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Research article

A class of time-varying differential equations for vibration research and application

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Abstract: As a pivotal branch within the realm of differential equations, the theory of oscillation holds a crucial position in the exploration of natural sciences and the construction of modern control theory frameworks. Despite the extensive research conducted globally, focusing on individual or combined analyses of key elements such as explicit damping terms, positive and negative coefficients, time-varying delays, and nonlinear neutral terms, systematic investigations into the oscillatory behavior of even-order differential equations that concurrently embody these four complex characteristics remain scarce. This paper, by establishing reasonable assumptions, innovatively presents two crucial criteria, aiming to preliminary delve into the oscillation patterns of even-order differential equations under specific complex settings. In the course of the study, a variety of mathematical techniques, such as Riccati transformation, calculus scaling methods, and partial integration, have been utilized by the researchers to perform the necessary derivations and confirmations.

Keywords: even-order differential equations; oscillation behavior; Riccati transformation; positive and negative coefficients; variable time delay
Mathematics Subject Classification: 34C10

1. Introduction

The study of vibrations in the realm of differential equations holds significant research value within the academic sphere. This field not only plays a crucial role in mathematical modeling but also finds application in the stability analysis of various systems [1–3]. Notable contributions include L. Fan et al. [1], who examined the stability challenges of switched stochastic systems with state-dependent delays; M. L. Xia et al. [2], who conducted a stability analysis for a particular class of stochastic differential equations; and Q. X. Zhu [3], who investigated the exponential stability of stochastic nonlinear delay systems. Consequently, it has emerged as a focal point of interest for numerous scholars seeking to deepen their understanding of these phenomena. S. R. Grace [4] introduced several novel criteria regarding the oscillatory behavior of fractional differential equations that utilize the Caputo derivative. Z. Došlá and P. Liška [5] developed a range of innovative criteria concerning the oscillatory characteristics and asymptotic behavior of solutions to third-order nonlinear neutral differential equations. They also conducted an in-depth investigation into the vibrations of third-order nonlinear neutral differential equations with time delays, offering significant insights for related research areas. R. Vimala et al. [6] employed the Riccati transformation along with comparison techniques to demonstrate the oscillatory behavior of higher-order differential equations. O. Özdemir [7] studied the vibration properties of differential equations with non-regular terms. T. X. Li et al. [8] investigated the oscillatory behavior of second-order nonlinear differential equations that include damping terms. Concurrently, T. X. Li and Y. V. Rogovchenko [9] applied inequality techniques to analyze the oscillation properties of a specific category of second-order neutral differential equations. Moreover, Y. F. Ge [10] investigated the stability properties of nonlinear fractional neutral differential equations that involve multiple variable time delays. In a related study, J. S. Yang et al. [11] proposed two novel criteria for determining the oscillatory behavior of second-order nonlinear neutral differential equations with time delays. Additionally, L. Jann and Y. C. Chih [12] focused on an integral criterion to assess the oscillation characteristics of nonlinear differential equations. J. R. Graef [13] examined the criteria for oscillatory behavior in nonlinear higher-order differential equations. S. Tamilvanan et al. [14] investigated the sufficient conditions that guarantee the behavior of all solutions for second-order nonlinear neutral differential equations characterized by sublinearity. Z. Oplustil [15] developed novel criteria for vibrations in second-order linear differential equations by employing the Riccati technique along with effective estimation methods for non-vibrational solutions. Z. F. Sun et al. [16] applied the Riccati transformation along with neutral delay systems to derive a novel vibration criterion for two-dimensional neutral time-delay power systems. Y. P. Zhao [17] utilized the Riccati transformation and various analytical techniques to establish sufficient conditions for the vibrations of second-order differential equations involving intermediate projects. X. Mi et al. [18] investigated a specific category of second-order nonlinear differential equations with damping terms, deriving new sufficient conditions for oscillations through the Riccati transformation technique. X. H. Deng et al. [19] explored the vibrational and non-vibrational behavior of third-order delay differential equations containing both positive and negative terms, formulating new oscillation criteria, including the Kamenev-type oscillation criterion. Lastly, S. Panigrahi and R. Basu [20] examined a class of nonlinear third-order neutral differential equations characterized by positive and negative coefficients.

Currently, researchers have employed the following differential equations in demonstrating a robust oscillation criterion: Second-order Emden-Fowler differential equations.

$$\left[r(t)|x'(t)|^{\alpha-1}x'(t)\right]' + Q(t)|x(t)|^{\alpha-1}x(t) = 0, \tag{1.1}$$

$$\{r(t)\left|[x(t) + p(t)x(\tau(t))]'\right|^{\lambda-1}[x(t) + p(t)x(\tau(t))]'\}' + q(t)|x(\delta(t))|^{\alpha-1}x(\delta(t)) = 0, \tag{1.2}$$

and second-order delayed differential equations with positive and negative coefficients,

$$\{r(t)[x(t) + p(t)x(t-\tau_0)]'\}' + Q(t)f(x(t-\sigma_0)) - R(t)g(x(t-\delta_0)) = 0. \tag{1.3}$$

Nonetheless, this investigation uniquely centers on the vibrational scrutiny of a distinct set of even-order differential equations amalgamating intricate features such as significant damping components, a blend of positive and negative coefficients, varying time delays, and nonlinear neutral terms.

$$\left(r(t)\varphi_\alpha\left(z'(t)\right)\right)' + g(t)\varphi_\alpha\left(z'(t)\right) + \sum_{i=1}^{m} Q_i(t) f_i\left(\varphi_\alpha\left(x\left(\sigma_i(t)\right)\right)\right) - \sum_{j=1}^{l} R_j(t) g_j\left(\varphi_\alpha\left(x\left(\delta_j(t)\right)\right)\right) = 0, \quad (1.4)$$

where ($t \geq t_0$). Currently, the exploration of its vibrational characteristics remains insufficient. In Eq (1.4), we have the expression $z(t) = x(t) + p(t)x(\tau(t))$, where $\varphi_\alpha(u) = |u|^{\alpha-1}u$, and the functions $r, g, p \in C^1([t_0,\infty),(0,\infty))$, $u \in \mathbb{R}$, $\tau, \sigma \in C^1([t_0,\infty),\mathbb{R})$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$, $\lim_{t\to\infty}\sigma(t) = \infty$, and $f \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$. Constants $\alpha > 0$ and $\beta > 0$ are specified, where $t_0 \geq 0$ and $\alpha > 0$ are real constants. Additionally, integers $m \geq 1$ and $l \geq 1$ are involved in the context.

Consider the following criteria:

(H1) $p(t) \in C([t_0,+\infty),[0,+\infty))$, $g(t) \in C([t_0,+\infty),[0,+\infty))$, $r(t) \in C([t_0,+\infty),[0,+\infty))$, $Q_i \in C([t_0,+\infty),[0,+\infty))$, $R_j \in C([t_0,+\infty),[0,+\infty))$, $f_i \in C(\mathbb{R},\mathbb{R})$, $g_j \in C(\mathbb{R},\mathbb{R})$, furthermore, $uf_i(u) > 0$ for $u \neq 0$, $ug_j(u) > 0$ for $u \neq 0$, where $i = 1,2,...,m$ and $j = 1,2,...,l$.

(H2) Retention function τ belongs to the set $C([t_0,+\infty),(0,+\infty))$, with the additional property that $\tau(t) \leq t$ for all t and $\lim_{t\to+\infty}\tau(t) = +\infty$.

(H3) The retention functions $\sigma_i(t)$ and $\delta_j(t)$ are both equal to $\sigma(t)$. Additionally, σ belongs to $C^1([t_0,+\infty),(0,+\infty))$ with $\sigma(t) \leq t$, $\lim_{t\to+\infty}\sigma(t) = +\infty$, and $\sigma'(t) > 0$.

(H4) For $u \neq 0$, there exist positive constants $\alpha_i > 0$ and $\beta_j > 0$ such that $f_i(u)/u \geq \alpha_i$, $g_j(u)/u \leq \beta_j$, and $\sum_{i=1}^{m}\alpha_i Q_i(t) - \sum_{j=1}^{l}\beta_j R_j(t) > 0$.

(H5) The conditions are as follows: $0 \leq p(t) \leq 1$, $r(t) \in C^1([t_0,+\infty),\mathbb{R})$ where $r(t) > 0$ and $r'(t) > 0$.

(H6) $\int_{t_0}^{+\infty}\left[\frac{1}{r(u)}\exp\left(-\int_{t_0}^{u}\frac{g(s)}{r(s)}ds\right)\right]^{\frac{1}{\alpha}}du = +\infty$.

As this paper is primarily concentrated on the vibrational characteristics of Eq (1.4), the analysis will predominantly revolve around these aspects. The motivation of this study lies in the desire to first understand how these complex characteristics influence the oscillatory behavior of the system. Secondly, it aims to explore the practical effects of time delays on the oscillation features, providing guidance for real-world applications. Furthermore, this research will contribute to the development of mathematical theories related to oscillation criteria, with broad interdisciplinary implications in fields such as engineering, physics, and biological systems. Ultimately, we hope to establish robust oscillation criteria that will facilitate deeper modeling and control of dynamic systems, laying a foundation for research and applications in related areas.

The study of vibrations encompasses a critical segment of even-order differential equations, with wide-ranging implications in the natural sciences and control theory, as demonstrated in Eq (1.4).

By employing the Riccati transformation, partial integration, and scaling methodologies, two novel criteria have been formulated, supported by meticulous mathematical proofs. Moreover, illustrative examples have been presented to support our findings, offering valuable insights that enrich the existing scholarly discourse.

2. Theoretical foundations of research

The function

$$\omega(t) = \exp\left(\int_{t_0}^{t} \frac{g(s)}{r(s)} ds\right), \tag{2.1}$$

plays a crucial role in the examination of dynamic systems and differential equations across a range of mathematical frameworks.

The Riccati transformation is a mathematical method employed to transform a Riccati differential equation into a second-order linear differential equation. Consider a Riccati differential equation represented as follows:

$$w^{'} = P(z) + Q(z)w + R(z)w^2,$$

where $w^{'} = \frac{dw}{dz}$. The transformation is defined in the following manner:

$$w = -\frac{y^{'}}{yR(z)},$$

where y is a function to be determined and $y^{'}$ is its derivative. This transformation seeks to transform the original nonlinear first-order equation into the following second-order linear homogeneous equation:

$$R(z)y^{''} - [R^{'}(z) + Q(z)R(z)]y' + [R(z)]^2 P(z)y = 0.$$

Lemma 1. *[10] Consider a function u that is positive and possesses an n-th order differentiability on the interval $[t_0, +\infty)$, with its n-th derivative, $u^{(n)}(t)$, existing and eventually settling to a definitive sign within this interval. Consequently, there exists a particular time point, t^*, such that $t^* \geq t_0$, corresponding to an integer l (where $0 \leq l \leq n$). If $u^{(n)}(t) \geq 0$, it follows that $n + l$ is even; conversely, if $u^{(n)}(t) \leq 0$, then $n + l$ is odd. Furthermore, this condition is maintained. In scenarios where $l > 0$, it is observed that $u^{(k)}(t) > 0$ for all $t \geq t^*$ and for $k = 0, 1$. When $l \leq n - 1$, it holds true that $(-1)^{l+k}u^{(k)}(t) > 0$ for all $t \geq t^*$ and for $k = 0, 1$.*

Lemma 2. *[10] Consider a function u that meets the conditions stipulated in Lemma 1, and for which the product $u'(t)u^{(2)}(t)$ remains non-positive for all $t \geq t^*$. Under these conditions, for any $\theta \in (0, 1)$, there exists a constant $M > 0$ such that for every sufficiently large t, the following inequality holds:*

$$u'(\theta t) \geq Mt^{n-2}u^{(n-1)}(t).$$

Lemma 3. *[10] (Holder inequality) The following inequality holds for integrals involving two functions $f(x)$ and $g(x)$ over a closed interval $[a, b]$:*

$$\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}}.$$

The inequality presented is constrained by the stipulations that $p > 0$, $q > 0$, and additionally mandates that $\frac{1}{p} + \frac{1}{q} = 1$ in order to hold true.

Lemma 4. *[8] Let $a > 0$, $b > 0$, $\gamma > 0$, there is $au - bu^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma a^{\gamma+1}}{(\gamma+1)^{\gamma+1} b^\gamma}$, $(u > 0)$.*

Lemma 5. *Upon satisfaction of conditions (H1)–(H6), the function $x(t)$ is recognized as the ultimate positive solution to Eq* (1.4)*. Consequently, it can be concluded that*

$$z(t) > 0, z'(t) > 0, z''(t) \le 0. \tag{2.2}$$

Proof. Given that $x(t)$ constitutes the final positive solution of Eq (1.4), it follows that the conditions are established as follows:

$$x(t) > 0, x(\tau(t)) > 0, x(\sigma_i(t)) = x(\delta_j(t)) = x(\sigma(t)) > 0, \text{for } t \ge T \ge t_0.$$

In the derivation, it is deduced that $z(t) = x(t) + p(t)x(\tau(t)) > 0$.
According to the formula (1.4), one has,

$$(r(t)\varphi_\alpha(z'(t)))' + g(t)\varphi_\alpha(z'(t)) \le -\left[\sum_{i=1}^{m} \alpha Q_i(t) - \sum_{i=1}^{l} \beta_j R_j(t)\right]\phi(x(\sigma(t))) < 0. \tag{2.3}$$

Further, derived from (2.1) the formula, get $\omega'(t) = \omega(t) g(t)/r(t)$.
Based on the above two conclusions, it can be deduced,

$$\begin{aligned}[\omega(t) r(t)\varphi_\alpha(z'(t))]' &= \omega(t)\frac{g(t)}{r(t)} \cdot r(t)\varphi_\alpha(z'(t)) + \omega(t)[r(t)\varphi_\alpha(z'(t))]' \\ &= \omega(t)\left\{[r(t)\varphi_\alpha(z'(t))]' + g(t)\varphi_\alpha(z'(t))\right\} < 0.\end{aligned} \tag{2.4}$$

Therefore $\omega(t) r(t)\varphi_\alpha(z'(t))$, when $t \ge T$, $\omega(t) r(t)\varphi_\alpha(z'(t))$ is a strictly decreasing trend, and $z'(t)$ the final number.

Hence

$$z'(t) > 0, t \ge T. \tag{2.5}$$

Counter evidence is utilized with the indication that $z'(t) < 0$, for $t \ge T$.
Following the information presented in Eq (2.4), the formula is derived as:

$$\omega(t)r(t)\varphi_\alpha(z'(t)) \le \omega(T)r(T)\varphi_\alpha(z'(T)) = -C.$$

Additionally, it is observed that $C = \omega(T)r(T)[-\varphi_\alpha(z'(T))] = \omega(T)r(T)|z'(T)|^{\alpha-1}(-z'(T)) > 0$, which remains constant.
So

$$z'(t) \le -C^{1/\alpha}\left[\frac{1}{r(t)}\exp\left(-\int_{t_0}^{t}\frac{g(s)}{r(s)}ds\right)\right]^{1/\alpha}.$$

Upon integration of both sides of the equation, it leads to the following outcome:

$$z(t) \le z(T) - C^{1/\alpha}\int_T^t\left[\frac{1}{r(u)}\exp\left(-\int_{t_0}^{u}\frac{g(s)}{r(s)}ds\right)\right]^{1/\alpha}du.$$

As t approaches $+\infty$, taking into account condition ($H6$), $\lim_{t\to+\infty} z(t) = -\infty$. This contradicts the assertion that $z(t) > 0$.

This can be confirmed to be true for Eq (2.5).

Derived from the formula (2.3),

$$\begin{aligned} 0 &\geq \left[r(t)\varphi_\alpha\left(z'(t)\right)\right]' \\ &= \left\{r(t)\left[z'(t)\right]^\alpha\right\}' \\ &= r'(t)\left[z'(t)\right]^\alpha + \alpha r(t)\left[z'(t)\right]^{\alpha-1} z''(t). \end{aligned} \tag{2.6}$$

As a result, it can be concluded that $z''(t) \leq 0\,(t \geq T)$. Given that 2 is an even number, according to Lemma 1, knowing that there is an odd number l, this condition is further derived when $t \geq T$, function $z'(t) > 0$. Therefore, it can be concluded that $z''(t) \leq 0$ for $t \geq T$. With the given even number being 2, and in accordance with Lemma 1, it is established that there exists an odd number l. This condition is further established for $t \geq T$ when the function $z'(t) > 0$. Lemma 5 is validated.

□

3. A class of time-varying differential equations for vibration studies

Consider the set D defined as $D = \{(t, s) \mid t \geq s \text{ and } s \geq t_0\}$, and the set D_0 defined as $D_0 = \{(t, s) \mid t > s \text{ and } s \geq t_0\}$. A binary function $H(t, s)$ is then introduced as an element of the function class Θ, denoted by $H \in \Theta$. This binary function must first satisfy the condition $H(t, s) \in C(D, \mathbb{R})$, and secondly, it must adhere to the following condition:

(i) For $t \geq t_0$, $H(t, t) = 0$, and for $(t, s) \in D_0$, $H(t, s) > 0$.

(ii) The function $H(t, s)$ possesses a continuous partial derivative with respect to s that is non-positive. This can be represented as $\frac{\partial H(t,s)}{\partial s} \leq 0$.

Theorem 1. *Given the fulfillment of the (H1)–(H6) conditions, let us consider that all requirements are satisfied. Assume the function* $\phi \in C^1([t_0, +\infty), (0, +\infty))$ *and the function* $H \in \Theta$*. Under these circumstances, the function can be derived as follows:*

$$\lim_{t\to+\infty} \sup \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)\phi(s)\left[\Phi(s) - \psi(s)\left|h(t, s)\right|^{\alpha+1}\right] ds = +\infty. \tag{3.1}$$

In this context, let it be known that a constant $\theta \in (0, 1)$ *and* $M > 0$ *are defined in accordance with Lemma 2. Consider the functions:*

$$\Phi(s) = \left[\sum_{i=1}^{m} \alpha_i Q_i(s) - \sum_{j=1}^{l} \beta_j R_j(s)\right]\left[1 - \eta p(\sigma(s))\right]^\alpha, \psi(s) = \frac{(\alpha+1)^{-(\alpha+1)} r(s)}{\left[\theta M \sigma(s)\sigma'(s)\right]^\alpha}. \tag{3.2}$$

Furthermore, with the function

$$h(t, s) = \frac{1}{H(t, s)} \frac{\partial H(t, s)}{\partial s} + \left(\frac{\phi'(s)}{\phi(s)} - \frac{g(s)}{r(s)}\right),$$

the Eq (1.4) *experiences oscillation.*

Proof. Counterevidence. Assuming Eq (1.4), there exists a solution $x(t)$ to the equation that does not exhibit vibration. Consider $x(t) > 0, x(\tau(t)) > 0, x(\sigma(t)) > 0, t \geq T \geq t_0$. By utilizing the formulas from (2.2) and Lemma 2, it can be shown that for any $0 < \theta < 1$, there exists a positive constant M such that

$$z'(\theta\sigma(t)) \geq M\sigma(t)z'(\sigma(t)) \geq M\sigma(t)z'(t). \tag{3.3}$$

Due to the relationship $z(t) = x(t) + p(t)x(\tau(t))$, it follows that $x(t) \leq z(t)$. Therefore, one has

$$\begin{aligned} z(t) &= x(t) + p(t)x(\tau(t)) \\ &\leq x(t) + p(t)z(\tau(t)) \\ &\leq x(t) + \eta p(t)z(t). \end{aligned}$$

Consequently, one has

$$x(t) \geq [1 - \eta p(t)]z(t) \geq 0. \tag{3.4}$$

Introducing the Riccati transform as

$$V(t) = \phi(t)\frac{r(t)\varphi_\alpha(z'(t))}{\varphi_\alpha(z(\theta\sigma(t)))} = \phi(t)\frac{r(t)[z'(t)]^\alpha}{[z(\theta\sigma(t))]^\alpha}, \quad t \geq T. \tag{3.5}$$

It can be concluded that for $t \geq T$, the function $V(t)$ is greater than zero. Leveraging Eqs (2.3), (3.3), and (3.4) along with (3.5), it can be deduced that

$$\begin{aligned} V'(t) &= \phi'(t)\frac{r(t)\varphi_\alpha(z'(t))}{\varphi_\alpha(z(\theta\sigma(t)))} + \phi(t)\frac{[r(t)\varphi_\alpha(z'(t))]'}{\varphi_\alpha(z(\theta\sigma(t)))} - \phi(t)\frac{r(t)[z'(t)]^\alpha}{[z(\theta\sigma(t))]^{\alpha+1}}\alpha\theta z'(\theta\sigma(t))\sigma'(t) \\ &\leq \frac{\phi'(t)}{\phi(t)}V(t) - \phi(t)\frac{g(t)\varphi_\alpha(z'(t)) + \left[\sum_{i=1}^m \alpha_i Q_i(t) - \sum_{j=1}^l \beta_j R_j(t)\right]\varphi_\alpha(x(\sigma(t)))}{\varphi_\alpha(z(\theta\sigma(t)))} \\ &\quad - \phi(t)r(t)\frac{[z'(t)]^{\alpha+1}}{[z(\theta\sigma(t))]^{\alpha+1}}\alpha\theta M\sigma(t)\sigma'(t) \\ &\leq \frac{\phi'(t)}{\phi(t)}V(t) - \frac{g(t)}{r(t)}V(t) - \phi(t)\left[\sum_{i=1}^m \alpha_i Q_i(t) - \sum_{j=1}^l \beta_j R_j(t)\right][1 - \eta p(\sigma(t))]^\alpha \\ &\quad - \frac{\alpha\theta M\sigma(t)\sigma'(t)[V(t)]^{(\alpha+1)/\alpha}}{[\phi(t)r(t)]^{1/\alpha}}. \end{aligned}$$

Given $\Phi(s) = \left[\sum_{i=1}^m \alpha_i Q_i(s) - \sum_{j=1}^l \beta_j R_j(s)\right][1 - \eta p(\sigma(s))]^\alpha$, the consequent implication when $t \geq T$ is delineated by the equation provided. On further analysis:

$$\phi(t)\Phi(t) \leq -V'(t) + \left[\frac{\phi'(t)}{\phi(t)} - \frac{g(t)}{r(t)}\right]V(t) - \frac{\alpha\theta M\sigma(t)\sigma'(t)[V(t)]^{(\alpha+1)/\alpha}}{[\phi(t)r(t)]^{1/\alpha}}. \tag{3.6}$$

Then, by replacing t with s in the previously mentioned equation and integrating both sides with respect to $H(t, s)$, the operation is concluded.

Because

$$h(t,s) = \frac{1}{H(t,s)}\frac{\partial H(t,s)}{\partial s} + \left(\frac{\phi'(s)}{\phi(s)} - \frac{g(s)}{r(s)}\right),$$

It can be concluded that

$$\begin{aligned}\int_T^t H(t,s)\phi(s)\Phi(s)\,ds &\le -\int_T^t H(t,s)V'(s)\,ds + \int_T^t H(t,s)\left[\frac{\phi'(s)}{\phi(s)} - \frac{g(s)}{r(s)}\right]V(s)\,ds \\ &\quad - \alpha\theta M\int_T^t H(t,s)\frac{\sigma(s)\sigma'(s)[V(s)]^{(\alpha+1)/\alpha}}{[\phi(s)r(s)]^{1/\alpha}}ds \\ &\le H(t,T)V(T) + \int_T^t\left[\frac{\partial H(t,s)}{\partial s} + H(t,s)\left(\frac{\phi'(s)}{\phi(s)} - \frac{g(s)}{r(s)}\right)\right]V(s)\,ds \\ &\quad - \alpha\theta M\int_T^t H(t,s)\frac{\sigma'(s)[V(s)]^{(\alpha+1)/\alpha}}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}}ds \\ &= H(t,T)V(T) + \int_t^T\{|h(t,s)|H(t,s)V(s) \\ &\quad - \frac{\alpha\theta MH(t,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}}[V(s)]^{\frac{\alpha+1}{\alpha}}\}ds.\end{aligned} \tag{3.7}$$

Apply the inequalities from Lemma 4 to the above equation and make use of functions

$$\psi(s) = \frac{(\alpha+1)^{-(\alpha+1)}r(s)}{[\theta M\sigma(s)\sigma'(s)]^{\alpha}}.$$

It is further obtained by the formula (3.7)

$$\begin{aligned}\int_T^t H(t,s)\phi(s)\Phi(s)\,ds &\le H(t,T)V(T) + \int_T^t\{\frac{\phi(s)r(s)|h(t,s)|^{\alpha+1}H(t,s)}{(\alpha+1)^{\alpha+1}[\alpha M\sigma(s)\sigma'(s)]^{\alpha}}\}ds \\ &= H(t,T)V(T) + \int_T^t \phi(s)\psi(s)|h(t,s)|^{\alpha+1}H(t,s)\,ds.\end{aligned} \tag{3.8}$$

So

$$\int_T^t H(t,s)\left[\phi(s) - \psi(s)|h(t,s)|^{\alpha+1}\right]\phi(s)\,ds \le H(t,T)V(T) \le H(t,t_0)V(T). \tag{3.9}$$

Therefore

$$\frac{1}{H(t,t_0)}\int_T^t H(t,s)\left[\Phi(s) - \psi(s)|h(t,s)|^{\alpha+1}\right]\phi(s)\,ds \le V(T).$$

Hence

$$\begin{aligned}&\frac{1}{H(t,t_0)}\int_{t_0}^t H(t,s)\left[\Phi(s) - \psi(s)|h(t,s)|^{\alpha+1}\right]\phi(s)\,ds \\ &= \frac{1}{H(t,t_0)}\left\{\int_{t_0}^T H(t,s)\left[\Phi(s) - \psi(s)|h(t,s)|^{\alpha+1}\right]\phi(s)\,ds\right\} \\ &\quad + \left\{\int_T^t H(t,s)\left[\Phi(s) - \psi(s)|h(t,s)|^{\alpha+1}\right]\phi(s)\,ds\right\} \\ &\le \int_{t_0}^T \frac{H(t,s)}{H(t,t_0)}\left[\Phi(s) - \psi(s)|h(t,s)|^{\alpha+1}\right]\phi(s)\,ds + V(T) \\ &\le \int_{t_0}^T \Phi(s)\phi(s)\,ds + V(T) \\ &= C,\end{aligned}$$

which constant $C = \int_{t_0}^{T} \Phi(s)\phi(s)\,ds + V(T)$. The above formula contradicts the conditions (3.1). The proof of Theorem 1 is now concluded. □

When the formula (3.1) is not true, then there are the following theorems.

Theorem 2. *The conditions (H1)–(H6) are valid if the function* $\phi \in C^1([t_0,+\infty),(0,+\infty)), H \in \Theta, \zeta \in L^2([t_0,+\infty),R)$ *, makes*

$$\lim_{t\to+\infty}\sup \frac{1}{H(t,T)}\int_T^t \phi(s)\psi(s)|h(t,s)|^{\alpha+1}H(t,s)\,ds < +\infty, \tag{3.10}$$

$$\lim_{t\to+\infty}\sup \frac{1}{H(t,u)}\int_u^t H(t,s)\left[\Phi(s) - \psi(s)|h(t,s)|^{\alpha+1}\right]\phi(s)\,ds \geq \zeta(u). \tag{3.11}$$

For all situations satisfying $u \geq T \geq t_0$, *it is assumed that the function* ζ *satisfies the following condition.*

$$\lim_{t\to+\infty}\inf \frac{1}{H(t,T)}\int_T^t \frac{H(t,s)\sigma'(s)[\zeta(s)]^{(\alpha+1)/\alpha}}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}}ds = +\infty. \tag{3.12}$$

Here, the explicit mention of the functions $\Phi(s)$, $\psi(s)$, *and* $h(t,s)$ *in their definitions, as stated in Theorem 1, leads to oscillations in Eq* (1.4).

In the aforementioned context, the functions $\Phi(s)$, $\psi(s)$, *and* $h(t,s)$ *are explicitly defined in accordance with Theorem 1. Under these circumstances, it is postulated that Eq* (1.4) *demonstrates oscillatory behavior.*

Proof. In accordance with Theorem 1, the derivations of Eqs (3.7) and (3.9) are established. From Eq (3.9), it can be deduced that for all $t \geq u \geq T \geq t_0$, the following conditions must hold true:

$$\frac{1}{H(t,u)}\int_u^t H(t,s)\left[\Phi(s) - \psi(s)|h(t,s)|^{\alpha+1}\right]\phi(s)\,ds \leq V(u).$$

Synthesizing the aforementioned formula with (3.11) yields: $\zeta(u) \leq V(u)$,

$$\lim_{t\to+\infty}\sup \frac{1}{H(t,T)}\int_T^t H(t,s)\Phi(s)\phi(s)\,ds \geq \zeta(T). \tag{3.13}$$

In accordance with the formulation presented in Eq (3.7), it is established that

$$\begin{aligned}&\frac{1}{H(t,T)}\int_T^t \left\{\frac{\alpha\theta M H(t,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}}[V(s)]^{\frac{\alpha+1}{\alpha}} - |h(t,s)|H(t,s)V(s)\right\}ds\\ &\leq \frac{-1}{H(t,T)}\int_T^t H(t,s)\phi(s)\Phi(s)\,ds + V(T),\end{aligned}$$

exploit (3.13), combine the above formula,

$$\lim_{t\to+\infty}\inf \frac{1}{H(t,T)}\int_T^t \left\{\frac{\alpha\theta M H(t,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}}[V(s)]^{\frac{\alpha+1}{\alpha}} - |h(t,s)|H(t,s)V(s)\right\}ds \leq C_0, \tag{3.14}$$

which constant $C_0 = -\zeta(T) + V(T)$, so

$$\lim_{t\to+\infty} \inf \frac{1}{H(t,T)} \int_T^t \frac{\alpha\theta MH(t,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}} [V(s)]^{\frac{\alpha+1}{\alpha}} ds < +\infty. \tag{3.15}$$

Contrast, there is a sequence $\{T_n\}_{n=1}^{\infty}$ ($T_n \in [T, +\infty)$) and $\lim_{n\to+\infty} T_n = +\infty$, such that,

$$\lim_{n\to\infty} \frac{1}{H(T_n,T)} \int_T^{T_n} \frac{\alpha\theta MH(T_n,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}} [V(s)]^{\frac{\alpha+1}{\alpha}} ds = +\infty. \tag{3.16}$$

And then by (3.14),

$$\lim_{n\to\infty} \frac{1}{H(T_n,T)} \int_T^{T_n} |h(T_n,s)| H(T_n,s) V(s) ds = +\infty. \tag{3.17}$$

In the combination of Eqs (3.14), (3.16), and (3.17), it can be noted that there exists a sufficiently large natural number N such that for all natural numbers n satisfying $n \geq N$, the following conditions hold:

$$\frac{1}{H(T_n,T)} \int_T^{T_n} \frac{\alpha\theta MH(T_n,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}} [V(s)]^{\frac{\alpha+1}{\alpha}} ds - \frac{1}{H(T_n,T)} \int_T^{T_n} |h(T_n,s)|H(T_n,s)V(s)ds < C_0 + 1.$$

Therefore, considering positive numbers ε, where ε is bounded by 0 and 1, specifically $0 < \varepsilon < 1$, when $n \geq N$, it is established that the following inequality is valid:

$$\frac{\int_T^{T_n} |h(T_n,s)|H(T_n,s)V(s)ds}{\int_T^{T_n} \frac{\alpha\theta MH(T_n,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}} [V(s)]^{\frac{\alpha+1}{\alpha}} ds} > 1 - \varepsilon > 0. \tag{3.18}$$

Consequently, by utilizing Lemma 3 and the function $\psi(s)$, it can be deduced that

$$\begin{aligned}
&\int_T^{T_n} |h(T_n,s)| H(T_n,s) V(s) ds \\
&= \int_T^{T_n} \left\{ \frac{\alpha\theta MH(T_n,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}} [V(s)]^{\frac{\alpha+1}{\alpha}} \right\}^{\frac{\alpha}{\alpha+1}} \cdot \frac{|h(T_n,s)| H(T_n,s)[\phi(s)r(s)]^{1/(\alpha+1)}}{[\alpha\theta M\sigma(s)H(T_n,s)\sigma'(s)]^{\alpha/(\alpha+1)}} ds \\
&\leq \left\{ \int_T^{T_n} \frac{\alpha\theta MH(T_n,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}} [V(s)]^{\frac{\alpha+1}{\alpha}} ds \right\}^{\frac{\alpha}{\alpha+1}} \cdot \left\{ \int_T^{T_n} \frac{[|h(T_n,s)| H(T_n,s)]^{\alpha+1} \phi(s) r(s)}{[\alpha\theta M\sigma(s)H(T_n,s)\sigma'(s)]^{\alpha}} ds \right\}^{\frac{1}{\alpha+1}} \\
&= \left\{ \int_T^{T_n} \frac{\alpha\theta MH(T_n,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}} [V(s)]^{\frac{\alpha+1}{\alpha}} ds \right\}^{\frac{\alpha}{\alpha+1}} \cdot \left\{ \int_T^{T_n} \frac{|h(T_n,s)|^{\alpha+1} H(T_n,s)\phi(s)\psi(s)}{\alpha^{\alpha}(\alpha+1) - (\alpha+1)} ds \right\}^{\frac{1}{\alpha+1}}.
\end{aligned}$$

Organizing the aforementioned formula and merging (3.10) with (3.18),

$$\begin{aligned}
0 &< \frac{(1-\varepsilon)^{\alpha}}{H(T_n,T)} \int_T^{T_n} |h(T_n,s)| H(T_n,s) V(s) ds \\
&< \frac{\{\int_T^{T_n} |h(T_n,s)| H(T_n,s) V(s) ds\}^{\alpha+1}}{H(T_n,T)\{\int_T^{T_n} \frac{\alpha\theta MH(T_n,s)\sigma'(s)}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}} [V(s)]^{\frac{\alpha+1}{\alpha}} ds\}^{\alpha}} \\
&\leq \frac{(\alpha+1)^{\alpha+1}}{\alpha^{\alpha}} \cdot \frac{1}{H(T_n,T)} \int_T^{T_n} \phi(s)\psi(s) |h(T_n,s)|^{\alpha+1} H(T_n,s) ds \\
&< +\infty.
\end{aligned}$$

Hence

$$0<\frac{1}{H(T_n,T)}\int_T^{T_n}|h(T_n,s)|H(T_n,s)V(s)ds<+\infty.$$

The formula presented above is in conflict with the condition (3.17), thus establishing the veracity of (3.15). By combining the initial formula for the utility (3.13) with (3.15), one deduces that

$$\begin{aligned}&\lim_{t\to+\infty}\inf\frac{1}{H(t,T)}\int_T^t\frac{H(t,s)\sigma'(s)[\zeta(s)]^{(\alpha+1)/\alpha}}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}}ds\\ &\le\lim_{t\to+\infty}\inf\frac{1}{H(t,T)}\int_T^t\frac{H(t,s)\sigma'(s)[V(s)]^{(\alpha+1)/\alpha}}{\sigma(s)[\phi(s)r(s)]^{1/\alpha}}ds\\ &<+\infty.\end{aligned}$$

The formula above conflicts with Eq (3.12). In conclusion, Theorem 2 is demonstrated. □

Corollary 1. *Based on the conditions (H1)–(H6) and the proofs of Theorems 1 and 2, the order of the second-order Eq* (1.4) *can be generalized to even-order, and the equation exhibits oscillatory behavior.*

4. Proven

Example 1. Investigation of a distinctive category of fourth-order time-delayed differential equations is executed, focusing on vibratory behavior, described by the following expression:

$$\begin{aligned}&\left\{t^{2/3}\left[(x(t)+1/4x(t/3))^{(3)}\right]^{5/3}\right\}'+t^{-2}\left[(x(t)+1/4x(t/3))^{(3)}\right]^{5/3}\\ &+\left(1/t+1/5\sqrt{t}\right)f\left((x(t/2))^{5/3}\right)-4/(3t)\,g\left((x(t/2))^{5/3}\right)\\ &=0,\end{aligned}$$

over the domain where $t\ge 1$, with the functions $f(u)=u[6+\ln^{\alpha}(1+u^2)]$ and $g(u)=\frac{3u}{1+2u^2+3u^4}$. The formula derived by Theorem 1 is expressed as follows:

$$\alpha=\frac{5}{3},t_0=1,r(t)=t^{\frac{2}{3}},g(t)=\frac{1}{t^2},p(t)=\frac{1}{4},\tau(t)=\frac{t}{3},\sigma(t)=\delta(t)=\frac{t}{2},Q(t)=\frac{1}{t}+\frac{1}{5\sqrt{t}},R(t)=\frac{4}{3t}.$$

Following the proof of the previously discussed theorem, we perform the subsequent calculations: $\frac{f(u)}{u}\ge 6,\quad \alpha=6\quad$ with $\quad u\neq 0,\ \frac{g(u)}{u}\le 3=\beta\quad (u\neq 0),\ \alpha Q(t)-\beta R(t)=\frac{2}{t}+\frac{6}{5\sqrt{t}}>0.$

From Theorem 1, the subsequent equation is obtained,

$$\Phi(s)=2^{-5/3}\left(2/t+6/5\sqrt{t}\right),\psi(s)=(8/3)^{-8/3}\frac{2^5}{(\theta M)^{5/3}}s^{-8/3}.$$

Simultaneously, as $t\to+\infty$, the following equation can be introduced:

$$\begin{aligned}&\int_{t_0}^t\left[\frac{1}{r(u)}\exp\left(-\int_{t_0}^u\frac{g(s)}{r(s)}ds\right)\right]^{\frac{1}{\alpha}}du\\ &=e^{-\frac{9}{25}}\int_1^t u^{-\frac{2}{5}}\exp\left(\frac{9}{25}u^{-\frac{5}{3}}\right)du\\ &\ge e^{-\frac{9}{25}}\int_1^t u^{-\frac{2}{5}}\left(1+\frac{9}{25}u^{-\frac{5}{3}}\right)du\to+\infty.\end{aligned}$$

Thus (H1)–(H6) is established.

With $\phi(t) = 1$ and $H(t,s) = t - s$ as $t \to +\infty$, the equation is as follows:

$$\frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \left[\Phi(s) - \psi(s) |h(t,s)|^{\alpha+1} \right] \phi(s)\, ds$$
$$= \frac{1}{t-1} \int_{1}^{t} (t-s)[2^{-\frac{5}{3}} \left(2/s + 6/5\sqrt{s} \right) - \left(\frac{8}{3}\right)^{-\frac{8}{3}} \frac{2^5}{(\theta M)^{\frac{5}{3}}} s^{-\frac{8}{3}}$$
$$\left| \frac{1}{(t-s)^3} + \left(\frac{\phi'(s)}{\phi(s)} - s^{-\frac{8}{3}} \right) \right|^{\frac{8}{3}}]\phi(s)\, ds \to +\infty.$$

Therefore, the vibration of the original equation is obtained from Theorem 1.

In summary, the complexity of even-order equations in canonical examples necessitates the use of Riccati transforms and partial integrals to deduce and apply the conclusions of the aforementioned theorems for the purpose of proof.

5. Conclusions

This research extends earlier work on even-order nonlinear differential equations, specifically concentrating on a distinct category of such equations represented by (1.4). Under appropriate conditions, we successfully derived two new criteria for the Eq (1.4). Notable features of these equations include the presence of pronounced damping terms, a mixture of positive and negative coefficients, variable time delays, and nonlinearity involving neutral terms. Using methods such as Riccati transformation, calculus scaling, and integration by parts, we carried out a thorough derivation and proof concerning the oscillatory characteristics of even-order differential equations. By leveraging established theoretic frameworks, we pursued an evolutionary extrapolation approach and assessed the effectiveness of this method through meticulous proofs.

Author contributions

Kai Zhou: Methodology; Duoduo Zhao, Xin Xu: writing-original draft preparation; Fengming Ye: supervision. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare there is no conflict of interest.

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