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Research article

Existence of solutions for Hadamard fractional nonlocal boundary value problems with mean curvature operator at resonance

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Abstract: This paper aims to study the existence of solutions for Hadamard fractional nonlocal boundary value problems with mean curvature operator at resonance. Based on the coincidence degree theory, some new results are established. Moreover, an example is given to verify our main results.

Keywords: Hadamard fractional order differential equation; boundary value problem; coincidence degree theory
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1. Introduction

In this paper, we are concerned with the following nonlocal fractional integral boundary value problem with mean curvature operator.

$$\begin{cases} {}^{H}D_{1^+}^{\beta}\left(\frac{\varphi_p({}^{H}D_{1^+}^{\alpha}u(t))}{\sqrt{1+|\varphi_p({}^{H}D_{1^+}^{\alpha}u(t))|^2}}\right) = f(t,u(t)),\ t\in(1,T),\\ u(1) = {}^{H}D_{1^+}^{\alpha}u(1) = 0,\ {}^{H}D_{1^+}^{\alpha-1}u(T) = \sum_{i=1}^{m}\zeta_i\, {}^{R}I_{1^+}^{\gamma_i}({}^{H}D_{1^+}^{\alpha-1}u(\xi_i)), \end{cases} \tag{1.1}$$

where ${}^{H}D_{1^+}^{\beta}, {}^{H}D_{1^+}^{\alpha}$ are Hadamard fractional derivatives and ${}^{R}I_{1^+}^{\gamma_i}$ are Riemann-Liouville fractional integrals, $1 < \alpha < 2$, $0 < \beta < 1$, $2 < \alpha+\beta < 3$, $\gamma_i > 0$, $\zeta_i > 0$, $1 < \xi_i < T$ $i = 1.2.3, ..., m$, m are positive integers, $f : [1, T]\times\mathbb{R} \to \mathbb{R}$ is continuous, φ_p is a p-Laplacian operator that for $s \in \mathbb{R}$ and $s \neq 0$, $\varphi_p(s) = |s|^{p-2}s$, $\varphi_p(0) = 0$, and its inverse is φ_q, $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

With the development of the fundamental theory of fractional calculus, fractional order differential equations have been applied in many practical problems (see [1,2]). For more research on the qualitative theory of fractional differential equations, it can be found in [3,4] and references therein. The Hadamard fractional order differential equation that comes from mechanical problems is one of the more important types of fractional order differential equations. Based on the Dhage-type fixed point theorem, Ahmad and Ntouyas [5] investigated the existence of solutions for a nonlocal initial value problem of Hadamard fractional hybrid differential equations as follows:

$$\begin{cases} {}^{H}D_{1^+}^{\alpha}(\frac{u(t)}{f(t,u(t))}) = g(t,u(t)),\ t \in [1,T], \\ {}^{H}I_{1^+}^{1-\alpha}u(t)\mid_{t=1} = \eta, \end{cases} \tag{1.2}$$

where ${}^{H}D_{1^+}^{\alpha}$ is Hadamard fractional derivative, ${}^{H}I_{1^+}^{1-\alpha}$ is Hadamard fractional integral, $0 < \alpha \leq 1$, $f \in C([1,T]\times\mathbb{R}, \mathbb{R}\setminus\{0\})$ and $g \in C([1,T]\times\mathbb{R},\mathbb{R})$. Moreover, in [6], they also considered the existence and uniqueness of solutions to a Hadamard fractional integral boundary value problem via some fixed-point theorems. Recently, Ahmad and Ntouyas [7] and Ahmad, Alsaedi, Ntouyas, Tariboon [8] further discussed the existence and uniqueness of solutions to the Hilfer-Hadamard and Hadamard fractional nonlocal boundary value problems. Subsequently, Pei, Wang, and Sun [9] studied the existence of solutions for the Hadamard fractional integro-differential equation on infinite domain with a nonlocal boundary condition by the monotone iterative method. Based on Mawhin's continuation theorem, Zhang and Liu [10] considered a Hadamard fractional integral boundary value on an infinite interval at resonance. Meanwhile, there are many scholars studying boundary value problems of Hadamard fractional order differential equations by some different methods such as fixed point index (see [11]), fixed point theorem (see [12–14]), coincidence degree theory (see [15]). Furthermore, for more papers on the qualitative analysis of fractional order models, please refer to [16–18] and references therein. On the other hand, in recent years, many scholars have paid more attention to second-order integer differential equations with mean curvature operators from the perspective of qualitative theory, which originates from relativity theory (see [19–21]). Therefore, this topic is very meaningful. Recently, Alves and Torres Ledesma [22] obtained the existence of infinite many solutions to the prescribed mean curvature equation on the smooth bounded domain via Clark's theorem. Subsequently, Torres Ledesma [23] considered the multiplicity of solutions for the following prescribed mean curvature equations with local conditions by variational methods.

$$\begin{cases} -\mathrm{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = f(x,u), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ that $N \geq 1$ is a smooth bounded domain, $f \in C(\Omega\times\mathbb{R},\mathbb{R})$ and satisfies the local conditions with respect to u at the origin.

Motivated by the works mentioned above, by the coincidence degree theory, which is a classical method and can be used to deal with the boundary value problems at resonance, the existence of solutions for a nonlocal boundary value problem of Hadamard fractional order differential equations with mean curvature operator at resonance (1.1) has been studied. The innovations of our paper are presented in the following aspects: To begin with, the mean curvature operator is a nonlinear operator, which is more complex than the linear case and brings some difficulties in the estimation of the boundedness of solutions. Moreover, our main results provide a perspective for future research of fractional order differential equations with mean curvature operators.

2. Preliminaries

For the basic definitions and properties of Hadamard fractional integral and derivative, one can refer to [1,2].

Definition 2.1. *[1,2] The Hadamard-type fractional integral of order* $\alpha > 0$ *of a function* $u : [1, T] \to \mathbb{R}$*, is defined by*

$$ {}^{H}I^{\alpha}_{1^+}u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t (ln\frac{t}{s})^{\alpha-1} u(s)\frac{ds}{s}, $$

provided the integral exists, where $\Gamma(\cdot)$ *means the well-known Gamma function.*

Definition 2.2. *[1,2] The Hadamard-type fractional derivative of order* $\alpha > 0$ *of a function* $u : [1, T] \to \mathbb{R}$*, is defined by*

$$ {}^{H}D^{\alpha}_{1^+}u(t) = \frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n \int_1^t (ln\frac{t}{s})^{n-\alpha-1} u(s)\frac{ds}{s}, $$

provided the integral exists that can be operated by $(t\frac{d}{dt})^n$*, where* $n = [\alpha] + 1$.

Lemma 2.3. *[1] Setting* $\alpha > 0$*,* $n = [\alpha] + 1$*, the equation* ${}^{H}D^{\alpha}_{1^+}u(t) = 0$ *is valid if and only if*

$$ u(t) = \sum_{i=1}^{n} c_i (lnt)^{\alpha-i}, $$

where $c_i \in \mathbb{R},\ i = 1, 2, ..., n$.

Lemma 2.4. *[1] Letting* $\alpha > 0$*,* $1 \le \gamma \le +\infty$*, then for* $u \in L^{\gamma}(1, T)$

$$ {}^{H}D^{\alpha}_{1^+}\,{}^{H}I^{\alpha}_{1^+}u = u. $$

And if ${}^{H}D^{\alpha}_{1^+}u \in L^{\gamma}(1, T)$*, one has*

$$ {}^{H}I^{\alpha}_{1^+}\,{}^{H}D^{\alpha}_{1^+}u(t) = u(t) + \sum_{i=1}^{n} c_i (lnt)^{\alpha-i}, $$

where $c_i \in \mathbb{R}, i = 1, 2, ..., n, n - 1 < \alpha < n$.

Next, some basic knowledge with regard to coincidence degree theory will be presented, which can be founded in [24]. Let X and Y be real Banach spaces, and $L : \text{dom } L \subset X \to Y$ be a Fredholm operator with index zero, which yields that there exist two continuous linear projectors $P : X \to X$, $Q : Y \to Y$ such that $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$, $X = \text{Ker } L \oplus \text{Ker } P$, $Y = \text{Im } L \oplus \text{Im } Q$ and $L|_{\text{dom}L \,\cap\, \text{Ker } P} : \text{dom } L \cap \text{Ker } P \to \text{Im } L$ is invertible. Let K_P represent the inverse of $L|_{\text{dom } L \,\cap\, \text{Ker } P}$.

Definition 2.5. *[24] Let* $L : dom\ L \subset X \to Y$ *be a Fredholm operator with index zero. Assuming that* $U \subset X$ *is an open bounded subset and* $dom\ L \cap \overline{U} \neq \emptyset$*, if* $QN : \overline{U} \to Y$ *is bounded and* $K_{P,Q}N := K_P(I - Q)N : \overline{U} \to X$ *is compact, the map* $N : X \to Y$ *is called* L*−compact on* $\overline{U}$.

Lemma 2.6. *[24] Let L : dom $L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be $L-$compact on $\overline{U}$. Assume that the following conditions are satisfied*

(i) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in [(dom\ L \setminus Ker\ L) \cap \partial U] \times (0, 1)$,

(ii) $Nu \notin Im\ L$ for every $u \in Ker\ L \cap \partial U$,

(iii) $deg(JQN|_{Ker\ L}, Ker\ L \cap U, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $Im\ L = Ker\ Q$, $J : Im\ Q \to Ker\ L$ is a homeomorphism. Then the equation $Lu = Nu$ admits at least one solution in dom $L \cap \overline{U}$.

3. Main results

Let $Y = C[1,T]$ with the norm $\|u\|_\infty = \max_{t\in[1,T]} |u(t)|$, $X = C^\alpha[1,T]$. Throughout this article, assume that $\sum_{i=1}^{m} \frac{\zeta_i(\xi_i-1)^{\gamma_i}}{\Gamma(\gamma_i+1)} = 1$ and $\sum_{i=1}^{m} \frac{\zeta_i}{\Gamma(\gamma_i)} \int_1^{\xi_i} (\xi_i - s)^{\gamma_i-1} \ln s\, ds < \ln T$. Since ${}^H D^\alpha_{1^+} u(1) = 0$, the problem (1.1) is equivalent to the following problem.

$$\begin{cases} {}^H D^\alpha_{1^+} u(t) = \varphi_q(\sqrt{1 + |\varphi_p({}^H D^\alpha_{1^+} u(t))|^2}\, {}^H I^\beta_{1^+} f(t, u(t))),\ t \in (1, T), \\ u(1) = 0,\ {}^H D^{\alpha-1}_{1^+} u(T) = \sum_{i=1}^{m} \zeta_i {}^R I^{\gamma_i}_{1^+} ({}^H D^{\alpha-1}_{1^+} u(\xi_i)). \end{cases} \tag{3.1}$$

In fact, on one hand, if ${}^H D^\alpha_{1^+} u(1) = 0$, by Lemma 2.4 and the invertibility of φ_p, we have

$${}^H D^\alpha_{1^+} u(t) = \varphi_q(\sqrt{1 + |\varphi_p({}^H D^\alpha_{1^+} u(t))|^2}\, {}^H I^\beta_{1^+} f(t, u(t))).$$

On the other hand, letting $t = 1$, one has ${}^H I^\beta_{1^+} f(t, u(t))\ |_{t=1} = 0$. By the above equation, we have ${}^H D^\alpha_{1^+} u(1) = 0$.

Let the operator L : dom $L \subset X \to Y$ be defined by

$$Lu = {}^H D^\alpha_{1^+} u, \tag{3.2}$$

where

$$\text{dom } L = \{u \in X | {}^H D^\alpha_{1^+} u \in Y,\ u(1) = 0,\ {}^H D^{\alpha-1}_{1^+} u(T) = \sum_{i=1}^{m} \zeta_i {}^R I^{\gamma_i}_{1^+} ({}^H D^{\alpha-1}_{1^+} u(\xi_i))\}.$$

Let $N : X \to Y$ be given by

$$Nu = \varphi_q(\sqrt{1 + |\varphi_p({}^H D^\alpha_{1^+} u)|^2}\, {}^H I^\beta_{1^+} f(t, u)).$$

Therefore, the problem (3.1) can be convert to the following operator equation.

$$Lu = Nu, \quad u \in \text{dom } L.$$

Next, some important conclusions will be given, which play an important role in proving the main results.

Lemma 3.1. *Let L be given by (3.2). Then*

$$\operatorname{Ker} L = \left\{ u \in X \,\middle|\, u(t) = c(\ln t)^{\alpha-1},\ c \in \mathbb{R} \right\}, \tag{3.3}$$

$$\operatorname{Im} L = \left\{ y \in Y \,\middle|\, \sum_{i=1}^{m} \zeta_i {}^{R}I_{1^+}^{\gamma_i}({}^{H}I_{1^+}^{1} y(\xi_i)) - {}^{H}I_{1^+}^{1} y(T) = 0 \right\}. \tag{3.4}$$

Proof. Based on $\sum\limits_{i=1}^{m} \frac{\zeta_i(\xi_i-1)^{\gamma_i}}{\Gamma(\gamma_i+1)} = 1$, it is not difficult to obtain (3.3). If $y \in \operatorname{Im} L$, it follows that there exists a function $u \in \operatorname{dom} L$ such that $y(t) = {}^{H}D_{1^+}^{\alpha} u(t)$. By $u(1) = 0$, one has

$$u(t) = {}^{H}I_{1^+}^{\alpha} y(t) + c(\ln t)^{\alpha-1},$$
$${}^{H}D_{1^+}^{\alpha-1} u(t) = {}^{H}I_{1^+}^{1} y(t) + c\Gamma(\alpha),$$

which implies that

$$\sum_{i=1}^{m} \zeta_i {}^{R}I_{1^+}^{\gamma_i}({}^{H}D_{1^+}^{\alpha-1} u(\xi_i)) = \sum_{i=1}^{m} \zeta_i {}^{R}I_{1^+}^{\gamma_i}({}^{H}I_{1^+}^{1} y(\xi_i)) + \sum_{i=1}^{m} \frac{c\Gamma(\alpha)\zeta_i(\xi_i-1)^{\gamma_i}}{\Gamma(\gamma_i+1)},$$

where $c \in \mathbb{R}$. Based on the boundary condition

$${}^{H}D_{1^+}^{\alpha-1} u(T) = \sum_{i=1}^{m} \zeta_i {}^{R}I_{1^+}^{\gamma_i}({}^{H}D_{1^+}^{\alpha-1} u(\xi_i)),$$

we have

$$\sum_{i=1}^{m} \zeta_i {}^{R}I_{1^+}^{\gamma_i}({}^{H}I_{1^+}^{1} y(\xi_i)) + \sum_{i=1}^{m} \frac{c\Gamma(\alpha)\zeta_i(\xi_i-1)^{\gamma_i}}{\Gamma(\gamma_i+1)} = {}^{H}I_{1^+}^{1} y(T) + c\Gamma(\alpha).$$

From $\sum\limits_{i=1}^{m} \frac{\zeta_i(\xi_i-1)^{\gamma_i}}{\Gamma(\gamma_i+1)} = 1$, it follows that

$$\sum_{i=1}^{m} \zeta_i {}^{R}I_{1^+}^{\gamma_i}({}^{H}I_{1^+}^{1} y(\xi_i)) - {}^{H}I_{1^+}^{1} y(T) = 0. \tag{3.5}$$

On the other hand, assuming that $y \in Y$ satisfies (3.5) and letting $u(t) = {}^{H}I_{1^+}^{\alpha} y(t)$, we have $u \in \operatorname{dom} L$ and $Lu(t) = {}^{H}D_{1^+}^{\alpha} u(t) = y(t)$, which implies that $y \in \operatorname{Im} L$. Thus, (3.4) holds. □

Let $P : X \to X$ and $Q : Y \to Y$ be the linear continuous operators given by

$$Pu(t) = \frac{1}{\Gamma(\alpha)} {}^{H}D_{1^+}^{\alpha-1} u(1)(\ln t)^{\alpha-1},$$
$$Qy(t) = \Lambda\Big(\sum_{i=1}^{m} \frac{\zeta_i}{\Gamma(\gamma_i)} \int_1^{\xi_i} (\xi_i - s)^{\gamma_i-1} \int_1^{s} y(\tau)\frac{d\tau}{\tau} ds - \int_1^{T} y(s)\frac{ds}{s}\Big),$$

where

$$\Lambda = \frac{1}{\sum_{i=1}^{m} \frac{\zeta_i}{\Gamma(\gamma_i)} \int_1^{\xi_i} (\xi_i - s)^{\gamma_i - 1} \ln s \, ds - \ln T}.$$

It implies that

$$P^2 u(t) = \frac{1}{\Gamma(\alpha)} {}^H D_{1^+}^{\alpha-1} (Pu(t)) \mid_{t=1} (\ln t)^{\alpha-1} = Pu(t),$$

$$Q^2 y(t) = \Lambda (\sum_{i=1}^{m} \frac{\zeta_i}{\Gamma(\gamma_i)} \int_1^{\xi_i} (\xi_i - s)^{\gamma_i - 1} \int_1^s Qy(\tau) \frac{d\tau}{\tau} ds - \int_1^T Qy(s) \frac{ds}{s}) = Qy(t).$$

So, the operators P and Q are idempotent, which yields that they are projector operators. It is obvious that $X = \text{Ker } L \oplus \text{Ker } P$, $Y = \text{Im } L \oplus \text{Im } Q$. Since $\dim \text{Im } Q = 1$, we can obtain $\dim \text{Ker } L = \text{codim Im } L = 1$, and L is a Fredholm operator of index zero. Define $K_P : \text{Im } L \to \text{dom } L \cap \text{Ker } P$ by $K_P y = {}^H I_{1^+}^{\alpha} y$, which implies that its inverse is $L|_{\text{dom } L \cap \text{ Ker } P}$. Based on the continuity of f and the standard arguments, it follows that N is L-compact on $\overline{U}$.

Theorem 3.2. *Assume that the following conditions hold.*

(G_1) *There exists a positive function* $\psi \in X$ *such that* $|f(t,u)| \leq \psi(t)$ *for any* $(t,u) \in [1,T] \times \mathbb{R}$.

(G_2) *For any* $u \in$ *dom L, there exist constants* $A > 0, 1 < \kappa < T$ *such that if* $|u(t)| > A$ *for any* $t \in [\kappa, T]$, *either*

$$sgn\{u(t)\} QN(u(t)) > 0,$$

or

$$sgn\{u(t)\} QN(u(t)) < 0.$$

Then the problem (1.1) admits at least one solution, provided that

$$\frac{(\ln T)^{2\beta} \|\psi\|_\infty^2}{(\Gamma(\beta+1))^2} < 1. \tag{3.6}$$

Proof. Define

$$U_1 = \{u \in \text{dom } L \setminus \text{Ker } L \mid Lu = \lambda Nu, \ \lambda \in (0,1)\}.$$

If $u \in U_1$, it follows $Nu \in \text{Im } L$. In view of $\text{Im } L = \text{Ker } Q$, one has $QNu = 0$. Based on (G_2), we can find a constant $\eta \in [\kappa, T]$ such that $|u(\eta)| \leq A$. Since $u(1) = 0$, it implies

$$u(t) = {}^H I_{1^+}^{\alpha} {}^H D_{1^+}^{\alpha} u(t) + c(\ln t)^{\alpha-1},$$

where $c \in \mathbb{R}$, which leads to

$$|c| \quad \leq \quad \frac{1}{(\ln \eta)^{\alpha-1}} [|u(\eta)| + \frac{1}{\Gamma(\alpha)} \int_1^{\eta} (\ln \eta - \ln s)^{\alpha-1} |{}^H D_{1^+}^{\alpha} u(s)| \frac{ds}{s}]$$

$$\leq \quad \frac{1}{(\ln\eta)^{\alpha-1}}(A+\frac{(\ln\eta)^{\alpha}}{\Gamma(\alpha+1)}\|^{H}D_{1^+}^{\alpha}u\|_{\infty}),$$

and

$$\|u\|_{\infty} \quad \leq \quad \frac{2(\ln T)^{\alpha}}{\Gamma(\alpha+1)}\|^{H}D_{1^+}^{\alpha}u\|_{\infty}+A(\frac{\ln T}{\ln\kappa})^{\alpha-1}. \tag{3.7}$$

In view of $Lu=\lambda Nu$, one has

$$^{H}D_{1^+}^{\alpha}u(t)=\lambda\varphi_q(\sqrt{1+|\varphi_p(^{H}D_{1^+}^{\alpha}u(t))|^2}\,{}^{H}I_{1^+}^{\beta}f(t,u(t))),$$

which leads to

$$|\varphi_p(^{H}D_{1^+}^{\alpha}u(t))|^2=\lambda^{2p-2}[(1+|\varphi_p(^{H}D_{1^+}^{\alpha}u(t))|^2)(^{H}I_{1^+}^{\beta}f(t,u(t)))^2]. \tag{3.8}$$

Since

$$|^{H}I_{1^+}^{\beta}f(t,u(t))|\leq\frac{(\ln T)^{\beta}\|\psi\|_{\infty}}{\Gamma(\beta+1)},$$

and $|\varphi_p(^{H}D_{1^+}^{\alpha}u(t))|^2=|^{H}D_{1^+}^{\alpha}u(t)|^{2p-2}$, from (3.8), it follows

$$|^{H}D_{1^+}^{\alpha}u(t)|^{2p-2}\leq(1+|^{H}D_{1^+}^{\alpha}u(t)|^{2p-2})\frac{(\ln T)^{2\beta}\|\psi\|_{\infty}^2}{(\Gamma(\beta+1))^2}. \tag{3.9}$$

Since $\frac{(\ln T)^{2\beta}\|\psi\|_{\infty}^2}{(\Gamma(\beta+1))^2}<1$, there exists a constant $r_1>0$ such that $\|^{H}D_{1^+}^{\alpha}u\|_{\infty}<r_1$. From (3.7), we have

$$\|u\|_{\infty} \quad \leq \quad \frac{2(\ln T)^{\alpha}r_1}{\Gamma(\alpha+1)}+A(\frac{\ln T}{\ln\kappa})^{\alpha-1}, \tag{3.10}$$

which means that U_1 is bounded.

Define

$$U_2=\{u|u\in\operatorname{Ker}L,\ Nu\in\operatorname{Im}L\}.$$

If $u\in U_2$, it follows $u(t)=c(\ln t)^{\alpha-1},\ c\in\mathbb{R}$ and $Nu\in\operatorname{Im}L$, which imply $QN[c(\ln t)^{\alpha-1}]=0$. By (G_2), we can find that $|c|\leq\frac{A}{(\ln\kappa)^{\alpha-1}}$. which means that U_2 is bounded.

Define $J^{-1}:\operatorname{Ker}L\to\operatorname{Im}Q$ by $J^{-1}(c(\ln t)^{\alpha-1})=c,\ c\in\mathbb{R}\,,t\in[1,T]$. Set

$$U_3=\{u\in\operatorname{Ker}L|\lambda J^{-1}u+(1-\lambda)QNu=0,\ \lambda\in[0,1]\}.$$

It follows

$$\lambda c+(1-\lambda)\Lambda[\sum_{i=1}^{m}\frac{\zeta_i}{\Gamma(\gamma_i)}\int_1^{\xi_i}(\xi_i-s)^{\gamma_i-1}\int_1^{s}\varphi_q(^{H}I_{1^+}^{\beta}f(\tau,c(\ln\tau)^{\alpha-1}))\frac{d\tau}{\tau}ds$$
$$-\int_1^{T}\varphi_q(^{H}I_{1^+}^{\beta}f(s,c(\ln s)^{\alpha-1}))\frac{ds}{s}]=0. \tag{3.11}$$

From the first assumption of (G_2), if $\lambda = 0$, one has $|c| \le \frac{A}{(\ln\kappa)^{\alpha-1}}$. Let $\lambda \in (0,1]$, it follows $|c| \le \frac{A}{(\ln\kappa)^{\alpha-1}}$. Otherwise, if $|c| > \frac{A}{(\ln\kappa)^{\alpha-1}}$, based on the first assumption of (G_2), we can obtain

$$\begin{aligned}&\lambda \mathrm{sgn}[c(\ln t)^{\alpha-1}]c\\ &+(1-\lambda)\Lambda \mathrm{sgn}[c(\ln t)^{\alpha-1}][\sum_{i=1}^{m}\frac{\zeta_i}{\Gamma(\gamma_i)}\int_1^{\xi_i}(\xi_i - s)^{\gamma_i-1}\int_1^{s}\varphi_q({}^{H}I^{\beta}_{1^+}f(\tau, c(\ln\tau)^{\alpha-1}))\frac{d\tau}{\tau}ds\\ &-\int_1^{T}\varphi_q({}^{H}I^{\beta}_{1^+}f(s, c(\ln s)^{\alpha-1}))\frac{ds}{s}] > 0, \end{aligned}\tag{3.12}$$

for any $t \in [\kappa, T]$. By choosing $t = T$, it is in direct contradiction to (3.11). So, U_3 is bounded.

Define

$$U_3' = \{u \in \mathrm{Ker}\, L| - \lambda J^{-1}u + (1-\lambda)QNu = 0,\ \lambda \in [0,1]\}.$$

By the same way, from the second assumption of (G_2), U_3' is bounded.

Define $U = \{u \in X| \|u\|_\infty < \frac{2(\ln T)^{\alpha} r_1}{\Gamma(\alpha+1)} + A(\frac{\ln T}{\ln\kappa})^{\alpha-1} + 1\}$, which implies that the conditions (i) and (ii) of Lemma 2.6 is satisfied. Let

$$F(u,\lambda) = \pm\lambda J^{-1}(u) + (1-\lambda)QNu.$$

It follows $F(U,\lambda) \ne 0$ for $U \in \mathrm{Ker}\, L \cap \partial U$. So, we have

$$\begin{aligned}\deg(JQN|_{\mathrm{Ker}\, L}, U\cap \mathrm{Ker}\, L, 0) &= \deg(F(\cdot,0), U\cap \mathrm{Ker}\, L, 0)\\ &= \deg(F(\cdot,1), U\cap \mathrm{Ker}\, L, 0)\\ &= \deg(\pm I, U\cap \mathrm{Ker}\, L, 0) \ne 0,\end{aligned}$$

which tell us that the condition (iii) of Lemma 2.6 is fulfilled. Hence, we know that $Lu = Nu$ admits at least one solution in dom $L \cap \overline{U}$. Then the problem (1.1) has at least one solution. □

Example. *The following example was given to verify our main results.*

$$\begin{cases} {}^{H}D^{\frac34}_{1^+}\left(\frac{{}^{H}D^{\frac32}_{1^+}u(t)}{\sqrt{1+|{}^{H}D^{\frac32}_{1^+}u(t)|^2}}\right) = \frac12\ln t + \frac{1}{24}\sin(u(t)) + \frac{1}{24},\ t\in(1,e),\\ u(1) = {}^{H}D^{\frac32}_{1^+}u(1) = 0,\ {}^{H}D^{\frac12}_{1^+}u(e) = {}^{R}I^{1}_{1^+}({}^{H}D^{\frac12}_{1^+}u(2)),\end{cases}\tag{3.13}$$

where $\alpha = \frac32$, $\beta = \frac34$, $T = e$, $m = 1$, $p = q = 2$, $\gamma_1 = 1$, $\zeta_1 = 1$, $\xi_1 = 2$, $f(t,u(t)) = \frac12\ln t + \frac{1}{24}\sin(u(t)) + \frac{1}{24}$. *It follows that*

$$\sum_{i=1}^{m}\frac{\zeta_i(\xi_i-1)^{\gamma_i}}{\Gamma(\gamma_i+1)} = \frac{1}{\Gamma(2)} = 1,$$

and

$$\sum_{i=1}^{m}\frac{\zeta_i}{\Gamma(\gamma_i)}\int_1^{\xi_i}(\xi_i - s)^{\gamma_i-1} lns\, ds = \int_1^2 lns\, ds = 2ln2 - 1 < 1.$$

Let $\psi(t) = \frac23\ln t + \frac{1}{12}$, $t\in[1,e]$, which implies that the (G_1) of Theorem 3.2 is satisfied and

$$\frac{(\ln T)^{2\beta}\|\psi\|_\infty^2}{(\Gamma(\beta+1))^2} = \frac{\frac{9}{16}}{(\Gamma(\frac74))^2} \approx 0.67 < 1.$$

Set $\kappa = \frac{3}{2}$. For $t \in [\kappa, e]$, we have

$$\begin{aligned} QN(u(t)) &= \frac{1}{2\ln 2 - 2}\Big(\int_1^2 \int_1^s \sqrt{1 + |^H D_{1^+}^{\frac{3}{2}} u(\tau)|^2}\, {}^H I_{1^+}^{\frac{3}{4}} \big(\frac{1}{2}\ln\tau + \frac{1}{24}\sin(u(\tau)) + \frac{1}{24}\big)\frac{d\tau}{\tau} ds \\ &\quad - \int_1^T \sqrt{1 + |^H D_{1^+}^{\frac{3}{2}} u(s)|^2}\, {}^H I_{1^+}^{\frac{3}{4}} \big(\frac{1}{2}\ln s + \frac{1}{24}\sin(u(s)) + \frac{1}{24}\big)\frac{ds}{s}\Big) \\ &= \frac{1}{2\ln 2 - 2}\Big(\int_1^2 \int_1^s \sqrt{1 + |^H D_{1^+}^{\frac{3}{2}} u(\tau)|^2}\, {}^H I_{1^+}^{\frac{3}{4}} \big(\frac{1}{2}\ln\tau + \frac{1}{24}\sin(u(\tau)) + \frac{1}{24}\big)\frac{d\tau}{\tau} ds \\ &\quad - \int_1^2 \int_1^e \sqrt{1 + |^H D_{1^+}^{\frac{3}{2}} u(\tau)|^2}\, {}^H I_{1^+}^{\frac{3}{4}} \big(\frac{1}{2}\ln\tau + \frac{1}{24}\sin(u(\tau)) + \frac{1}{24}\big)\frac{d\tau}{\tau} ds\Big) > 0. \end{aligned}$$

Letting $A = 2$, if $u(t) > 2$, the first part of (G_2) in Theorem 3.2 is satisfied. Moreover, if $u(t) < -2$, the second part of (G_2) in Theorem 3.2 is fulfilled. Thus, from Theorem 3.2, the problem (3.13) admits at least one solution.

4. Conclusions

In this paper, we are concerned with the existence of solutions for Hadamard fractional nonlocal boundary value problems with mean curvature operator at resonance via the coincidence degree theory. By constructing the continuous linear projectors and performing spatial decomposition, we obtained some new results. Formally, we have extended the form of the integer order equation in [22,23] to the Hadamard fractional order case. Methodologically, we use the coincidence degree theory to study the existence of solutions to nonlocal boundary problems of Hadamard fractional order mean curvature equations, while [22,23] use the variational method to study the existence of solutions to Dirichlet boundary value problems of integer order mean curvature equations. Moreover, our results may provide a perspective for future research of fractional order differential equations with mean curvature operators. Furthermore, in the future, we will attempt to investigate the multiplicity and stability of solutions to such fractional boundary value problems with mean curvature operators.

Author contributions

Tengfei Shen: Conceptualization, investigation, methodology, writing-review and editing; Jiangen Liu: Conceptualization, investigation, methodology, writing-review and editing; Xiaohui Shen: Conceptualization, investigation, methodology, writing-review and editing. All authors of this article have been contributed equally. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that there is no conflict of interest.

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