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Research article

A new approach for fixed point theorems for C-class functions in Hilbert C^*-modules

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Abstract: In this paper, we introduced a new contraction principle via altering distance and C-class functions with rational forms which extends and generalizes the existing version provided by Hasan Ranjbar et al. [H. Ranjbar, A. Niknam, A fixed point theorem in Hilbert C^*-modules, *Korean J. Math.*, **30** (2022), 297–304]. Specifically, the rational forms involved in the contraction condition we presented involve the p-th power of the displacements which can exceed the second power mentioned in Hasan Ranjbar et al.'s paper. Moreover, we also proved a fixed point theorem for this type of contraction in the Hilbert C^*-module. Some adequate examples were provided to support our results. As an application, we applied our result to prove the existence of a unique solution to an integral equation and a second-order (p, q)-difference equation with integral boundary value conditions.

Keywords: fixed point; Hilbert C^*-modules; C-class function
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1. Introduction

Fixed point theory, which emerged in the 20th century, has significantly contributed to nonlinear analysis across various disciplines. A key result of this theory is the Banach contraction principle (BPC), asserting the existence of a unique fixed point for self-mappings exhibiting contraction properties in complete metric spaces. The BPC has been extended in different directions, particularly focusing on contraction conditions and the characteristics of the underlying space, see [1,2]. In 1965, Browder demonstrated a fixed point theorem in Hilbert spaces in [3], asserting that every Hilbert space exhibits the fixed point property (FPP). This property is characterized by the presence of a fixed point for every nonexpansive self-mapping defined on a nonempty, closed, convex, bounded subset within the Hilbert space. In the same year, Browder [4] and Göhde [5] researched fixed point theorems in convex Banach spaces, while Kirk [6] established similar results in reflexive Banach spaces. Additionally, a study by Ma [7] delved into operator-valued metric spaces, presenting fixed point theorems. Following this, Ma et al. [8] introduced the concept of C^*-algebra-valued metric spaces, which extend the notions of metric and operator-valued metric spaces. They demonstrated various fixed point theorems for self-maps under contractive or expansive conditions in these spaces, emphasizing the use of positive elements from a unital C^*-algebra instead of real numbers. Ma and Jiang (2017) [9] introduced a novel class of metric spaces known as C^*-algebra-valued b-metric spaces, extending the concepts of b-metric spaces and operator-valued metric spaces. They established fundamental fixed point theorems for self-mappings satisfying contractive conditions in this new space. Building upon this work, Batul and Kamran (2018) [10] defined continuity in C^*-algebra-valued metric spaces and demonstrated the continuity of C^*-valued contraction mappings based on this notion. They also relaxed the contractive conditions for C^*-valued contraction mappings proposed by Ma et al. [8], proposing a fixed point theorem for such mappings. It is worth noting that the initial Banach spaces with the fixed point property (FPP) were the Hilbert spaces. Conversely, Hilbert C-modules, serving as a valuable tool in operator and operator algebra theory, generalize the concept of Hilbert spaces [3]. Recently, Ranjbar and Niknam (2020) [11] achieved fixed point results for continuous self-mappings in Hilbert C^*-modules under rational conditions.

In 2014 the concept of C-class functions (see Definition 1) was introduced by A. H. Ansari in [12] stated in the following.

Definition 1.1. *[12]* A mapping $G : [0, +\infty)^2 \to \mathbb{R}$ is called a *C-class* function if it is continuous and satisfies the following axioms:

(1) $G(s, t) \leq s$;

(2) $G(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, +\infty)$.

Note that, for some G, we have that $G(0, 0) = 0$. We denote the set of C-class functions by $\mathcal{C}$.

Example 1.2. *[13]* The following functions $G : [0, +\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, +\infty)$:

(1) $G(s, t) = s - t$;

(2) $G(s, t) = ms$ for some $m \in (0, 1)$;

(3) $G(s, t) = \frac{s}{(1+t)^r}$ for some $r \in (0, +\infty)$;

(4) $G(s, t) = \lg(t + a^s)/(1 + t)$ for some $a > 1$;

(5) $G(s, t) = \ln(1 + a^s)/2$ for $1 < a < e$;

(6) $G(s, t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1$ for $r \in (0, +\infty)$;

(7) $G(s, t) = s \log_{t+a} a$, $a > 1$. Indeed, $F(s, t) = s \Rightarrow s = 0$ or $t = 0$;

(8) $G(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$;

(9) $G(s, t) = s\beta(s)$, where $\beta : [0, +\infty) \to (0, 1)$ is a continuous function;

(10) $G(s, t) = s - \frac{t}{k+t}$;

(11) $G(s, t) = s - \varphi(s)$ where $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow$

$t = 0$;

(12) $G(s,t) = sh(s,t)$ where $h : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$;

(13) $G(s,t) = s - (\frac{2+t}{1+t})t$;

(14) $G(s,t) = \sqrt[n]{\ln(1+s^n)}$, $F(s,t) = s \Rightarrow s = 0$;

(15) $G(s,t) = \phi(s)$ where $\phi : [0,+\infty) \to [0,+\infty)$ is an upper semi-continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$;

(16) $G(s,t) = \frac{s}{(1+s)^r}$ for some $r \in (0,+\infty)$. For more examples, see [14, 15].

Definition 1.3. *[16]* A function $\psi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is nondecreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote the set of altering distance functions by Ψ.

Definition 1.4. *[12]* Let Φ_u denote the class of the functions $\varphi : [0,+\infty) \to [0,+\infty)$ which satisfy the following conditions:

(i) φ is continuous;

(ii) $\varphi(t) > 0$, $t > 0$, and $\varphi(0) \geq 0$.

Definition 1.5. *[12]* An ultra-altering distance function is a continuous, nondecreasing mapping $\varphi : [0,+\infty) \to [0,+\infty)$ such that $\varphi(t) > 0$, $t > 0$, and $\varphi(0) \geq 0$.

Definition 1.6. *[12]* A tripled (ψ, φ, G) where $\psi \in \Psi$, $\varphi \in \Phi_u$, and $G \in C$ is said to be monotone if for any $x, y \in [0,+\infty)$

$$x \leqslant y \text{ implies } G(\psi(x), \varphi(x)) \leqslant G(\psi(y), \varphi(y)).$$

Example 1.7. *[12]* Let $G(s,t) = s - t, \varphi(x) = \sqrt{x}$,

$$\psi(x) = \begin{cases} \sqrt{x}, & \text{if} \quad 0 \leq x \leq 1, \\ x^2, & \text{if} \quad x > 1, \end{cases}$$

then (ψ, φ, G) is monotone.

Example 1.8. *[12]* Let $G(s,t) = s - t, \varphi(x) = x^2$,

$$\psi(x) = \begin{cases} \sqrt{x}, & \text{if} \quad 0 \leq x \leq 1, \\ x^2, & \text{if} \quad x > 1, \end{cases}$$

then (ψ, φ, G) is not monotone.

Let A be a C^*-algebra. A (right) inner-product A-module is a linear space E, which is a right A-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for all $x \in E$, $a \in A$, $\lambda \in \mathbb{C}$ together with an inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ satisfying the following conditions:

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(ii) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$;

(iii) $\langle x, ya \rangle = \langle x, y \rangle a$;

(iv) $\langle x, y \rangle^* = \langle y, x \rangle$, for all $x, y, z \in E, a \in A$, and $\lambda \in \mathbb{C}$.

A Hilbert A-module (Hilbert C^*-module) is an inner product A-module E which is complete under the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

Lemma 1.9. *[17]* Let E be a Hilbert C^*-module and $\{x_n\}$ be a sequence in E such that $\|x_n - x_{n+1}\| \to 0$ as $n \to +\infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

(i) $\lim_{k\to+\infty} \left\|x_{m(k)-1} - x_{n(k)+1}\right\| = \varepsilon$;
(ii) $\lim_{k\to+\infty} \left\|x_{m(k)} - x_{n(k)}\right\| = \varepsilon$;
(iii) $\lim_{k\to+\infty} \left\|x_{m(k)-1} - x_{n(k)}\right\| = \varepsilon$;
(iv) $\lim_{k\to+\infty} \left\|x_{m(k)+1} - x_{n(k)+1}\right\| = \varepsilon$;
(v) $\lim_{k\to+\infty} \left\|x_{m(k)} - x_{n(k)-1}\right\| = \varepsilon$.

Example 1.10. *[18]* Let E be a Hilbert C^*-module. Let $\{x_n\}$ be a sequence in E such that $\|x_n - x_{n+1}\| \to 0$ as $n \to +\infty$. If $\{x_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences tend to ε^+ when $k \to +\infty$:$\left\|x_{2n_k} - x_{2m_k}\right\|$, $\left\|x_{2n_k} - x_{2m_k+1}\right\|$, $\left\|x_{2n_k-1} - x_{2m_k}\right\|$, $\left\|x_{2n_k-1} - x_{2m_k+1}\right\|$.

Motivated by the work going on in this direction, in this paper, we aim to introduce a contraction principle via altering distance and C-class functions and prove a fixed point theorem for this type of contraction in the Hilbert C^*-module.

2. Fixed point results

Now, we will present our main theorem in the following.

Theorem 2.1. Let S be a nonempty closed subset of a Hilbert C^*-module E. Let T be a self-mapping on S such that

$$\psi(\|Tx - Ty\|) \leq G(\psi(M(x, y)), \varphi(M(x, y))), \tag{2.1}$$

where

$$\begin{aligned} M(x, y) = {} & a\left(\frac{\|Tx - x\|^p + \|Ty - y\|^p + \|Ty - x\|^p + \|Tx - y\|^p}{(\|Tx - x\| + \|Ty - y\| + \|Ty - x\| + \|Tx - y\|)^{p-1}}\right) \\ & + b\left(\frac{\|Tx - x\|^p + \|Ty - y\|^p + \|y - x\|^p}{(\|Tx - x\| + \|Ty - y\| + \|y - x\|)^{p-1}}\right) \\ & + c\left(\frac{\|Ty - x\|^p + \|Tx - y\|^p}{(\|Ty - x\| + \|Tx - y\|)^{p-1}}\right) + d\,\|x - y\|, \end{aligned}$$

for all $x, y \in S$ and $a, b, c, d \geq 0, p \in N, p \geq 2$ with $4a + 3b + 2c + d = 1$ and $\psi \in \Psi, \varphi \in \Phi_u$, $G \in C$. Assume that the tripled (ψ, φ, G) is monotone. Then T has a unique fixed point.

Proof. Let x_0 be any arbitrary point in S. We define a sequence $x_{n+1} = Tx_n$, for all $n \geq 1$. From (2.1), we have

$$\begin{aligned} \psi(\|x_{n+1} - x_n\|) &= \psi(\|Tx_n - Tx_{n-1}\|) \\ &\leq G(\psi(M(x_n, x_{n-1})), \varphi(M(x_n, x_{n-1}))) \\ &\leq \psi(M(x_n, x_{n-1})), \end{aligned} \tag{2.2}$$

where

$$M(x_n, x_{n-1}) \tag{2.3}$$

$$
\begin{aligned}
&= a\left(\frac{\|x_n - Tx_n\|^p + \|x_{n-1} - Tx_{n-1}\|^p + \|x_n - Tx_{n-1}\|^p + \|x_{n-1} - Tx_n\|^p}{(\|x_n - Tx_n\| + \|x_{n-1} - Tx_{n-1}\| + \|x_n - Tx_{n-1}\| + \|x_{n-1} - Tx_n\|)^{p-1}}\right)\\
&\quad + b\left(\frac{\|x_n - Tx_n\|^p + \|x_{n-1} - Tx_{n-1}\|^p + \|x_{n-1} - x_n\|^p}{(\|x_n - Tx_n\| + \|x_{n-1} - Tx_{n-1}\| + \|x_{n-1} - x_n\|)^{p-1}}\right)\\
&\quad + c\left(\frac{\|x_n - Tx_{n-1}\|^p + \|x_{n-1} - Tx_n\|^p}{(\|x_n - Tx_{n-1}\| + \|x_{n-1} - Tx_n\|)^{p-1}}\right) + d\,\|x_{n-1} - x_n\|)\\
&= a\left(\frac{\|x_n - x_{n+1}\|^p + \|x_{n-1} - x_n\|^p + \|x_n - x_n\|^p + \|x_{n-1} - x_{n+1}\|^p}{(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_n - x_n\| + \|x_{n-1} - x_{n+1}\|)^{p-1}}\right)\\
&\quad + b\left(\frac{\|x_n - x_{n+1}\|^p + \|x_{n-1} - x_n\|^p + \|x_{n-1} - x_n\|^p}{(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_{n-1} - x_n\|)^{p-1}}\right)\\
&\quad + c\left(\frac{\|x_n - x_n\|^p + \|x_{n-1} - x_{n+1}\|^p}{(\|x_n - x_n\| + \|x_{n-1} - x_{n+1}\|)^{p-1}}\right) + d\,\|x_{n-1} - x_n\|)\\
&\le a\,(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_{n-1} - x_{n+1}\|)\\
&\quad + b\,(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_{n-1} - x_n\|) + c\,\|x_{n-1} - x_{n+1}\|\\
&+ d\,\|x_{n-1} - x_n\|\\
&\le a\,(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|)\\
&+ b\,(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_{n-1} - x_n\|)\\
&+ c\,(\|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|) + d\,\|x_{n-1} - x_n\|\,. \qquad (2.4)
\end{aligned}
$$

Owing to the monotonicity assumption on ψ, we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\le M\,(x_n, x_{n-1})\\
&\le a\,(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|)\\
&+ b\,(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| + \|x_{n-1} - x_n\|)\\
&+ c\,(\|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|) + d\,\|x_{n-1} - x_n\|\,,
\end{aligned}
$$

and

$$(1 - 2a - b - c)\,\|x_{n+1} - x_n\| \le (2a + 2b + c + d)\,\|x_{n-1} - x_n\|\,.$$

So, we have

$$\|x_{n+1} - x_n\| \le \|x_{n-1} - x_n\|\,. \qquad (2.5)$$

Inequality (2.5) implies that $\{\|x_{n+1} - x_n\|\}$ is a monotone decreasing sequence. Consequently, there exists $r \ge 0$ such that

$$\|x_{n+1} - x_n\| \to r \quad \text{as} \quad n \to +\infty.$$

Letting $n \to +\infty$ in (2.2) and (2.4), we obtain that

$$
\begin{aligned}
\psi\,(r) \quad &\le G(\psi\,((4a + 3b + 2c + d)r)\,, \varphi\,(r(4a + 3b + 2c + d)))\\
&= G(\psi\,(r)\,, \varphi\,(r)).
\end{aligned}
$$

From Definition 1.1, we get $r = 0$. Hence,

$$\lim_{n\to+\infty} \|x_{n+1} - x_n\| = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence, then, by Lemma 1.9, there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$\begin{aligned}\lim_{k\to+\infty}\left\|x_{m(k)-1}-x_{n(k)+1}\right\| &= \lim_{k\to+\infty}\left\|x_{m(k)}-x_{n(k)}\right\| \\ &= \lim_{k\to+\infty}\left\|x_{m(k)-1}-x_{n(k)}\right\| \\ &= \epsilon. \end{aligned}\tag{2.6}$$

Now, consider the (2.6) and (2.1) with letting $k \to +\infty$, and we have

$$\begin{aligned}&M\left(x_{m(k)}, x_{n(k)}\right) \\ &= a\left(\frac{\left\|x_{m(k)}-Tx_{m(k)}\right\|^p+\left\|x_{n(k)}-Tx_{n(k)}\right\|^p+\left\|x_{m(k)}-Tx_{n(k)}\right\|^p+\left\|x_{n(k)}-Tx_{m(k)}\right\|^p}{(\left\|x_{m(k)}-Tx_{m(k)}\right\|+\left\|x_{n(k)}-Tx_{n(k)}\right\|+\left\|x_{m(k)}-Tx_{n(k)}\right\|+\left\|x_{n(k)}-Tx_{m(k)}\right\|)^{p-1}}\right) \\ &\quad + b\left(\frac{\left\|x_{m(k)}-Tx_{m(k)}\right\|^p+\left\|x_{n(k)}-Tx_{n(k)}\right\|^p+\left\|x_{n(k)}-x_{m(k)}\right\|^p}{(\left\|x_{m(k)}-Tx_{m(k)}\right\|+\left\|x_{n(k)}-Tx_{n(k)}\right\|+\left\|x_{n(k)}-x_{m(k)}\right\|)^{p-1}}\right) \\ &\quad + c\left(\frac{\left\|x_{m(k)}-Tx_{n(k)}\right\|^p+\left\|x_{n(k)}-Tx_{m(k)}\right\|^p}{(\left\|x_{m(k)}-Tx_{n(k)}\right\|+\left\|x_{n(k)}-Tx_{m(k)}\right\|)^{p-1}}\right)+d(\left\|x_{n(k)}-x_{m(k)}\right\|) \\ &\to a\left(\frac{2\varepsilon^p}{(2\varepsilon)^{p-1}}\right)+b\left(\varepsilon\right)+c\left(\frac{2\varepsilon^p}{(2\varepsilon)^{p-1}}\right)+d\varepsilon \\ &= a\left(\frac{\varepsilon}{2^{p-2}}\right)+b\left(\varepsilon\right)+c\left(\frac{\varepsilon}{2^{p-2}}\right)+d\varepsilon \le (a+b+c+d)\varepsilon.\end{aligned}$$

So

$$\begin{aligned}\psi\left(\epsilon\right) &\le G(\psi\left((a+b+c+d)\varepsilon\right), \varphi\left((a+b+c+d)\varepsilon\right)) \\ &\le \psi\left((a+b+c+d)\varepsilon\right) \\ &\le \epsilon.\end{aligned}\tag{2.7}$$

From Definition 1.1, we get $\varepsilon = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since S is a closed subset of Hilbert C^*-module E, the sequence $\{x_n\}$ converges to a point $z \in X$.

Now we show that z is a fixed point of T. We assume on the contrary that $Tz \neq z$. It follows from (2.1) that

$$\begin{aligned}\psi(|\|Tx_n - z\| - \|z - Tz\||) &\le \psi(\|Tx_n - Tz\|) \\ &\le G(\psi\left(M(x_n, z)\right), \varphi\left(M(x_n, z)\right)) \\ &\le \psi(M(x_n, z)),\end{aligned}$$

where with taking the limit as $n \to +\infty$,

$$\begin{aligned}M\left(x_n, z\right) &= a\left(\frac{\|x_n - Tx_n\|^p + \|z - Tz\|^p + \|x_n - Tz\|^p + \|z - Tx_n\|^p}{(\|x_n - Tx_n\| + \|z - Tz\| + \|x_n - Tz\| + \|z - Tx_n\|)^{p-1}}\right) \\ &\quad + b\left(\frac{\|x_n - Tx_n\|^p + \|z - Tz\|^p + \|x_n - z\|^p}{(\|x_n - Tx_n\| + \|z - Tz\| + \|x_n - Tz\|)^{p-1}}\right)\end{aligned}$$

$$
\begin{aligned}
&+ c\left(\frac{\|x_n - Tz\|^p + \|z - Tx_n\|^p}{(\|x_n - Tz\| + \|z - Tx_n\|)^{p-1}}\right) + d\,\|z - x_n\|)\\
&= a\left(\frac{\|x_n - x_{n+1}\|^p + \|z - Tz\|^p + \|x_n - Tz\|^p + \|z - x_{n+1}\|^p}{(\|x_n - x_{n+1}\| + \|z - Tz\| + \|x_n - Tz\| + \|z - x_{n+1}\|)^{p-1}}\right)\\
&+ b\left(\frac{\|x_n - x_{n+1}\|^p + \|z - Tz\|^p + \|z - x_n\|^p}{(\|x_n - x_{n+1}\| + \|z - Tz\| + \|z - x_n\|)^{p-1}}\right)\\
&+ c\left(\frac{\|x_n - Tz\|^p + \|z - x_{n+1}\|^p}{(\|x_n - Tz\| + \|z - x_{n+1}\|)^{p-1}}\right) + d\,\|z - x_n\|)\\
&\leq 2a\,\|Tz - z\| + b\,\|z - Tz\| + c\,\|Tz - z\|.
\end{aligned}
$$

So,

$$
\begin{aligned}
\psi(\|z - Tz\|) &\leq G(\psi((2a + b + c)\,\|z - Tz\|), \varphi((2a + b + c)\,\|z - Tz\|))\\
&\leq \psi((2a + b + c)\,\|z - Tz\|)\\
&\leq \psi(\|z - Tz\|).
\end{aligned}
$$

From Definition 1.1, we get $\|z - Tz\| = 0$. Thus $z = Tz$.

Now we prove that T has a unique fixed point. Assume that z and w are two distinct fixed points of T. From (2.1), we have

$$
\psi(\|Tw - Tz\|) \leq G(\psi\,(M(w, z)), \varphi\,(M(w, z))) \leq \psi(M(w, z)),
$$

where

$$
\begin{aligned}
M\,(w, z) &= a\left(\frac{\|w - Tw\|^p + \|z - Tz\|^p + \|w - Tz\|^p + \|z - Tw\|^p}{(\|w - Tw\| + \|z - Tz\| + \|w - Tz\| + \|z - Tw\|)^{p-1}}\right)\\
&+ b\left(\frac{\|w - Tw\|^p + \|z - Tz\|^p + \|z - w\|^p}{(\|w - Tw\| + \|z - Tz\| + \|z - w\|)^{p-1}}\right)\\
&+ c\left(\frac{\|w - Tz\|^p + \|z - Tw\|^p}{(\|w - Tz\| + \|z - Tw\|)^{p-1}}\right) + d\,\|z - w\|)\\
&= a\left(\frac{\|w - w\|^p + \|z - z\|^p + \|w - z\|^p + \|z - w\|^p}{(\|w - w\| + \|z - z\| + \|w - z\| + \|z - w\|)^{p-1}}\right)\\
&+ b\left(\frac{\|w - w\|^p + \|z - z\|^p + \|z - w\|^p}{(\|w - w\| + \|z - z\| + \|z - w\|)^{p-1}}\right) + c\left(\frac{\|w - z\|^p + \|z - w\|^p}{(\|w - z\| + \|z - w\|)^{p-1}}\right)\\
&+ d\,\|z - w\|)\\
&= (2a + b + c + d)\,\|z - w\|,
\end{aligned}
$$

which is a contradiction. So $z = w$.

Taking $a = b = c = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. Let S be a nonempty closed subset of a Hilbert C^*-module space E. Let T be a self-mapping on S satisfying

$$
\psi(\|Tx - Ty\|) \leq G\,(\psi\,(\|x - y\|)\,, \varphi\,(\|x - y\|))
$$

for all $x, y \in S$, $\psi \in \Psi, \varphi \in \Phi_u$, and $G \in C$. Then T has a unique fixed point.

Taking $G(s,t) = s$, $\psi(t) = t$ in Theorem 2.1 with tiny modifications, we have the following corollary.

Corollary 2.3. Let S be a nonempty closed subset of a Hilbert C^*-module E. Let T be a self-mapping on S satisfying

$$\begin{aligned}\|Tx - Ty\| \leq\ & a\left(\frac{\|Tx - x\|^p + \|Ty - y\|^p + \|Ty - x\|^p + \|Tx - y\|^p}{(\|Tx - x\| + \|Ty - y\| + \|Ty - x\| + \|Tx - y\|)^{p-1}}\right)\\ &+ b\left(\frac{\|Tx - x\|^p + \|Ty - y\|^p + \|y - x\|^p}{(\|Tx - x\| + \|Ty - y\| + \|y - x\|)^{p-1}}\right)\\ &+ c\left(\frac{\|Ty - x\|^p + \|Tx - y\|^p}{(\|Ty - x\| + \|Tx - y\|)^{p-1}}\right) + d\,\|x - y\|,\end{aligned}$$

for all $x, y \in S$ and $a, b, c, d \geq 0$, with $4a + 3b + 2c + d < 1$. Then T has a unique fixed point.

Taking $a = b = c = 0$ in Corollary 2.3, it reduces to the following corollary.

Corollary 2.4. Let S be a nonempty closed subset of a Hilbert C^*-module E. Let T be a self-mapping on S satisfying

$$\|Tx - Ty\| \leq d\,\|x - y\|,$$

for all $x, y \in S$ and $d \geq 0$, with $d < 1$. Then T has a unique fixed point.

Here are two examples to support the validity of Corollary 2.3.

Example 2.5. Let $S = C([0, 1], \mathbb{R})$ with

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx, \|f\|^2 = \int_0^1 f(x)^2 dx.$$

Define the mapping $T : S \to S$ by $T(f(x)) = \frac{f(x)}{5} + 2x$. Let the function $\psi : [0, +\infty) \to [0, +\infty)$ be defined by $\psi(t) = t, t \geq 0$. Obviously, $\psi \in \Psi$. Let $G(s,t) = \frac{10}{11}s$. We have

$$\begin{aligned}\psi(\|Tf - Tg\|) = \|Tf - Tg\| = \frac{1}{5}\|f - g\| \leq\ & \frac{3}{11}\|f - g\|\\ \leq\ & a\left(\frac{\|Tf - f\|^p + \|Tg - g\|^p + \|Tg - f\|^p + \|Tf - g\|^p}{(\|Tf - f\| + \|Tg - g\| + \|Tg - f\| + \|Tf - g\|)^{p-1}}\right)\\ &+ b\left(\frac{\|Tf - f\|^p + \|Tg - g\|^p + \|g - f\|^p}{(\|Tf - f\| + \|Tg - g\| + \|g - f\|)^{p-1}}\right)\\ &+ c\left(\frac{\|Tg - f\|^p + \|Tf - g\|^p}{(\|Tg - f\| + \|Tf - g\|)^{p-1}}\right) + d\,\|f - g\|,\end{aligned}$$

where $p \in N$, $p \geq 2$, and $a, b, c \geq 0$, $\frac{3}{11} < d < 1$ with $4a + 2b + 2c + d < 1$. So all the assumptions of Corollary 2.3 are satisfied. Then the mapping T has a fixed point, that is, $f(x) = \frac{5}{2}x$.

Example 2.6. Let $G(s,t) = \frac{s}{2}$, and $S = M_{2\times 2}(\mathbb{R})$ with $\langle A, B\rangle = A \times B^*$. Define the mappings $T : S \to S$ by $T(A) = \frac{A}{4} + \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. Consider the function $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ defined as $\psi(t) = \frac{9t}{8}$, where ψ and φ are continuous, monotone, and nondecreasing. Then for all $A, B \in M_{2\times 2}(\mathbb{R})$, we have

$$\psi(\|TA - TB\|) = \frac{9}{8} \cdot \frac{1}{2}\|A - B\| \leq \frac{9}{8}M(A, B) = G(\psi(M(A, B)), \varphi(M(A, B))),$$

where $a, b, c \geq 0$, and $\frac{1}{2} < d < 1$ with $4a + 2b + 2c + d < 1$. So all the assumptions of Corollary 2.3 are satisfied. Then the mapping T has a fixed point, that is, $\begin{bmatrix} \frac{4}{3} & \frac{8}{3} \\ 4 & \frac{20}{3} \end{bmatrix}$.

Taking $a = b = d = 0$ in Corollary 2.3, it reduces to the following corollary.

Corollary 2.7. Let S be a nonempty closed subset of a Hilbert C^*-module E. Let T be a self-mapping on S satisfying

$$\|Tx - Ty\| \leq c\left(\frac{\|Ty - x\|^p + \|Tx - y\|^p}{(\|Ty - x\| + \|Tx - y\|)^{p-1}}\right),$$

for all $x, y \in S$ and $c \geq 0$, with $c < \frac{1}{2}$. Then T has a unique fixed point.

Taking $G(s, t) = s - t$ in Theorem 2.1, we have the following corollary.

Corollary 2.8. Let S be a nonempty closed subset of a Hilbert C^*-module space E. Let T be a self-mapping on S satisfying

$$\psi(\|Tx - Ty\|) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where

$$\begin{aligned} M(x, y) = a&\left(\frac{\|Tx - x\|^p + \|Ty - y\|^p + \|Ty - x\|^p + \|Tx - y\|^p}{(\|Tx - x\| + \|Ty - y\| + \|Ty - x\| + \|Tx - y\|)^{p-1}}\right) \\ &+ b\left(\frac{\|Tx - x\|^p + \|Ty - y\|^p + \|y - x\|^p}{(\|Tx - x\| + \|Ty - y\| + \|y - x\|)^{p-1}}\right) \\ &+ c\left(\frac{\|Ty - x\|^p + \|Tx - y\|^p}{(\|Ty - x\| + \|Tx - y\|)^{p-1}}\right) + d\,\|x - y\| \end{aligned}$$

for all $x, y \in S$ and $a, b, c, d \geq 0, p \in N, p \geq 2$ with $4a + 3b + 2c + d = 1$ and $\psi \in \Psi, \varphi \in \Phi_u$, the tripled $(\psi, \varphi, G(s, t))$ is monotone. Then T has a unique fixed point.

Taking $G(s, t) = \frac{s}{1+t}$ in Theorem 2.1, we have the following corollary.

Corollary 2.9. Let S be a nonempty closed subset of a Hilbert C^*-module space E. Let T be a self-mapping on S satisfying

$$\psi(\|Tx - Ty\|) \leq \frac{\psi(M(x, y))}{1 + \varphi(M(x, y))},$$

where

$$\begin{aligned} M(x, y) = a&\left(\frac{\|Tx - x\|^p + \|Ty - y\|^p + \|Ty - x\|^p + \|Tx - y\|^p}{(\|Tx - x\| + \|Ty - y\| + \|Ty - x\| + \|Tx - y\|)^{p-1}}\right) \\ &+ b\left(\frac{\|Tx - x\|^p + \|Ty - y\|^p + \|y - x\|^p}{(\|Tx - x\| + \|Ty - y\| + \|y - x\|)^{p-1}}\right) \\ &+ c\left(\frac{\|Ty - x\|^p + \|Tx - y\|^p}{(\|Ty - x\| + \|Tx - y\|)^{p-1}}\right) + d\,\|x - y\|, \end{aligned}$$

for all $x, y \in S$ and $a, b, c, d \geq 0, p \in N, p \geq 2$ with $4a + 3b + 2c + d = 1$ and $\psi \in \Psi, \varphi \in \Phi_u$, and the tripled $(\psi, \varphi, G(s, t))$ is monotone. Then T has a unique fixed point.

Taking $G(s, t) = s \log_{t+q} q, q > 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.10. Let S be a nonempty closed subset of a Hilbert C^*-module space E. Let T be a self-mapping on S satisfying

$$\psi(\|Tx - Ty\|) \leq \psi(M(x, y)) \times \log_{q+\varphi(M(x,y))} q,$$

where

$$M(x,y) = a\left(\frac{\|Tx-x\|^p + \|Ty-y\|^p + \|Ty-x\|^p + \|Tx-y\|^p}{(\|Tx-x\| + \|Ty-y\| + \|Ty-x\| + \|Tx-y\|)^{p-1}}\right) + b\left(\frac{\|Tx-x\|^p + \|Ty-y\|^p + \|y-x\|^p}{(\|Tx-x\| + \|Ty-y\| + \|y-x\|)^{p-1}}\right) + c\left(\frac{\|Ty-x\|^p + \|Tx-y\|^p}{(\|Ty-x\| + \|Tx-y\|)^{p-1}}\right) + d\,\|x-y\|,$$

for all $x, y \in S$ and $a, b, c, d \geq 0, p \in N, p \geq 2$ with $4a + 3b + 2c + d = 1$ and $\psi \in \Psi, \varphi \in \Phi_u$, the tripled $(\psi, \varphi, G(s,t))$ is monotone. Then T has a unique fixed point.

In the next theorem, we will present a common fixed point result a Hilbert C^*-module space E under a similar contraction condition as the one in Theorem 2.1.

Theorem 2.11. Let S be a nonempty closed subset of a Hilbert C^*-module space E. Let T_1 and T_2 be self-mappings on S satisfying

$$\psi(\|T_1x - T_2y\|) \leq G(\psi(M(x,y)), \varphi(M(x,y))), \tag{2.8}$$

where

$$M(x,y) = a\left(\frac{\|T_1x-x\|^p + \|T_2y-y\|^p + \|T_2y-x\|^p + \|T_1x-y\|^p}{(\|T_1x-x\| + \|T_2y-y\| + \|T_2y-x\| + \|T_1x-y\|)^{p-1}}\right) + b\left(\frac{\|T_1x-x\|^p + \|T_2y-y\|^p + \|y-x\|^p}{(\|T_1x-x\| + \|T_2y-y\| + \|y-x\|)^{p-1}}\right) + c\left(\frac{\|T_2y-x\|^p + \|T_1x-y\|^p}{(\|T_2y-x\| + \|T_1x-y\|)^{p-1}}\right) + d\,\|x-y\|,$$

for all $x, y \in S$ and $a, b, c, d \geq 0, p \in N, p \geq 2$ with $4a + 3b + 2c + d = 1$ and $\psi \in \Psi, \varphi \in \Phi_u$, $G \in C$, and the tripled $(\psi, \varphi, G(s,t))$ is monotone. Then T_1 and T_2 have a unique common fixed point.

Proof. Let x_0 be any arbitrary point in S. We define a sequence x_n in S by $x_{2n+1} = T_1x_{2n}$ and $x_{2n+2} = T_2x_{2n+1}$ for all $n \geq 1$.

From the contractive condition, we have

$$\begin{aligned}\psi(\|x_{2n+1} - x_{2n}\|) &= \psi(\|T_1x_{2n} - T_2x_{2n-1}\|)\\ &\leq G(\psi(M(x_{2n}, x_{2n-1})), \varphi(M(x_{2n}, x_{2n-1})))\\ &\leq \psi(M(x_{2n}, x_{2n-1})),\end{aligned} \tag{2.9}$$

where

$$M(x_{2n}, x_{2n-1}) = a\left(\frac{\|x_{2n} - T_1x_{2n}\|^p + \|x_{2n-1} - T_2x_{2n-1}\|^p + \|x_{2n} - T_2x_{2n-1}\|^p + \|x_{2n-1} - T_1x_{2n}\|^p}{(\|x_{2n} - T_1x_{2n}\| + \|x_{2n-1} - T_2x_{2n-1}\| + \|x_{2n} - T_2x_{2n-1}\| + \|x_{2n-1} - T_1x_{2n}\|)^{p-1}}\right)$$

$$+ b\left(\frac{\|x_{2n} - T_1x_{2n}\|^p + \|x_{2n-1} - T_2x_{2n-1}\|^p + \|x_{2n-1} - x_{2n}\|^p}{(\|x_{2n} - T_1x_{2n}\| + \|x_{2n-1} - T_2x_{2n-1}\| + \|x_{2n-1} - x_{2n}\|)^{p-1}}\right)$$

$$
\begin{aligned}
&+ c\left(\frac{\|x_{2n} - T_2x_{2n-1}\|^p + \|x_{2n-1} - T_1x_{2n}\|^p}{(\|x_{2n} - T_2x_{2n-1}\| + \|x_{2n-1} - T_1x_{2n}\|)^{p-1}}\right) + d\,\|x_{2n-1} - x_{2n}\|)\\
&= a\left(\frac{\|x_{2n} - x_{2n+1}\|^p + \|x_{2n-1} - x_{2n}\|^p + \|x_{2n} - x_{2n}\|^p + \|x_{2n-1} - x_{2n+1}\|^p}{(\|x_{2n} - x_{2n+1}\| + \|x_{2n-1} - x_{2n}\| + \|x_{2n} - x_{2n}\| + \|x_{2n-1} - x_{2n+1}\|)^{p-1}}\right)\\
&+ b\left(\frac{\|x_{2n} - x_{2n+1}\|^p + \|x_{2n-1} - x_{2n}\|^p + \|x_{2n-1} - x_{2n}\|^p}{(\|x_{2n} - x_{2n+1}\| + \|x_{2n-1} - x_{2n}\| + \|x_{2n-1} - x_{2n}\|)^{p-1}}\right)\\
&+ c\left(\frac{\|x_{2n} - x_{2n}\|^p + \|x_{2n-1} - x_{2n+1}\|^p}{(\|x_{2n} - x_{2n}\| + \|x_{2n-1} - x_{2n+1}\|)^{p-1}}\right) + d\,\|x_{2n-1} - x_{2n}\|)\\
&\le a\,(\|x_{2n} - x_{2n+1}\| + \|x_{2n-1} - x_{2n}\| + \|x_{2n-1} - x_{2n+1}\|)\\
&+ b\,(\|x_{2n} - x_{2n+1}\| + \|x_{2n-1} - x_{2n}\| + \|x_{2n-1} - x_{2n}\|)\\
&+ c\,\|x_{2n-1} - x_{2n+1}\| + d\,\|x_{2n-1} - x_{2n}\|\\
&\le a\,(\|x_{2n} - x_{2n+1}\| + \|x_{2n-1} - x_{2n}\| + \|x_{2n-1} - x_{2n}\| + \|x_{2n} - x_{2n+1}\|)\\
&+ b\,(\|x_{2n} - x_{2n+1}\| + \|x_{2n-1} - x_{2n}\| + \|x_{2n-1} - x_{2n}\|)\\
&+ c\,(\|x_{2n-1} - x_{2n}\| + \|x_{2n} - x_{2n+1}\|) + d\,\|x_{2n-1} - x_{2n}\|\,.
\end{aligned}
$$

Therefore from the monotone property of ψ, we have

$$
\begin{aligned}
\|x_{2n+1} - x_{2n}\| &\le M(x_{2n}, x_{2n-1})\\
&\le a\,(\|x_{2n} - x_{2n+1}\| + \|x_{2n-1} - x_{2n}\| + \|x_{2n-1} - x_{2n}\| + \|x_{2n} - x_{2n+1}\|)\\
&+ b\,(\|x_{2n} - x_{2n+1}\| + \|x_{2n-1} - x_{2n}\| + \|x_{2n-1} - x_{2n}\|)\\
&+ c\,(\|x_{2n-1} - x_{2n}\| + \|x_{2n} - x_{2n+1}\|) + d\,\|x_{2n-1} - x_{2n}\|\,,
\end{aligned}
$$

and

$$(1 - 2a - b - c)\,\|x_{2n+1} - x_{2n}\| \le (2a + 2b + c + d)\,\|x_{2n-1} - x_{2n}\|\,.$$

So, we have

$$\|x_{2n+1} - x_{2n}\| \le \|x_{2n-1} - x_{2n}\|\,. \tag{2.10}$$

The inequality (2.10) implies that $\{\|x_{2n+1} - x_{2n}\|\}$ is a monotone decreasing sequence. Consequently, there exists $r \ge 0$ such that

$$\|x_{2n+1} - x_{2n}\| \to r \ \ as\ n \to +\infty. \tag{2.11}$$

Letting $n \to +\infty$ in (2.8) and using (2.11), we obtain that

$$
\begin{aligned}
\psi(r) \quad &\le G(\psi((4a + 3b + 2c + d)r)\,, \varphi(r(4a + 3b + 2c + d)))\\
&= G(\psi(r)\,, \varphi(r)).
\end{aligned}
$$

From Definition 1.1, we get $r = 0$. Hence,

$$\lim_{n\to+\infty} \|x_{2n+1} - x_{2n}\| = 0,$$

and

$$\lim_{n\to+\infty} \|x_{n+1} - x_n\| = 0.$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence. Then, using Lemma 1.10, we get that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences tend to ε^+ when $k \to +\infty$:

$$\left\|x_{2n_k} - x_{2m_k}\right\|, \left\|x_{2n_k} - x_{2m_k+1}\right\|, \left\|x_{2n_k-1} - x_{2m_k}\right\|, \left\|x_{2n_k-1} - x_{2m_k+1}\right\|. \tag{2.12}$$

Now, consider (2.12) and (2.8) with $k \to +\infty$, and we have

$$\begin{aligned}
&M\left(x_{2m_k}, x_{2n_k}\right) \\
&= a\left(\frac{\left\|x_{2m_k} - Tx_{2m_k}\right\|^p + \left\|x_{2n_k} - Tx_{2n_k}\right\|^p + \left\|x_{2m_k} - Tx_{2n_k}\right\|^p + \left\|x_{n(k)} - Tx_{2m_k}\right\|^p}{(\left\|x_{2m_k} - Tx_{2m_k}\right\| + \left\|x_{2n_k} - Tx_{2n_k}\right\| + \left\|x_{2m_k} - Tx_{2n_k}\right\| + \left\|x_{n(k)} - Tx_{2m_k}\right\|)^{p-1}}\right) \\
&+ b\left(\frac{\left\|x_{2m_k} - Tx_{2m_k}\right\|^p + \left\|x_{2n_k} - Tx_{2n_k}\right\|^p + \left\|x_{2m_k} - x_{2n_k}\right\|^p}{(\left\|x_{2m_k} - Tx_{2m_k}\right\| + \left\|x_{2n_k} - Tx_{2n_k}\right\| + \left\|x_{2m_k} - x_{2n_k}\right\|)^{p-1}}\right) \\
&+ c\left(\frac{\left\|x_{2m_k} - Tx_{2n_k}\right\|^p + \left\|x_{n(k)} - Tx_{2m_k}\right\|^p}{(\left\|x_{2m_k} - Tx_{2n_k}\right\| + \left\|x_{n(k)} - Tx_{2m_k}\right\|)^{p-1}}\right) + d(\left\|x_{2m_k} - x_{2n_k}\right\|) \\
&\to a\left(\frac{2\varepsilon^p}{(2\varepsilon)^{p-1}}\right) + b\left(\frac{\varepsilon^p}{\varepsilon^{p-1}}\right) + c\left(\frac{2\varepsilon^p}{(2\varepsilon)^{p-1}}\right) + d\varepsilon = (a+b+c+d)\varepsilon.
\end{aligned}$$

So

$$\begin{aligned}
\psi\left(\epsilon\right) &\le G(\psi\left((a+b+c+d)\varepsilon\right), \varphi\left((a+b+c+d)\varepsilon\right)) \\
&\le \psi\left((a+b+c+d)\varepsilon\right) \\
&\le \epsilon.
\end{aligned} \tag{2.13}$$

From Definition 1.1, we get $\varepsilon = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since S is a closed subset of Hilbert C^*-module E, the sequence $\{x_n\}$ converges to a point $z \in X$.

Now we show that z is common fixed point of T_1 and T_2. We assume on the contrary that $T_1 z \neq z$. From (2.8),

$$\begin{aligned}
\psi(|\|T_2 x_{2n+1} - z\| - \|z - T_1 z\||) &\le \psi(\|T_2 x_{2n+1} - T_1 z\|) \\
&\le G(\psi\left(M(z, x_{2n+1})\right), \varphi\left(M(z, x_{2n+1})\right)) \\
&\le \psi(M(z, x_{2n+1})),
\end{aligned}$$

whereby taking the limit as $n \to +\infty$, we have

$$\begin{aligned}
&M\left(z, x_{2n+1}\right) \\
&= a\left(\frac{\|z - T_1 z\|^p + \|x_{2n+1} - T_2 x_{2n+1}\|^p + \|z - T_2 x_{2n+1}\|^p + \|x_{2n+1} - T_1 z\|^p}{(\|z - T_1 z\| + \|x_{2n+1} - T_2 x_{2n+1}\| + \|z - T_2 x_{2n+1}\| + \|x_{2n+1} - T_1 z\|)^{p-1}}\right) \\
&+ b\left(\frac{\|z - T_1 z\|^p + \|x_{2n+1} - T_2 x_{2n+1}\|^p + \|z - x_{2n+1}\|^p}{(\|z - T_1 z\| + \|x_{2n+1} - T_2 x_{2n+1}\| + \|z - x_{2n+1}\|)^{p-1}}\right) \\
&+ c\left(\frac{\|z - T_2 x_{2n+1}\|^p + \|x_{2n+1} - T_1 z\|^p}{(\|z - T_2 x_{2n+1}\| + \|x_{2n+1} - T_1 z\|)^{p-1}}\right) + d\,\|z - x_{2n+1}\|)
\end{aligned}$$

$$
\begin{aligned}
&= a\left(\frac{\|z-T_1z\|^p+\|x_{2n+1}-x_{2n+2}\|^p+\|z-x_{2n+2}\|^p+\|x_{2n+1}-T_1z\|^p}{(\|z-T_1z\|+\|x_{2n+1}-x_{2n+2}\|+\|z-x_{2n+2}\|+\|x_{2n+1}-T_1z\|)^{p-1}}\right)\\
&+ b\left(\frac{\|z-T_1z\|^p+\|x_{2n+1}-x_{2n+2}\|^p+\|z-x_{2n+1}\|^p}{(\|z-T_1z\|+\|x_{2n+1}-x_{2n+2}\|+\|z-x_{2n+1}\|)^{p-1}}\right)\\
&+ c\left(\frac{\|z-x_{2n+2}\|^p+\|x_{2n+1}-T_1z\|^p}{(\|z-x_{2n+2}\|+\|x_{2n+1}-T_1z\|)^{p-1}}\right)+d\,\|z-x_{2n+1}\|)\\
&\leq 2a\,\|z-T_1z\|+b\,\|z-T_1z\|+c\,\|z-T_1z\|.
\end{aligned}
$$

So,

$$
\begin{aligned}
\psi(\|z-T_1z\|) &\leq F(\psi((2a+b+c)\,\|z-T_1z\|),\varphi((2a+b+c)\,\|z-T_1z\|))\\
&\leq \psi((2a+b+c)\,\|z-T_1z\|)\leq \psi(\|z-T_1z\|).
\end{aligned}
$$

From Definition 1.1, we get $\|z-T_1z\|=0$. Thus $z=T_1z$. Similarly, we can show $z=T_2z$.

Now we prove that T_1 and T_2 have a unique common fixed points. Assume z and w are two distinct common fixed points of T_1 and T_2. From (2.8),

$$
\begin{aligned}
\psi(\|T_1w-T_2z\|) &\leq G(\psi\,(M(w,z)),\varphi\,(M(w,z)))\\
&\leq \psi(M(w,z)),
\end{aligned}
$$

where

$$
\begin{aligned}
M\,(w,z) &= a\left(\frac{\|w-T_1w\|^p+\|z-T_2z\|^p+\|w-T_2z\|^p+\|z-T_1w\|^p}{(\|w-T_1w\|+\|z-T_2z\|+\|w-T_2z\|+\|z-T_1w\|)^{p-1}}\right)\\
&+ b\left(\frac{\|w-T_1w\|^p+\|z-T_2z\|^p+\|z-w\|^p}{(\|w-T_1w\|+\|z-T_2z\|+\|z-w\|)^{p-1}}\right)\\
&+ c\left(\frac{\|w-T_2z\|^p+\|z-T_1w\|^p}{(\|w-T_2z\|+\|z-T_1w\|)^{p-1}}\right)+d\,\|z-w\|)\\
&= a\left(\frac{\|w-w\|^p+\|z-z\|^p+\|w-z\|^p+\|z-w\|^p}{(\|w-w\|+\|z-z\|+\|w-z\|+\|z-w\|)^{p-1}}\right)\\
&+ b\left(\frac{\|w-w\|^p+\|z-z\|^p+\|z-w\|^p}{(\|w-w\|+\|z-z\|+\|z-w\|)^{p-1}}\right)\\
&+ c\left(\frac{\|w-z\|^p+\|z-w\|^p}{(\|w-z\|+\|z-w\|)^{p-1}}\right)+d\,\|z-w\|)\\
&\leq (2a+b+2c+d)\,\|z-w\|.
\end{aligned}
$$

So,

$$
\begin{aligned}
\psi(\|z-w\|) &= \psi(\|T_1w-T_2z\|)\\
&\leq G((2a+b+2c+d)\,\|z-w\|),\psi((2a+b+2c+d)\,\|z-w\|)))\\
&\leq \psi(((2a+b+2c+d)\,\|z-w\|)\leq \psi(\|z-w\|).
\end{aligned}
$$

From Definition 1, we get $\|z-w\|=0$. So $z=w$, which shows that T_1 and T_2 have a unique common fixed point.

Taking $a = b = c = 0$ in Theorem 2.11, we have the following corollary.

Corollary 2.12. Let S be a nonempty closed subset of a Hilbert C^*-module space E. Let T_1 and T_2 be self-mappings on S satisfying

$$\psi(\|T_1x - T_2y\|) \leq G\left(\psi\|x - y\|, \varphi\left(\|x - y\|\right)\right) \tag{2.14}$$

for all $x, y \in S$ and $\psi \in \Psi, \varphi \in \Phi_u$. Then T_1 and T_2 have a unique common fixed point.

Taking $G(s,t) = s - t$ in Theorem 2.11, we have the following corollary.

Corollary 2.13. Let S be a nonempty closed subset of a Hilbert C^*-module space E. Let T_1 and T_2 be two self-mappings on S satisfying

$$\psi(\|T_1x - T_2y\|) \leq \psi(M(x, y)) - \varphi\left(M(x, y)\right),$$

where

$$\begin{aligned} M(x,y) = a&\left(\frac{\|T_1x - x\|^p + \|T_2y - y\|^p + \|T_2y - x\|^p + \|T_1x - y\|^p}{(\|T_1x - x\| + \|T_2y - y\| + \|T_2y - x\| + \|T_1x - y\|)^{p-1}}\right) \\ &+ b\left(\frac{\|T_1x - x\|^p + \|T_2y - y\|^p + \|y - x\|^p}{(\|T_1x - x\| + \|T_2y - y\| + \|y - x\|)^{p-1}}\right) \\ &+ c\left(\frac{\|T_2y - x\|^p + \|T_1x - y\|^p}{(\|T_2y - x\| + \|T_1x - y\|)^{p-1}}\right) + d\|x - y\| \end{aligned}$$

for all $x, y \in S$ and $a, b, c, d \geq 0, p \in N, p \geq 2$ with $4a + 3b + 2c + d = 1$ and $\psi \in \Psi, \varphi \in \Phi_u$, and the tripled $(\psi, \varphi, G(s,t))$ is monotone. Then T_1 and T_2 have a unique common fixed point.

Taking $G(s,t) = \frac{s}{1+t}$ in Theorem 2.11, we have the following corollary.

Corollary 2.14. Let S be a nonempty closed subset of a Hilbert C^*-module space E. Let T_1 and T_2 be two self-mappings on S satisfying

$$\psi(\|T_1x - T_2y\|) \leq \frac{\psi(M(x, y))}{1 + \varphi\left(M(x, y)\right)},$$

where

$$\begin{aligned} M(x,y) = a&\left(\frac{\|T_1x - x\|^p + \|T_2y - y\|^p + \|T_2y - x\|^p + \|T_1x - y\|^p}{(\|T_1x - x\| + \|T_2y - y\| + \|T_2y - x\| + \|T_1x - y\|)^{p-1}}\right) \\ &+ b\left(\frac{\|T_1x - x\|^p + \|T_2y - y\|^p + \|y - x\|^p}{(\|T_1x - x\| + \|T_2y - y\| + \|y - x\|)^{p-1}}\right) \\ &+ c\left(\frac{\|T_2y - x\|^p + \|T_1x - y\|^p}{(\|T_2y - x\| + \|T_1x - y\|)^{p-1}}\right) + d\|x - y\|, \end{aligned}$$

for all $x, y \in S$ and $a, b, c, d \geq 0, p \in N, p \geq 2$ with $4a + 3b + 2c + d = 1$ and $\psi \in \Psi, \varphi \in \Phi_u$, and the tripled $(\psi, \varphi, G(s,t) = \frac{s}{1+t})$ is monotone. Then T_1 and T_2 have a unique common fixed point.

Taking $G(s,t) = s\log_{t+q} q, q > 1$ in Theorem 2.11, we have the following corollary.

Corollary 2.15. Let S be a nonempty closed subset of a Hilbert C^*-module space E. Let T_1 and T_2 be two self-mappings on S satisfying

$$\psi(\|T_1x - T_2y\|) \leq \psi(M(x, y)) \times \log_{p+\varphi(M(x,y))} p,$$

where

$$M(x,y) = a\left(\frac{\|T_1x-x\|^p + \|T_2y-y\|^p + \|T_2y-x\|^p + \|T_1x-y\|^p}{(\|T_1x-x\| + \|T_2y-y\| + \|T_2y-x\| + \|T_1x-y\|)^{p-1}}\right) + b\left(\frac{\|T_1x-x\|^p + \|T_2y-y\|^p + \|y-x\|^p}{(\|T_1x-x\| + \|T_2y-y\| + \|y-x\|)^{p-1}}\right) + c\left(\frac{\|T_2y-x\|^p + \|T_1x-y\|^p}{(\|T_2y-x\| + \|T_1x-y\|)^{p-1}}\right) + d\,\|x-y\|,$$

for all $x, y \in S$ and $a, b, c, d \geq 0, p \in N, p \geq 2$ with $4a + 3b + 2c + d = 1$ and $\psi \in \Psi, \varphi \in \Phi_u$, and the tripled $(\psi, \varphi, G(s,t) = s\log_{t+q} q, q > 1)$ is monotone. Then T_1 and T_2 have a unique common fixed point.

Taking $G(s,t) = s$, $\psi(t) = t$ in Theorem 2.11 with tiny modifications, we have the following corollary.

Corollary 2.16. Let S be a nonempty closed subset of a Hilbert C^*-module space E. Let T_1 and T_2 be two self-mappings on S satisfying

$$\|T_1x - T_2y\| \leq a\left(\frac{\|T_1x-x\|^p + \|T_2y-y\|^p + \|T_2y-x\|^p + \|T_1x-y\|^p}{(\|T_1x-x\| + \|T_2y-y\| + \|T_2y-x\| + \|T_1x-y\|)^{p-1}}\right) + b\left(\frac{\|T_1x-x\|^p + \|T_2y-y\|^p + \|y-x\|^p}{(\|T_1x-x\| + \|T_2y-y\| + \|y-x\|)^{p-1}}\right) + c\left(\frac{\|T_2y-x\|^p + \|T_1x-y\|^p}{(\|T_2y-x\| + \|T_1x-y\|)^{p-1}}\right) + d\,\|x-y\|$$

for all $x, y \in S$ and $a, b, c, d \geq 0, p \in N, p \geq 2$ with $4a + 3b + 2c + d < 1$. Then T_1 and T_2 have a unique common fixed point.

3. Application

In this section, we will prove some existence results of the integral/differential equation solution via our main result. First, let us consider the following integral equation:

$$x(t) = g(t) + \int_0^1 K(s, x(s))\,ds, \quad t \in [0,1] \tag{3.1}$$

and consider $S = C[0,1]$ with norm $\|.\|_\infty$.

Theorem 3.1. Consider the integral Eq (3.1) and suppose:

(i) $K : [0,1] \times S \to S$ is continuous and $g \in S$,

(ii) $|K(s, x(s)) - K(s, y(s))| \leq |x(s) - y(s)|$, for all $s \in [0,1]$,

(iii) there exist $\psi \in \Psi, \varphi \in \Phi_u, G \in C$ such that

$$\psi\left(\max_{t\in[0,1]} \int_0^t |K(s, x(s)) - K(s, y(s))|\,ds\right) \leq G(\psi(|x(s) - y(s)|), \varphi(|x(s) - y(s)|)),$$

for all $s \in [0,1]$. Then integral Eq (3.1) has a unique solution.

Proof. Let $T : S \to S$ be defined by

$$T(x)(t) = \int_0^t K(s, x(s))\, ds + g(t), \quad t \in [0, 1].$$

From the definition of integral Eq (3.1),

$$\begin{aligned}
\psi\left(\|Tx - Ty\|_{+\infty}\right) &= \psi\left(\max_{t\in[0,1]} |Tx(t) - Ty(t)|\right) \\
&\le \psi\left(\max_{t\in[0,1]} \int_0^t |K(s, x(s)) - K(s, y(s))|\, ds\right) \\
&\le \max_{s\in[0,1]} G\left(\psi\left(|x(s) - y(s)|\right), \varphi\left(|x(s) - y(s)|\right)\right) \\
&\le G\left(\psi\left(\|x - y\|_{+\infty}\right), \varphi\left(\|x - y\|_{+\infty}\right)\right).
\end{aligned}$$

Thus, by Corollary 2.2, T has a unique fixed point which implies that integral Eq (3.1) has a unique solution.

Next, another application to the second-order (p, q)-difference equations with integral boundary value conditions is provided as follows.

We will prove an existence and uniqueness theorem for a second-order (p, q)-difference equation with integral boundary value conditions of the form stated as follows:

$$\begin{cases} D^2_{p,q}u(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \\ u(0) = \int_0^1 u(t) d_{p,q}t, \\ u(1) = \int_0^1 tu(t) d_{p,q}t, \end{cases} \tag{3.2}$$

where p, q are such that $0 < q < p \le 1, f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given a function, and $D_{p,q}$ is the (p, q)-difference operator, defined as follows (see [19]). Assume $u : [0, T] \to \mathbb{R}, T > 0$ is a given function and p, q are such that $0 < q < p \le 1, p + q \ne 1$.The (p, q)-derivative of u, denoted by $D_{p,q}u(t)$, is defined by

$$D_{p,q}u(t) = \frac{u(pt) - u(qt)}{(p - q)t}, \quad \text{if} \quad t \ne 0, \tag{3.3}$$

and

$$D_{p,q}u(0) = \lim_{t\to 0} D_{p,q}u(t), \quad \text{if} \quad t = 0. \tag{3.4}$$

One can see that $D_{p,q}u(t)$ is defined on the large interval $[0, T/p]$, which includes the interval $[0, T]$ on which u is defined.

We say that the function u is (p, q)-differentiable if $D_{p,q}u(t)$ exists for all $t \in [0, T/p]$.

The (p, q)-integral of u, denoted by $\int_0^t u(s) d_{p,q}s$, is defined by

$$\int_0^t u(s) d_{p,q}s = \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} u\left(\frac{q^n}{p^{n+1}} t\right), \tag{3.5}$$

whenever the series in the right-hand side of (3.5) converges.

In contrast with the case of the (p,q)-derivative of u, the (p,q)-integral of u is defined on the interval $[0, pT]$ which is included in the interval $[0, T]$. To see the properties of (p,q)-differentiation and (p,q)-integration, one can refer to Theorems 2.3 and 2.4 and Lemma 2.8 in [20].

First, we state the following lemma for the sake of completeness, which will be used in our next discussion.

Lemma 3.2. *[20]* For any $h \in C([0,1],\mathbb{R})$, the boundary value problem

$$\begin{cases} D^2_{p,q}u(t) + h(t) = 0, \quad t \in (0,1), \\ u(0) = \int_0^1 u(t)d_{p,q}t, \\ u(1) = \int_0^1 tu(t)d_{p,q}t, \end{cases} \tag{3.6}$$

is equivalent to the integral equation

$$\begin{aligned} u(t) = &-\frac{1}{p}\int_0^t (t-qs)h(\frac{s}{p})d_{p,q}s + \frac{1}{p}\cdot\frac{p+q}{p+q-1}\int_0^1 (1-qs)h(\frac{s}{p})d_{p,q}s \\ &- \frac{p^2-q^2}{p^3(p+q-1)(p^2+pq+q^2)}\int_0^1 (s-qs^2) \\ &\Big[(p^2+pq+q^2)(1-t)(p+q-1) + s + p^2 + (p+q)(q-1)\Big]h(\frac{s}{p^2})d_{p,q}s. \end{aligned} \tag{3.7}$$

We can now state our main result in this section. To this end, we denote for brevity

$$\delta = \frac{1}{p} + \frac{p-q}{p\,(p^2+pq+q^2)} + \frac{1}{(p+q-1)\,(p^2+pq+q^2)} + \frac{(p-q)\Big(p^2+(p+q)(q-1)\Big)}{p(p+q-1)\,(p^2+pq+q^2)}.$$

Theorem 3.3. Suppose that:

(i) $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function;
(ii) $|f(t,u) - f(t,v)| \le |u-v|$ for all $t \in [0,1]$ and $u, v \in \mathbb{R}$;
(iii) there exist $\psi \in \Psi$, $\phi \in \Phi_u$ and $G \in C$ such that

$$\psi(|f(t,u) - f(t,v)|) \le G(\psi(|u-v|), \phi(|u-v|)). \tag{3.8}$$

Then the boundary value problem (3.2) has a unique solution $u^* \in C([0,1],\mathbb{R})$.

Proof. Define a mapping T by

$$\begin{aligned} Tu = &-\frac{1}{p}\int_0^t (t-qs)h\left(\frac{s}{p}\right)d_{p,q}s + \frac{1}{p}\cdot\frac{p+q}{p+q-1}\int_0^1 (1-qs)h\left(\frac{s}{p}\right)d_{p,q}s \\ &- \frac{p^2-q^2}{p^3(p+q-1)\,(p^2+pq+q^2)}\int_0^1 \left(s-qs^2\right) \\ &\Big[\left(p^2+pq+q^2\right)(1-t)(p+q-1) + s + p^2 + (p+q)(q-1)\Big]h\left(\frac{s}{p^2}\right)d_{p,q}s. \end{aligned} \tag{3.9}$$

In view of Lemma 3.2, the boundary value problem (3.2) is equivalent to the fixed point problem

$$x = Tx, \tag{3.10}$$

where $T: X \to X$ is the integral operator defined by the right-hand side of (3.7) and $X = C([0,1], \mathbb{R})$.
It is well known that X is a Banach space concerning the sup norm, defined by

$$\|u\| = \sup_{t\in[0,1]} |u(t)|, \text{ for all } u \in X.$$

It follows from assumptions (i) and (ii) that

$$\begin{aligned}
\psi(\|Tu - Tv\|) &= \psi(\sup_{t\in[0,1]} |(Tu)(t) - (Tv)(t)|) \\
&\leq \psi(\frac{1}{p} \sup_{t\in[0,1]} \int_0^t |t - qs| \cdot |f(s, u(ps)) - f(s, v(ps))| d_{p,q}s \\
&+ \frac{1}{p} \cdot \frac{p+q}{p+q-1} \sup_{t\in[0,1]} \int_0^1 |1 - qs| \cdot |f(s, u(ps)) - f(s, v(ps))| d_{p,q}s \\
&+ \frac{p^2 - q^2}{p^3(p+q-1)(p^2+pq+q^2)})
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{t\in[0,1]} \int_0^1 [|s - qs^2|[(p^2+pq+q^2)(1-t)(p+q-1) \\
&+ s + p^2 + (p+q)(q-1)]] \cdot |f(s, u(p^2 s)) - f(s, v(p^2 s))| d_{p,q}s \\
&\leq G[\psi(\sup_{t\in[0,1]} (|u - v|)), \phi(\sup_{t\in[0,1]} (|u - v|))] \\
&= G[\psi(\|u - v\|), \phi(\|u - v\|)].
\end{aligned}$$

Thus all assumptions of Corollary 2.2 are satisfied. Hence, T has a unique fixed point, that is, (3.2) has a unique solution.

Author contributions

M. Zhou, A. H. Ansari, C. Park, S. Maksimović, and Z. D. Mitrović: Conceptualization; A. H. Ansari and M. Zhou: Investigation; M. Zhou, A. H. Ansari, C. Park, S. Maksimović, and Z. D. Mitrović: Investigation; A. H. Ansari and M. Zhou: Writing review and editing; M. Zhou: Writing review and editing. M. Zhou: Funding acquisition. All authors read and approved the final manuscript.

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Conflict of interest

The authors declare that they have no competing interests.

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