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Research article

Estimates related to Caputo derivatives using generalized modified h-convex functions

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Abstract: In the present work, we have established some new fractional integral inequalities for functions whose kth-derivatives are generalized modified h-convex and symmetric about the midpoint involving the Caputo fractional derivatives. Many particular cases are obtained by using the findings.

Keywords: modified h-convexity; convexity; Caputo fractional derivatives
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1. Introduction

The fractional derivatives with constant or variable order [3, 9] are excellent mathematical tools for the description of memory and the hereditary properties of various processes and materials [12, 19]. In fractional calculus,these derivatives are defined through fractional integrals. There are several approaches to fractional derivatives including Riemann-Liouville [10, 14, 15], Caputo, Hadamard derivatives, [4, 6, 13, 17].

Efforts have been dedicated to generalizations concerning mappings of bounded variation, absolute continuity, various classes of convex functions, and their extension to fractional calculus, involving Riemann-Liouville integrals and their generalizattions as referenced in [1, 2, 12, 15].

In [8], the author proved some integral inequalities for functions whose kth ($k \in \mathbb{N}$) derivatives are convex involving Caputo derivatives and obtain the following results for $a, \Delta \in I, a < \Delta$, $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 1$, and $\psi : I \to \mathbb{R}$:

- If $\psi^{(k)}$ $(k \in \mathbb{N})$ exists and is positive and convex, then

$$\Gamma(k-\alpha+1)^C D_{a_+}^{\alpha-1}\psi(\xi) + (-1)^k\Gamma(k-\beta+1)^C D_{\Delta-}^{\beta-1}\psi(\xi) \leq (\xi-a)^{k-\alpha+1}\frac{\psi^{(k)}(a)+\psi^{(k)}(\xi)}{2} + (\Delta-\xi)^{k-\beta+1}\frac{\psi^{(k)}(\Delta)+\psi^{(k)}(\xi)}{2}. \tag{1.1}$$

- If $\psi^{(k)}$ exists and is positive, convex and symmetric about $\frac{a+\Delta}{2}$, then

$$\frac{1}{2}\left(\frac{1}{k-\alpha+1}+\frac{1}{k-\beta+1}\right)\psi^{(k)}\left(\frac{a+\Delta}{2}\right) \leq \frac{\Gamma(k-\beta+1)^C D_{\Delta_-}^{\beta-1}\psi(\alpha)}{2(\Delta-a)^{k-\beta+1}} + (-1)^k\frac{\Gamma(k-\alpha+1)^C D_{a_+}^{\alpha-1}\psi(\Delta)}{2(\Delta-a)^{k-\alpha+1}} \leq \frac{\psi^{(k)}(\Delta)+\psi^{(k)}(a)}{2}. \tag{1.2}$$

In [11], the authors gave a version of Hadamard's inequality using the Caputo derivative. In [7], the authors proved Hadamard inequalities for strongly α, m-convex functions via Caputo fractional derivatives. In this paper, we consider the Caputo derivatives of a real valued function ψ whose derivatives $\psi^{(k)}$ $(k \in \mathbb{N})$ are genaralized modified h-convex. Some Caputo fractional versions of Hermite-Hadamard inequalities are obtained. From which particular cases are revealed, we have also established a new integral inequality between Caputo derivatives ${}^C D^{\alpha}_{\cdot}\psi$ and the Riemann-Liouville integrals $R^{k-\alpha}_{\cdot}(\psi^{(k)})^2$. By deriving new differential inequalities in this context, we aim to extend the applicability of fractional calculus to problems involving generalized convex functions. These results have significance in various fields, including mathematics, physics, and engineering, where fractional calculus plays a crucial role in modeling complex phenomena with memory and long-range dependence.sts Our results generalize those cited in [8] and unify several classes of functions,like convex and s-convex functions.

2. Preliminaries

This section deals with some definitions of convexity [2, 5, 8], generalized h-convexity [20], fractional integrals and derivatives [6, 18].

Let $I \subset \mathbb{R}$ be an interval and $h : [0, 1] \to (0, \infty)$, $\psi : I \to (0, \infty)$ be two real valued functions, then

- ψ is said to be h-convex, if

$$\psi(\rho c + (1-\rho)d) \leq h(\rho)\psi(c) + h(1-\rho)\psi(d) \tag{2.1}$$

  holds for all $c, d \in I$ and $\rho \in (0, 1]$. If (2.1) is reversed, then ψ is said to be h-concave.
- The function ψ is said to be modified h-convex if

$$\psi(\rho c + (1-\rho)d) \leq h(\rho)\psi(c) + (1-h(\rho))\psi(d). \tag{2.2}$$

- The function ψ is said to be generalized modified h-convex if

$$\psi(\rho c + (1-\rho)d) \leq \psi(d) + h(\rho)\theta(\psi(c), \psi(d)). \tag{2.3}$$

Definition 2.1 (Additivity). *[20] A continuous bifunction θ is said to be additive, if*

$$\theta(a_1, b_1) + \theta(a_2, b_2) = \theta(a_1 + a_2, b_1 + b_2), \quad \forall a_1, a_2, b_1, b_2 \in \mathbb{R}.$$

Definition 2.2 (Nonnegative homogeneity). *[20] A continuous bifunction θ is said to be nonnegatively homogeneous if, for all $\lambda > 0$,*

$$\theta(\lambda a_1, \lambda a_2) = \lambda \theta(a_1, a_2), \quad \forall a_1, a_2 \in \mathbb{R}.$$

Remark 2.1. *For different functions h, θ one can obtain various classes of generalized modified convex functions:*

- *By taking in (2.1) $h(z) = z^s (0 < s \leq 1)$, we have the definition of modified generalized s-convex functions.*
- *If, we take $\theta(r, z) = r - z$, then we obtain the definition of a modified h-convex function.*

Let $[a, \Delta]$ $(-\infty < a < \Delta < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. For any function $\psi \in L_1([a, \Delta])$, the Riemann-Liouville fractional integrals $R^{\alpha}_{a_+}$ and $R^{\alpha}_{\Delta_-}$ of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ of ψ are defined by

$$R^{\alpha}_{a_+}\psi(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (s - t)^{\alpha - 1} \psi(t) dt, \quad s > a \quad (left) \tag{2.4}$$

and

$$R^{\alpha}_{\Delta_-}\psi(s) = \frac{1}{\Gamma(\alpha)} \int_s^{\Delta} (t - s)^{\alpha - 1} \psi(t) dt, \quad s < \Delta \quad (right), \tag{2.5}$$

respectively. Here $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha - 1} e^{-t} \, dt$, $\alpha > 0$ is the gamma function. We set $R^0_{a+}\psi = R^0_{\Delta-}\psi = \psi$.

Let $[a, \Delta]$ be a finite interval of the real line $\mathbb{R}$. Let $\alpha > 0, k \in \mathbb{N}$, $k = [\alpha] + 1$ and $\psi \in AC^k([a, \Delta])$ ($AC^k([a, \Delta])$ means the space of complex-valued functions $\psi(x)$ which have continuous derivatives up to order $k-1$ on $[a, b]$ such that $\psi^{(k-1)}(x) \in AC([a, \Delta])$: i.e., absolutely continuous) see Lemma 2.4 [18]. The left and right Caputo fractional derivatives of order α $(\alpha \geq 0)$ of ψ are given by the following formulas (see [1, 4, 10, 13])

$$^{C}D^{\alpha}_{a_+}\psi(\xi) = \frac{1}{\Gamma(k - \alpha)} \int_a^{\xi} \psi^{(k)}(t)(\xi - t)^{k - \alpha - 1} dt, \qquad \xi > a$$

and

$$^{C}D^{\alpha}_{\Delta_-}\psi(\xi) = \frac{(-1)^k}{\Gamma(k - \alpha)} \int_{\xi}^{\Delta} \psi^{(k)}(t)(t - \xi)^{k - \alpha - 1} dt, \qquad \xi < \Delta,$$

respectively.

If $\alpha = k \in \mathbb{N}$, then

$$^{C}D^{\alpha}_{a+}\psi(\xi) = \psi^{(k)}(\xi) \quad \text{and} \quad ^{C}D^{\alpha}_{\Delta-}\psi(\xi) = (-1)^k \psi^{(k)}(\xi).$$

In particular, if $k = 1$, $\alpha = 0$, then

$$^{C}D^{0}_{a_+}\psi(\xi) = ^{C}D^{0}_{\Delta_-}\psi(\xi) = \psi(\xi).$$

Lemma 2.1. *[16] The following formulas for Caputo fractional derivatives of order $\alpha > 0, k-1 < \alpha < k\,(k \in \mathbb{N})$ of a power function at $t = a$ and $t = b$ hold*

$$ {}^{C}D^{\alpha}_{a_+}(t-a)^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(t-a)^{p-\alpha}, \qquad t > a \tag{2.6} $$

and

$$ {}^{C}D^{\alpha}_{b_-}(b-t)^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(b-t)^{p-\alpha}, \qquad t < b. \tag{2.7} $$

Our objective in this work, is to prove some fractional integral inequalities for functions whose kth ($k \in \mathbb{N}$) derivatives are generalized modified h-convex functions involving the Caputo derivative operator.

3. Results

Theorem 3.1. *Let I be an interval of $\mathbb{R}$, $a, \Delta \in I, a < \Delta$ and $\alpha, \beta > 0$, such that $k-1 < \alpha, \beta < k, k \in \mathbb{N}$. Let $\psi : I \to \mathbb{R}$ be differentiable function. If, $\psi^{(k)}(k \in \mathbb{N})$ exists and is a positive generalized modified h-convex function and θ is a continuous bifunction, then the following integral inequality*

$$ \begin{aligned} &\Gamma(k-\alpha+1)\left({}^{C}D^{\alpha-1}_{a_+}\psi\right)(\xi) + (-1)^k\Gamma(k-\beta+1)\left({}^{C}D^{\beta-1}_{\Delta_-}\psi\right)(\xi) \\ &\leq (\Delta-\xi)^{k-\beta+1}\left[\psi^{(k)}(\xi) + \theta\left(\psi^{(k)}(\Delta), \psi^{(k)}(\xi)\right)\int_0^1 h(z)dz\right] \\ &\quad + (\xi-a)^{k-\alpha+1}\left[\psi^{(k)}(\xi) + \theta\left(\psi^{(k)}(a), \psi^{(k)}(\xi)\right)\int_0^1 h(z)dz\right] \end{aligned} \tag{3.1} $$

holds.

Proof. For all $\xi \in [a, \Delta]$ and for all $t \in [a, \xi]$, we have

$$ (\xi - t)^{k-\alpha} \leq (\xi - a)^{k-\alpha}, \tag{3.2} $$

and

$$ t = \frac{\xi - t}{\xi - a}a + \frac{t-a}{\xi - a}\xi. $$

Since $\psi^{(k)}$ is generalized modified h-convex, (2.3) implies that

$$ \psi^{(k)}(t) \leq \psi^{(k)}(\xi) + h\left(\frac{\xi - t}{\xi - a}\right)\theta\left(\psi^{(k)}(a), \psi^{(k)}(\xi)\right). \tag{3.3} $$

Multiplying inequalities (3.2) and (3.3) on both side and integrating, we obtain

$$ \begin{aligned} &\int_a^{\xi}(\xi - t)^{k-\alpha}\,\psi^{(k)}(t)dt \\ &\leq \int_a^{\xi}(\xi-a)^{k-\alpha} \times \left[\psi^{(k)}(\xi) + h\left(\frac{\xi - t}{\xi - a}\right)\theta\left(\psi^{(k)}(a), \psi^{(k)}(\xi)\right)\right]dt. \end{aligned} \tag{3.4} $$

That is

$$\Gamma(k-\alpha+1)\left({}^{C}D_{a_+}^{\alpha-1}\psi\right)(\xi) \le (\xi-a)^{k-\alpha+1} \times \left[\psi^{(k)}(\xi) + \theta\left(\psi^{(k)}(a), \psi^{(k)}(\xi)\right)\int_0^1 h(z)dz\right]. \tag{3.5}$$

Let $\xi \in [a, \Delta], t \in [\xi, \Delta]$, thus

$$(t-\xi)^{k-\beta} \le (\Delta-\xi)^{k-\beta}. \tag{3.6}$$

We have

$$t = \frac{t-\xi}{\Delta-\xi}\Delta + \frac{\Delta-t}{\Delta-\xi}\xi.$$

Since $\psi^{(k)}$ is generalized modified h-convex on $[\alpha, \Delta]$, then

$$\psi^{(k)}(t) \le \psi^{(k)}(\xi) + h\left(\frac{t-\xi}{\Delta-\xi}\right)\theta\left(\psi^{(k)}(\Delta), \psi^{(k)}(\xi)\right). \tag{3.7}$$

Similarly, we obtain

$$(-1)^k\Gamma(k-\beta+1)\left({}^{C}D_{\Delta_-}^{\beta-1}\psi\right)(\xi) \le (\Delta-\xi)^{k-\beta+1} \times \left[\psi^{(k)}(\xi) + \theta\left(\psi^{(k)}(\Delta), \psi^{(k)}(\xi)\right)\int_0^1 h(z)dz\right]. \tag{3.8}$$

Adding (3.5) and (3.8), the claim follows. □

Corollary 3.1. *If, we set $\alpha = \beta$ in (3.1), then we obtain*

$$\begin{aligned}
&\Gamma(k-\alpha+1)\left[\left({}^{C}D_{a_+}^{\alpha-1}\psi\right)(\xi) + (-1)^k\left({}^{C}D_{\Delta_-}^{\alpha-1}\psi\right)(\xi)\right] \\
&\le (\Delta-\xi)^{k-\alpha+1}\left[\psi^{(k)}(\xi) + \theta\left(\psi^{(k)}(\Delta), \psi^{(k)}(\xi)\right)\int_0^1 h(z)dz\right] \\
&+ (\xi-a)^{k-\alpha+1}\left[\psi^{(k)}(\xi) + \theta\left(\psi^{(k)}(a), \psi^{(k)}(\xi)\right)\int_0^1 h(z)dz\right].
\end{aligned}$$

Corollary 3.2. *By setting $\theta(r,z) = r-z, h(t) = t^s, s \in [0,1]$ in (3.1), we obtain*

$$\begin{aligned}
&\Gamma(k-\alpha+1)\left({}^{C}D_{a_+}^{\alpha-1}\psi\right)(\xi) + (-1)^k\Gamma(k-\beta+1)\left({}^{C}D_{\Delta_-}^{\beta-1}\psi\right)(\xi) \\
&\le \frac{(\Delta-\xi)^{k-\beta+1}\,\psi^{(k)}(\Delta) + (\xi-a)^{k-\alpha+1}\,\psi^{(k)}(a)}{s+1} \\
&+ \frac{(\xi-a)^{k-\alpha+1} + (\Delta-\xi)^{k-\beta+1}}{s+1}\, s\,\psi^{(k)}(\xi).
\end{aligned} \tag{3.9}$$

In particular, if $h(z) = z$, then we have

$$\begin{aligned}
&\Gamma(k-\alpha+1)\left({}^{C}D_{\alpha_+}^{\alpha-1}\psi\right)(\xi) + (-1)^k\Gamma(k-\beta+1)\left({}^{C}D_{\Delta_-}^{\beta-1}\psi\right)(\xi) \\
&\le \frac{(\Delta-\xi)^{k-\beta+1}\,\psi^{(k)}(\Delta) + (\xi-a)^{k-\alpha+1}\,\psi^{(k)}(a)}{2} \\
&+ \frac{(\xi-a)^{k-\alpha+1} + (\Delta-\xi)^{k-\beta+1}}{2}\,\psi^{(k)}(\xi).
\end{aligned} \tag{3.10}$$

Taking $\alpha = \beta$ in (3.10), we obtain

$$\begin{aligned}
&\Gamma(k-\alpha+1)\left[\left({}^{C}D_{a_+}^{\alpha-1}\psi\right)(\xi)+(-1)^k\left({}^{C}D_{\Delta_-}^{\alpha-1}\psi\right)(\xi)\right]\\
&\le \frac{(\Delta-\xi)^{k-\alpha+1}\,\psi^{(k)}(\Delta)+(\xi-a)^{k-\alpha+1}\psi^{(k)}(a)}{2}\\
&+\frac{(\xi-a)^{k-\alpha+1}+(\Delta-\xi)^{k-\alpha+1}}{2}\,\psi^{(k)}(\xi).
\end{aligned} \tag{3.11}$$

Example 3.1. *Let $\psi : [a,\Delta] \to [0,\infty)$, $\psi(\xi) = \frac{2}{(k+2)!}(\xi-a)^{k+2}$, $a < \xi \le \Delta$. Let $h : [0,1] \to (0,\infty)$, $h(t) \ge t$, $\theta(x,y) = 2x + y$. We verify easly that $\psi^{(k)}(\xi) = (\xi-a)^2$ is generalized modified h-convex on $[a,\Delta]$. From Corollary 3.1 and Lemma 2.1, we obtain*

$$lhs := \Gamma(k-\alpha+1)\left({}^{C}D_{a_+}^{\alpha-1}\psi\right)(\xi) = \frac{2(\xi-a)^{k-\alpha+3}}{(k-\alpha+1)(k-\alpha+2)(k-\alpha+3)}, \tag{3.12}$$

and

$$\begin{aligned}
rhs :&= (\xi-a)^{k-\alpha+1}\left[(\xi-a)^2+(0+(\xi-a)^2)\int_0^1 h(z)dz\right]\\
&= (\xi-a)^{k-\alpha+3}\left(1+\int_0^1 h(z)dz\right).
\end{aligned} \tag{3.13}$$

For the right derivative $\left({}^{C}D_{\Delta_-}^{\alpha-1}\psi\right)(\xi)$, we consider the function $\psi(\xi) = \frac{2(\Delta-\xi)^{k+2}}{(k+2)!}$, $a \le \xi < \Delta$.

$$(-1)^k\Gamma(k-\alpha+1)\left({}^{C}D_{\Delta_-}^{\alpha-1}\psi\right)(\xi) = \frac{2(\Delta-\xi)^{k-\alpha+3}}{(k-\alpha+1)(k-\alpha+2)(k-\alpha+3)} \tag{3.14}$$

and

$$rhs := (\Delta-\xi)^{k-\alpha+3}\left(1+\int_0^1 h(z)dz\right). \tag{3.15}$$

Now let I be an interval of $\mathbb{R}$, $a,\Delta \in I$, $(a < \Delta)$ and $\alpha,\beta > 0$, such that $k-1 < \alpha,\beta < k$, $(k \in \mathbb{N})$. Let $\psi : I \to \mathbb{R}$. Assume that $|\psi^{(k+1)}|$ is generalized modified h- convex on $[a,\Delta]$.

It is clear that for all $\xi \in [a,\Delta]$, $t \in [a,\xi]$, we have

$$(\xi-t)^{k-\alpha} \le (\xi-a)^{k-\alpha}, \qquad t \in [a,\xi]. \tag{3.16}$$

Since $|\psi^{(k+1)}|$ is generalized modified h-convex, we have for $t \in [a,\xi]$,

$$\begin{aligned}
\textbf{Lhs}= &-\left[|\psi^{(k+1)}(\xi)|+\theta\left(|\psi^{(k+1)}(a)|,|\psi^{(k+1)}(\xi)|\right)h\left(\frac{t-a}{\xi-a}\right)\right]\\
&\le |\psi^{(k+1)}(t)| \le |\psi^{(k+1)}(\xi)|+\theta\left(|\psi^{(k+1)}(a)|,|\psi^{(k+1)}(\xi)|\right)h\left(\frac{t-a}{\xi-a}\right) = \textbf{Rhs}.
\end{aligned} \tag{3.17}$$

Multiplying (3.16) by the **Rhs** of inequality (3.17) and integrating the resulting inequality over $[a,\xi]$, we obtain

$$\int_a^{\xi}(\xi-t)^{k-\alpha}\,\psi^{(k+1)}(t)dt \le (\xi-a)^{k-\alpha}\left(|\psi^{(k+1)}(\xi)|+\theta\left(|\psi^{(k+1)}(a)|,|\psi^{(k+1)}(\xi)|\right)\int_0^1 h(z)dz\right), \tag{3.18}$$

by integration by parts, we have

$$\int_a^{\xi} (\xi - t)^{k-\alpha} \psi^{(k+1)}(t)dt = \psi^{(k)}(t)(\xi - t)^{k-\alpha} \mid_a^{\xi} + (k-\alpha) \int_a^{\xi} (\xi - t)^{k-\alpha-1} \psi^{(k)}(t)dt$$
$$= \Gamma(k-\alpha+1) \left({}^C D_{a+}^{\alpha} \psi\right)(\xi) - \psi^{(k)}(a)(\xi - a)^{k-\alpha}.$$

Hence

$$\Gamma(k-\alpha+1) \left({}^C D_{a_+}^{\alpha} \psi\right)(\xi) - \psi^{(k)}(a)(\xi - a)^{k-\alpha}$$
$$\leq \left(\psi^{(k+1)}(\xi) + \theta\left(|\psi^{(k+1)}(a)|, |\psi^{(k+1)}(\xi)|\right) \int_0^1 h(z)dz\right)(\xi - a)^{k-\alpha}. \quad (3.19)$$

In a similar way, if we proceed with the **Lhs** of (3.17) as we did for the **Rhs**, it follows that

$$\psi^{(k)}(a)(\xi - a)^{k-\alpha} - \Gamma(k-\alpha+1) \left({}^C D_{a_+}^{\alpha} \psi\right)(\xi)$$
$$\leq \left(|\psi^{(k+1)}(\xi)| + \theta\left(|\psi^{(k+1)}(a)|, |\psi^{(k+1)}(\xi)|\right) \int_0^1 h(z)dz\right)(\xi - a)^{k-\alpha}. \quad (3.20)$$

From (3.19) and (3.20), we obtain

$$|\Gamma(k-\alpha+1) \left({}^C D_{a_+}^{\alpha} \psi\right)(\xi) - \psi^{(k)}(a)(\xi - a)^{k-\alpha}|$$
$$\leq \left(|\psi^{(k+1)}(\xi)| + \theta\left(|\psi^{(k+1)}(a)|, |\psi^{(k+1)}(\xi)|\right) \int_0^1 h(z)dz\right)(\xi - a)^{k-\alpha}. \quad (3.21)$$

Doing the same for $t \in [\xi, \Delta]$ and $\beta > 0, k-1 < \beta < k$, and taking into acount that $|\psi^{(k+1)}|$ is generalized modified h-convex, we have

$$\textbf{Lhs=} -\left[|\psi^{(k+1)}(\xi)| + \theta\left(|\psi^{(k+1)}(\Delta)|, |\psi^{(k+1)}(\xi)|\right) h\left(\frac{t-\xi}{\Delta - \xi}\right)\right]$$
$$\leq \psi^{(k+1)}(t) \leq \psi^{(k+1)}(\xi) + \theta\left(|\psi^{(k+1)}(\Delta)|, |\psi^{(k+1)}(\xi)|\right) h\left(\frac{t-\xi}{\Delta - \xi}\right) = \textbf{Rhs}. \quad (3.22)$$

Hence

$$|\Gamma(k-\beta+1) \left({}^C D_{\Delta_-}^{\beta} \psi\right)(\xi) - \psi^{(k)}(\Delta)(\Delta - \xi)^{k-\beta}|$$
$$\leq (\Delta - \xi)^{k-\beta} \times \left[|\psi^{(k+1)}(\xi)| + \theta\left(|\psi^{(k+1)}(\Delta)|, |\psi^{(k+1)}(\xi)|\right) \int_0^1 h(z)dz\right]. \quad (3.23)$$

Combine (3.21) and (3.23) via triangular inequality, and we obtain the double inequality

$$\Big|\Gamma(k-\alpha+1) \left({}^C D_{a_+}^{\alpha} \psi\right)(\xi) + \Gamma(k-\beta+1) \left({}^C D_{\Delta_-}^{\beta} \psi\right)(\xi)$$
$$- \left(\psi^{(k)}(a)(\xi - a)^{k-\alpha} + \psi^{(k)}(\Delta)(\Delta - \xi)^{k-\beta}\right)\Big|$$
$$\leq (\Delta - \xi)^{k-\beta} \left[|\psi^{(k+1)}(\xi)| + \theta\left(|\psi^{(k+1)}(\Delta)|, |\psi^{(k+1)}(\xi)|\right) \int_0^1 h(z)dz\right]$$
$$+ (\xi - a)^{k-\alpha} \left[|\psi^{(k+1)}(\xi)| + \theta(|\psi^{(k+1)}(a)|, |\psi^{(k+1)}(\xi)|) \int_0^1 h(z)dz\right]. \quad (3.24)$$

Which leads to the following result:

Theorem 3.2. *Let I be an interval of $\mathbb{R}$, $a, \Delta \in I\,(a < \Delta)$ and $\alpha, \beta > 0$, such that $k-1 < \alpha, \beta < k$, $(k \in \mathbb{N})$. Let $\psi : I \to \mathbb{R}$ be a function such that $\psi \in AC^{k+1}$. Assume that $|\psi^{(k+1)}|$ is a generalized modified h-convex function and θ a continuous bifunction, then*

$$\begin{aligned}&\Big|\Gamma(k-\alpha+1)\left({}^{C}D^{\alpha}_{a_+}\psi\right)(\xi)+\Gamma(k-\beta+1)\left({}^{C}D^{\beta}_{\Delta_-}\psi\right)(\xi)\\&\qquad-\left(\psi^{(k)}(a)(\xi-a)^{k-\alpha}+\psi^{(k)}(\Delta)(\Delta-\xi)^{k-\beta}\right)\Big|\\&\le(\Delta-\xi)^{k-\beta}\left[|\psi^{(k+1)}(\xi)|+\theta\left(|\psi^{(k+1)}(\Delta)|,|\psi^{(k+1)}(\xi)|\right)\int_0^1 h(z)dz\right]\\&\quad+(\xi-a)^{k-\alpha}\left[|\psi^{(k+1)}(\xi)|+\theta\left(|\psi^{(k+1)}(a)|,|\psi^{(k+1)}(\xi)|\right)\int_0^1 h(z)dz\right]\end{aligned}\tag{3.25}$$

holds.

As a consequences, we have

Corollary 3.3. *If in* (3.25)*, we set $\alpha = \beta$, then*

$$\begin{aligned}&\left|\Gamma(k-\alpha+1)({}^{C}D^{\alpha}_{a_+}\psi(\xi)+{}^{C}D^{\alpha}_{\Delta_-}\psi(\xi))-\left(\psi^{(k)}(a)(\xi-a)^{k-\alpha}+\psi^{(k)}(\Delta)(\Delta-\xi)^{k-\alpha}\right)\right|\\&\le(\Delta-\xi)^{k-\alpha}\left[|\psi^{(k+1)}(\xi)|+\theta\left(|\psi^{(k+1)}(\Delta)|,|\psi^{(k+1)}(\xi)|\right)\int_0^1 h(z)dz\right]\\&\quad+(\xi-a)^{k-\alpha}\left[|\psi^{(k+1)}(\xi)|+\theta\left(|\psi^{(k+1)}(a)|,|\psi^{(k+1)}(\xi)|\right)\int_0^1 h(z)dz\right]\end{aligned}\tag{3.26}$$

holds.

Corollary 3.4. *By taking $\theta(z,r) = z - r, h(t) = t^s$, $s \in [0,1]$ in* (3.26)*, we obtain*

$$\begin{aligned}&|\Gamma(k-\alpha+1)[\left({}^{C}D^{\alpha}_{a_+}\psi\right)(\xi)+\left({}^{C}D^{\alpha}_{\Delta_-}\psi\right)(\xi)]-\left(\psi^{(k)}(a)(\xi-a)^{k-\alpha}+\psi^{(k)}(\Delta)(\Delta-\xi)^{k-\alpha}\right)|\\&\qquad\le\frac{s\left((\xi-a)^{k-\alpha}+(\Delta-\xi)^{k-\alpha}\right)|\psi^{(k+1)}(\xi)|}{s+1}+\frac{(\xi-a)^{k-\alpha}|\psi^{(k+1)}(a)|+(\Delta-\xi)^{k-\alpha}|\psi^{(k+1)}(\Delta)|}{s+1}.\end{aligned}\tag{3.27}$$

In particular for $s = 1$, we have

$$\begin{aligned}&|\Gamma(k-\alpha+1)[\left({}^{C}D^{\alpha}_{a_+}\psi\right)(\xi)+\left({}^{C}D^{\alpha}_{\Delta_-}\psi\right)(\xi)]-\left(\psi^{(k)}(a)(\xi-a)^{k-\alpha}+\psi^{(k)}(\Delta)(\Delta-\xi)^{k-\alpha}\right)|\\&\qquad\le\frac{\left((\xi-a)^{k-\alpha}+(\Delta-\xi)^{k-\alpha}\right)|\psi^{(k+1)}(\xi)|}{2}+\frac{(\xi-a)^{k-\alpha}|\psi^{(k+1)}(a)|+(\Delta-\xi)^{k-\alpha}|\psi^{(k+1)}(\Delta)|}{2}.\end{aligned}\tag{3.28}$$

Example 3.2. *Let ψ, h, θ as in the Example 3.1. We verify easily that $\psi^{(k+1)}(\xi) = 2(\xi - a)$ is generalized modified h-convex on $[a, \Delta]$. From Corollary 3.3 and Lemma 2.1, we obtain*

$$lhs := \Gamma(k-\alpha+1)\,{}^{C}D^{\alpha}_{a_+}\psi(\xi)=\frac{2(\xi-a)^{k-\alpha+2}}{(k-\alpha+1)(k-\alpha+2)},\tag{3.29}$$

and

$$rhs := (\xi-a)^{k-\alpha}\left[2(\xi-a)+(0+2(\xi-a))\int_0^1 h(z)dz\right]=2\,(\xi-a)^{k-\alpha+1}\left(1+\int_0^1 h(z)dz\right).$$

For the right derivative ${}^{C}D^{\alpha}_{\Delta_-}\psi(\xi)$, *we have*

$$lhs := (-1)^k\Gamma(k-\alpha+1)^C D^{\alpha}_{\Delta_-}\psi(\xi) = \frac{2(\Delta-\xi)^{k-\alpha+2}}{(k-\alpha+1)(k-\alpha+2)} \tag{3.30}$$

and

$$rhs := 2\,(\Delta-\xi)^{k-\alpha+1}\left(1+\int_0^1 h(z)dz\right). \tag{3.31}$$

Now suppose that $\psi : [a,\Delta] \to (0,\infty)$ is a generalized modified h-convex function and symmetric about $\dfrac{a+\Delta}{2}$, then for all $\xi \in [a,\Delta]$ the inequality

$$\psi\left(\frac{a+\Delta}{2}\right) \le \psi(\xi)\left(1+h\left(\frac{1}{2}\right)\theta(1,1)\right) \tag{3.32}$$

is valid. Here θ is assumed to be nonnegatively homogeneous. Indeed, set

$$r = a\frac{\xi-a}{\Delta-a} + \Delta\frac{\Delta-\xi}{\Delta-a}, \quad z = \Delta\frac{\xi-a}{\Delta-a} + \alpha\frac{\Delta-\xi}{\Delta-a}.$$

Hence

$$\frac{a+\Delta}{2} = \frac{r}{2} + \frac{z}{2}.$$

Since ψ is generalized modified h-convex, symmetric about $\frac{a+\Delta}{2}$, and the bifunction θ is assumed to be nonnegatively homogeneous, it results in

$$\begin{aligned}\psi\left(\frac{a+\Delta}{2}\right) &= \psi\left(\frac{r}{2}+\frac{z}{2}\right)\\ &\le \psi(z) + h\left(\frac{1}{2}\right)\theta(\psi(r),\psi(z))\\ &= \psi(\xi) + h\left(\frac{1}{2}\right)\theta(\psi(\xi),\psi(\xi))\\ &= \psi(\xi)\left(1+h\left(\frac{1}{2}\right)\theta(1,1)\right).\end{aligned}$$

Theorem 3.3. *Let I be an interval of $\mathbb{R}$, $a,\Delta \in I$ $(a < \Delta)$ and $\alpha,\beta \ge 1$, $k-1 < \alpha,\beta < k, k \in \mathbb{N}$. Let $\psi : I \to \mathbb{R}$ be a real valued function such that $\psi \in AC^k$. If $\psi^{(k)}$ is a positive, generalized modified h-convex and symmetric about $\frac{a+\Delta}{2}$ and furthermore the bifunction θ is nonnegatively homogeneous, then the following inequality holds*

$$\begin{aligned}N_\theta^{-1}\left\{\frac{\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{k-\beta+1} + \frac{\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{k-\alpha+1}\right\} &\le \frac{\Gamma(k-\beta+1)\left({}^{C}D^{\beta-1}_{\Delta_-}\psi\right)(a)}{(\Delta-a)^{k-\beta+1}} + \frac{\Gamma(k-\alpha+1)\left({}^{C}D^{\alpha-1}_{a_+}\psi\right)(\Delta)}{(\Delta-a)^{k-\alpha+1}}\\ &\le \psi^{(k)}(\Delta) + \psi^{(k)}(a) + \left[\theta(\psi^{(k)}(\Delta),\psi^{(k)}(a)) + \theta(\psi^{(k)}(a),\psi^{(k)}(\Delta))\right]\int_0^1 h(z)dz. \quad (3.33)\end{aligned}$$

If, furthermore , θ is additive, then

$$N_\theta^{-1}\left\{\frac{\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{k-\beta+1}+\frac{\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{k-\alpha+1}\right\}\leq\frac{\Gamma(k-\beta+1)\left({}^{C}D_{\Delta_-}^{\beta-1}\psi\right)(\alpha)}{(\Delta-a)^{k-\beta+1}}+\frac{\Gamma(k-\alpha+1)\left({}^{C}D_{a_+}^{\alpha-1}\psi\right)(\Delta)}{(\Delta-a)^{k-\alpha+1}}$$
$$\leq M_\theta(\psi^{(k)}(\Delta)+\psi^{(k)}(a)) \tag{3.34}$$

holds. Here

$$N_\theta=1+h\left(\frac{1}{2}\right)\theta(1,1),\ M_\theta=1+\theta(1,1)\int_0^1 h(z)dz.$$

Proof. For all $\xi\in[a,\Delta], k-1<\alpha<k$, we have $\xi=\frac{\Delta-\xi}{\Delta-a}a+\frac{\xi-a}{\Delta-a}\Delta$ and

$$(\xi-\alpha)^{k-\alpha}\leq(\Delta-a)^{k-\alpha} \tag{3.35}$$

and $\psi^{(k)}$ satisfies

$$\psi^{(k)}(\xi)\leq\psi^{(k)}(a)+h\left(\frac{\xi-a}{\Delta-a}\right)\theta\left(\psi^{(k)}(\Delta),\psi^{(k)}(a)\right). \tag{3.36}$$

Multiplying (3.35) and (3.36) and proceeding as above, we obtain

$$\Gamma(k-\alpha+1)\left({}^{C}D_{\Delta-}^{\alpha-1}\psi\right)(a)$$
$$\leq\left[\psi^{(k)}(a)+\theta\left(\psi^{(k)}(\Delta),\psi^{(k)}(a)\right)\int_0^1 h(z)dz\right]\times(\Delta-\alpha)^{k-\alpha+1}. \tag{3.37}$$

Also, we have for $\xi\in[a,\Delta], k-1<\beta<k$,

$$(\Delta-\xi)^{k-\beta}\leq(\Delta-a)^{k-\beta} \tag{3.38}$$

and

$$\psi^{(k)}(\xi)\leq\psi^{(k)}(\Delta)+h\left(\frac{\Delta-\xi}{\Delta-a}\right)\theta(\psi^{(k)}(a),\psi^{(k)}(\Delta)). \tag{3.39}$$

Multiplying (3.39) and (3.38) and integrating over $[a,\Delta]$, we get

$$\Gamma(k-\beta+1)\left({}^{C}D_{\Delta_-}^{\beta-1}\psi\right)(a)\leq\left[\psi^{(k)}(\Delta)+\theta\left(\psi^{(k)}(a),\psi^{(k)}(\Delta)\right)\int_0^1 h(z)dz\right](\Delta-a)^{k-\beta+1}. \tag{3.40}$$

Adding (3.37) and (3.40), we obtain

$$\frac{\Gamma(k-\beta+1)\left({}^{C}D_{\Delta_-}^{\beta-1}\psi\right)(\alpha)}{(\Delta-a)^{k-\beta+1}}+\frac{\Gamma(k-\alpha+1)\left({}^{C}D_{a_+}^{\alpha-1}\psi\right)(\Delta)}{(\Delta-a)^{k-\alpha+1}} \tag{3.41}$$

$$\leq\psi^{(k)}(\Delta)+\psi^{(k)}(a)+\left[\theta(\psi^{(k)}(\Delta),\psi^{(k)}(a))+\theta(\psi^{(k)}(a),\psi^{(k)}(\Delta))\right]\int_0^1 h(z)dz. \tag{3.42}$$

Set $N_\theta=1+h\left(\frac{1}{2}\right)\theta(1,1)$, thus (3.32) is written as

$$\psi^{(k)}\left(\frac{a+\Delta}{2}\right)\leq N_\theta\,\psi^{(k)}(\xi),\quad \xi\in[a,\Delta]. \tag{3.43}$$

Multiplying by $(\xi - a)^{k-\alpha}$ on both sides of (3.43) and integrating the result over $[a, \Delta]$, it results that

$$N_\theta^{-1} \frac{\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{k-\alpha+1} \leq \frac{\Gamma(k-\alpha+1)\left({}^C D_{\Delta_-}^{\alpha-1}\psi\right)(a)}{(\Delta-a)^{k-\alpha+1}}. \tag{3.44}$$

Multiplying (3.43) by $(\Delta - \xi)^{k-\beta}$, and integrating over $[a, \Delta]$, we obtain

$$N_\theta^{-1} \frac{\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{k-\beta+1} \leq \frac{\Gamma(k-\beta+1)\left({}^C D_{a_+}^{\beta-1}\psi\right)(\Delta)}{(\Delta-a)^{k-\beta+1}}. \tag{3.45}$$

Adding (3.44) and (3.45), we obtain the first inequality. By combining the resulting inequality with (3.41), we obtain (3.33). Using the fact that θ is additive and nonnegatively homogeneous (3.34) results. That proves the claim. □

Corollary 3.5. *By taking $\alpha = \beta$ in (3.33), then*

$$\begin{aligned} N_\theta^{-1} \frac{2\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{k-\alpha+1} &\leq \frac{\Gamma(k-\alpha+1)\left({}^C D_{a_+}^{\alpha-1}\psi(\Delta) + {}^C D_{\Delta_-}^{\alpha-1}\psi(a)\right)}{(\Delta-a)^{k-\alpha+1}} \\ &\leq \psi^{(k)}(\Delta) + \psi^{(k)}(a) \\ &+ \left[\theta(\psi^{(k)}(\Delta), \psi^{(k)}(a)) + \theta(\psi^{(k)}(a), \psi^{(k)}(\Delta))\right] \int_0^1 h(z)dz \end{aligned} \tag{3.46}$$

holds.

If, θ is additive, then

$$\begin{aligned} \frac{2N_\theta^{-1}\,\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{k-\alpha+1} &\leq \frac{\Gamma(k-\alpha+1)\left({}^C D_{a_+}^{\alpha-1}\psi(\Delta) + {}^C D_{\Delta_-}^{\alpha-1}\psi(a)\right)}{(\Delta-a)^{k-\alpha+1}} \\ &\leq M_\theta\,(\psi^{(k)}(\Delta) + \psi^{(k)}(a)). \end{aligned} \tag{3.47}$$

Corollary 3.6. *By setting $h(t) = t^s$, $s \in [0, 1]$ in (3.47), it results that*

$$\begin{aligned} &\frac{2^s\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{(2^s+\theta(1,1))(k-\alpha+1)} \\ &\leq \frac{\Gamma(k-\alpha+1)\left[\left({}^C D_{\Delta_-}^{\alpha+1}\psi\right)(a) + \left({}^C D_{a_+}^{\alpha+1}\psi\right)(\Delta)\right]}{(\Delta-a)^{k-\alpha+1}} \\ &\leq \frac{\psi^{(k)}(a) + \psi^{(k)}(\Delta)}{s+1}(s+1+\theta(1,1)). \end{aligned}$$

In particular, if $h(t) = t$, then

$$
\begin{aligned}
&\frac{2\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{(2+\theta(1,1))(k-\alpha+1)}\\
&\leq \frac{\Gamma(k-\alpha+1)\left[\left({}^{C}D^{\alpha+1}_{\Delta_-}\psi\right)(a)+\left({}^{C}D^{\alpha+1}_{a_+}\psi\right)(\Delta)\right]}{(\Delta-a)^{k-\alpha+1}}\\
&\leq \frac{\psi^{(k)}(a)+\psi^{(k)}(\Delta)}{2}(2+\theta(1,1)).
\end{aligned}
$$

Theorem 3.4. *Let $\psi \in AC^k(a,\Delta)$, $k \in N; k-1<\alpha<k$. Assume that $\psi^{(k)}$ is positive, generalized modified h-convex on $[a,\Delta]$ and symmetric to $\dfrac{a+\Delta}{2}$. Assume that θ is nonnegatively homogeneous. Then*

$$
\frac{\psi^{(k)}\left(\dfrac{a+\Delta}{2}\right)}{1+h(\frac{1}{2})\theta(1,1)}\left[\left({}^{C}D^{\alpha}_{\Delta_-}\psi\right)(a)+\left({}^{C}D^{\alpha}_{a_+}\psi\right)(\Delta)\right] \leq R^{k-\alpha}_{\Delta-}(\psi^{(k)})^2(a)+R^{k-\alpha}_{a+}(\psi^{(k)})^2(\Delta) \tag{3.48}
$$

holds. Where $R^{k-\alpha}_{\cdot}$ is the Riemann-Liouville integral operator of order $k-\alpha$.

Proof. Since $\psi^{(k)}$ is generalized modified h-convex and θ is nonnegatively homogeneous, then we have for $\mu \in [0,1]$

$$
\begin{aligned}
\psi^{(k)}\left(\frac{a+\Delta}{2}\right) &= \psi^{(k)}\left(\frac{\mu\Delta+(1-\mu)a+\mu a+(1-\mu)\Delta}{2}\right)\\
&\leq \psi^{(k)}(\mu\Delta+(1-\mu)a)+h(\frac{1}{2})\,\theta\,(\psi^{(k)}(\mu a+(1-\mu)\Delta),\psi^{(k)}(\mu\Delta+(1-\mu)a))\\
&= (\psi^{(k)})^2(\mu\Delta+(1-\mu)a)\left[1+h\left(\frac{1}{2}\right)\theta(1,1)\right].
\end{aligned} \tag{3.49}
$$

Multiplying (3.49) by $\mu^{k-\alpha-1}\psi^{(k)}(\mu\Delta+(1-\mu)a)$ and integrating over $[0,1]$, with respect to μ, we obtain

$$
\psi^{(k)}\left(\frac{a+\Delta}{2}\right)\int_0^1 \mu^{k-\alpha-1}\psi^{(k)}(\mu\Delta+(1-\mu)a)d\mu = \frac{\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{(\Delta-a)^{k-\alpha}}\Gamma(k-\alpha)({}^{C}D^{\alpha}_{a_+}\psi)(\Delta),
$$

and

$$
\begin{aligned}
&\left[1+h\left(\frac{1}{2}\right)\theta(1,1)\right]\int_0^1 \mu^{k-\alpha-1}(\psi^{(k)})^2(\mu\Delta+(1-\mu)a)d\mu\\
&= \frac{1+h(\frac{1}{2})\theta(1,1)}{(\Delta-a)^{k-\alpha}}\int_a^\Delta (x-a)^{k-\alpha-1}(\psi^{(k)})^2(x)dx\\
&= \left[1+h(\frac{1}{2})\theta(1,1)\right]\frac{\Gamma(k-\alpha)}{(\Delta-a)^{k-\alpha}}R^{k-\alpha}_{a_+}(\psi^{(k)})^2(\Delta).
\end{aligned}
$$

Hence

$$
\frac{\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{1+h(\frac{1}{2})\theta(1,1)}({}^{C}D^{\alpha}_{a_+}f)(\Delta) \leq R^{k-\alpha}_{a_+}(\psi^{(k)})^2(\Delta). \tag{3.50}
$$

And similarly

$$\psi^{(k)}\left(\frac{a+\Delta}{2}\right) \leq \psi^{(k)}(\mu a+(1-\mu)\Delta)\left[1+h\left(\frac{1}{2}\right)\theta(1,1)\right] \tag{3.51}$$

by multiplying (3.51) by $\mu^{k-\alpha-1}\psi^{(k)}(\mu a+(1-\mu)\Delta)$, integration yields to

$$\psi^{(k)}\left(\frac{a+\Delta}{2}\right)\int_0^1 \mu^{k-\alpha-1}\psi^{(k)}(\mu a+(1-\mu)\Delta)d\mu = \frac{\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{(\Delta-a)^{k-\alpha}}\Gamma(k-\alpha)({}^C D^{\alpha}_{\Delta-}\psi)(a)$$

and

$$\begin{aligned}&\left[1+h\left(\frac{1}{2}\right)\theta(1,1)\right]\int_0^1 \mu^{k-\alpha-1}(\psi^{(k)})^2(\mu a+(1-\mu)\Delta)d\mu \\ &= \left[1+h\left(\frac{1}{2}\right)\theta(1,1)\right]\frac{\Gamma(k-\alpha)}{(\Delta-a)^{k-\alpha}}R^{k-\alpha}_{\Delta_-}(\psi^{(k)})^2(a),\end{aligned} \tag{3.52}$$

it results that

$$\frac{\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{1+h(\frac{1}{2})\,\theta(1,1)}({}^C D^{\alpha}_{\Delta-}\psi)(a) \leq R^{k-\alpha}_{\Delta_-}(\psi^{(k)})^2(a). \tag{3.53}$$

By adding (3.50) and (3.53), we get (3.48). That proves the claim. □

Corollary 3.7. *Under the same assumptions as Theorem 3.4, if* $h(t) = t^s$, $s \in [0,1]$, *then*

$$\frac{2^s\,\psi^{(k)}\left(\frac{a+\Delta}{2}\right)}{2^s+\theta(1,1)}\left[\left({}^C D^{\alpha}_{\Delta-}\psi\right)(a)+\left({}^C D^{\alpha}_{a_+}\psi\right)(\Delta)\right] \leq R^{k-\alpha}_{\Delta_-}(\psi^{(k)})^2(a)+R^{k-\alpha}_{a+}(\psi^{(k)})^2(\Delta).$$

If $\theta(u,v) = -\theta(v,u)$, *then*

$$\psi^{(k)}\left(\frac{a+\Delta}{2}\right)\left({}^C D^{\alpha}_{\Delta_-}\psi(a)+{}^C D^{\alpha}_{a_+}\psi(\Delta)\right) \leq R^{k-\alpha}_{\Delta_-}(\psi^{(k)})^2(\alpha)+R^{k-\alpha}_{a_+}(\psi^{(k)})^2(\Delta) \tag{3.54}$$

is valid.

4. Conclusions

In this work, we have established some estimates including once the derivatives of Caputo and another time the integrals of Riemann-Liouville and the derivatives of Caputo for a function whose derivative order kth ($k \in \mathbb{N}$) is generalized modified h-convex and symmetrical in the middle. Estimates of consequences for special classes of convex functions and s-convex functions in $[0,1]$ were obtained. The estimates we have just made are compared to those presented in the results [8].

Future research could focus on extending these results to variable order or other types of convex functions or exploring inequalities for functions that do not necessarily have symmetry. Furthermore, the application of derived inequalities to concrete problems in applied mathematics, physics, or engineering could still validate the practical significance of our theoretical contributions. Taking these limitations into account could lead to a more complete understanding and wider applicability of fractional inequalities.

Author contributions

HB: conceptualization, writing original draft preparation, writing review and editing, supervision; MSS: conceptualization, writing original draft preparation, writing review and editing, supervision; HG: conceptualization, writing review and editing, supervision; UFG: funding, writing review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no competing interests.

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