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Research article

Problems involving combinations of coefficients for the inverse of some complex-valued analytical functions

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Abstract: Inequalities are essential in solving mathematical problems in many different areas of mathematics. Among these, problems involving coefficient combinations that occurred in the Taylor–Maclaurin series of the inverse of complex-valued analytic functions are the challenging ones to solve. In the current article, our aim is to study certain coefficient-related problems that construct from coefficients of the inverse of specific analytic functions. These problems include the Zalcman and Fekete–Szegö inequalities, as well as sharp estimates of the second and third-order Hankel determinants with inverse function coefficients. Also, one of the obtained results gives an improvement of the problem that has been recently published in the journal "AIMS Mathematics".

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1. Introduction

This research focuses on the comprehensive connection among analytic functions and their inverses, which provides new ideas for investigating coefficient estimates and inequalities. The outcome of the present study is particularly relevant in the framework of geometric function theory (GFT), where particular geometric properties are established for analytic functions employing methods specific to this domain of research, but also could offer applications in other related fields such as partial differential equation theory, engineering, fluid dynamics, and electronics. Tremendous impact in the development of GFT was given by the Bieberbach's conjecture, an essential problem related to coefficient estimates for functions that lie within the family $\mathcal{S}$ of univalent functions. This conjecture suggests that for $f \in \mathcal{S}$, expressed through the Taylor–Maclaurin series expansion:

$$f(\upsilon) = \upsilon + \sum_{k=2}^{\infty} d_k \upsilon^k, \quad \upsilon \in \mathbb{D}, \tag{1.1}$$

where $\mathbb{D} := \{\upsilon \in \mathbb{C} : |\upsilon| < 1\}$, the coefficients inequality $|d_k| \leq k$ holds for all $k \geq 2$. The family of such analytic functions with the series representation provided in (1.1) is represented by $\mathcal{A}$. It is worth mentioning that Koebe first introduced $\mathcal{S}$ as a subclass of $\mathcal{A}$ in 1907. Bieberbach [1] originally proposed this conjecture in 1916, initially verifying it for the case $k = 2$. Subsequent advancements by researchers including Löwner [2], Garabedian and Schiffer [3], Pederson and Schiffer [4], and Pederson [5] offered partial proofs for cases up to $k = 6$. However, the complete proof for $k \geq 7$ remained unsolved until 1985, when de-Branges [6] utilized hypergeometric functions to establish it for $k \geq 2$.

In 1960, Lawrence Zalcman postulated the functional $\left|d_k^2 - d_{2k-1}\right| \leq (k-1)^2$ with $k \geq 2$ for $f \in \mathcal{S}$ in order to establish the Bieberbach hypothesis. This has led to the publication of several papers [7–9] on the Zalcman hypothesis and its generalized form $\left|\lambda d_k^2 - d_{2k-1}\right| \leq \lambda k^2 - 2k + 1$ with $\lambda \geq 0$ for various subfamilies of the family $\mathcal{S}$. This hypothesis remained unproven for a long time until Krushkal's breakthrough in 1999, when he proved it in [10] for $k \leq 6$ and solved it by utilizing the holomorphic homotopy of univalent functions in an unpublished manuscript [11] for $k \geq 2$. It was also demonstrated that $\left|d_k^l - d_2^{l(k-1)}\right| \leq 2^{l(k-1)} - k^l$ with $k, l \geq 2$ for $f \in \mathcal{S}$. The Bieberbach conjecture landscape is further enhanced by other conjectures, such as the one presented by Ma [12] in 1999, which is

$$\left|d_j d_k - d_{j+k-1}\right| \leq (j-1)(k-1), \quad j, k \geq 2.$$

He restricted his proof to a subclass of $\mathcal{S}$. The challenge for class $\mathcal{S}$ remains available.

Now, let us recall the concept of subordination, which essentially describes a relationship between analytic functions. An analytic function g_1 is subordinate to g_2 if there exists a Schwarz function ω such that $g_1(\upsilon) = g_2(\omega(\upsilon))$ and it is mathematically represented as $g_1 \prec g_2$. If g_2 is univalent in $\mathbb{D}$, then

$$g_1(\upsilon) \prec g_2(\upsilon), \quad (\upsilon \in \mathbb{D}),$$

if and only if

$$g_1(0) = g_2(0) \;\;\&\;\; g_1(\mathbb{D}) \sqsubseteq g_2(\mathbb{D}).$$

In essence, this relationship helps us understand how one function is "contained" within another, providing insights into their behavior within the complex plane. The family of univalent functions comprises three classic subclasses $\mathcal{C}$, $\mathcal{S}^*$, and $\mathcal{K}$, each distinguished by its unique properties. These subclasses are commonly known as convex functions, starlike functions, and close-to-convex functions, respectively. Let us define each class:

$$\mathcal{C} := \left\{ f \in \mathcal{S} : \frac{(\upsilon f'(\upsilon))'}{f'(\upsilon)} \prec \frac{1+\upsilon}{1-\upsilon}, \quad \upsilon \in \mathbb{D} \right\},$$

$$\mathcal{S}^* := \left\{ f \in \mathcal{S} : \frac{\upsilon f'(\upsilon)}{f(\upsilon)} \prec \frac{1+\upsilon}{1-\upsilon}, \quad \upsilon \in \mathbb{D} \right\}$$

and

$$\mathcal{K} := \left\{ f \in \mathcal{S} : \frac{\upsilon f'(\upsilon)}{h(\upsilon)} \prec \frac{1+\upsilon}{1-\upsilon}, \quad \upsilon \in \mathbb{D} \right\},$$

for some $h \in \mathcal{S}^*$. The above family $\mathcal{K}$ may be reduced to the family of bounded turning functions $\mathcal{BT}$ by choosing $h(\upsilon) = \upsilon$. Moreover, a number of intriguing subfamilies of class $\mathcal{S}$ were examined by replacing $\frac{1+\upsilon}{1-\upsilon}$ by other special functions. For the reader's benefit, a few of them are included below:

(i). $\mathcal{S}_e^* \equiv \mathcal{S}^*(e^z)$ and $\mathcal{C}_e \equiv \mathcal{C}(e^z)$ [13], $\mathcal{S}_{SG}^* \equiv \mathcal{S}^*\left(\frac{2}{1+e^{-z}}\right)$ and $\mathcal{C}_{SG} \equiv \mathcal{C}\left(\frac{2}{1+e^{-z}}\right)$ [14],
(ii). $\mathcal{S}_{cr}^* \equiv \mathcal{S}^*\left(z+\sqrt{1+z^2}\right)$ and $\mathcal{C}_{cr} \equiv \mathcal{C}\left(z+\sqrt{1+z^2}\right)$ [15], $\mathcal{S}_{Ne}^* \equiv \mathcal{S}^*\left(1+z-\frac{1}{3}z^3\right)$ [16],
(iii). $\mathcal{S}_{(n-1)\mathcal{L}}^* \equiv \mathcal{S}^*(\Psi_{n-1}(z))$ [17] with $\Psi_{n-1}(z) = 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n$ for $n \geq 2$.
(iv). $\mathcal{S}_{\sinh}^* \equiv \mathcal{S}^*(1+\sinh(\lambda z))$ with $0 \leq \lambda \leq \ln\left(1+\sqrt{2}\right)$ [18].

It is observed that a significant area of mathematics is the study of the inverse functions for the functions in various subclasses of $\mathcal{S}$. The well-known Koebe's 1/4 theorem states that there exists the inverse f^{-1} for every univalent function f defined in $\mathbb{D}$, at least on the disk with a radius of 1/4, which has Taylor's series form

$$f^{-1}(\omega) := \omega + \sum_{n=2}^{\infty} B_n \omega^n, \quad |\omega| < 1/4. \tag{1.2}$$

Employing the formula $f\left(f^{-1}(\omega)\right) = \omega$, we acquire

$$B_2 = -d_2, \tag{1.3}$$
$$B_3 = 2d_2^2 - d_3, \tag{1.4}$$
$$B_4 = 5d_2d_3 - 5d_2^3 - d_4, \tag{1.5}$$
$$B_5 = 14d_2^4 + 3d_3^2 - 21d_2^2d_3 + 6d_2d_4 - d_5. \tag{1.6}$$

We consider the Hankel determinant of f^{-1} given by

$$\hat{H}_{\lambda,n}\left(f^{-1}\right) = \begin{vmatrix} B_n & B_{n+1} & \dots & B_{n+\lambda-1} \\ B_{n+1} & B_{n+2} & \dots & B_{n+\lambda} \\ \vdots & \vdots & \dots & \vdots \\ B_{n+\lambda-1} & B_{n+\lambda} & \dots & B_{n+2\lambda-2} \end{vmatrix}.$$

Specifically, the second and third-order Hankel determinants of f^{-1} are defined as the following determinants, respectively:

$$\hat{H}_{2,2}\left(f^{-1}\right) = \begin{vmatrix} B_2 & B_3 \\ B_3 & B_4 \end{vmatrix} = B_2B_4 - B_3^2,$$
$$\hat{H}_{3,1}\left(f^{-1}\right) = \begin{vmatrix} 1 & B_2 & B_3 \\ B_2 & B_3 & B_4 \\ B_3 & B_4 & B_5 \end{vmatrix} = B_3\left(B_2B_4 - B_3^2\right) - B_4\left(B_4 - B_2B_3\right) + B_5\left(B_3 - B_2^2\right).$$

As it is seen, f^{-1} is also not necessary to be univalent. Thus, this concept is also a natural generalization of the Hankel determinant with coefficients of $f \in \mathcal{S}$ as entries. There are very few publications in the literature that address coefficient-related problems of the inverse function, particularly determinants as stated above. Due to such a reason, the researchers motivated, and so this led to the publication of some good articles [19–23] on the above-stated Hankel determinants.

The key mathematical concept in this study is the Hankel determinant $\hat{H}_{\lambda,n}(f)$, where $n, \lambda \in \{1, 2, \ldots\}$. This concept was introduced by Pommerenke [24, 25]. It is composed of the coefficients of the function $f \in \mathcal{S}$ and is expressed as

$$\hat{H}_{\lambda,n}(f) := \begin{vmatrix} d_n & d_{n+1} & \ldots & d_{n+\lambda-1} \\ d_{n+1} & d_{n+2} & \ldots & d_{n+\lambda} \\ \vdots & \vdots & \ldots & \vdots \\ d_{n+\lambda-1} & d_{n+\lambda} & \ldots & d_{n+2\lambda-2} \end{vmatrix}.$$

This determinant is utilized in both pure mathematics and applied sciences, including non-stationary signal theory in the Hamburger moment problem, Markov process theory, and a variety of other fields. There are relatively few publications on the estimates of the Hankel determinant for functions in the general class $\mathcal{S}$. Hayman established the best estimate for $f \in \mathcal{S}$ in [26] by asserting that $\left|\hat{H}_{2,n}(f)\right| \leq |\eta|$, where η is a constant. Moreover, it was demonstrated in [27] that $\left|\hat{H}_{2,2}(f)\right| \leq \eta$ where $0 \leq \eta \leq 11/3$ for $f \in \mathcal{S}$. The two determinants $\hat{H}_{2,1}(f)$ and $\hat{H}_{2,2}(f)$ for different subfamilies of univalent functions have been thoroughly examined in the literature. Notable work was done by Janteng et al. [28], Lee et al. [29], Ebadian et al. [30], and Cho et al. [31], who determine the sharp estimates of the second-order Hankel determinant for certain subclasses of $\mathcal{S}$.

The sharp estimate of the third-order Hankel determinant $\hat{H}_{3,1}(f)$ for some analytic univalent functions is mathematically more difficult to find than the second-order Hankel determinant. Numerous articles on the third-order Hankel determinant have been published in the literature in which nonsharp limits of this determinant for the fundamental subclasses of analytic functions are determined. Following these arduous investigations, a few scholars were eventually able to obtain sharp bounds of this determinant for the classes $\mathcal{C}$, $\mathcal{BT}$, and $\mathcal{S}^*$, as reported in the recently published works [32–34], respectively. These estimates are given by

$$\left|\hat{H}_{3,1}(f)\right| \leq \begin{cases} \frac{4}{135} & \text{for} \quad f \in \mathcal{C}, \\ \frac{1}{4} & \text{for} \quad f \in \mathcal{BT}, \\ \frac{4}{9} & \text{for} \quad f \in \mathcal{S}^*. \end{cases}$$

Later on, Lecko et al. [35] established the sharp estimate for $\left|\hat{H}_{3,1}(f)\right|$ by utilizing similar approaches, specifically for functions that belong to the $\mathcal{S}^*(1/2)$ class. Also, the articles [36–38] provide more investigations on the exact bounds of this third-order Hankel determinant.

Now, let us consider the three function classes defined respectively by

$$\mathcal{S}^*_{\mathfrak{S}\mathfrak{g}} := \left\{ f \in \mathcal{S} : \frac{2\upsilon f'(\upsilon)}{f(\upsilon) - f(-\upsilon)} \prec \frac{2}{1 + e^{-\upsilon}}, \quad \upsilon \in \mathbb{D} \right\},$$

$$\mathcal{S}^*_{3l,s} := \left\{ f \in \mathcal{S} : \frac{2\upsilon f'(\upsilon)}{f(\upsilon) - f(-\upsilon)} \prec 1 + \frac{4}{5}\upsilon + \frac{1}{5}\upsilon^4, \quad \upsilon \in \mathbb{D} \right\}$$

and

$$\mathcal{SK}_{\exp} := \left\{ f \in \mathcal{S} : \frac{2\left(\upsilon f'\left(\upsilon\right)\right)'}{\left(f\left(\upsilon\right) - f\left(-\upsilon\right)\right)'} \prec e^{\upsilon}, \quad \upsilon \in \mathbb{D} \right\}.$$

These classes have been studied by Faisal et al. [39], Tang et al. [40], and Mendiratta et al. [13] respectively. In this paper, we improved the bound of the third-order Hankel determinant $\left|\hat{H}_{3,1}\left(f^{-1}\right)\right|$, which was determined by Hu and Deng [41] and published recently in AIMS Mathematics. Furthermore, we obtain the bounds of the initial three inverse coefficients together with the sharp bounds of Krushkal, Zalcman, and Fekete–Szegö functionals along with the Hankel determinants $\left|\hat{H}_{2,2}\left(f^{-1}\right)\right|$ and $\left|\hat{H}_{3,1}\left(f^{-1}\right)\right|$ upper bounds.

2. A set of lemmas

Let $\mathcal{B}_0$ be the class of Schwarz functions. It is noted that $\omega \in \mathcal{B}_0$ can be written as

$$\omega\left(\upsilon\right) = \sum_{n=1}^{\infty} \sigma_n \upsilon^n, \qquad \upsilon \in \mathbb{D}. \tag{2.1}$$

We require the following lemmas to prove our main results.

Lemma 2.1. [42] Let $\omega\left(\upsilon\right)$ be a Schwarz function. Then, for any real numbers ϱ and ς such that

$$\begin{aligned}
(\varrho, \varsigma) &= \left\{ |\varrho| \leq \frac{1}{2}, \ -1 \leq \varsigma \leq 1 \right\} \cup \left\{ \frac{1}{2} \leq |\varrho| \leq 2, \ \frac{4}{27}\left(|\varrho| + 1\right)^3 - \left(|\varrho| + 1\right) \leq \varsigma \leq 1 \right\}, \\
(\varrho, \varsigma) &= \left\{ 2 \leq |\varrho| \leq 4, \ \frac{2\left(|\varrho| + 1\right)|\varrho|}{4 + |\varrho|^2 + 2|\varrho|} \leq \varsigma \leq \frac{1}{12}\left(\varrho^2 + 8\right) \right\}, \\
(\varrho, \varsigma) &= \left\{ \frac{1}{2} \leq |\varrho| \leq 2, \ -\frac{2}{3}\left(1 + |\varrho|\right) \leq \varsigma \leq \frac{4}{27}\left(1 + |\varrho|\right)^3 - \left(1 + |\varrho|\right) \right\},
\end{aligned}$$

the following sharp estimate holds:

$$\left|\sigma_3 + \varrho\sigma_1\sigma_2 + \varsigma\sigma_1^3\right| \leq 1.$$

Lemma 2.2. [43] If $\omega\left(\upsilon\right)$ be a Schwarz function, then

$$|\sigma_n| \leq 1, \qquad n \geq 1. \tag{2.2}$$

Moreover, for $\tau \in \mathbb{C}$, the following inequality holds

$$\left|\sigma_2 + \tau\sigma_1^2\right| \leq \max\left\{1, |\tau|\right\}. \tag{2.3}$$

Lemma 2.3. [44] Let $\omega\left(\upsilon\right)$ be a Schwarz function. Then

$$|\sigma_2| \leq 1 - |\sigma_1|^2, \tag{2.4}$$

$$|\sigma_3| \leq 1 - |\sigma_1|^2 - \frac{|\sigma_2|^2}{1 + |\sigma_1|}, \tag{2.5}$$

$$|\sigma_4| \leq 1 - |\sigma_1|^2 - |\sigma_2|^2. \tag{2.6}$$

Lemma 2.4. [45] Let $\omega\left(\upsilon\right)$ be a Schwarz function. Then

$$\left|\sigma_1\sigma_3 - \sigma_2^2\right| \leq 1 - |\sigma_1|^2$$

and

$$\left|\sigma_4 + \left(1 + \Lambda\right)\sigma_1\sigma_3 + \sigma_2^2 + \left(1 + 2\Lambda\right)\sigma_1^2\sigma_2 + \Lambda\sigma_1^4\right| \leq \max\left\{1, |\Lambda|\right\}, \qquad \Lambda \in \mathbb{C}. \tag{2.7}$$

3. Third-order Hantel determinant

In this section, we will improve the bound of the third-order Hankel determinant $\left|\hat{H}_{3,1}\left(f^{-1}\right)\right|$ with inverse coefficient entries for functions belonging to the class $\mathcal{S}^{*}_{\mathfrak{S}\mathfrak{g}}$.

Theorem 3.1. Let f^{-1} be the inverse of the function $f \in \mathcal{S}^{*}_{\mathfrak{S}\mathfrak{g}}$ and has the form (1.2). Then

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| < 0.0317.$$

Proof. Let $f \in \mathcal{S}^{*}_{\mathfrak{S}\mathfrak{g}}$. Then, by subordination relationship, it implies

$$\frac{2\upsilon f'(\upsilon)}{f(\upsilon) - f(-\upsilon)} = \frac{2}{1 + e^{-\omega(\upsilon)}}, \qquad \upsilon \in \mathbb{D} \tag{3.1}$$

and also assumes that

$$\omega(\upsilon) = \sigma_1\upsilon + \sigma_2\upsilon^2 + \sigma_3\upsilon^3 + \sigma_4\upsilon^4 + \cdots. \tag{3.2}$$

Using (1.1), we obtain

$$\frac{2\upsilon f'(\upsilon)}{f(\upsilon) - f(-\upsilon)} := 1 + 2d_2\upsilon + 2d_3\upsilon^2 + (-2d_2d_3 + 4d_4)\upsilon^3 + \left(-2d_3^2 + 4d_5\right)\upsilon^4 + \cdots. \tag{3.3}$$

By some easy calculation and utilizing the series expansion of (3.2), we achieve

$$\frac{2}{1 + e^{-\omega(\upsilon)}} = 1 + \frac{1}{2}\sigma_1\upsilon + \frac{1}{2}\sigma_2\upsilon^2 + \left(-\frac{1}{24}\sigma_1^3 + \frac{1}{2}\sigma_3\right)\upsilon^3 + \left(-\frac{1}{8}\sigma_1^2\sigma_2 + \frac{1}{2}\sigma_4\right)\upsilon^4 + \cdots. \tag{3.4}$$

Now by comparing (3.3) and (3.4), we obtain

$$d_2 = \frac{1}{4}\sigma_1, \tag{3.5}$$
$$d_3 = \frac{1}{4}\sigma_2, \tag{3.6}$$
$$d_4 = -\frac{1}{96}\sigma_1^3 + \frac{1}{8}\sigma_3 + \frac{1}{32}\sigma_1\sigma_2, \tag{3.7}$$
$$d_5 = \frac{1}{8}\sigma_4 + \frac{1}{32}\sigma_2^2 - \frac{1}{32}\sigma_1^2\sigma_2. \tag{3.8}$$

Putting (3.5)–(3.8) in (1.3)–(1.6), we obtain

$$B_2 = -\frac{1}{4}\sigma_1, \tag{3.9}$$
$$B_3 = -\frac{1}{4}\sigma_2 + \frac{1}{8}\sigma_1^2, \tag{3.10}$$
$$B_4 = -\frac{13}{192}\sigma_1^3 - \frac{1}{8}\sigma_3 + \frac{9}{32}\sigma_1\sigma_2, \tag{3.11}$$
$$B_5 = \frac{5}{32}\sigma_2^2 - \frac{1}{4}\sigma_1^2\sigma_2 + \frac{5}{128}\sigma_1^4 - \frac{1}{8}\sigma_4 + \frac{3}{16}\sigma_1\sigma_3. \tag{3.12}$$

The determinant $\left|\hat{H}_{3,1}\left(f^{-1}\right)\right|$ can be reconfigured as follows:

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| = \left|2B_2B_3B_4 - B_4^2 - B_2^2B_5 - B_3^3 + B_3B_5\right|.$$

From (3.9)–(3.12), we easily write

$$\begin{aligned}\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| &= \frac{1}{64}\left|-\sigma_3^2 + \left(\frac{1}{2}\sigma_2 + \frac{1}{6}\sigma_1^2\right)\sigma_1\sigma_3 + \frac{5}{576}\sigma_1^6 - \frac{3}{2}\sigma_2^3 + 2\sigma_2\sigma_4\right.\\ &\quad\left. -\frac{5}{48}\sigma_1^4\sigma_2 + \frac{5}{16}\sigma_1^2\sigma_2^2 - \frac{1}{2}\sigma_1^2\sigma_4\right|.\end{aligned}$$

Now we begin by utilizing Lemma 2.1 with $\varrho = -\frac{1}{2}$ and $\varsigma = -\frac{1}{6}$ that

$$\left|\sigma_3\left[\sigma_3 + \left(-\frac{1}{2}\right)\sigma_1\sigma_2 + \left(-\frac{1}{6}\right)\sigma_1^3\right]\right| \le |\sigma_3|$$

and also by using Lemma 2.3, we have

$$|\sigma_3| \le 1 - \frac{|\sigma_2|^2}{1 + |\sigma_1|} - |\sigma_1|^2 \le 1 - \frac{|\sigma_2|^2}{2} - |\sigma_1|^2.$$

Applying it and also using $|\sigma_4| \le 1 - |\sigma_2|^2 - |\sigma_1|^2$, we achieve

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| \le \frac{1}{64}E\left(|\sigma_1|, |\sigma_2|\right),$$

where

$$\begin{aligned}E(\sigma, t) &= 1 - \sigma^2 - \frac{1}{2}t^2 + \frac{5}{576}\sigma^6 + \frac{3}{2}t^3 + 2t\left(1 - \sigma^2 - t^2\right) + \frac{5}{48}\sigma^4 t + \frac{5}{16}\sigma^2 t^2\\ &\quad + \frac{1}{2}\sigma^2\left(1 - \sigma^2 - t^2\right), \quad \sigma = |\sigma_1|,\ t = |\sigma_2|.\end{aligned}$$

But E is a decreasing function of the variable σ; consequently,

$$E(\sigma, t) \le E(0, t) = 1 - \frac{1}{2}t^2 - \frac{1}{2}t^3 + 2t.$$

The function $E(0, t)$ reaches its maximum value in $[0, 1]$ if $t = -\frac{1}{3} + \frac{1}{3}\sqrt{13}$, so $E(0, t) \le 2.0322$, which completes the proof. □

Conjecture 3.2. If the inverse of $f \in \mathcal{S}^*_{\mathfrak{S}\mathfrak{g}}$ is of the form (1.2), then

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| \le \frac{1}{64}.$$

Equality will be obtained by using (1.3)–(1.6) together with

$$\frac{2\upsilon f'(\upsilon)}{f(\upsilon) - f(-\upsilon)} = 1 + \frac{1}{2}\upsilon^3 - \frac{1}{24}\upsilon^9 + \cdots.$$

4. Inverse coefficients for $\mathcal{S}^*_{3l,s}$

We begin this section by computing the estimates of the first three initial inverse coefficients for functions in the family $\mathcal{S}^*_{3l,s}$.

Theorem 4.1. Let the inverse function of $f \in \mathcal{S}^*_{3l,s}$ has the series form (1.2). Then

$$\begin{aligned} |B_2| &\leq \frac{2}{5}, \\ |B_3| &\leq \frac{2}{5}, \\ |B_4| &\leq \frac{1}{5}. \end{aligned}$$

The equality can easily be obtained by utilizing (1.3) up to (1.5) together with

$$\frac{2\upsilon f'(\upsilon)}{f(\upsilon)-f(-\upsilon)} = 1+\frac{1}{2}\upsilon^m - \frac{1}{24}\upsilon^{4m} + \cdots, \text{ for } m = 1,2,3. \tag{4.1}$$

Proof. Let $f \in \mathcal{S}^*_{3l,s}$. Then we easily write

$$\frac{2\upsilon f'(\upsilon)}{f(\upsilon)-f(-\upsilon)} = 1+\frac{4}{5}\omega(\upsilon)+\frac{1}{5}(\omega(\upsilon))^4, \qquad \upsilon \in \mathbb{D},$$

and here ω represents the Schwarz function. Also, let us assume that

$$\omega(\upsilon) = \sigma_1\upsilon + \sigma_2\upsilon^2 + \sigma_3\upsilon^3 + \sigma_4\upsilon^4 + \cdots. \tag{4.2}$$

Using (1.1) , we obtain

$$\begin{aligned} \frac{2\upsilon f'(\upsilon)}{f(\upsilon)-f(-\upsilon)} &= 1+2d_2\upsilon+2d_3\upsilon^2+(4d_4-2d_2d_3)\upsilon^3 \\ &\quad + \left(4d_5-2d_3^2\right)\upsilon^4\cdots. \end{aligned} \tag{4.3}$$

By some easy calculation and utilizing the series expansion of (4.2), we have

$$1+\frac{4}{5}\omega(\upsilon)+\frac{1}{5}(\omega(\upsilon))^4 = 1+\frac{4}{5}\sigma_1\upsilon+\frac{4}{5}\sigma_2\upsilon^2+\frac{4}{5}\sigma_3\upsilon^3+\left(\frac{4}{5}\sigma_4+\frac{1}{5}\sigma_1^4\right)\upsilon^4+\cdots. \tag{4.4}$$

Now, by comparing (4.3) and (4.4), we obtain

$$d_2 = \frac{2}{5}\sigma_1, \tag{4.5}$$

$$d_3 = \frac{2}{5}\sigma_2, \tag{4.6}$$

$$d_4 = \frac{1}{5}\sigma_3+\frac{2}{25}\sigma_1\sigma_2, \tag{4.7}$$

$$d_5 = \frac{1}{5}\sigma_4+\frac{1}{20}\sigma_1^4+\frac{2}{25}\sigma_2^2. \tag{4.8}$$

Substituting (4.5)–(4.8) in (1.3)–(1.6), we obtain

$$B_2 = -\frac{2}{5}\sigma_1, \tag{4.9}$$

$$B_3 = \frac{8}{25}\sigma_1^2 - \frac{2}{5}\sigma_2, \tag{4.10}$$

$$B_4 = -\frac{8}{25}\sigma_1^3 - \frac{1}{5}\sigma_3 + \frac{18}{25}\sigma_1\sigma_2, \tag{4.11}$$

$$B_5 = \frac{771}{2500}\sigma_1^4 - \frac{144}{125}\sigma_1^2\sigma_2 + \frac{12}{25}\sigma_1\sigma_3 + \frac{2}{5}\sigma_2^2 - \frac{1}{5}\sigma_4. \tag{4.12}$$

Using (2.2) in (4.9), we achieve

$$|B_2| \leq \frac{2}{5}.$$

To prove the second inequality, we can write (4.10) as

$$|B_3| = \frac{2}{5}\left|\sigma_2 + \left(-\frac{4}{5}\right)\sigma_1^2\right|.$$

Applying (2.3) in the above equation, we achieve

$$|B_3| \leq \frac{2}{5}.$$

From (4.11) , we deduce that

$$|B_4| = \frac{1}{5}\left|\sigma_3 + \left(-\frac{18}{5}\right)\sigma_1\sigma_2 + \frac{8}{5}\sigma_1^3\right|.$$

Comparing it with Lemma 2.1, we note that

$$\varrho = -\frac{18}{5} \qquad \text{and} \qquad \varsigma = \frac{8}{5}.$$

It is clear that $2 \leq |\varrho| \leq 4$ with

$$\frac{2|\varrho|(|\varrho|+1)}{|\varrho|^2 + 2|\varrho| + 4} = \frac{45}{151} \leq \varsigma \qquad \text{and} \qquad \varsigma \leq \frac{1}{12}\left(\varrho^2 + 8\right) = \frac{131}{75}.$$

All the conditions of Lemma 2.1 are satisfied. Therefore

$$|B_4| \leq \frac{1}{5}.$$

The required proof is thus completed. □

Now, we compute the Fekete–Szegö functional bound for the inverse function of $f \in \mathcal{S}^*_{3l,s}$.

Theorem 4.2. If f^{-1} is the inverse of the function $f \in \mathcal{S}^*_{3l,s}$ with series expansion (1.2), then

$$\left|B_3 - \tau B_2^2\right| \leq \max\left\{\frac{2}{5}, \left|\frac{4\tau - 8}{25}\right|\right\}, \qquad \tau \in \mathbb{C}.$$

This functional bound is sharp.

Proof. Putting (4.9) and (4.10), we obtain

$$\begin{aligned}\left|B_3-\tau B_2^2\right| &= \left|-\frac{2}{5}\sigma_2-\frac{4\tau}{25}\sigma_1^2+\frac{8}{25}\sigma_1^2\right| \\ &= \frac{2}{5}\left|\sigma_2+\left(\frac{2\tau-4}{5}\right)\sigma_1^2\right|.\end{aligned}$$

Application of (2.3) leads us to

$$\left|B_3-\tau B_2^2\right| \leq \max\left\{\frac{2}{5},\left|\frac{4\tau-8}{25}\right|\right\}.$$

The bound of the above functional is best possible, and it can easily be checked by (1.3), (1.4), and (4.1) with $m=2$. □

By replacing $\tau=1$ in Theorem 4.2, we arrive at the below result.

Corollary 4.3. If the inverse of the function $f\in\mathcal{S}^*_{3l,s}$ is f^{-1} with series expansion (1.2), then

$$\left|B_3-B_2^2\right| \leq \frac{2}{5}.$$

This estimate is sharp, and equality will be obtained by using (1.3), (1.4), and (4.1) with $m=2$.

Next, we investigate the Zalcman functional upper bound for $f^{-1}\in\mathcal{S}^*_{3l,s}$.

Theorem 4.4. If $f\in\mathcal{S}^*_{3l,s}$ and its inverse function f^{-1} have the form (1.2), then

$$|B_2B_3-B_4| \leq \frac{1}{5}.$$

The above estimate is sharp.

Proof. Taking use of (4.9)–(4.11) , we achieve

$$|B_2B_3-B_4| = \frac{1}{5}\left|\sigma_3+\left(-\frac{14}{5}\right)\sigma_1\sigma_2+\frac{24}{25}\sigma_1^3\right|.$$

From Lemma 2.1, let

$$\varrho=-\frac{14}{5} \qquad \text{and} \qquad \varsigma=\frac{24}{25}.$$

It is clear that $2\leq|\varrho|\leq 4$ with

$$\frac{2|\varrho|(|\varrho|+1)}{|\varrho|^2+2|\varrho|+4}=\frac{35}{109}\leq\varsigma \qquad \text{and} \qquad \varsigma\leq\frac{1}{12}\left(\varrho^2+8\right)=\frac{33}{25}.$$

Thus, all the conditions of Lemma 2.1 are satisfied. Hence

$$|B_2B_3-B_4| \leq \frac{1}{5}.$$

The required estimate is best possible and will easily be obtained by using (1.3)–(1.5), and (4.1) with $m=3$. □

Further, we intend to compute the Krushkal functional bound for the family $\mathcal{S}^{*}_{3l,s}$.

Theorem 4.5. If $f \in \mathcal{S}^{*}_{3l,s}$ and its inverse function f^{-1} have the form (1.2), then

$$\left|B_4 - B_2^3\right| \leq \frac{1}{5}.$$

This estimate is sharp.

Proof. Putting (4.9) and (4.11), we obtain

$$\left|B_4 - B_2^3\right| = \frac{1}{5}\left|\sigma_3 + \left(-\frac{18}{5}\right)\sigma_1\sigma_2 + \left(\frac{32}{25}\right)\sigma_1^3\right|.$$

From Lemma 2.1, let

$$\varrho = -\frac{18}{5} \qquad \text{and} \qquad \varsigma = \frac{32}{25}.$$

It is clear that $2 \leq |\varrho| \leq 4$ with

$$\frac{2|\varrho|(|\varrho|+1)}{|\varrho|^2 + 2|\varrho| + 4} = \frac{45}{151} \leq \varsigma \qquad \text{and} \qquad \varsigma \leq \frac{1}{12}\left(\varrho^2 + 8\right) = \frac{131}{75}.$$

Thus, all the conditions of Lemma 2.1 are satisfied. Hence

$$\left|B_4 - B_2^3\right| \leq \frac{1}{5}.$$

This estimate is best possible and will be confirmed by using (1.3), (1.5), and (4.1) with $m = 3$. □

In the upcoming result, we will investigate the estimate of $\hat{H}_{2,2}\left(f^{-1}\right)$ for the family $\mathcal{S}^{*}_{3l,s}$.

Theorem 4.6. Let the inverse function of $f \in \mathcal{S}^{*}_{3l,s}$ has the series expansion (1.2). Then

$$\left|\hat{H}_{2,2}\left(f^{-1}\right)\right| \leq \frac{4}{25}.$$

This inequality is sharp, and equality will easily be achieved by using (1.3)–(1.5) and (4.1) with $m = 2$.

Proof. The determinant $\hat{H}_{2,2}\left(f^{-1}\right)$ can be reconfigured as follows:

$$\begin{aligned}\hat{H}_{2,2}\left(f^{-1}\right) &= B_2B_4 - B_3^2.\\ &= -d_3^2 + d_2d_4 - d_2^2d_3 + d_2^4.\end{aligned}$$

Substituting (4.9)–(4.11), we achieve

$$\begin{aligned}\left|\hat{H}_{2,2}\left(f^{-1}\right)\right| &= \frac{4}{25}\left|-\frac{4}{25}\sigma_1^4 + \frac{1}{5}\sigma_1^2\sigma_2 - \frac{1}{2}\sigma_1\sigma_3 + \sigma_2^2\right|\\ &= \frac{4}{25}\left|\frac{1}{2}\left(\sigma_2^2 - \sigma_1\sigma_3\right) + \frac{1}{2}\left(-\frac{8}{25}\sigma_1^4 + \frac{2}{5}\sigma_1^2\sigma_2 + \sigma_2^2\right)\right|\\ &\leq \frac{4}{50}\left|\sigma_2^2 - \sigma_1\sigma_3\right| + \frac{4}{50}\left|-\frac{8}{25}\sigma_1^4 + \frac{2}{5}\sigma_1^2\sigma_2 + \sigma_2^2\right|\end{aligned}$$

$$= \frac{4}{50}Y_1 + \frac{4}{50}Y_2,$$

where

$$Y_1 = \left|\sigma_2^2 - \sigma_1\sigma_3\right|,$$

and

$$Y_2 = \left|-\frac{8}{25}\sigma_1^4 + \frac{2}{5}\sigma_1^2\sigma_2 + \sigma_2^2\right|.$$

Utilizing Lemma 2.4, we acquire $Y_1 \leq 1$. Applying (2.4) along with triangle inequality for Y_2, we have

$$|Y_2| \leq \left(1 - |\sigma_1|^2\right)^2 + \frac{8}{25}|\sigma_1|^4 + \frac{2}{5}\left(1 - |\sigma_1|^2\right)|\sigma_1|^2.$$

By setting $|\sigma_1| = \varkappa$ with $\varkappa \in (0, 1]$, we obtain

$$|Y_2| \leq -\frac{23\varkappa^4}{25} - \frac{8\varkappa^2}{5} + 1 = N(\varkappa).$$

Clearly $N'(\varkappa) \leq 0$, $N(\varkappa)$ is a decreasing function of $\varkappa$, indicating that it achieves its maxima at $\varkappa = 0$, that is,

$$|Y_2| \leq 1.$$

Therefore

$$\left|\hat{H}_{2,2}\left(f^{-1}\right)\right| \leq \frac{4}{50}Y_1 + \frac{4}{50}Y_2 \leq \frac{4}{25}$$

and so the required proof is accomplished. □

Theorem 4.7. Let f^{-1} be the inverse of $f \in \mathcal{S}^*_{3l,s}$ with series expansion (1.2). Then

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| < 0.11600.$$

Proof. The determinant $\left|\hat{H}_{3,1}\left(f^{-1}\right)\right|$ is described as follows:

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| = \left|2B_2B_3B_4 - B_4^2 - B_2^2B_5 - B_3^3 + B_3B_5\right|.$$

From (4.9)–(4.12), we easily write

$$\begin{aligned}\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| &= \frac{1}{25}\left|-\sigma_3^2 + \frac{4}{5}\sigma_2\sigma_1\sigma_3 - \frac{61}{625}\sigma_1^6 - \frac{12}{5}\sigma_2^3 - \frac{67}{250}\sigma_1^4\sigma_2\right.\\ &\left.\quad + \frac{52}{25}\sigma_1^2\sigma_2^2 - \frac{4}{5}\sigma_1^2\sigma_4 + 2\sigma_2\sigma_4\right|.\end{aligned}$$

The below inequality follows easily by using Lemma 2.1 with $\varrho = -\frac{4}{5}$ and $\varsigma = 0$

$$\left|\sigma_3\left[\sigma_3 + \left(-\frac{4}{5}\right)\sigma_1\sigma_2 + (0)\,\sigma_1^3\right]\right| \leq |\sigma_3|$$

and also by virtue of Lemma 2.3, we have

$$|\sigma_3| \leq 1 - \frac{|\sigma_2|^2}{1+|\sigma_1|} - |\sigma_1|^2 \leq 1 - \frac{|\sigma_2|^2}{2} - |\sigma_1|^2 .$$

Applying it and also using $|\sigma_4| \leq 1 - |\sigma_2|^2 - |\sigma_1|^2$, we achieve

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| \leq \frac{1}{25} E\left(|\sigma_1|, |\sigma_2|\right),$$

where

$$\begin{aligned} E(\sigma,t) &= 1 - \sigma^2 - \frac{1}{2}t^2 + \frac{61}{625}\sigma^6 + \frac{12}{5}t^3 + \frac{67}{250}\sigma^4 t + \frac{52}{25}\sigma^2 t^2 + \frac{4}{5}\sigma^2\left(1-\sigma^2-t^2\right) \\ &\quad +2t\left(1-\sigma^2-t^2\right), \qquad \sigma = |\sigma_1|,\ t = |\sigma_2|. \end{aligned}$$

But E is an increasing function of the variable σ; consequently,

$$E(\sigma,t) \leq E(0,t) = 1 - \frac{1}{2}t^2 + \frac{2}{5}t^3 + 2t.$$

The function $E(0,t)$ reaches its maximum value in $[0,1]$ if $t = 1$, so $E(0,t) \leq \frac{29}{10}$, which completes the proof. □

Conjecture 4.8. If the inverse of $f \in \mathcal{S}^*_{3l,s}$ is of the form (1.2), then

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| \leq \frac{1}{25}.$$

This result is best possible.

5. Initial coefficients for $\mathcal{SK}_{\exp}$

Next, we begin this section by determining the estimates of the first four initial coefficients for functions in the family $f \in \mathcal{SK}_{\exp}$.

Theorem 5.1. If the function $f \in \mathcal{SK}_{\exp}$ has the series form (1.1), then

$$\begin{aligned} |d_2| &\leq \frac{1}{4}, \\ |d_3| &\leq \frac{1}{6}, \\ |d_4| &\leq \frac{1}{16}, \\ |d_5| &\leq \frac{1}{20}. \end{aligned}$$

The equality is attained by the following extremal functions:

$$\frac{2\upsilon f'(\upsilon)}{f(\upsilon) - f(-\upsilon)} = 1 + \upsilon^m + \frac{1}{2}\upsilon^{2m} + \cdots, \text{ for } m = 1,2,3,4. \tag{5.1}$$

Proof. Let $f \in \mathcal{SK}_{\exp}$. Then there exists a Schwarz function w such that

$$\frac{2\left(\upsilon f'\left(\upsilon\right)\right)'}{\left(f\left(\upsilon\right)-f\left(-\upsilon\right)\right)'}=e^{\omega(\upsilon)}, \qquad \upsilon \in \mathbb{D}. \tag{5.2}$$

Also, assuming that

$$\omega\left(\upsilon\right)=\sigma_1\upsilon+\sigma_2\upsilon^2+\sigma_3\upsilon^3+\sigma_4\upsilon^4+\cdots. \tag{5.3}$$

Using (1.1), we obtain

$$\frac{2\left(\upsilon f'\left(\upsilon\right)\right)'}{\left(f\left(\upsilon\right)-f\left(-\upsilon\right)\right)'}:=1+4d_2\upsilon+6d_3\upsilon^2+\left(-12d_2d_3+16d_4\right)\upsilon^3+\left(-18d_3^2+20d_5\right)\upsilon^4+\cdots. \tag{5.4}$$

By some easy calculation and utilizing the series expansion of (5.3), we achieve

$$\begin{aligned} e^{\omega(\upsilon)} &= 1+\sigma_1\upsilon+\left(\sigma_2+\frac{1}{2}\sigma_1^2\right)\upsilon^2+\left(\sigma_3+\sigma_1\sigma_2+\frac{1}{6}\sigma_1^3\right)\upsilon^3 \\ &\quad +\left(\sigma_4+\sigma_1\sigma_3+\frac{1}{2}\sigma_2^2+\frac{1}{24}\sigma_1^4+\frac{1}{2}\sigma_1^2\sigma_2\right)\upsilon^4+\cdots. \end{aligned} \tag{5.5}$$

Now by comparing (5.4) and (5.5) , we have

$$d_2 = \frac{1}{4}\sigma_1, \tag{5.6}$$

$$d_3 = \frac{1}{6}\sigma_2+\frac{1}{12}\sigma_1^2, \tag{5.7}$$

$$d_4 = \frac{1}{16}\sigma_3+\frac{5}{192}\sigma_1^3+\frac{3}{32}\sigma_1\sigma_2, \tag{5.8}$$

$$d_5 = \frac{1}{20}\sigma_2^2+\frac{1}{20}\sigma_4+\frac{1}{20}\sigma_1^2\sigma_2+\frac{1}{20}\sigma_1\sigma_3+\frac{1}{120}\sigma_1^4. \tag{5.9}$$

Using (2.2) in (5.6), we obtain

$$|d_2| \leq \frac{1}{4}.$$

Rearranging of (5.7), we obtain

$$|d_3| = \frac{1}{6}\left|\sigma_2+\frac{1}{2}\sigma_1^2\right|.$$

Applying (2.3) in the above equation, we achieve

$$|d_3| \leq \frac{1}{6}.$$

For d_4, we can write (5.8), as

$$|d_4| = \frac{1}{16}\left|\sigma_3+\frac{3}{2}\sigma_1\sigma_2+\frac{5}{12}\sigma_1^3\right|.$$

From Lemma 2.1, let

$$\varrho = \frac{3}{2} \qquad \text{and} \qquad \varsigma = \frac{5}{12}.$$

It is clear that $\frac{1}{2} \leq |\varrho| \leq 2$ with

$$\frac{4}{27}(1+|\varrho|)^3 - (1+|\varrho|) = -\frac{5}{27} \leq \varsigma \qquad \text{and} \qquad \varsigma \leq 1.$$

Hence the conditions of Lemma 2.1 are satisfied. Therefore

$$|d_4| \leq \frac{1}{16}.$$

From (5.9), we deduce that

$$|d_5| = \frac{1}{20}\left|\frac{1}{2}\left(2\sigma_1\sigma_3 + \sigma_4 + \sigma_2^2 + \sigma_1^4 + 3\sigma_1^2\sigma_2\right) + \frac{1}{2}\left(\sigma_4 + \sigma_2^2 - \frac{2}{3}\sigma_1^4 - \sigma_1^2\sigma_2\right)\right|. \tag{5.10}$$

The initial segment is estimated by $\frac{1}{2}$ by utilizing (2.7) with $\Lambda = 1$. Lemma 2.3 uses for the estimation of the second segment in the following:

$$\begin{aligned} \frac{1}{2}\left|-\frac{2}{3}\sigma_1^4 + \sigma_4 - \sigma_1^2\sigma_2 + \sigma_2^2\right| &\leq \frac{1}{2}\left[-|\sigma_2|^2 - |\sigma_1|^2 + 1 + \frac{2}{3}|\sigma_1|^4 + |\sigma_1|^2\left(1 - |\sigma_1|^2\right) + |\sigma_2|^2\right]. \\ &= -\frac{1}{6}\frac{|\sigma_1|^4}{} + \frac{1}{2} \leq \frac{1}{2}. \end{aligned}$$

By adding the bounds of the segments of (5.10) , we achieve

$$|d_5| \leq \frac{1}{20}.$$

Thus, the proof is completed. □

6. Inverse coefficients for $\mathcal{SK}_{\exp}$

Lastly, we will investigate the estimates of first three initial inverse coefficients for functions in the family $\mathcal{SK}_{\exp}$.

Theorem 6.1. If the inverse function of $f \in \mathcal{SK}_{\exp}$ is of the form (1.2), then

$$\begin{aligned} |B_2| &\leq \frac{1}{4}, \\ |B_3| &\leq \frac{1}{6}, \\ |B_4| &\leq \frac{1}{16}. \end{aligned}$$

Equalities hold in these bounds and will be confirmed by using (1.3)–(1.5) and (5.1) with $m = 1, 2, 3$.

Proof. Applying (5.6)–(5.9) in (1.3)–(1.6), we achieve

$$B_2 = -\frac{1}{4}\sigma_1, \tag{6.1}$$

$$
\begin{aligned}
B_3 &= \frac{1}{24}\sigma_1^2 - \frac{1}{6}\sigma_2, && (6.2)\\
B_4 &= \frac{11}{96}\sigma_1\sigma_2 - \frac{1}{16}\sigma_3, && (6.3)\\
B_5 &= -\frac{1}{320}\sigma_1^4 - \frac{43}{960}\sigma_1^2\sigma_2 + \frac{7}{160}\sigma_1\sigma_3 + \frac{1}{30}\sigma_2^2 - \frac{1}{20}\sigma_4. && (6.4)
\end{aligned}
$$

Using (2.2) in (6.1), we obtain

$$|B_2| \leq \frac{1}{4}.$$

For B_3, we can write (6.2), as

$$|B_3| = \frac{1}{6}\left|\sigma_2 + \left(-\frac{1}{4}\right)\sigma_1^2\right|.$$

Applying (2.3) in the above equation, we achieve

$$|B_3| \leq \frac{1}{6}.$$

For B_4, we consider

$$|B_4| = \frac{1}{16}\left|\sigma_3 + \left(-\frac{11}{6}\right)\sigma_1\sigma_2 + (0)\,\sigma_1^3\right|.$$

From Lemma 2.1, let

$$\varrho = -\frac{11}{6} \qquad \text{and} \qquad \varsigma = 0.$$

It is clear that $\frac{1}{2} \leq |\varrho| \leq 2$ with

$$-\frac{2}{3}\left(|\varrho| + 1\right) = -\frac{17}{9} \leq \varsigma \qquad \text{and} \qquad \varsigma \leq \frac{4}{27}\left(1 + |\varrho|\right)^3 - \left(1 + |\varrho|\right) = \frac{391}{729}.$$

This shows that all conditions of Lemma 2.1 are satisfied. Thus

$$|B_4| \leq \frac{1}{16}.$$

Thus, the required proof is completed. □

Theorem 6.2. If $f \in \mathcal{SK}_{\exp}$ has inverse function f^{-1} with a series form (1.2) , then

$$\left|B_3 - \tau B_2^2\right| \leq \max \frac{1}{6}\left\{1, \left|\frac{-2 + 3\tau}{8}\right|\right\}, \qquad \tau \in \mathbb{C}.$$

This inequality is sharp.

Proof. Employing (6.1) and (6.2), we have

$$
\begin{aligned}
\left|B_3 - \tau B_2^2\right| &= \frac{1}{6}\left|\sigma_2 - \frac{1}{4}\sigma_1^2 + \frac{3\tau}{8}\sigma_1^2\right|.\\
&= \frac{1}{6}\left|\sigma_2 + \left(\frac{3\tau - 2}{8}\right)\sigma_1^2\right|.
\end{aligned}
$$

Implementation of Lemma 2.2 along with triangle inequality leads us to

$$\left|B_3 - \tau B_2^2\right| \le \max \frac{1}{6}\left\{1, \left|\frac{-2+3\tau}{8}\right|\right\}.$$

The functional bound is sharp and will be obtained from (1.3), (1.4), and (5.1) with $m = 2$. □

By replacing $\tau = 1$ in Theorem 6.2, we arrive at the below result.

Corollary 6.3. If $f \in \mathcal{SK}_{\exp}$ has the inverse function with a series form (1.2), then

$$\left|B_3 - B_2^2\right| \le \frac{1}{6}.$$

The functional bound is sharp. Equality will be achieved by utilizing (1.3), (1.4), and (5.1) with $m = 2$.

Theorem 6.4. If the inverse of the function $f \in \mathcal{SK}_{\exp}$ is expressed in (1.2), then

$$|B_4 - B_2 B_3| \le \frac{1}{16}.$$

This outcome is sharp, and it will be confirmed easily by using (1.3)–(1.5), and (5.1) with $m = 3$.

Proof. From (6.1)–(6.3), we have

$$|B_4 - B_2 B_3| = \frac{1}{16}\left|\sigma_3 - \frac{7}{6}\sigma_1\sigma_2 - \frac{1}{6}\sigma_1^3\right|.$$

From Lemma 2.1, let

$$\varrho = -\frac{7}{6} \qquad \text{and} \qquad \varsigma = -\frac{1}{6}.$$

It is clear that $\frac{1}{2} \le |\varrho| \le 2$ with

$$\frac{4}{27}(1+|\varrho|)^3 - (1+|\varrho|) = -\frac{481}{729} \le \varsigma \qquad \text{and} \qquad \varsigma \le 1.$$

Thus, all the conditions of Lemma 2.1 are satisfied. Hence

$$|B_4 - B_2 B_3| \le \frac{1}{16}$$

and hence the proof is completed. □

Theorem 6.5. If the inverse function of $f \in \mathcal{SK}_{\exp}$ is provided in (1.2) , then

$$\left|B_4 - B_2^3\right| \le \frac{1}{16}.$$

Equality will be held by using (1.3), (1.5), and (5.1) with $m = 3$.

Proof. Putting (6.1) and (6.3), we have

$$\left|B_4 - B_2^3\right| = \frac{1}{16}\left|\sigma_3 + \left(-\frac{11}{6}\right)\sigma_1\sigma_2 + \left(-\frac{1}{4}\right)\sigma_1^3\right|.$$

From Lemma 2.1, let

$$\varrho = -\frac{11}{6} \qquad \text{and} \qquad \varsigma = -\frac{1}{4}.$$

It is clear that $\frac{1}{2} \leq |\varrho| \leq 2$ with

$$-\frac{2}{3}\left(|\varrho| + 1\right) = -\frac{17}{9} \leq \varsigma \qquad \text{and} \qquad \varsigma \leq \frac{4}{27}\left(1 + |\varrho|\right)^3 - \left(1 + |\varrho|\right) = \frac{391}{729}.$$

Thus, all the conditions of Lemma 2.1 are satisfied. Hence

$$\left|B_4 - B_2^3\right| \leq \frac{1}{16}.$$

□

Theorem 6.6. Let f^{-1} be the inverse of $f \in \mathcal{SK}_{\exp}$ as defined in (1.2). Then

$$\left|\hat{H}_{2,2}\left(f^{-1}\right)\right| = \left|B_2B_4 - B_3^2\right| \leq \frac{1}{36}.$$

Equality will be achieved by using (1.3)–(1.5) and (5.1) with $m = 2$.

Proof. From (6.1)–(6.3), we have

$$\begin{aligned}
\left|\hat{H}_{2,2}\left(f^{-1}\right)\right| &= \frac{1}{36}\left|\frac{1}{16}\sigma_1^4 + \frac{17}{32}\sigma_1^2\sigma_2 - \frac{9}{16}\sigma_1\sigma_3 + \sigma_2^2\right| \\
&= \frac{1}{36}\left|\frac{1}{2}\left(\sigma_2^2 - \sigma_1\sigma_3\right) + \frac{1}{2}\left(\frac{1}{8}\sigma_1^4 - \frac{1}{8}\sigma_1\sigma_3 + \frac{17}{16}\sigma_1^2\sigma_2 + \sigma_2^2\right)\right| \\
&\leq \frac{1}{72}\left|\sigma_2^2 - \sigma_1\sigma_3\right| + \frac{1}{72}\left|\frac{1}{8}\sigma_1^4 - \frac{1}{8}\sigma_1\sigma_3 + \frac{17}{16}\sigma_1^2\sigma_2 + \sigma_2^2\right| \\
&= \frac{1}{72}R_1 + \frac{1}{72}R_2,
\end{aligned}$$

where

$$R_1 = \left|\sigma_2^2 - \sigma_1\sigma_3\right|$$

and

$$R_2 = \left|\frac{1}{8}\sigma_1^4 - \frac{1}{8}\sigma_1\sigma_3 + \frac{17}{16}\sigma_1^2\sigma_2 + \sigma_2^2\right|.$$

Utilizing Lemma 2.4, we obtain $R_1 \leq 1$. Also, by virtue of Lemma 2.3 for R_2, we achieve

$$\begin{aligned}
|R_2| &\leq \frac{17}{16}|\sigma_1|^2|\sigma_2| + \frac{|\sigma_1|^4}{8} + |\sigma_2|^2 + \frac{|\sigma_1|}{8}\left(-\frac{|\sigma_2|^2}{(|\sigma_1| + 1)} - |\sigma_1|^2 + 1\right) \\
&\leq \frac{|\sigma_1|^4}{8} + \frac{17|\sigma_2||\sigma_1|^2}{16} + \left(-\frac{|\sigma_1|}{8(|\sigma_1| + 1)} + 1\right)|\sigma_2|^2 - \frac{|\sigma_1|^3}{8} + \frac{|\sigma_1|}{8}. \qquad (6.5)
\end{aligned}$$

Since $\left(-\frac{|\sigma_1|}{8(|\sigma_1|+1)}+1\right)>0$. Thus, we can substitute (2.4) in (6.5), and we easily obtain

$$
\begin{aligned}
|R_2| &\leq -\frac{|\sigma_1|^3}{8}+\frac{17}{16}|\sigma_1|^2\left(1-|\sigma_1|^2\right)+\left(-\frac{|\sigma_1|}{8\left(|\sigma_1|+1\right)}+1\right)\left(1-|\sigma_1|^2\right)^2 \\
&\quad +\frac{|\sigma_1|}{8}+\frac{|\sigma_1|^4}{8}.
\end{aligned}
$$

The basic computation of maximum and minimum leads us to

$$|R_2|\leq 1.$$

Hence

$$\left|\hat{H}_{2,2}\left(f^{-1}\right)\right|\leq\frac{1}{72}R_1+\frac{1}{72}R_2\leq\frac{1}{36}.$$

The proof is thus accomplished. □

Theorem 6.7. Let f^{-1} be the inverse function of $f\in\mathcal{SK}_{\exp}$ and is expressed in (1.2). Then

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right|<0.006671.$$

Proof. The determinant $\left|\hat{H}_{3,1}\left(f^{-1}\right)\right|$ can be expressed as follows:

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right|=\left|2B_2B_3B_4-B_4^2-B_2^2B_5-B_3^3+B_3B_5\right|.$$

From (6.1)–(6.4), we easily write

$$
\begin{aligned}
\left|\hat{H}_{3,1}\left(f^{-1}\right)\right| &= \frac{1}{256}\left|-\sigma_3^2+\left(\frac{7}{15}\sigma_2+\frac{1}{10}\sigma_1^2\right)\sigma_1\sigma_3-\frac{1}{540}\sigma_1^6-\frac{1}{60}\sigma_1^4\sigma_2-\frac{13}{180}\sigma_1^2\sigma_2^2\right. \\
&\quad \left.+\frac{4}{15}\sigma_1^2\sigma_4-\frac{32}{135}\sigma_2^3+\frac{32}{15}\sigma_2\sigma_4\right|.
\end{aligned}
$$

At the beginning, it should be noted that

$$\left|\sigma_3\left[\sigma_3+\left(-\frac{7}{15}\right)\sigma_1\sigma_2+\left(-\frac{1}{10}\right)\sigma_1^3\right]\right|\leq|\sigma_3|,$$

where we have used Lemma 2.1 with $\varrho=-\frac{7}{15}$ and $\varsigma=-\frac{1}{10}$. Also, by using Lemma 2.3, we have

$$|\sigma_3|\leq 1-\frac{|\sigma_2|^2}{1+|\sigma_1|}-|\sigma_1|^2\leq 1-\frac{|\sigma_2|^2}{2}-|\sigma_1|^2.$$

Applying it and also using $|\sigma_4|\leq 1-|\sigma_2|^2-|\sigma_1|^2$, we achieve

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right|\leq\frac{1}{256}E\left(|\sigma_1|,|\sigma_2|\right),$$

where

$$E(\sigma,t) = 1-\sigma^2-\frac{1}{2}t^2+\frac{1}{540}\sigma^6+\frac{1}{60}\sigma^4t+\frac{13}{180}\sigma^2t^2+\frac{4}{15}\sigma^2\left(1-\sigma^2-t^2\right)$$

$$+\frac{32}{135}t^3+\frac{32}{15}t\left(1-\sigma^2-t^2\right), \qquad \sigma=|\sigma_1|,\ t=|\sigma_2|.$$

But E is a decreasing function of the variable σ; consequently,

$$E(\sigma,t)\leq E(0,t)=1-\frac{1}{2}t^2-\frac{256}{135}t^3+\frac{32}{15}t.$$

The function $E(0,t)$ reaches its maximum value in $[0,1]$ if $t=-\frac{45}{512}+\frac{1}{512}\sqrt{100329}$, so $E(0,t)\leq 1.7079$, which completes the proof. □

Conjecture 6.8. If the inverse function of $f\in\mathcal{SK}_{\exp}$ is of the form (1.2), then the sharp bounds

$$\left|\hat{H}_{3,1}\left(f^{-1}\right)\right|\leq\frac{1}{256}.$$

Equality will be achieved by using (1.3)–(1.5) and (5.1) with $m=3$.

7. Conclusions

The study of Hankel determinant bounds is of great importance in the research community due to its vast applications in mathematical science. In the current article, we have considered the Hankel determinant involving the coefficients of inverse functions for various subclasses of analytic functions. This generalizes the classical definition of the Hankel determinant and could provide more knowledge of inverse functions. The main focus of this article that we have studied is the coefficient-related problems along with Hankel determinants for the inverse function of the functions that belong to the families of symmetric starlike and symmetric convex functions associated with three different image domains. In particular, these problems include the sharp estimates of some initial inverse coefficients, the Zalcman, Fekete–Szegö, and Krushkal inequalities, along with the sharp estimation of second and third Hankel determinants containing inverse coefficients for functions in the mentioned families by using the concept of a Schwarz function. Also, we have given some conjectures that strongly support our obtained results. Our research introduces a new framework for analyzing the Hankel determinant, emphasizing the importance of inverse coefficients in analytic functions, potentially promoting more attention to coefficient-related problems. This study may be applied to meromorphic analytic functions, and the same methodology can be used to examine higher-order Hankel determinants, as studied in articles [46–48].

Author contributions

Huo Tang: Funding acquisition, Methodology, Project administration; Muhammad Abbas: Investigation, Writing-original draft; Reem K. Alhefthi: Formal analysis, Supervision, Writing-review and editing; Muhammad Arif: Formal analysis, Supervision, Writing-review and editing. All authors read and approved the final manuscript.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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