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Research article

Some new characterizations of spheres and Euclidean spaces using conformal vector fields

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Abstract: Given a conformal vector field X defined on an n-dimensional Riemannian manifold (N^n, g), naturally associated to X are the conformal factor σ, a smooth function defined on N^n, and a skew symmetric $(1, 1)$ tensor field Ω, called the associated tensor, that is defined using the 1-form dual to X. In this article, we prove two results. In the first result, we show that if an n-dimensional compact and connected Riemannian manifold (N^n, g), $n > 1$, of positive Ricci curvature admits a nontrivial (non-Killing) conformal vector field X with conformal factor σ such that its Ricci operator Rc and scalar curvature τ satisfy

$$Rc\,(X) = -(n-1)\nabla\sigma \quad \text{and} \quad X(\tau) = 2\sigma\,(n(n-1)c - \tau)$$

for a constant c, necessarily $c > 0$ and (N^n, g) is isometric to the sphere S^n_c of constant curvature c. The converse is also shown to be true. In the second result, it is shown that an n-dimensional complete and connected Riemannian manifold (N^n, g), $n > 1$, admits a nontrivial conformal vector field X with conformal factor σ and associated tensor Ω satisfying

$$Rc\,(X) = -div\Omega \quad \text{and} \quad \Omega\,(X) = 0,$$

if and only if (N^n, g) is isometric to the Euclidean space $(E^n, \langle , \rangle)$.

Keywords: conformal field; conformal factor; isometric to sphere; isometric to Euclidean space
Mathematics Subject Classification: 53C21, 53C24

1. Introduction

Conformal geometry is one of the oldest branches of differential geometry, as one can follow it through [1] as old as 1959. It is evolving with time and getting enriched constantly till one can find the most recent work in [2]. The main topic in the conformal geometry is about studying the influence of a conformal vector field X on an n-dimensional Riemannian manifold (N^n, g). We shall abbreviate a conformal vector field X by *CONFVF X* for the sake of convenience. There is a smooth function σ that is naturally associated to a *CONFVFX* on (N^n, g) called the conformal factor satisfying

$$\frac{1}{2}\pounds_X g = \sigma g, \tag{1.1}$$

where £ is the Lie derivative operator. A *CONFVF X* is said to be Killing if the conformal factor $\sigma = 0$, and consequently, a nontrivial *CONFVF X* must have a conformal factor $\sigma \neq 0$. There is naturally associated a $(1, 1)$ skew symmetric tensor field Ω to a *CONFVF X* on (N^n, g) called the associated tensor of *CONFVF X*, defined by

$$\frac{1}{2}d\eta(E, F) = g\left(\Omega E, F\right), \tag{1.2}$$

for smooth vector fields E, F on N^n, where η is the 1-form dual to X. This associated tensor Ω plays a crucial role in studying the impact of a *CONFVF X* on the geometry of (N^n, g) (see [2]).

The sphere S^n_c of constant curvature as a hypersurface of the Euclidean space $\left(E^{n+1}, \langle,\rangle\right)$, where $\langle,\rangle$ is the Euclidean metric, has unit normal ζ, induced metric g, and the Weingarten map

$$S = -\sqrt{c}I.$$

Choosing a unit constant vector field Z on the Euclidean space $\left(E^{n+1}, \langle,\rangle\right)$, its tangential component X to the sphere S^n_c satisfies

$$\nabla_E X = -\sqrt{c}\rho E \quad \text{and} \quad \nabla\rho = \sqrt{c}X, \tag{1.3}$$

where

$$\rho = \langle Z, \zeta\rangle$$

and $\nabla\rho$ is the gradient of ρ on S^n_c with respect to the induced metric g. Thus, we see that X is a *CONFVF* on the sphere S^n_c with conformal factor

$$\sigma = -\sqrt{c}\rho.$$

This *CONFVF X* is closed, and therefore the associated tensor $\Omega = 0$.

Also, using complex structure J on $\left(E^{2n}, \langle,\rangle\right)$, define a unit vector field $\xi = J\zeta$, which has covariant derivative

$$\nabla_E \xi = (JE)^T, \tag{1.4}$$

where $(JE)^T$ is the tangential component of JE to S^{2n-1} and ξ is a Killing vector field, that is,

$$\pounds_\xi g = 0.$$

Now, define a vector field

$$\overline{X} = X + \xi,$$

we obtain a *CONFVF*$\overline{X}$ with conformal factor σ that is not closed and indeed has an associated tensor

$$\Omega E = (JE)^T.$$

Non-closed *CONFVF* are in abundance, for instance on the Euclidean space $\left(E^{2n}, \langle,\rangle\right)$, if ξ is the position vector field on E^{2n}, then

$$X = \xi + J\xi,$$

where J is the complex structure, is a *CONFVF* on $\left(E^{2n}, \langle,\rangle\right)$ that is not closed and has an associated tensor

$$\Omega E = JE.$$

Riemannian manifolds that admit closed CONFVF have been extensively studied [3–5]. For related work on conformal vector fields that are not necessarily closed, refer to [6–9]. We see that while studying the impact of a *CONFVF* X on the geometry of the Riemannian manifold (N^n, g), on which it is defined, the associated tensor Ω offers some resistance in analyzing the geometry. Therefore, this study becomes smooth once one assumes that the *CONFVF* X is closed, which forces the associated tensor Ω to vanish. The study of the impact of the existence of a non-closed *CONFVF* on Riemannian manifolds is relatively difficult, and therefore this difficulty is softened by imposing geometric restrictions on Riemannian manifolds, such as the scalar curvature τ being constant. Riemannian manifolds admitting non-closed *CONFVF* have been studied in [8, 10, 11]. Moreover, apart from the fact that the presence of a *CONFVF* on a Riemannian manifold influences its geometry, they are also used in the theory of relativity [12–14].

Observe that the Ricci operator Rc on a Riemannian manifold (N^n, g) is related to the Ricci tensor Ric by

$$Ric\,(E, F) = g\,(RcE, F)\,,$$

and for the sphere S^n_c the Ricci operator is given by

$$Rc = (n-1)cI.$$

Moreover, the conformal factor σ of the *CONFVFX* on S^n_c described in Eq (1.3) satisfies

$$Rc\,(X) = -(n-1)\nabla\sigma. \tag{1.5}$$

It naturally raises a question: Under what conditions is a compact and connected Riemannian manifold (N^n, g) admitting a nontrivial *CONFVF* X with conformal factor σ satisfying Eq (1.5) is isometric to the sphere S^n_c?

The reader may refer to the following sources, as well as the references in [10, 15–17], for additional information on this question. In this article, we answer this question and indeed find a new characterization of the sphere S^n_c (see Theorem 1). Finally, in this paper, we find a characterization of the Euclidean space $(E^n, \langle,\rangle)$ using a nontrivial *CONFVF* X with conformal factor σ on a complete and connected Riemannian manifold (N^n, g) (see Theorem 2).

2. Preliminaries

Let X be a nontrivial *CONFVF* on an n-dimensional Riemannian manifold (N^n, g) with conformal factor σ. Then employing Eqs (1.1) and (1.2) in Koszul's formula, we have

$$2\,(\nabla_E X, F) = (\pounds_X g)\,(E, F) + d\eta\,(E, F)\,,$$

where ∇_E is the covariant derivative with respect to the Riemannian connection on (N^n, g) and E, F are smooth vector fields on N^n. Consequently, we have

$$\nabla_E X = \sigma E + \Omega E, \tag{2.1}$$

where Ω is the associated tensor associated with *CONFVF X*.

Using Eq (2.1), we find the following expression for the curvature tensor field R

$$R(E, F)X = E(\sigma)F - F(\sigma)E + (\nabla_E\Omega)(F) - (\nabla_F\Omega)(E), \tag{2.2}$$

for smooth vector fields E, F on N^n. Choosing a local frame $\{E_1, \ldots, E_n\}$ on (N^n, g) in the above equation in order to compute the Ricci tensor Ric, we obtain

$$Ric(E, X) = -(n-1)E(\sigma) - g(E, div\Omega), \tag{2.3}$$

where

$$div\Omega = \sum_j \left(\nabla_{E_j}\Omega\right)(E_j).$$

Thus, for the *CONFVF X* on an n-dimensional Riemannian manifold (N^n, g) with conformal factor σ, on using Eq (2.3), for Ricci operator Rc, we have

$$Rc(X) = -(n-1)\nabla\sigma - div\Omega. \tag{2.4}$$

On a Riemannian manifold (N^n, g), the scalar curvature $\tau = Tr.Rc$ satisfies the following [18, 19]

$$\frac{1}{2}\nabla\tau = divRc, \tag{2.5}$$

where

$$divRc = \sum_j \left(\nabla_{E_j}Rc\right)(E_j).$$

We see that for the *CONFVF X* on an n-dimensional Riemannian manifold (N^n, g) with conformal factor σ, on using Eq (2.1), we have

$$divX = n\sigma. \tag{2.6}$$

Also, we compute the divergence of the vector field $Rc(X)$ as follows:

$$\begin{aligned} divRc(X) &= \sum_j g\left(\nabla_{E_j}Rc(X), E_i\right) \\ &= \sum_j \left(\left(\nabla_{E_j}Rc\right)(X) + Rc\left(\nabla_{E_j}X\right), E_j\right), \end{aligned}$$

which, on using the symmetry of Rc and Eqs (2.1) and (2.5), yields

$$divRc(X) = \frac{1}{2}X(\tau) + \sum_j g\left(Rc\left(E_j\right), \sigma E_j + \Omega E_j\right),$$

and Ω being skew symmetric, we reach at

$$divRc\,(X) = \frac{1}{2}X\,(\tau) + \sigma\tau. \tag{2.7}$$

Similarly, on using Eq (2.1) and the skew symmetry of the associated operator Ω for the *CONFVF X* on an n-dimensional Riemannian manifold (N^n, g) with conformal factor σ, we find

$$div\,(\Omega X) = -\,\|\Omega\|^2 - g\,(X, div\Omega)\,, \tag{2.8}$$

where

$$\|\Omega\|^2 = \sum_j g\left(\Omega E_j, \Omega E_j\right).$$

Now, for the conformal factor σ of the *CONFVF X* on an n-dimensional Riemannian manifold (N^n, g), the Hessian $Hess(\sigma)$ of σ is the symmetric bilinear form defined by

$$Hess(\sigma)\,(E, F) = E\,(F\sigma) - (\nabla_E F)\,(\sigma)\,,$$

and the Hessian operator $\mathcal{H}^\sigma$ of σ is defined by

$$Hess(\sigma)\,(E, F) = g\,(\mathcal{H}^\sigma E, F)$$

for smooth vector fields E, F on N^n. The Laplacian $\Delta\sigma$ of σ is defined by

$$\Delta\sigma = div\,(\nabla\sigma)\,,$$

which is also given by

$$\Delta\sigma = Tr.\mathcal{H}^\sigma.$$

If (N^n, g) is a compact Riemannian manifold, then we have the following Bochner's formula

$$\int_{N^n} Ric\,(\nabla\sigma, \nabla\sigma) = \int_{N^n}\left((\Delta\sigma)^2 - \|\mathcal{H}^\sigma\|^2\right), \tag{2.9}$$

where for a local frame $\{E_1, .., E_n\}$ on N^n

$$\|\mathcal{H}^\sigma\|^2 = \sum_j g\left(\mathcal{H}^\sigma E_j, \mathcal{H}^\sigma E_j\right).$$

3. Characterizing spheres using conformal vector fields

Let X be a nontrivial *CONFVF* on an n-dimensional Riemannian manifold (N^n, g) with conformal factor σ. In this section, we shall answer the questions raised in the introduction. Indeed, first we prove the following:

Theorem 1. *An n-dimensional compact and connected Riemannian manifold (N^n, g), $n > 1$, of positive Ricci curvature and scalar curvature τ admits a nontrivial CONFVF X with conformal factor σ satisfying*

$$Rc\,(X) = -(n-1)\nabla\sigma \quad and \quad X\,(\tau) = 2\sigma\,(n(n-1)c - \tau)\,,$$

for a constant c, if and only if $c > 0$ and (N^n, g) is isometric to the sphere S^n_c.

Proof. Suppose X is a nontrivial *CONFVF* on an n-dimensional compact and connected Riemannian manifold (N^n, g), $n > 1$, of positive Ricci curvature with conformal factor σ, which satisfies

$$Rc\,(X) = -(n-1)\nabla\sigma \tag{3.1}$$

and

$$X\,(\tau) = 2\sigma\,(n(n-1)c - \tau)\,, \tag{3.2}$$

where c is a constant. Using Eqs (2.4) and (3.1), we obtain $div\Omega = 0$ and inserting it in Eq (2.8) yields

$$div\,(\Omega X) = -\,\|\Omega\|^2\,. \tag{3.3}$$

Now, in Eq (3.1), taking the inner product with X, provides

$$Ric\,(X, X) = -(n-1)X\,(\sigma)\,,$$

which, in light of Eq (2.6) used in the formula

$$\begin{aligned} div\,(\sigma X) &= X\,(\sigma) + \sigma div X \\ &= X\,(\sigma) + n\sigma^2, \end{aligned}$$

takes the form

$$Ric\,(X, X) = n(n-1)\sigma^2 - (n-1)div\,(\sigma X)\,.$$

Integrating the above equation leads to

$$\int_{N^n} Ric\,(X, X) = n(n-1)\int_{N^n}\sigma^2. \tag{3.4}$$

Again, using Eq (3.1), we immediately have

$$\int_{N^n} Ric\,(X, \nabla\sigma) = -(n-1)\int_{N^n}\|\nabla\sigma\|^2\,. \tag{3.5}$$

Next, taking divergence on both sides of Eq (3.1) and making use of Eq (2.7), we get

$$-(n-1)\Delta\sigma = \frac{1}{2}X\,(\tau) + \sigma\tau,$$

which, on treating with Eq (3.2), reduces to

$$\Delta\sigma = -nc\sigma. \tag{3.6}$$

Multiplying the above equation by σ and then integrating leads to

$$\int_{N^n}\|\nabla\sigma\|^2 = nc\int_{N^n}\sigma^2. \tag{3.7}$$

Also, we have

$$Ric\,(\nabla\sigma + cX, \nabla\sigma + cX) = Ric\,(\nabla\sigma, \nabla\sigma) + 2cRic\,(\nabla\sigma, X) + c^2 Ric\,(X, X)\,.$$

Integrating the above equation and using Eqs (2.9), (3.4), and (3.5), we arrive at

$$\int_{N^n} Ric\,(\nabla\sigma + cX, \nabla\sigma + cX) = \int_{N^n} \left\{(\Delta\sigma)^2 - \|\mathcal{H}^\sigma\|^2 - 2(n-1)c\,\|\nabla\sigma\|^2 + n(n-1)c^2\sigma^2\right\},$$

which is rearranged as

$$\int_{N^n} Ric\,(\nabla\sigma + cX, \nabla\sigma + cX) = \int_{N^n} \left\{-\left(\|\mathcal{H}^\sigma\|^2 - \frac{1}{n}(\Delta\sigma)^2\right) + \frac{n-1}{n}(\Delta\sigma)^2 \right.$$
$$\left. -2(n-1)c\,\|\nabla\sigma\|^2 + n(n-1)c^2\sigma^2\right\}.$$

Now, inserting Eqs (3.6) and (3.7) in the above equation reveals

$$\int_{N^n} Ric\,(\nabla\sigma + cX, \nabla\sigma + cX) + \int_{N^n} \left(\|\mathcal{H}^\sigma\|^2 - \frac{1}{n}(\Delta\sigma)^2\right) = 0. \tag{3.8}$$

Observe that the integrand in the second integral in Eq (3.8) is non-negative by Schwarz inequality, and the hypothesis requires that the Ricci curvatures of the Riemannian manifold (N^n, g) are positive. This traps Eq (3.8) to come forward with only the following solutions:

$$\nabla\sigma + cX = 0, \quad \|\mathcal{H}^\sigma\|^2 = \frac{1}{n}(\Delta\sigma)^2. \tag{3.9}$$

Interestingly, both equations in Eq (3.9) reach to the same conclusions. First, take the equation

$$\nabla\sigma + cX = 0,$$

which on differentiation with respect to a vector field E on N^n and using Eq (2.1), leads to the equation

$$\mathcal{H}^\sigma E + c\sigma E = -c\Omega E, \tag{3.10}$$

where the left-hand side is symmetric and the right-hand side is skew symmetric. Hence, we have both

$$\mathcal{H}^\sigma E + c\sigma E = 0$$

and

$$c\Omega E = 0$$

for arbitrary E. Thus, we have two choices: either $c = 0$ or $\Omega = 0$. If $c = 0$, Eq (3.6) will imply σ is a constant, and then the integral of Eq (2.6) will produce $\sigma = 0$: and that is contrary to the fact that X is nontrivial. Hence, $\Omega = 0$ and Eq (3.10) reduces to

$$\mathcal{H}^\sigma = -c\sigma I, \tag{3.11}$$

where σ has to be a non-constant function, for σ a constant implies $\sigma = 0$, which is forbidden by X being nontrivial. The second equation in Eq (3.9) also reaches the same conclusion as Eq (3.11). For it is the equality in Schwarz's inequality

$$\|\mathcal{H}^\sigma\|^2 \geq \frac{1}{n}(\Delta\sigma)^2,$$

which holds if and only if

$$\mathcal{H}^{\sigma} = \frac{\Delta\sigma}{n} I,$$

and combining it with Eq (3.6), gives Eq (3.11). As seen in the above paragraph, the constant $c \neq 0$, indeed, as σ is a non-constant function, Eq (3.7) reveals that $c > 0$. Thus, we have reached the conclusion that Eq (3.11) is Obata's differential equation [9, 17]. Hence, (N^n, g) is isometric to the sphere S_c^n.

Conversely, if (N^n, g) is isometric to the sphere S_c^n, then through Eq (1.3), we have a *CONFVF* X on S_c^n with conformal factor

$$\sigma = -\sqrt{c}\rho.$$

First, we claim that X is a nontrivial *CONFVF* on S_c^n. For if $\sigma = 0$, that is, $\rho = 0$, which on using Eq (1.3) would imply $X = 0$, and in turn it would give

$$Z = X + \rho\zeta = 0$$

a contradiction to the fact that Z is a unit vector. Hence, X is a nontrivial *CONFVF* on S_c^n. Also, for S_c^n, we have

$$Rc = (n-1)cI,$$

which, on treating with Eq (1.3), implies

$$Rc\,(X) = (n-1)cX = (n-1)\sqrt{c}\nabla\rho = -(n-1)\nabla\sigma,$$

that is, Eq (3.1) holds. Also, the scalar curvature τ of S_c^n is

$$\tau = n(n-1)c$$

a constant, and therefore, Eq (3.2) holds. Finally, S_c^n is compact and has positive Ricci curvature. Hence, converse is true. □

It is worth noticing that the condition (3.2) is essential in the above characterization of S_c^n, as there are compact manifolds admitting a nontrivial *CONFVF* that are not isometric to S_c^n and on them the Eq (3.2) does not hold.

For example, consider the compact Riemannian manifold (N^n, g), where

$$N^n = S^1 \times_\rho S_c^{(n-1)}$$

is the warped product, with ρ a smooth positive function on the unit circle S^1 and the warped product metric

$$g = d\theta^2 + \rho^2\overline{g},$$

θ is a coordinate function on S^1 and $\overline{g}$ is the canonical metric on the sphere $S_c^{(n-1)}$ of constant curvature c. Then the vector field

$$X = \rho\frac{\partial}{\partial\theta}$$

on (N^n, g) satisfies [19]

$$\nabla_E X = \rho' E,$$

where E is any vector field on N^n. Thus, we obtain

$$\frac{1}{2}\pounds_X g = \sigma g,$$

that is, X is a *CONFVF* (N^n, g) with conformal factor $\sigma = \rho'$. We have the following expression for the Ricci operator Rc on the warped product manifold (N^n, g) [19]

$$Rc\,(E) = -\frac{n-1}{\rho}\mathcal{H}^{\rho}E \tag{3.12}$$

for a horizontal vector field E on S^1 and

$$Rc\,(V) = (n-2)cV - \left(\frac{\rho''}{\rho} + (n-2)\left(\frac{\rho''}{\rho}\right)^2\right)V$$

for vertical vector field V on S^n_c. As the *CONFVF* X is horizontal, we see by Eq (3.12) that

$$Rc\,(X) = -(n-1)\mathcal{H}^{\rho}\left(\frac{\partial}{\partial\theta}\right) = -(n-1)\nabla\sigma,$$

that is, the condition (3.1) holds for the *CONFVF* X on the compact warped product manifold (N^n, g). The scalar curvature τ of (N^n, g) is given by

$$\tau = -\frac{n-1}{\rho^2}\left(2\rho\rho'' + (n-2)\rho'^2 - (n-2)c\right),$$

which does not satisfy Eq (3.2).

4. Characterizing Euclidean spaces by conformal vector fields

On the Euclidean space $(E^n, \langle,\rangle)$, there are finitely many nontrivial conformal vector fields. For instance, the position vector field ξ

$$\xi = \sum_j u^j \frac{\partial}{\partial u^j} \tag{4.1}$$

satisfies

$$\frac{1}{2}\pounds_\xi g = g,$$

that is, ξ is a *CONFVF* on $(E^n, \langle,\rangle)$ with conformal factor $\sigma = 1$. However, ξ is closed, and therefore its associated tensor $\Omega = 0$. Next, we construct a non-closed nontrivial *CONFVL* on $(E^n, \langle,\rangle)$. Define a vector field X on E^n, $n > 2$, by

$$X = \xi + u^2\frac{\partial}{\partial u^1} - u^1\frac{\partial}{\partial u^2}.$$

Then, we see that

$$\nabla_E X = E + \Omega E, \tag{4.2}$$

where

$$\Omega E = E\left(u^2\right)\frac{\partial}{\partial u^1} - E\left(u^1\right)\frac{\partial}{\partial u^2},$$

and it follows that

$$\langle \Omega E, F\rangle = -\langle E, \Omega F\rangle,$$

that is, Ω is a skew symmetric $(1,1)$ tensor on the Euclidean space $(E^n, \langle,\rangle)$. Using Eq (4.2), one confirms that

$$\frac{1}{2}\pounds_X g = g,$$

that is, X is a nontrivial *CONFVF* on $(E^n, \langle,\rangle)$ with conformal factor $\sigma = 1$, and it is not a closed vector field. Moreover, we see that there are finitely many of these types of nontrivial conformal vector fields on the Euclidean space $(E^n, \langle,\rangle)$.

In this section, we find the following characterization for a Euclidean space.

Theorem 2. *Let X be a nontrivial CONFVF on an n-dimensional complete and connected Riemannian manifold (N^n, g), $n > 1$, with conformal factor σ and associated tensor Ω. Then the following conditions hold:*

$$Rc(X) = -div\Omega, \quad \Omega(X) = 0,$$

if and only if (N^n, g) is isometric to the Euclidean space $(E^n, \langle,\rangle)$.

Proof. Suppose an n-dimensional complete and connected Riemannian manifold (N^n, g) admits a nontrivial *CONFVL* X with conformal factor σ and associated tensor Ω satisfying

$$Rc(X) = -div\Omega \tag{4.3}$$

and

$$\Omega(X) = 0. \tag{4.4}$$

Using Eqs (2.4), (4.3), and $n > 1$, we reach the conclusion that σ is a constant. It is clear that the constant $\sigma \neq 0$ due to the fact that X is nontrivial. Next, define a smooth function α by

$$2\alpha = \|X\|^2.$$

Then, using Eqs (2.1) and (4.4), we find the gradient of the smooth function α is given by

$$\nabla\alpha = \sigma X. \tag{4.5}$$

Since the *CONFVF* X is nontrivial and the constant $\sigma \neq 0$, the above equation confirms that the function α is not a constant. Now, differentiating Eq (4.5) with respect to a vector field E on N^n and using Eq (2.1), we arrive at

$$\mathcal{H}^\alpha E = \sigma^2 E + \sigma\Omega E,$$

and taking the inner product in the above equation by E, yields

$$Hess(\alpha)(E, E) = \sigma^2 g(E, E).$$

On polarizing the above equation, we conclude

$$Hess(\alpha) = \sigma^2 g, \tag{4.6}$$

where α is a non-constant function and σ^2 is a nonzero constant. Equation (4.6) guarantees that (N^n, g) is isometric to the Euclidean space $(E^n, \langle,\rangle)$.

The converse is trivial, for the position vector field ξ given in (4.1) is a nontrivial *CONFVF* on the Euclidean space $(E^n, \langle,\rangle)$ with conformal factor $\sigma = 1$, and it being closed, we have associated tensor $\Omega = 0$. Also, for the Euclidean space $(E^n, \langle,\rangle)$ being flat, we have Ricci operator $Rc = 0$, and we see that the conditions (4.3) and (4.4) are automatically satisfied. Hence, the converse. □

5. Conclusions

It is needless to mention that one of the most interesting and most sought-out questions in differential geometry is to find different characterizations of the model spaces, namely, the sphere S_c^n of constant curvature c, the Euclidean space E^n, and the hyperbolic space H_{-c}^n of constant curvature $-c$ $(c > 0)$. In the present paper, we witnessed that an n-dimensional compact and connected Riemannian manifold (N^n, g), $n > 1$, of positive Ricci curvature and scalar curvature admits a nontrivial *CONFVF* X with conformal factor σ satisfying the conditions

$$Ric\,(X) = -(n-1)\nabla\sigma \quad \text{and} \quad X\,(\tau) = 2\sigma\,(n(n-1)c - \tau)\,, \tag{5.1}$$

for a constant c, if and only if $c > 0$ and (N^n, g) is isometric to the sphere S_c^n. In the process of the proof, we have seen that the above conditions together with the assumption that (N^n, g) has positive Ricci curvature lead us to

$$\nabla\sigma = -cX. \tag{5.2}$$

Since X is a *CONFVF*, this indicates that σ is not constant, and furthermore, the compactness of N^n forces the constant c to be positive. Finally, combining Eq (5.2) with the definition of the *CONFVF* X leads us to Obata's differential equation, concluding that (N^n, g) is isometric to the sphere S_c^n.

Note that the *CONFVF* X on the sphere S_c^n with conformal factor σ, by virtue of Eq (1.3), satisfies

$$Ric\,(\nabla\sigma) = \frac{\tau}{n}\nabla\sigma, \tag{5.3}$$

where

$$\tau = n(n-1)c$$

is the scalar curvature of the sphere S_c^n. This naturally leads to the question: Under what conditions a compact and connected Riemannian manifold (N^n, g) with Ricci operator Ric scalar curvature τ, admitting a nontrivial *CONFVF* X with conformal factor σ satisfying Eq (5.3), that is

$$Ric\,(\nabla\sigma) = \frac{\tau}{n}\nabla\sigma$$

is isometric to the sphere S_c^n? We shall be interested in taking up this question in our future studies.

Author contributions

S. Deshmukh: conceptualization, investigation, methodology, validation, writing-original draft, formal analysis, validation, inspection; M. Guediri: conceptualization, investigation, methodology, formal analysis, writing-review and editing, validation, inspection. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker Inc., 1970.
2. S. Hwang, G. Yun, Conformal vector fields and their applications to Einstein-type manifolds, *Results Math.*, **79** (2024), 45. https://doi.org/10.1007/s00025-023-02070-7
3. A. Caminha, The geometry of closed conformal vector fields on Riemannian spaces, *Bull. Braz. Math. Soc. New Ser.*, **42** (2011), 277–300. https://doi.org/10.1007/s00574-011-0015-6
4. J. F. da S. Filho, Critical point equation and closed conformal vector fields, *Math. Nach.*, **293** (2020), 2299–2305. https://doi.org/10.1002/mana.201900316
5. S. Tanno, W. Weber, Closed conformal vector fields, *J. Differ. Geom.*, **3** (1969), 361–366. https://doi.org/10.4310/JDG/1214429058
6. W. Kuhnel, H. B. Rademacher, Conformal diffeomorphisms preserving the Ricci tensor, *Proc. Amer. Math. Soc.*, **123** (1995), 2841–2848.
7. W. Kuhnel, H. B. Rademacher, Einstein spaces with a conformal group, *Results Math.*, **56** (2009), 421. https://doi.org/10.1007/s00025-009-0440-7
8. W. Kühnel, H. B. Rademacher, Conformal vector fields on pseudo-Riemannian spaces, *Differ. Geom. Appl.*, **7** (1997), 237–250. https://doi.org/10.1016/S0926-2245(96)00052-6
9. M. Obata, The conjectures about conformal transformations, *J. Differ. Geom.*, **6** (1971), 247–258. https://doi.org/10.4310/JDG/1214430407
10. S. Deshmukh, Characterizing spheres and Euclidean spaces by conformal vector field, *Ann. Mat. Pura Appl.*, **196** (2017), 2135–2145. https://doi.org/10.1007/s10231-017-0657-0
11. K. Yano, T. Nagano, Einstein spaces admitting a one-parameter group of conformal transformations, *North-holland Math. Stud.*, **70** (1982), 219–229. https://doi.org/10.1016/S0304-0208(08)72248-5
12. B. Y. Chen, A simple characterization of generalized Robertson-Walker spacetimes, *Gen. Relat. Gravit.*, **46** (2014), 1833. https://doi.org/10.1007/s10714-014-1833-9
13. G. S. Hall, Conformal vector fields and conformal-type collineations in space-times, *Gen. Relat. Gravit.*, **32** (2000), 933–941. https://doi.org/10.1023/A:1001941209388
14. G. S. Hall, J. D. Steele, Conformal vector fields in general relativity, *J. Math. Phys.*, **32** (1991), 1847. https://doi.org/10.1063/1.529249
15. S. Deshmukh, M. Guediri, Characterization of Euclidean spheres, *AIMS Math.*, **6** (2021), 7733–7740. https://doi.org/10.3934/math.2021449
16. M. Guediri, S. Deshmukh, Hypersurfaces in a Euclidean space with a Killing vector field, *AIMS Math.*, **9** (2024), 1899–1910. https://doi.org/10.3934/math.2024093

17. M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, *J. Math. Soc. Jpn.*, **14** (1962), 333–340. https://doi.org/10.2969/JMSJ/01430333

18. A. L. Besse, *Einstein manifolds*, Springer-Verlag, 1987. https://doi.org/10.1007/978-3-540-74311-8

19. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, 1983.

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