &\leq |t-s|^{2H} \int_E \int_0^{\frac{s}{t-s}} \int_0^{\frac{s}{t-s}} d\xi\, dv\, du\, |u-v|^{2H-2} \\ &\times \left[e^{-(t-s)(2+u+v)\|\xi\|^2} e^{-2a^\alpha (t-s)\|\xi\|^2} - 2e^{-(t-s)(1+u+v)\|\xi\|^2} e^{-a^\alpha (t-s)\|\xi\|^2} + e^{-(t-s)(u+v)\|\xi\|^2} \right] \\ &\leq |t-s|^{2H} \int_E \int_0^{+\infty} \int_0^{+\infty} d\xi\, dv\, du\, |u-v|^{2H-2} \\ &\times \left[e^{-(t-s)(2(1+a^\alpha)+u+v)\|\xi\|^2} - 2e^{-(t-s)(1+a^\alpha+u+v)\|\xi\|^2} + e^{-(t-s)(u+v)\|\xi\|^2} \right] \\ &\leq |t-s|^{2H-\frac{d}{2}} \int_0^{+\infty} \int_0^{+\infty} |u-v|^{2H-2} \\ &\times \int_{\mathbb{R}^d} \left[e^{-(2(1+a^\alpha)+u+v)\|z\|^2} - 2e^{-(1+a^\alpha+u+v)\|z\|^2} + e^{-(u+v)\|z\|^2} \right] d\xi\, dv\, du. \end{aligned}$$

Therefore, by the changes of variables $\xi = (2(1+a^\alpha)+u+v)^{1/2}z$, $\xi = (1+a^\alpha+u+v)^{1/2}z$, and $\xi = (u+v)^{1/2}z$, we obtain

$$\begin{aligned} &C_{2,2}^{a,b}(t,s,x) \leq Cte\, |t-s|^{2H-\frac{d}{\alpha}} \int_{\mathbb{R}^d} e^{-\|\xi\|^2} d\xi \\ &\times \int_0^{+\infty} \int_0^{+\infty} |u-v|^{2H-2} \left[(2(1+a^\alpha)+u+v)^{-\frac{d}{2}} - 2(1+a^\alpha+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}} \right] dv\, du. \end{aligned}$$

The Gaussian integral $\int_{\mathbb{R}^d} e^{-\|\xi\|^2} d\xi$ is finite. Applying Lemma 4.1 with $\lambda = a^\alpha$ and $\beta = 2$, we also get that the improper double integral above is finite. It follows that

$$C_{2,2}^{a,b}(t,s,x) \leq Cte\, |t-s|^{2H-\frac{d}{\alpha}}. \tag{4.19}$$

Gathering (4.16), (4.18), and (4.19) we get

$$A_{3,2}^{a,b}(t,s,x) \leq Cte\, |t-s|^{2H-\frac{d}{\alpha}}. \tag{4.20}$$

This, with (4.15) allows us to achieve the proof of Theorem 3.

As an immediate consequence of Theorem 3, applying Kolmogorov's criterion of continuity, we get:

Corollary 4.2. *Let $u^{a,b}$ be the mild solution to Equation (1.4) and assume that $\lambda_d^{H,\alpha} > 0$. Then, for every $x \in \mathbb{R}^d$, the process $t \to u^{a,b}(t,x)$ is Hölder continuous of order $\delta \in \left(0, \dfrac{\lambda_d^{H,\alpha}}{2}\right)$.*

The following interesting theorem will allow us to show the non-differentiability of the trajectories of the process $u^{a,b}(.,x)$.

Theorem 4. *Let $u^{a,b}$ be the mild solution to Equation (1.4) and assume that $\lambda_d^{H,\alpha} > 0$. There exists a positive constant Cte such that*

$$Cte\, |t-s|^{\lambda_d^{H,2}} \leq \mathbb{E}[|u^{a,b}(t,x) - u^{a,b}(s,x)|^2] \tag{4.21}$$

for any $(t,s) \in [0,T]^2$ and $x \in \mathbb{R}^d$.

Proof. Denoting: $c_1(\theta,\nu)=\frac{(\theta+\nu)^2}{2}\mathbf{1}_{\{\theta\nu\le 0\}}+\frac{(\theta-\nu)^2}{2}\mathbf{1}_{\{\theta\nu\ge 0\}}$, $c_2(\theta,\nu)=-\theta\nu\mathbf{1}_{\{\theta\nu\le 0\}}$, and $c_3(\theta,\nu)=\theta\nu\mathbf{1}_{\{\theta\nu\ge 0\}}$, using (4.7) and (3.4), we get

$$\mathbb{E}[\left|u^{a,b}(t,x)-u^{a,b}(s,x)\right|^2]=\sum_{i=1}^{3}T_i^{a,b}(t,s,x),$$

with

$$\begin{aligned}T_i^{a,b}(t,s,x) &= c_i(\theta,\nu)\int_0^T\int_0^T\frac{\partial^2 R_Z^{H,i}}{\partial u\partial v}(u,v)\int_{\mathbb{R}^d}[G_{a,b}(a_{t,u},x,z)\,\mathbf{1}_{(0,t)}(u)-G_{a,b}(a_{s,u},x,z)\,\mathbf{1}_{(0,s)}(u)]\\ &\times\ [G_{a,b}(a_{t,v},x,z)\,\mathbf{1}_{(0,t)}(v)-G_{a,b}(a_{s,v},x,z)\,\mathbf{1}_{(0,s)}(v)]\,dz\,du\,dv,\end{aligned}$$

for every $i\in\{1,2,3\}$, with $R_Z^{H,1}=R_Z^{H,1,0}$, $R_Z^{H,2}=R_Z^{H,1,-1}$, and $R_Z^{H,3}=R_Z^{H,1,1}$ as the covariance functions of $Z^H(1,0)$, $Z^H(1,-1)$, and $Z^H(1,1)$, respectively (see Example 3.1).

The three terms $T_i^{a,b}$, $\forall i\in\{1,2,3\}$, are nonnegative. Indeed, for every $i\in\{1,2,3\}$, we have

$$T_i^{a,b}(t,s,x)=\ c_i(\theta,\nu)\,\mathbb{E}[\left(V_i^{a,b}(t,x)-V_i^{a,b}(s,x)\right)^2];$$

$$V_i^{a,b}(t,x)-V_i^{a,b}(s,x)=\int_0^T\int_{\mathbb{R}^d}[G_{a,b}(a_{t,u},x,z)\,\mathbf{1}_{(0,t)}(u)-G_{a,b}(a_{s,u},x,z)\,\mathbf{1}_{(0,s)}(u)]\,W_i^H(dz,du)$$

where $W_i^H=\{W_i^H(t,A);(t,A)\in[0,T]\times\mathcal{B}_b(\mathbb{R}^d)\}$ is a centered Gaussian field with covariance:

$$\mathbf{E}(W_i^H(t,A)W_i^H(s,B))=R_Z^{H,i}(t,s)\lambda^d(A\cap B).\tag{4.22}$$

Therefore, $\mathbb{E}[\left(u^{a,b}(t,x)-u^{a,b}(s,x)\right)^2]\ge T_1^{a,b}(t,s,x)$. Since $Z^H(1,0)$ is none other than the fBm, by the known transfer formula (see, e.g., Proposition 2.4 in [22]), we obtain:

$$\begin{aligned}&V_1^{a,b}(t,x)-V_1^{a,b}(s,x)\\ =\ &\int_{\mathbb{R}^d}\int_{\mathbb{R}}\left(\int_0^T[G_{a,b}(a_{t,u},x,y)\mathbf{1}_{(0,t)}(u)-G_{a,b}(a_{s,u},x,y)\mathbf{1}_{(0,s)}(u)]\,(u-z)_+^{H-\frac{3}{2}}du\right)W(dy,dz),\end{aligned}$$

where the process $W=\{W(t,A);t\in[0,T],A\in\mathcal{B}_b(\mathbb{R}^d)\}$ is a space time white Noise with covariance

$$\mathbf{E}(W(t,A)W(s,B))=(t\wedge s)\,\lambda^d(A\cap B).\tag{4.23}$$

Therefore, by Wiener isometry, we get

$$\begin{aligned}&\mathbb{E}[\left(V_1^{a,b}(t,x)-V_1^{a,b}(s,x)\right)^2]\\ =\ &\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}}\mathbf{1}_{(0,T)}(u)\,[G_{a,b}(a_{t,u},x,y)\mathbf{1}_{(0,t)}(u)-G_{a,b}(a_{s,u},x,y)\mathbf{1}_{(0,s)}(u)]\,(a_{u,z})_+^{H-\frac{3}{2}}du\right)^2dydz\\ \ge\ &\int_s^t\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}}\mathbf{1}_{(0,T)}(u)\,[G_{a,b}(a_{t,u},x,y)\mathbf{1}_{(0,t)}(u)-G_{a,b}(a_{s,u},x,y)\mathbf{1}_{(0,s)}(u)]\,(a_{u,z})_+^{H-\frac{3}{2}}du\right)^2dydz\\ =\ &\int_s^t\int_{\mathbb{R}^d}\left(\int_z^t G_{a,b}(a_{t,u},x,y)\,a_{u,z}^{H-\frac{3}{2}}du\right)^2dy\,dz\\ =\ &\int_s^t\int_{\mathbb{R}^d}\left(\int_z^t G_{a,b}(a_{t,u},x,y)\,a_{u,z}^{H-\frac{3}{2}}du\right)\left(\int_z^t G_{a,b}(a_{t,v},x,y)\,a_{v,z}^{H-\frac{3}{2}}dv\right)dy\,dz\\ =\ &\int_s^t du\int_s^t dv\int_{\mathbb{R}^d}dy\,G_{a,b}(a_{t,u},x,y)\,G_{a,b}(a_{t,v},x,y)\int_s^{u\wedge v}a_{v,z}^{H-\frac{3}{2}}\,a_{u,z}^{H-\frac{3}{2}}dz.\end{aligned}$$

By the change of variables $Z = \dfrac{a_{v,z}}{a_{u,z}}\mathbf{1}_{v<u} + \dfrac{a_{u,z}}{a_{v,z}}\mathbf{1}_{v>u}$, we see that

$$\int_s^{u\wedge v} a_{v,z}^{H-\frac{3}{2}}\, a_{u,z}^{H-\frac{3}{2}} dz = |u-v|^{2H-2} \int_0^{\frac{a_{v,s}\wedge a_{u,s}}{a_{v,s}\vee a_{u,s}}} (1-Z)^{1-2H}\, Z^{H-\frac{3}{2}} dZ.$$

This, with (3.10), allows us to obtain:

$$\begin{aligned}
\mathbb{E}[\left|u^{a,b}(t,x)-u^{a,b}(s,x)\right|^2] \;&\geq\; Cte \int_s^t \int_s^t du\, dv\, |u-v|^{2H-2} \int_0^{\frac{a_{v,s}\wedge a_{u,s}}{a_{v,s}\vee a_{u,s}}} (1-z)^{1-2H}\, z^{H-\frac{3}{2}} dz \\
&\times\; \int_{\mathbb{R}^d} dy\, a_{t,u}^{-d/2}\, a_{t,v}^{-d/2} \exp\left(-\frac{c_2\|x-y\|^2}{a_{t,u}}\right) \exp\left(-\frac{c_2\|x-y\|^2}{a_{t,v}}\right) dy \\
&=\; Cte \int_s^t \int_s^t du\, dv\, |u-v|^{2H-2} \int_0^{\frac{(v-s)\wedge(u-s)}{(v-s)\vee(u-s)}} (1-z)^{1-2H}\, z^{H-\frac{3}{2}} dz \\
&\times\; \left((t-u)^{-1/2}\,(t-v)^{-1/2} \int_{\mathbb{R}} \exp\left(-\frac{c_2|x_1-y_1|^2}{(t-u)}\right) \exp\left(-\frac{c_2|x_1-y_1|^2}{(t-v)}\right) dy_1\right)^d.
\end{aligned}$$

By the change variable $Y = (x_1-y_1)\sqrt{\dfrac{c_2(2t-u-v)}{(t-u)(t-v)}}$, we get $\displaystyle\int_{\mathbb{R}} \exp\left(-\frac{c_2(x_1-y_1)^2(2t-u-v)}{(t-u)(t-v)}\right) dy_1 = \sqrt{\frac{\pi(t-u)(t-v)}{c_2(2t-u-v)}}$. This, with the changes of variables $U = a_{u,s}$ and $V = a_{v,s}$, then $\tilde{u} = \frac{U}{a_{t,s}}$ and $\tilde{v} = \frac{V}{a_{t,s}}$, we get:

$$\begin{aligned}
&\mathbb{E}[\left|u^{a,b}(t,x)-u^{a,b}(s,x)\right|^2] \\
\geq\;& Cte \int_s^t \int_s^t du\, dv\, |u-v|^{2H-2}\, (2t-u-v)^{-d/2} \int_0^{\frac{(v-s)\wedge(u-s)}{(v-s)\vee(u-s)}} (1-z)^{1-2H}\, z^{H-\frac{3}{2}} dz \\
\geq\;& Cte\, |t-s|^{2H-\frac{d}{2}} \int_0^1 \int_0^1 du\, dv\, |u-v|^{2H-2} (2-u-v)^{-d/2} \int_0^{\frac{v\wedge u}{v\vee u}} (1-z)^{1-2H}\, z^{H-\frac{3}{2}} dz \\
\geq\;& Cte\, |t-s|^{2H-\frac{d}{2}},
\end{aligned}$$

where in the last line we used that the last double integral is finite because $4H > 2\alpha H > d$. □

Corollary 4.3. *Let $u^{a,b}$ be the mild solution to Equation (1.4) and assume that $\lambda_d^{H,\alpha} > 0$. For every fixed $x \in \mathbb{R}^d$, we have*

$$\lim_{\epsilon\to 0} \sup_{t\in[t_0-\epsilon,t_0+\epsilon]} \left|\frac{u^{a,b}(t,x)-u^{a,b}(t_0,x)}{t-t_0}\right| = +\infty$$

with probability one for every t_0. Consequently, the trajectories of the process $u^{a,b}(.,x)$ are not differentiable.

Proof. The corollary can be obtained by applying Theorem 4 and by proceeding as in the proof of Theorem 3.3, page 88, in [16]. □

Remark 4.2. *(1) The particular case where $a = 1$ and $b = 0$, $\mathcal{L}^{a,b}$ reduced to $\mathcal{L}^{1,0} = \Delta^{\alpha/2}$, which has been examined by various authors, such as in [10, 11].*

(2) The case where the noise is fBm or sfBM can be directly derived from this paper, as it presents a specific instance of gfBm.

5. Discussion and future works

In this paper, we have introduced and analyzed a novel stochastic FPDE that integrates a mixed operator, combining the standard Laplacian, fractional Laplacian, and gradient operator. This approach provides an effective framework for modeling complex phenomena where standard and fractional diffusion processes interact with spatially dependent randomness.

Our investigation allows us to analyze the complex behaviors of the solution under different random noise structures. The explicit form of the covariance function derived from our analysis reveals how the stochastic component affects the solution's properties. This result is crucial for understanding the interplay between the deterministic and stochastic elements in such FPDEs.

The specific case of noise which behaves as a Wiener process with respect to the space variable, and as a gfBm with respect to the time variable offers additional insights into the regularity of sample paths. By focusing on this case, we explore how the fractional nature of the noise influences the solution's smoothness and continuity. This examination is particularly relevant for applications where the underlying random processes exhibit long-range dependencies or memory effects.

Our results provide a foundation for further exploration into more complex scenarios and applications. For instance, the mixed operator FPDEs could be applied to areas such as turbulence modeling, financial mathematics, and environmental sciences where both local and nonlocal effects are significant. Future work could extend our results by considering more general forms of noise or by developing numerical methods to simulate such FPDEs effectively.

6. Conclusions

In summary, we have developed and analyzed a novel stochastic FPDE incorporating a mixed operator with standard, fractional, and gradient components. Our study successfully derived an explicit covariance function for solutions influenced by spatially-dependent random noise. Additionally, by considering noise which behaves as a Wiener process with respect to the space variable, and as a gfBm with respect to the time variable, we provided insights into how fractional noise affects the regularity of solutions. These contributions advance the field of stochastic FPDEs and offer a robust framework for future research in complex systems with both local and nonlocal dynamics.

A visual summary of our research contributions is presented below.

Feature	Existing research	Contribution of this research
Operator	Often restricted to $\Delta + \Delta^{\alpha/2}$, Δ or $\Delta^{\alpha/2}$	General mixed fractional operator $\Delta + a^{\alpha}\,\Delta^{\alpha/2} + b\,.\nabla$
Random Noise	White-space Gaussian field with temporal-covariance measure structure restricted to some particular gaussian processes as e.g. fBm, sfBm, etc.	White-space Gaussian field with temporal covariance measure structure with focus on the more general gaussian process: gfBm, extending both fBm and sfBm
Results	Characterization of the solution especially in fBm and sfBm cases	Groundbreaking investigation of the generalized gfBm case

Author contributions

M. Zili: Conceptualization, formal analysis, investigation, resources, writing-original draft and editing. E. Zougar: Conceptualization, formal analysis and methodology, investigation, resources, writing-original draft, review and editing. M. Rhaima: Methodology, funding acquisition, project administration, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that they have no conflict of interest.

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