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Research article

Utilization of Ramanujan-type Eisenstein series in the generation of differential equations

H. C. Vidya[1] **and B. A. Rao**[2,*]

[1] Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal 576104, Karnataka, India

[2] Computer Science and Engineering Department, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal 576104, Karnataka, India

* **Correspondence:** Email: ashwath.rao.b@gmail.com.

Abstract: In his lost notebook, Ramanujan presented unique categories of remarkable infinite series, known as the Ramanujan-type Eisenstein series. The objective of this paper is to generate various differential identities related to classical η-functions and h-functions with the help of the Ramanujan-type Eisenstein series. Moreover, we present particular identities that incorporate Eisenstein series of different levels and h-functions, thereby establishing links between class one infinite series and h-functions.

Keywords: Eisenstein series; Dedekind η-function; class one infinite series; differential equations
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1. Introduction

Differential equations are essential in applied mathematics, serving as fundamental instruments in the progress of various fields including clinical, engineering, physics, chemistry, and many more. As a tribute to Ramanujan's contributions, we aim to develop ordinary differential equations by utilizing the derived relationships of Eisenstein series of various levels. Considerable research efforts have been dedicated to ordinary differential equations satisfied by modular forms. In his notebook, as referenced in [1], Ramanujan devoted significant attention to Eisenstein series, notably P, Q, and R, and presented numerous captivating differential identities involving infinite series and theta functions. A comprehensive study of h-functions and several modular equations for h has been conducted by Naika et al. [2]. In their research, Cooper and Ye [3] conducted an extensive examination of the h-function and demonstrated its application in the formation of differential equations. Additionally, Cooper [4] has documented specific relationships between Eisenstein series of different levels and

h-functions. Ramanujan [1] recorded certain differential equations related to η-functions, while Berndt et al. [5] massively highlighted the importance of constructing differential equations that incorporate both η-functions and Eisenstein series. In their recent work [6, 7], Vidya and Rao demonstrated elegant linkages between Eisenstein series and theta functions. These relationships were later used to construct differential identities and evaluate certain convolutional identities. Additionally, Vidya and Kumar, as cited in [8], derived differential equations that incorporate identities associated with the η-function. They emphasized the importance of developing these equations to facilitate the generation of incomplete integrals involving η-functions.

Additionally, on page 188 of his lost notebook [9], Ramanujan recorded specific formulas linking the class one infinite series $T_{2r}(q)$, where $r = 1, 2, \ldots, 6$, with the Eisenstein series P, Q, and R. The main proof of six formulas for $T_{2r}(q)$ was presented in a research paper by Berndt and Yee [10]. An additional proof of these formulas appeared in Liu's paper [11].

These authors took great delight in exploring the inherent beauty of mathematics through their work on constructing differential identities that are satisfied by h-functions. The convergence of the Eisenstein series to an infinite product is a key result that underscores the deep connections between modular forms and infinite series. Motivated by the work of these researchers, we have showcased differential identities satisfied by h-functions, utilizing the derived relations among infinite series and h-functions. In addition, we have shown that the sum of an infinite series converges to an infinite product.

The main focus of this paper is the development of various differential identities involving η-functions and the infinite products known as h-functions. First, we extract results of [4] and establish a relationship between Ramanujan-type Eisenstein series and h-functions. These results can be found in Section 3. These derived relations are further used in Sections 4 and 5 to construct interesting differential equations involving h-functions and to establish relations among two elegant infinite series. Section 2 is dedicated to documenting initial findings that supports in achieving the main goals.

2. Preliminaries

Ramanujan provided the following definition of a general theta function in his notebook [5, p.35]. Let a, b, and q be any complex number with $|ab| < 1$,

$$\begin{aligned} f(a,b) &:= \sum_{i=-\infty}^{\infty} a^{i(i+1)/2} b^{i(i-1)/2} \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \end{aligned}$$

where,

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i), \quad |q| < 1.$$

The following instances are particular cases of theta functions as defined by Ramanujan [5, p.35]:

$$\varphi(q) := f(q,q) = \sum_{i=-\infty}^{\infty} q^{i^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{i=0}^{\infty} q^{i(i+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$
$$f(-q) := f(-q,-q^2) = \sum_{i=-\infty}^{\infty} (-1)^i q^{i(3i-1)/2} = (q;q)_\infty = q^{-1/24}\eta(\tau), \tag{2.1}$$

where

$$q = e^{2\pi i\tau}.$$

We denote

$$f(-q^n) = f_n.$$

Definition 2.1. Cooper [4] documented certain relations involving η-functions in his notebook, as described below:

$$r_a(q) = \frac{\eta_2\eta_6^5}{\eta_1^5\eta_3}, \quad r_b(q) = \frac{\eta_1^2\eta_6^4}{\eta_2^4\eta_3^2}, \quad r_c(q) = \frac{\eta_1\eta_6^3}{\eta_2\eta_3^3},$$
$$1 + 8r_a(q) = \frac{\eta_2^4\eta_3^8}{\eta_1^8\eta_6^4}, \quad 1 + 9r_a(q) = \frac{\eta_2^9\eta_3^3}{\eta_1^9\eta_6^3},$$
$$1 - 3r_b(q^2) = \frac{\eta_1^3\eta_{12}}{\eta_3\eta_4^3}, \quad 1 + r_b(q^2) = \frac{\eta_2^2\eta_3^3\eta_{12}}{\eta_1\eta_4^3\eta_6^2}.$$

Definition 2.2. Ramanujan [9] presented the class one infinite series,

$$T_{2k}(q) := 1 + \sum_{i=1}^{\infty} (-1)^i \left[(6i-1)^{2k} q^{\frac{i(3i-1)}{2}} + (6i+1)^{2k} q^{\frac{i(3i+1)}{2}}\right], \tag{2.2}$$

and expressed $T_{2k}(q)$ for $k = 1, 2, \ldots, 6$ in terms of Ramanujan-type Eisenstein series:

$$P(q) := 1 - 24\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j} = 1 - 24\sum_{j=1}^{\infty} \delta_1(j)q^j.$$

Throughout, we denote

$$P(q^n) = P_n.$$

In addition, a fascinating relationship was made by Berndt [10]:

$$T_2(q) = (q;q)_\infty P(q). \tag{2.3}$$

Definition 2.3. [4] For $|q| < 1$, the h-function is defined by

$$h := h(q) := q\prod_{i=1}^{\infty} \frac{(1-q^{12i-1})(1-q^{12i-11})}{(1-q^{12i-5})(1-q^{12i-7})}.$$

The weight two modular form y_{12} in terms of h-function is defined by

$$y_{12} = q\frac{d}{dq} logh = 1 - \sum_{s=1}^{\infty} \chi_{12}(s)\frac{sq^s}{1-q^s},$$

where

$$\chi_{12}(s) = \begin{cases} 1, & \text{if s=1 or 11 (mod 12)}, \\ -1, & \text{if s=5 or 7 (mod 12)}, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4. *[4] The relation among an infinite series and h-functions is given by*

$$\begin{pmatrix} P(q) \\ P(q^2) \\ P(q^3) \\ p(q^4) \\ P(q^6) \\ P(q^{12}) \end{pmatrix} = \begin{pmatrix} 6 & 2 & 3 & -6 & 2 & 0 \\ 3 & 2 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 2 & -2 & \frac{5}{3} & \frac{2}{3} & 0 & \frac{2}{3} \\ \frac{3}{2} & \frac{5}{4} & -\frac{5}{3} & \frac{3}{4} & \frac{1}{2} & 0 \\ 1 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{6} & \frac{5}{12} & 0 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} \frac{h(dy_{12})}{dh} \\ \frac{(1-h^2)}{(1+h^2)}y_{12} \\ \frac{(1-h^2)}{(1-h+h^2)}y_{12} \\ \frac{(1-h^2)}{(1-4h+h^2)}y_{12} \\ \frac{(1-h^2)}{(1-2h+h^2)}y_{12} \\ \frac{(1-h^2)}{(1+2h+h^2)}y_{12} \end{pmatrix}.$$

3. Relations among Eisenstein series and h-functions

Theorem 3.1. *Let*

$$h + \frac{1}{h} = u$$

and

$$-h + \frac{1}{h} = v.$$

Then the connections among the Ramanujan-type Eisenstein series and h-function holds:

$$(i) \quad -P_1 - 2P_2 - 3P_3 + 18P_6 = \frac{4v(2u^3 - 5u^2 + 16u - 4)}{(u-1)(u-2)(u-4)(u+2)}y_{12},$$

$$(ii) \quad -7P_1 + 10P_2 + 3P_3 + 6P_6 = \frac{12v(u^3 + 10u^2 - 20u)}{(u-1)(u-4)(u-2)(u+2)}y_{12},$$

$$(iii) \quad -P_1 - 3P_3 + 12P_6 = \frac{2v(-2u-4)}{u(u-2)}y_{12},$$

$$(iv) \quad -3P_1 + 2P_2 + 3P_3 + 6P_6 = \frac{4v(2u^2 + 8u - 18)}{u(u-4)(u+2)}y_{12},$$

$$(v) \quad 3P_1 - 10P_2 - 15P_3 + 42P_6 = \frac{4v(5u^3 - 18u^2 - 12u + 16)}{u(u-1)(u-2)(u+2)}y_{12},$$

$$(vi) \quad P_1 - 6P_2 + 3P_3 + 6P_6 = \frac{4v(u^3 - 6u^2 + 12u - 16)}{u(u-1)(u-2)(u+2)}y_{12},$$

$$(vii) \quad -13P_1+10P_2+21P_3+6P_6=\frac{24v(u^2+8u-6)}{u(u-4)(u-2)}y_{12},$$

$$(viii) \quad -P_1+2P_2+9P_3-18P_6=\frac{8v(-u+4)}{u(u-1)(u-2)}y_{12},$$

$$(ix) \quad -P_1+3P_3=\frac{v(u^4-2u^2-32u+52)}{3u(u-1)(u-4)(u-2)(u+2)}y_{12},$$

$$(x) \quad -5P_1+8P_2+15P_3-24P_6=\frac{6v(7u^4-16u^3-32u^2+96u-64)}{u(u-1)(u-4)(u-2)(u+2)}y_{12},$$

$$(xi) \quad -7P_1+6P_2+15P_3-6P_6=\frac{3v(u^2+16u-32)}{u(u-4)(u-2)}y_{12},$$

$$(xii) \quad -9P_1+10P_2+33P_3-42P_6=\frac{8v(-u^2+32u-64)}{u(u-4)(u-2)}y_{12},$$

$$(xiii) \quad -P_1+3P_2+P_3-2P_4-3P_6+2P_{12}=\frac{4v}{(u-4)(u+2)}y_{12},$$

$$(xiv) \quad -2P_1+3P_2+2P_3+2P_4-3P_6-2P_{12}=\frac{4v(12u+6)}{(u-1)(u-4)(u+2)}y_{12},$$

$$(xv) \quad -5P_1+11P_2+9P_3-6P_4-15P_6+6P_{12}=\frac{12v(10u+8)}{u(u-4)(u+22)}y_{12},$$

$$(xvi) \quad -4P_1+7P_2+6P_4-3P_6-6P_{12}=\frac{12v(8u^2+2u+8)}{u(u-1)(u-4)(u+2)}y_{12},$$

$$(xvii) \quad P_1+2P_2+3P_3-12P_4-6P_6+12P_{12}=\frac{12v(-2u-4)}{u(u-1)(u-4)}y_{12},$$

$$(xviii) \quad P_1-P_2-3P_3+P_6=6\frac{(4v)}{u(u-4)}y_{12},$$

$$(xix) \quad -7P_1+10P_2+3P_3+6P_6=\frac{12v(u^3+10u^2-20u)}{(u-1)(u-4)(u-2)(u+2)}y_{12},$$

$$(xx) \quad -P_1+4P_2+3P_3-12P_6=\frac{6v(-u^2+6u-8)}{(u-1)(u-4)((u+2)}y_{12},$$

$$(xxi) \quad -7P_1+10P_2+3P_3+6P_6=\frac{12v(u^3+10u^2-20u)}{(u-1)(u-4)(u-2)(u+2)}y_{12},$$

$$(xxii) \quad -11P_1+26P_2+15P_3-42P_6=\frac{12v(-u^3+26u^2-52u)}{(u-1)(u-4)(u-2)(u+2)}y_{12},$$

$$(xxiii) \quad -P_1+2P_2=\frac{v(u^4-20u^3-48u^2+32u-30)}{u(u-1)(u-4)(u-2)(u+2)}y_{12},$$

$$(xxiv) \quad -5P_1+10P_2+9P_3-18P_6=\frac{8v(u^4+20u^3-48u^2+32u-30)}{u(u-1)(u-4)(u-2)(u+2)}y_{12},$$

$$(xxv) \quad -P_1+P_2-3P_3-6P_4+15P_6+6p_{12}=\frac{12v(u^2-4u-10)}{(u-1)(u-4)(u-2)}y_{12},$$

$$(xxvi) \quad 4P_1-P_2-6P_4-15P_6+6P_{12}=\frac{12v(-u^2-2u+6)}{(u-1)(u-4)(u-2)}y_{12},$$

$$(xxvii)\quad 5P_1 - 8P_2 - 9P_3 - 12P_4 + 24P_6 + 12P_{12} = \frac{12v(u^3 - 14u^2 + 16u - 24)}{(u-1)(u-4)(u-2)(u+2)}y_{12},$$

$$(xxviii)\quad P_1 + 8P_2 + 3P_3 - 12P_4 - 24P_6 + 12P_{12} = \frac{12v(-u^3 + 2u^2 - 16u + 24)}{(u-1)(u-4)(u-2)(u+2)}y_{12},$$

$$(xxix)\quad 2P_1 - P_2 - 6P_3 - 6P_4 + 9P_6 + 6P_{12} = \frac{4v(u^3 - 18u^2 + 42u - 16)}{u(u-1)(u-4)(u-2)}y_{12},$$

$$(xxx)\quad P_1 + P_2 + 3P_3 - 6P_4 - 9P_6 + 6P_{12} = \frac{4v(2u^2 - 5)}{u(u-1)(u-2)}y_{12},$$

$$(xxxi)\quad -P_1 + P_2 + P_3 - 2P_4 - 3P_6 + 2P_{12} = \frac{4v(u^3 - 6u + 8)}{u(u-1)(u-2)}y_{12},$$

$$(xxxii)\quad 2P_1 + P_2 + 6P_3 - 6P_4 - 21P_6 + 6P_{12} = \frac{12v(-u^3 + 2u^2 - 6u + 8)}{(u-1)(u-4)(u-2)}y_{12},$$

$$(xxxiii)\quad P_1 - 4P_2 + 3P_3 - 12P_4 + 12P_6 + 12P_{12} = \frac{12v(u^3 - 2u^2 - 8)}{u(u-1)(u-2)(u+2)}y_{12},$$

$$(xxxiv)\quad -P_1 + 4P_2 + 5P_3 - 4P_4 - 12P_6 + 4P_{12} = \frac{4v(-u^3 + 6u^2 - 8)}{u(u-1)(u-2)(u+2)}y_{12}.$$

Proof. Through the utilization of Lemma 2.4 and subsequent simplification using Maple, we derive the requisite relationships. □

4. Formation of differential identities involving η-functions and h-functions

In Theorems 4.1–4.4, we formulate differential equations in consequence of Eisenstein series of different levels that associates h-functions and the derivative of weight two modular forms.

Theorem 4.1. *If*

$$S(q) = r_a(q)r_b(q),$$

then the following differential identity holds:

$$q\frac{ds}{dq} - \left[\frac{v(2u^3 - 5u^2 + 16u - 4)}{2(u-1)(u-4)(u-2)(u+2)}y_{12}\right]S = 0, \tag{4.1}$$

where

$$h + \frac{1}{h} = u, \quad -h + \frac{1}{h} = v.$$

Proof. We note from (2.1) that, $S(q)$ may be reformulated in terms of η-functions given by

$$S(q) = \frac{\eta_6^9}{\eta_1^3\eta_2^3\eta_3^3}.$$

Rewriting the expression for $S(q)$ using the theta-function,

$$S(q) = \frac{q^{3/2}f_6^9}{f_1^3f_2^3f_3^3}.$$

Further expressing the above in terms of q-series notation and then logarithmically differentiating with respect to q, we reach the following outcome:

$$\frac{1}{S}\frac{dS}{dq}=\frac{3}{2q}+\frac{3}{q}\left[\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}+\sum_{r=1}^{\infty}\frac{2rq^{2r}}{1-q^{2r}}+\sum_{r=1}^{\infty}\frac{3rq^{3r}}{1-q^{3r}}-\sum_{r=1}^{\infty}\frac{18rq^{6r}}{1-q^{6r}}\right].$$

By utilizing the definition of the Eisenstein series and performing simplifications, we obtain the following result:

$$\frac{q}{S}\frac{dS}{dq}=\frac{1}{8}[-P_1-2P_2-3P_3+18P_6].$$

Finally, incorporating Lemma 2.4, and representing Eisenstein series in terms of h-functions, followed by simplification, we derive

$$\frac{q}{S}\frac{dS}{dq}=\frac{(1-h^2)(2h^6-5h^5+22h^4-14h^3+22h^2-5h+2)}{(1-h+h^2)(1+2h+h^2)(1-4h+h^2)(1-2h+h^2)}y_{12}.$$

Further, denoting

$$h+\frac{1}{h}=u,\quad -h+\frac{1}{h}=v$$

and then simplifying, we derive expression (4.1).

By applying the same methodology used to derive the previously mentioned result, we can formulate the following set of differential equations:

$$(i)\quad \text{If } S(q)=\frac{r_a(q)}{r_b(q)}=\frac{q^{1/2}f_2^5f_3f_6}{f_1^7},\ \text{then } q\frac{ds}{dq}-\left[\frac{v(u^3+10u^2-20u)}{2(u-1)(u-4)(u-2)(u+2)}y_{12}\right]S=0;$$

$$(ii)\quad \text{If } S(q)=r_a(q)r_c(q)=\frac{q^{4/3}f_6^8}{f_1^4f_3^4},\ \text{then } q\frac{ds}{dq}+\left[\frac{v(2u+4)}{3u(u-2)}y_{12}\right]S=0;$$

$$(iii)\quad \text{If } S(q)=\frac{r_a(q)}{r_c(q)}=\frac{q^{2/3}f_2^2f_3^2f_6^2}{f_1^6},\ \text{then } q\frac{ds}{dq}-\left[\frac{v(2u^2+8u-18)}{3u(u-4)(u+2)}y_{12}\right]S=0;$$

$$(iv)\quad \text{If } S(q)=r_b(q)r_c(q)=\frac{q^{5/6}f_1^3f_6^7}{f_2^5f_3^5},\ \text{then } q\frac{ds}{dq}-\left[\frac{v(5u^3-18u^2-12u+16)}{6u(u-1)(u-2)(u+2)}y_{12}\right]S=0;$$

$$(v)\quad \text{If } S(q)=\frac{r_b(q)}{r_c(q)}=\frac{q^{1/6}f_1f_3f_6}{f_2^3},\ \text{then } q\frac{ds}{dq}-\left[\frac{v(u^3-6u^2+12u-16)}{6u(u-1)(u-2)(u+2)}y_{12}\right]S=0;$$

where

$$h+\frac{1}{h}=u,\quad -h+\frac{1}{h}=v.$$

Therefore, the proof is completed. □

Theorem 4.2. *If*

$$S(q)=(1+8r_a(q))r_a(q),$$

then the following differential identity holds:

$$q\frac{ds}{dq}-\left[\frac{v(u^2+8u-6)}{u(u-2)(u-4)}y_{12}\right]S=0, \tag{4.2}$$

where

$$h+\frac{1}{h}=u, \quad -h+\frac{1}{h}=v.$$

Proof. We note from (2.1) that $S(q)$ may be reformulated in terms of η-functions given by

$$S(q)=\frac{\eta_2^5\eta_3^7\eta_6}{\eta_1^{13}}.$$

Rewriting the expression for $S(q)$ using the theta-function,

$$S(q)=\frac{q^{3/2}f_2^5f_3^7f_6}{f_1^{13}}.$$

Further expressing the above by q-series notation and then by logarithmic differentiation with respect to q, we get the following result:

$$\frac{1}{S}\frac{dS}{dq}=\frac{3}{2q}+\frac{1}{q}\left[\sum_{r=1}^{\infty}\frac{13rq^r}{1-q^r}-\sum_{r=1}^{\infty}\frac{10rq^{2r}}{1-q^{2r}}-\sum_{r=1}^{\infty}\frac{21rq^{3r}}{1-q^{3r}}-\sum_{r=1}^{\infty}\frac{6rq^{6r}}{1-q^{6r}}\right].$$

By utilizing the definition of the Eisenstein series and performing simplifications, we obtain the following result:

$$\frac{q}{S}\frac{dS}{dq}=\frac{1}{24}[-13P_1+10P_2+21P_3+6P_6].$$

Finally, incorporating Lemma 2.4, and expressing the Eisenstein series in terms of h-functions, followed by simplification, we derive

$$\frac{q}{S}\frac{dS}{dq}=\frac{(1-h^2)(h^4+8h^3-4h^2+8h+1)}{(1+h^2)(1-4h+h^2)(1-2h+h^2)}y_{12}.$$

Further, denoting

$$h+\frac{1}{h}=u, \quad -h+\frac{1}{h}=v,$$

and then simplifying, we derive expression (4.2).

Applying the same methodology used to derive the aforementioned result, we can construct the following set of differential equations: for

$$h+\frac{1}{h}=u, \quad -h+\frac{1}{h}=v:$$

(i) If $S(q)=\dfrac{(1+8r_a(q))}{r_a(q)}=\dfrac{f_2^3f_3^9}{qf_1^3f_6^9}$, then $q\dfrac{ds}{dq}+\left[\dfrac{v(u-4)}{u(u-1)(u-2)}y_{12}\right]S=0;$

(ii) If $S(q)=(1+8r_a(q))r_b(q)=\dfrac{q^{1/2}f_3^6}{f_1^6}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(u^4-2u^2-32u+52)}{12u(u-1)(u-4)(u-2)(u+2)}y_{12}\right]S=0;$

(iii) If $S(q)=\dfrac{(1+8r_a(q))}{r_b(q)}=\dfrac{f_2^8f_3^{10}}{q^{1/2}f_1^{10}f_6^8}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(7u^4-16u^3-32u^2+96u-64)}{2u(u-1)(u-4)(u-2)(u+2)}y_{12}\right]S=0;$

(iv) If $S(q) = (1 + 8r_a(q))r_c(q) = \dfrac{q^{1/3} f_2^3 f_3^5}{f_1^7 f_6}$, then $q\dfrac{ds}{dq} - \left[\dfrac{v(u^2 + 16u - 32)}{8u(u-4)(u-2)} y_{12}\right] S = 0;$

(v) If $S(q) = \dfrac{(1 + 8r_a(q))}{r_c(q)} = \dfrac{f_2^5 f_3^{11}}{q^{1/3} f_1^9 f_6^7}$, then $q\dfrac{ds}{dq} + \left[\dfrac{v(u^2 - 32u + 64)}{3u(u-4)(u-2)} y_{12}\right] S = 0.$

The proof is completed. □

Theorem 4.3. *If*

$$S(q) = (1 + 9r_a(q))(1 - 3r_b(q^2)),$$

then the following differential identity holds:

$$q\frac{ds}{dq} - \left[\frac{v}{(u-4)(u+2)} y_{12}\right] S = 0, \tag{4.3}$$

where

$$h + \frac{1}{h} = u, \quad -h + \frac{1}{h} = v.$$

Proof. According to the information provided in [4, p.189], it appears that $S(q)$ can be expressed using theta functions as

$$S(q) = \frac{f_2^9 f_3^2 f_{12}}{q^{1/8} f_1^6 f_4^3 f_6^3}.$$

By reformulating the expression for $S(q)$ in q-series notation and subsequently logarithmically differentiating with respect to q, we arrive at the following result:

$$\frac{1}{S}\frac{dS}{dq} = -\frac{1}{8q} + \frac{6}{q}\left[\sum_{r=1}^{\infty} \frac{rq^r}{1-q^r} - \sum_{r=1}^{\infty} \frac{3rq^{2r}}{1-q^{2r}} - \sum_{r=1}^{\infty} \frac{rq^{3r}}{1-q^{3r}} + \sum_{r=1}^{\infty} \frac{3rq^{6r}}{1-q^{6r}} - \sum_{r=1}^{\infty} \frac{2rq^{12r}}{1-q^{12r}}\right].$$

Upon applying the definition of the Eisenstein series and conducting simplifications, we arrive at the following outcome:

$$\frac{q}{S}\frac{dS}{dq} = \frac{1}{4}[-P_1 + 3P_2 + P_3 - 2P_4 - 3P_6 + 2P_{12}].$$

Finally, incorporating Lemma 2.4 and representing Eisenstein series in terms of h-functions, followed by simplification, we derive expression (4.3).

Employing the methodology utilized in deriving the aforementioned results, we can formulate the following set of differential equations:

(i) If $S(q) = \dfrac{(1 + 9r_a(q))}{(1 - 3r_b(q^2))} = \dfrac{f_2^9 f_3^4 f_4^3}{f_1^{12} f_6^3 f_{12}}$, then $q\dfrac{ds}{dq} - \left[\dfrac{v(12u + 6)}{(u-1)(u-4)(u+2)} y_{12}\right] S = 0;$

(ii) If $S(q) = (1 + 9r_a(q))(1 + r_b(q^2)) = \dfrac{f_2^{11} f_3^6 f_{12}}{f_1^{10} f_4^3 f_6^5}$, then $q\dfrac{ds}{dq} - \left[\dfrac{v(10u + 8)}{u(u-4)(u+22)} y_{12}\right] S = 0;$

(iii) If $S(q) = \dfrac{(1 + 9r_a(q))}{(1 + r_b(q^2))} = \dfrac{f_2^7 f_3^3}{f_1^8 f_6 f_{12}}$, then $q\dfrac{ds}{dq} - \left[\dfrac{v(8u^2 + 2u + 8)}{u(u-1)(u-4)(u+2)} y_{12}\right] S = 0;$

(iv) If $S(q) = (1 - 3r_b(q^2))(1 + r_b(q^2)) = \dfrac{f_1^2 f_2^2 f_3^2 f_{12}^2}{q^{1/6} f_4^6 f_6^2}$, then $q\dfrac{ds}{dq} + \left[\dfrac{v(2u + 4)}{u(u-1)(u-4)} y_{12}\right] S = 0;$

(v) If $S(q) = \dfrac{(1-3r_b(q^2))}{(1+r_b(q^2))} = \dfrac{f_1^4 f_6^2}{f_2^2 f_3^4}$, then $q\dfrac{ds}{dq} - \left[\dfrac{4v}{u(u-4)}y_{12}\right]S = 0;$

where

$$h + \frac{1}{h} = u, \quad -h + \frac{1}{h} = v.$$

Therefore, the proof is complete. □

Theorem 4.4. *If*

$$S(q) = (1 + 9r_a(q))r_b(q),$$

then the following differential identity holds:

$$q\frac{ds}{dq} - \left[\frac{v(u^3 + 10u^2 - 20u)}{(u-1)(u-2)(u-4)(u+2)}y_{12}\right]S = 0, \tag{4.4}$$

where

$$h + \frac{1}{h} = u, \quad -h + \frac{1}{h} = v.$$

Proof. Based on the details outlined in [4, p.189], it appears that the expression of $S(q)$ can be represented utilizing theta functions as

$$S(q) = \frac{q f_2^{10} f_3^2 f_6^2}{f_1^{14}}.$$

Through rephrasing the expression for $S(q)$ in q-series notation and subsequently applying logarithmic differentiation with respect to q, we obtain the following outcome:

$$\frac{1}{S}\frac{dS}{dq} = \frac{1}{q} + \frac{2}{q}\left[\sum_{r=1}^{\infty}\frac{7rq^r}{1-q^r} - \sum_{r=1}^{\infty}\frac{10rq^{2r}}{1-q^{2r}} - \sum_{r=1}^{\infty}\frac{3rq^{3r}}{1-q^{3r}} + \sum_{r=1}^{\infty}\frac{6rq^{6r}}{1-q^{6r}}\right].$$

After employing the definition of the Eisenstein series and performing simplifications, we reach the following result:

$$\frac{q}{S}\frac{dS}{dq} = \frac{1}{12}[-7P_1 + 10P_2 + 3P_3 - 6P_6].$$

Ultimately, by incorporating Lemma 2.4 and expressing Eisenstein series in the context of h-functions, subsequent simplification yields the expression (4.4).

Employing the methodology utilized in deriving the aforementioned results, we can formulate the following set of differential equations: for

$$h + \frac{1}{h} = u, \quad -h + \frac{1}{h} = v:$$

(i) If $S(q)=\dfrac{(1+9r_a(q))}{r_a(q))}=\dfrac{f_2^8f_3^4}{qf_1^4f_6^8}$, then $q\dfrac{ds}{dq}+\left[\dfrac{v(u^2-6u+8)}{(u-1)(u-4)((u+2)}y_{12}\right]S=0;$

(ii) If $S(q)=(1+9r_a(q))r_b(q)=\dfrac{f_2^5f_3f_6}{qf_1^7}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(u^3+10u^2-20u)}{2(u-1)(u-4)(u-2)(u+2)}y_{12}\right]S=0;$

(iii) If $S(q)=\dfrac{(1+9r_a(q))}{r_b(q)}=\dfrac{f_2^{13}f_3^5}{q^{1/2}f_1^{11}f_6^7}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(-u^3+26u^2-52u)}{2(u-1)(u-4)(u-2)(u+2)}y_{12}\right]S=0;$

(iv) If $S(q)=(1+9r_a(q))(r_c(q))=\dfrac{q^{1/3}f_2^8}{f_1^8}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(u^4-20u^3-48u^2+32u-30)}{3u(u-1)(u-4)(u-2)(u+2)}y_{12}\right]S=0;$

(v) If $S(q)=\dfrac{(1+9r_a(q))}{r_c(q)}=\dfrac{f_2^{10}f_3^6}{q^{1/3}f_1^{10}f_6^6}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(u^4+20u^3-48u^2+32u-30)}{3u(u-1)(u-4)(u-2)(u+2)}y_{12}\right]S=0;$

(vi) If $S(q)=(1-3r_b(q^2))r_a(q)=\dfrac{qf_2f_6^5f_{12}}{f_1^2f_3^2f_4^3}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(u^2-4u-10)}{(u-1)(u-4)(u-2)}y_{12}\right]S=0;$

(vii) If $S(q)=\dfrac{(1-3r_b(q^2))}{r_a(q)}=\dfrac{f_1^8f_{12}}{qf_2f_4^3f_6^5}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(-u^2-2u+6)}{(u-1)(u-4)(u-2)}y_{12}\right]S=0;$

$(viii)$ If $S(q)=(1-3r_b(q^2))r_b(q)=\dfrac{q^{1/2}f_1^5f_6^4f_{12}}{f_2^4f_3^3f_4^3}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(u^3-14u^2+16u-24)}{2(u-1)(u-4)(u-2)(u+2)}y_{12}\right]S=0;$

(ix) If $S(q)=\dfrac{(1-3r_b(q^2))}{r_a(q)}=\dfrac{f_1f_2^4f_3f_{12}}{q^{1/2}f_4^3f_6^4}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(-u^3+2u^2-16u+24)}{2(u-1)(u-4)(u-2)(u+2)}y_{12}\right]S=0;$

(x) If $S(q)=(1-3r_b(q^2))r_c(q)=\dfrac{q^{1/3}f_1^4f_6^3f_{12}}{f_2f_3^4f_4^3}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(u^3-18u^2+42u-16)}{3u(u-1)(u-4)(u-2)}y_{12}\right]S=0;$

(xi) If $S(q)=\dfrac{(1-3r_b(q^2))}{r_c(q)}=\dfrac{f_1^2f_2f_3^2f_{12}}{q^{1/3}f_4^3f_6^3}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(2u^2-5)}{3u(u-1)(u-2)}y_{12}\right]S=0;$

(xii) If $S(q)=(1+r_b(q^2))r_a(q)=\dfrac{qf_2^3f_3^2f_6^3f_{12}}{f_1^6f_4^3}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(u^3-6u+8)}{u(u-1)(u-2)}y_{12}\right]S=0;$

$(xiii)$ If $S(q)=\dfrac{(1+r_b(q^2))}{r_a(q)}=\dfrac{f_1^4f_2f_3^4f_{12}}{qf_4^3f_6^7}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(-u^3+2u^2-6u+8)}{(u-1)(u-4)(u-2)}y_{12}\right]S=0;$

(xiv) If $S(q)=(1+r_b(q^2))r_b(q)=\dfrac{q^{1/2}f_1f_3f_6^2f_{12}}{f_2^2f_4^3}$, then $q\dfrac{ds}{dq}-\left[\dfrac{v(u^3-2u^2-8)}{2u(u-1)(u-2)(u+2)}y_{12}\right]S=0;$

(xv) If $S(q)=\dfrac{(1+r_b(q^2))}{r_b(q)}=\dfrac{f_2^6f_3^5f_{12}}{q^{1/2}f_1^3f_4^3f_6^6}$, then $q\dfrac{ds}{dq}+\left[\dfrac{v(u^3-6u^2+8)}{2u(u-1)(u-2)(u+2)}y_{12}\right]S=0.$

Therefore, the proof is complete. □

5. Relations among class one infinite series and *h*-functions

A representation of the Eisenstein series in terms of the classical class one infinite series was developed by Vidya and Rao [6], drawing inspiration from the work of Berndt [10]. Making use of this representation, we obtain interesting formulas that relate the class one infinite series to h-functions. A few are listed in Theorem 5.2.

Lemma 5.1. *[6] For every positive integer $n \geq 2$, the subsequent relationship among the two distinct series holds:*

$$P(q^n) = 1 + nq^{n-1}\left[\frac{T_2(q^n)+1}{(q^n;q^n)_\infty} - 1\right]. \tag{5.1}$$

Theorem 5.2. *Denoting*

$$h + \frac{1}{h} = u, \quad -h + \frac{1}{h} = v \quad \text{and} \quad h\frac{dy_{12}}{dh} = w,$$

we have the series expansion among Class one infinite series and h-functions:

$$\begin{aligned} i) \quad & -\frac{T_2(q)}{f_1} - 4q\frac{T_2(q^2)}{f_2} - 9q^2\frac{T_2(q^3)}{f_3} + 108q^5\frac{T_2(q^6)}{f_6} + 4q\left(1 - \frac{1}{f_2}\right) \\ & + 9q^2\left(1 - \frac{1}{f_3}\right) - 108q^5\left(1 - \frac{1}{f_6}\right) - \frac{4v(2u^3 - 5u^2 + 16u - 4)}{(u-1)(u-2)(u-4)(u+2)}y_{12} + 13 = 0, \end{aligned}$$

$$\begin{aligned} ii) \quad & -7\frac{T_2(q)}{f_1} + 20q\frac{T_2(q^2)}{f_2} + 9q^2\frac{T_2(q^3)}{f_3} + 36q^5\frac{T_2(q^6)}{f_6} - 20q\left(1 - \frac{1}{f_2}\right) \\ & - 9q^2\left(1 - \frac{1}{f_3}\right) - 36q^5\left(1 - \frac{1}{f_6}\right) - \frac{12v(u^3 + 10u^2 - 20u)}{(u-1)(u-4)(u-2)(u+2)}y_{12} + 65 = 0, \end{aligned}$$

$$\begin{aligned} iii) \quad & -\frac{T_2(q)}{f_1} - 9q^2\frac{T_2(q^3)}{f_3} + 72q^5\frac{T_2(q^5)}{f_5} + 9q^2\left(1 - \frac{1}{f_3}\right) \\ & - 72q^5\left(1 - \frac{1}{f_6}\right) - \frac{2v(-2u-4)}{u(u-2)}y_{12} + 9 = 0, \end{aligned}$$

$$\begin{aligned} iv) \quad & -3\frac{T_2(q)}{f_1} + 4q\frac{T_2(q^2)}{f_2} + 9q^2\frac{T_2(q^3)}{f_3} + 36q^5\frac{T_2(q^6)}{f_6} - 4q\left(1 - \frac{1}{f_2}\right) \\ & - 9q^2\left(1 - \frac{1}{f_3}\right) - 36q^5\left(1 - \frac{1}{f_6}\right) - \frac{4v(2u^2 + 8u - 18)}{u(u-4)(u+2)}y_{12} + 11 = 0, \end{aligned}$$

$$\begin{aligned} v) \quad & 3\frac{T_2(q)}{f_1} - 20q\frac{T_2(q^2)}{f_2} - 45q^2\frac{T_2(q^3)}{f_3} + 252q^5\frac{T_2(q^6)}{f_6} - 20q\left(1 - \frac{1}{f_2}\right) \\ & + 45q^2\left(1 - \frac{1}{f_3}\right) - 252q^5\left(1 - \frac{1}{f_6}\right) - \frac{4v(5u^3 - 18u^2 - 12u + 16)}{u(u-1)(u-2)(u+2)}y_{12} + 17 = 0, \end{aligned}$$

$$\begin{aligned} vi) \quad & \frac{T_2(q)}{f_1} - 12q\frac{T_2(q^2)}{f_2} + 9q^3\frac{T_2(q^3)}{f_4} + 36q^{11}\frac{T_2(q^6)}{f_{12}} + 12q\left(1 - \frac{1}{f_2}\right) \\ & - 9q^3\left(1 - \frac{1}{f_4}\right) - 36q^{11}\left(1 - \frac{1}{f_{12}}\right) - \frac{24v}{(u-1)(u-4)}y_{12} + 3 = 0, \end{aligned}$$

$$vii)\quad -13\frac{T_2(q)}{f_1}+20q\frac{T_2(q^2)}{f_2}+63q\frac{T_2(q^3)}{f_3}+36q^3\frac{T_2(q^6)}{f_4}-20q\left(1-\frac{1}{f_2}\right)$$
$$-63q^2\left(1-\frac{1}{f_3}\right)-36q^5\left(1-\frac{1}{f_6}\right)-\frac{24v}{u(u-1)}y_{12}+24=0,$$

$$viii)\quad -\frac{T_2(q)}{f_1}+4q\frac{T_2(q^2)}{f_2}+27q^2\frac{T_2(q^3)}{f_3}-108q^5\frac{T_2(q^6)}{f_6}-4q\left(1-\frac{1}{f_2}\right)$$
$$-27q^2\left(1-\frac{1}{f_3}\right)+108q^5\left(1-\frac{1}{f_6}\right)-\frac{2v(-u^4+16u^3-24u^2-32u+32)}{u(u-1)(u-2)(u-4)(u+2)}y_{12}-7=0,$$

$$ix)\quad -\frac{T_2(q)}{f_1}+9q^2\frac{T_2(q^3)}{f_3}-9q^2\left(1-\frac{1}{f_3}\right)-\frac{3(5u^2+13u-20)v}{u(u-1)(u-4)}y_{12}+3=0,$$

$$x)\quad -5\frac{T_2(q)}{f_1}+16q\frac{T_2(q^2)}{f_2}+45q^2\frac{T_2(q^3)}{f_3}-144q^5\frac{T_2(q^6)}{f_4}-16q\left(1-\frac{1}{f_2}\right)$$
$$-45q^2\left(1-\frac{1}{f_3}\right)+144q^5\left(1-\frac{1}{f_6}\right)-\frac{3(-u^4-2u^3+12u^2-32u+32)v}{u(u-1)(u-2)(u-4)(u+4)}y_{12}-1=0,$$

$$xi)\quad -7\frac{T_2(q)}{f_1}+16q\frac{T_2(q^2)}{f_2}+45q^2\frac{T_2(q^3)}{f_3}-36q^5\frac{T_2(q^6)}{f_6}-16q\left(1-\frac{1}{f_2}\right)$$
$$-45q^2\left(1-\frac{1}{f_3}\right)+36q^5\left(1-\frac{1}{f_6}+\right)-\frac{24v(17u^2+8u-16)}{u(u-4)(u+2)}y_{12}+17=0,$$

$$xii)\quad -9\frac{T_2(q)}{f_1}+20q\frac{T_2(q^2)}{f_2}+99q^2\frac{T_2(q^3)}{f_3}-252q^5\frac{T_2(q^6)}{f_6}-20q\left(1-\frac{1}{f_2}\right)$$
$$-99q^2\left(1-\frac{1}{f_3}\right)+252q^5\left(1-\frac{1}{f_6}+\right)-\frac{24v(2-u)}{u(u-1)(u+2)}y_{12}+1=0,$$

$$xiii)\quad -\frac{T_2(q)}{f_1}+6q\frac{T_2(q^2)}{f_2}+3q^2\frac{T_2(q^3)}{f_3}-8q^3\frac{T_2(q^4)}{f_4}-18q^5\frac{T_2(q^6)}{f_6}+24q^{12}\frac{T_2(q^{12})}{f_{12}}$$
$$-6q\left(1-\frac{1}{f_2}\right)-3q^2\left(1-\frac{1}{f_3}\right)+8q^3\left(1-\frac{1}{f_4}\right)+18q^5\left(1-\frac{1}{f_6}\right)$$
$$-24q^{11}\left(1-\frac{1}{f_{12}}\right)-\frac{24(-u^3+16u^2-34u+16)v}{u(u-1)(u-4)(u-2)}y_{12}+1=0,$$

$$xiv)\quad -2\frac{T_2(q)}{f_1}+6q\frac{T_2(q^2)}{f_2}+6q^2\frac{T_2(q^3)}{f_3}+8q^3\frac{T_2(q^4)}{f_4}-18q^5\frac{T_2(q^6)}{f_6}$$

$$- 24q^{11}\frac{T_2(q^{12})}{f_{12}} - 6q\left(1 - \frac{1}{f_2}\right) - 6q^2\left(1 - \frac{1}{f_3}\right) - 8q^3\left(1 - \frac{1}{f_4}\right)$$
$$+ 18q^5\left(1 - \frac{1}{f_6}\right) - 24q^{11}\left(1 - \frac{1}{f_{12}}\right) - \frac{36v(3u-4)}{u(u-1)(u-4)}y_{12} + 2 = 0,$$

$xv)$
$$- 5\frac{T_2(q)}{f_1} + 22q\frac{T_2(q^2)}{f_2} + 27q^2\frac{T_2(q^3)}{f_3} - 24q^3\frac{T_2(q^6)}{f_4} - 90q^5\frac{T_2(q^2)}{f_6}$$
$$+ 72q^{11}\frac{T_2(q^3)}{f_{12}} - 22q\left(1 - \frac{1}{f_2}\right) - 27q^2\left(1 - \frac{1}{f_3}\right) + 24q^3\left(1 - \frac{1}{f_4}\right)$$
$$+ 90q^5\left(1 - \frac{1}{f_6}\right) - 72q^{11}\left(1 - \frac{1}{f_{12}}\right) - \frac{24v(7u-4)}{u(u-1)(u-4)}y_{12} + 7 = 0.$$

Proof. By utilizing Eqs (2.3) and (5.1) within the context of Theorem 3.1 (i) to (xv), and subsequently streamlining the expressions, we arrive at equivalences involving $P(q^n)$ expressed in relation to $T_2(q^n)$ and h-functions, as detailed above. □

6. Conclusions

The function $h = h(q)$ has been explored recently by several researchers. These authors have derived continued fractions for the h-function and provided numerous modular equations related to it. Notably, we succeeded in deriving several differential identities using Ramanujan-type Eisenstein series, that involves h-functions. These identities could prove valuable in the development of new mathematical functions or in constructing incomplete elliptic integrals. The new differential identities could serve as a foundation for further studies and discoveries in mathematics. Additionally, we successfully connected two different series derived by Srinivasa Ramanujan. By linking the Ramanujan-type series with a class one infinite series, we have demonstrated that the sum of the class one infinite series converges to an infinite product that incorporates h-functions. By relating these series, this work deepens the understanding of Ramanujan's contributions and their connections to modern mathematical theories.

Author contributions

H. C. Vidya: conceptualization, methodology, writing-original draft; B. A. Rao: conceptualization, software, writing-original draft. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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