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Research article

Existence and analysis of Hilfer-Hadamard fractional differential equations in RLC circuit models

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Abstract: This paper explores a fractional integro-differential equation with boundary conditions that incorporate the Hilfer-Hadamard fractional derivative. We model the RLC circuit using fractional calculus and define weighted spaces of continuous functions. The existence and uniqueness of solutions are established, along with their Ulam-Hyers and Ulam-Hyers-Rassias stability. Our analysis employs Schaefer's fixed-point theorem and Banach's contraction principle. An illustrative example is presented to validate our findings.

Keywords: fractional differential equations; Hilfer-Hadamard fractional derivative; non-local boundary conditions; existence; fixed-point theorem
Mathematics Subject Classification: 34A05, 34B15, 47H10

Abbreviations

The following abbreviations are used in this manuscript:

BVPs	Boundary Value Problems
HHFDEs	Hilfer-Hadamard Fractional-order Differential Equations
HFIs	Hadamard Fractional Integrals
HHFDs	Hilfer-Hadamard Fractional Derivatives
CFDs	Caputo Fractional Derivatives
HFDs	Hilfer Fractional Derivatives
HFDEs	Hilfer Fractional Differential Equations
HFDs	Hadamard Fractional Derivatives (HFDs)
CHFDs	Caputo-Hadamard Fractional Derivatives (CHFDs)

1. Introduction

Fractional calculus extends classical calculus by generalizing derivatives and integrals from integer orders to arbitrary orders. In this field, various definitions of integrals and derivatives exist, with the Caputo and Riemann-Liouville (R-L) formulations being widely recognized. These formulations have driven extensive research, extending differential equations from integer to fractional orders. Recently, Hilfer introduced a generalized R-L derivative, known as the Hilfer derivative, which bridges the gap between Caputo and R-L derivatives and has attracted significant attention. Recent studies have investigated the Ulam stability and existence results of differential equations using fractional Hadamard and Hilfer derivatives, marking a notable advancement in fractional calculus and opening new avenues for mathematical exploration and application [2, 5, 8, 14].

The researchers focus on stability analysis of fractional differential equations using different types of fractional derivatives and make an important contribution to the understanding of stability properties in fractional calculus [1,6,25,26]. Beginning with studies on the stability of solutions to linear differential equations with fractional Caputo derivatives [25], subsequent research investigated the existence and Ulam stability of solutions to equations characterized by the Hilfer-Hadamard type [1] and extended the analysis to include the new Caputo-Fabrizio fractional derivative and advance the discourse on stability in fractional calculus [6]. In addition, studies in [26] investigated the Hyers-Ulam stability of nonlinear differential equations subject to fractional integrable momentum, expanding the scope of stability analysis in fractional calculus applications.

Fixed-point theorems are crucial for establishing both the existence and uniqueness of solutions in various mathematical contexts. Their application also extends to examining the attractivity of solutions within fractional calculus, facilitating advancements in both theoretical understanding and practical applications across a broad range of scientific disciplines. For instance, the authors in [7] studied the complex interplay between nonlinear Caputo fractional derivatives and nonlocal Riemann-Liouville fractional integral conditions, offering new insights through fixed-point theorems. Similarly, [24] expanded our understanding of positive solutions for fractional differential equations with derivative terms, introducing a novel fixed-point theorem to address this challenging problem. In [3], the authors explored the complexities of fractional boundary value problems, particularly focusing on mixed boundary conditions, which are essential for modeling various physical phenomena. Additionally, [10] extended the applicability of mathematical tools, such as Mönch's fixed-point theorem, to analyze intricate systems of Hilfer-type fractional differential equations, opening avenues for further theoretical developments. Lastly, [22] investigated the existence and uniqueness of nonlocal boundary conditions in the context of Hilfer-Hadamard-type fractional differential equations, bridging theoretical insights with practical applications.

The study by Fan et al. [11] investigates the synchronization of fractional-order multi-link memristive neural networks (MNNs) with time delays, proposing a hybrid impulsive feedback control strategy to achieve drive-response synchronization in these complex systems. This approach offers valuable insights into the stability and synchronization dynamics in neural networks and chaotic circuits. In contrast, Li et al. [18] focus on the stability analysis of fractional differential equations (FDEs) with non-instantaneous impulses and multi-point boundary conditions, providing new stability criteria for systems with delayed impulses. Their work has broad implications for biological systems and engineering problems influenced by fractional dynamics. Both articles contribute significantly to the understanding of control and stability in fractional-order systems, with distinct applications in neural computation and biological modeling.

In 1892, Hadamard introduced the new concept of Hadamard fractional derivatives, using a logarithmic function with an arbitrary exponent at its [9]. Building on this foundation, subsequent research, exemplified by notable works such as [4, 16, 20], has explored various extensions, including the study of Hilfer Hadamard-type fractional derivatives and Caputo-Hadamard fractional derivatives. It is important to note that Hadamard fractional derivatives and Caputo-Hadamard fractional derivatives are specific instances of the broader Hilfer-Hadamard type fractional derivatives framework, distinguished by the parameter β with values of $\beta = 0$ and $\beta = 1$, respectively. This rich line of research has led to investigations into the existence and properties of solutions to Hilfer-Hadamard type fractional differential equations, particularly with respect to non-local integro-multi-point boundary conditions.

Existence results for a Hilfer-Hadamard type fractional differential equations with nonlocal integro-multipoint boundary conditions were derived in [21],

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}_1^{\vartheta,\beta}x(\mathrm{t}) = f(\mathrm{t}, x(\mathrm{t})), \quad \mathrm{t} \in [1, T], \\ x(1) = 0, \quad \sum_{i=1}^{m} \mathcal{X}_i x(\xi_i) = \lambda^H \mathcal{I}^\delta x(\eta), \end{cases} \tag{1.1}$$

here $\vartheta \in (1, 2]$, $\beta \in \lfloor 0, 1 \rfloor$, $\mathcal{X}_i, \lambda \in \mathbb{R}$, $\eta, \xi_i \in (1, T)$ $(i = 1, 2, ..., m)$, ${}^H\mathcal{I}^\delta$ is the HFI of order $\delta > 0$, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Problem (1.1) represents a non-coupled system.

Integer order integro-differential equations find applications in various domains of science and engineering, including circuit analysis. According to Kirchhoff's second law, the total voltage drop across a closed loop is equal to the applied voltage, denoted as $\mathbb{E}(\mathrm{t})$. This principle essentially stems from the law of energy conservation. Consequently, an RLC circuit equation has the form

$$\frac{d\mathbb{I}(\mathrm{t})}{d\mathrm{t}} + \mathbb{R}\mathbb{I}(\mathrm{t}) + \frac{1}{\mathbb{C}} \int_0^{\mathrm{t}} \mathbb{I}(s)\, ds = \mathbb{E}(\mathrm{t}).$$

This paper explores the practical application of fractional derivatives in modeling various electrical circuits, including $\mathbb{RC}$, $\mathbb{RL}$, and $\mathbb{RLC}$ configurations, as well as power electronic devices and nonlinear loads. The $\mathbb{RLC}$ circuit serves as a fundamental component in the assembly of more intricate electrical circuits and networks. Illustrated in Figure 1, it comprises a resistor with a resistance of $\mathbb{R}$ ohms, an inductor with an inductance of $\mathbb{L}$ henries, and a capacitor with a capacitance of $\mathbb{C}$ farads, all arranged in series with an electromotive force source (like a battery or a generator) providing a voltage of $\mathbb{E}(\mathrm{t})$ volts at time t.

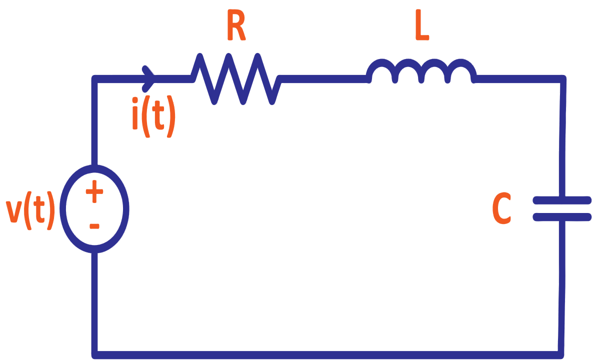

Figure 1. Diagram of a $\mathbb{RLC}$ circuit.

In [19], Malarvizhi et al. discussed the transient analysis of an RLC circuit in the RK4 order method. In [12], Gomez-Aguilar et al.studied the electrical circuits RC and RL for the Atangana-Beleanu-Caputo (ABC) fractional bi-order system:

$$^{ABC}D^{\beta}\mathbb{V}(t) = \delta\mathbb{E}(\mathrm{t}) - \delta\mathbb{V}.$$

In 2016 in [13], researchers derived analytical and numerical solutions of electrical circuits employing fractional derivatives. By substituting traditional time derivatives with fractional derivatives like Riemann-Liouville, Grünwald-Letnikov, Liouville-Caputo, and the recently introduced Caputo and Fabrizio fractional definitions, the authors derived equations capturing the dynamic behavior of these circuits. Motivated by [13], we have considered the following Hilfer-Hadamard fractional derivative equation with the RLC model:

$$^{\mathcal{HH}}\mathcal{D}_1^{\vartheta,\beta}\mathbb{I}(\mathrm{t}) = \frac{\mathbb{E}(\mathrm{t})}{\mathbb{L}} - \frac{\mathbb{R}}{\mathbb{L}}\mathbb{I}(\mathrm{t}) - \frac{1}{\mathbb{CL}}\int_1^{\mathrm{t}}\mathbb{I}(\varsigma)d\varsigma, \quad \mathrm{t}\in\mathcal{J} = \lfloor 1, \mathfrak{T}\rfloor, \tag{1.2}$$

$$\mathcal{X}(1) = 0, \quad \mathcal{X}(\mathfrak{T}) = \sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\mathcal{X}(\xi_i) + \sum_{\mathrm{j}=1}^{\text{ŧ}}\varrho_{\mathrm{j}}\mathcal{I}^{\nu_{\mathrm{j}}}\mathcal{X}(\zeta_{\mathrm{j}}), \quad \nu_{\mathrm{j}} > 0, \eta_i, \varrho_{\mathrm{j}} \in \mathbb{R}, \ \zeta_{\mathrm{j}}, \xi_i \in \mathcal{J}. \tag{1.3}$$

The Problems (1.2) and (1.3) exhibit nonlocal coupling with integral and multi-point boundary conditions. The $\mathbb{RLC}$ circuit system is shown in Figure 1.

The primary contribution of this endeavor can be outlined as follows:

(1) The existence, uniqueness, and stability of the solution of the Hilfer-Hadamard fractional multi point integro-differential equation for the RLC circuit model have been investigated via the fixed-point approach.

(2) We apply a novel hypothesis to verify the existence, uniqueness, and Ulam-Hyers stability of the solution to the RLC circuit Eqs (1.2) and (1.3). We additionally, the paper is structured as follows:

Section 2 introduces definitions and properties of fractional derivatives, along with an investigation into the existence of solutions for the boundary value problem. Section 3, we give the main results. Sections 3.1 and 3.2 focus on the existence and uniqueness of solutions. Section 4 examines Ulam stability, while Section 5 presents examples illustrating the developed theorems. Finally, Section 6 provides concluding insights.

2. Preliminaries

Definition 2.1. *[17] The Hadamard fractional integral of order $\vartheta \in \mathbb{R}$ for the function $\mathcal{X} : \lfloor a, \infty) \to \mathbb{R}$ is defined as follows:*

$$ {}^{\mathcal{H}}\mathcal{I}^{\vartheta}_{a^+}\mathcal{X}(\mathfrak{t}) = \frac{1}{\Gamma(\vartheta)}\int_b^{\mathfrak{t}} \Big(\log\frac{\mathfrak{t}}{\mathfrak{a}}\Big)^{\vartheta-1}\frac{\mathcal{X}(\mathfrak{t})}{\mathfrak{a}}d\mathfrak{a}, \quad \mathfrak{a} > b, \tag{2.1} $$

provided the integral exists, where $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. *[17] For a continuous function $\mathcal{X} : \lfloor a, \infty) \to \mathbb{R}$, the HFD of order $\vartheta > 0$ is given by*

$$ {}^{\mathcal{H}}\mathcal{D}^{\vartheta}_{\mathfrak{a}^+}\mathcal{X}(\mathfrak{t}) = \mathfrak{p}^n({}^{\mathcal{H}}\mathcal{I}^{n-\vartheta}_{\mathfrak{a}^+}\mathcal{X})(\mathfrak{t}), \ \ n = \lfloor\vartheta\rfloor + 1, \tag{2.2} $$

where $\mathfrak{p}^n = \mathfrak{t}^n\frac{d^n}{dt^n}$ and $\lfloor\vartheta\rfloor$ represent the integer parts of the real number ϑ.

Lemma 2.1. *[17] If $\vartheta, \gamma > 0$ and $0 < \mathfrak{a} < \mathfrak{b} < \infty$ then*

$$ (1) \quad \Big({}^{\mathcal{H}}\mathcal{I}^{\vartheta}_{a^+}\Big(\log\frac{\mathfrak{t}}{\mathfrak{a}}\Big)^{\gamma-1}\Big)(\mathcal{X}) = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\vartheta)}\Big(\log\frac{\mathfrak{t}}{\mathfrak{a}}\Big)^{\gamma+\vartheta-1}; $$

$$ (2) \quad \Big({}^{\mathcal{H}}\mathcal{D}^{\vartheta}_{a^+}\Big(\log\frac{\mathfrak{t}}{\mathfrak{a}}\Big)^{\gamma-1}\Big)(\mathcal{X}) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\vartheta)}\Big(\log\frac{\mathfrak{t}}{\mathfrak{a}}\Big)^{\gamma-\vartheta-1}. $$

In particular, if $\gamma = 1$, then the following is the case:

$$ \Big({}_{\mathcal{H}}\mathcal{D}^{\vartheta}_{a^+}\Big)(1) = \frac{1}{\Gamma(1-\vartheta)}\Big(\log\frac{\mathfrak{t}}{\mathfrak{a}}\Big)^{-\vartheta} \neq 0, 0 < \vartheta < 1. $$

Definition 2.3. *[15] For $n-1 < \mathfrak{p} < n$ and $0 \le \mathfrak{q} \le 1$, the HHFD of order mathfrakp and the type $\mathfrak{q}$ for $\mathfrak{f} \in \mathcal{L}'(\mathfrak{a}, \mathfrak{b})$ is defined as*

$$ \begin{aligned} ({}^{\mathcal{H}\mathcal{H}}\mathcal{I}^{\delta,\gamma}_{\mathfrak{a}^+}) =& ({}^{\mathcal{H}}\mathcal{I}^{\gamma(\mathfrak{n}-\delta)}_{\mathfrak{a}^+}\mathfrak{p}^{\mathfrak{n}\mathcal{H}}\mathcal{I}^{(\mathfrak{n}-\delta)(1-\gamma)}_{\mathfrak{a}^+}\mathcal{X})(\mathfrak{t}) \\ =& ({}^{\mathcal{H}}\mathcal{I}^{\gamma(\mathfrak{n}-\delta)}_{\mathfrak{a}^+}\mathfrak{p}^{\mathfrak{n}\mathcal{H}}\mathcal{I}^{(\mathfrak{n}-\mathfrak{q})}_{\mathfrak{a}^+}\mathcal{X})(\mathfrak{t}) \\ =& ({}^{\mathcal{H}}\mathcal{I}^{\gamma(\mathfrak{n}-\delta)}_{\mathfrak{a}^+}\mathfrak{p}^{\mathfrak{n}\mathcal{H}}\mathcal{D}^{\mathfrak{q}}_{\mathfrak{a}^+}\mathcal{X})(\mathfrak{t}), \ \ \mathfrak{q} = \delta + \mathfrak{n}\gamma - \delta\gamma, \end{aligned} $$

where ${}^{\mathcal{H}}\mathcal{I}^{(\cdot)}_{\mathfrak{a}^+}$ and ${}^{\mathcal{H}}\mathcal{D}^{(\cdot)}_{\mathfrak{a}^+}$ are given in Definitions 2.1 and 2.2, respectively.

Lemma 2.2. *[23] If $\varphi \in \mathcal{L}^1(\mathfrak{a}, \mathfrak{b}), 0 < \mathfrak{a} < \mathfrak{b} < \infty$, and $\Big({}^{\mathcal{H}}\mathcal{I}^{n-\mathfrak{q}}_{a^+}\varphi\Big)(\mathfrak{t}) \in \mathcal{AC}^n_{\mathfrak{p}}[\mathfrak{a}, \mathfrak{b}]$, then*

$$ \begin{aligned} {}^{\mathcal{H}}\mathcal{D}^{\vartheta}_{a^+}\Big({}^{\mathcal{H}\mathcal{H}}\mathcal{D}^{\delta,\gamma}_{a^+}\varphi\Big)(\mathfrak{t}) =& {}^{\mathcal{H}}\mathcal{I}^{\mathfrak{q}}_{a^+}\Big({}^{\mathcal{H}\mathcal{H}}\mathcal{D}^{\mathfrak{q}}_{a^+}\varphi\Big)(\mathfrak{t}) \\ =& \varphi(\mathfrak{t}) - \sum_{j=o}^{n-1}\frac{(\mathfrak{p}^{(n-j-1)}({}^{\mathcal{H}}\mathcal{D}^{\vartheta}_{a^+}\varphi))(\mathfrak{a})}{\Gamma(\mathfrak{q}-j)}\Big(\log\frac{\mathfrak{t}}{\mathfrak{a}}\Big)^{\mathfrak{q}-j-1}, \end{aligned} $$

where $\delta > 0, 0 \le \gamma \le 1$ and $\mathfrak{q} = \delta + n\gamma - \delta\gamma, n = \lfloor\delta\rfloor + 1$. Observe that $\Gamma(\mathfrak{q} - j)$ exists for all $j = 1, 2, \cdots, n-1$ and $\mathfrak{q} \in \lfloor\delta, n\rfloor$.

2.1. Problem formulation

Let us consider the general structure of the Hilfer-Hadamard fractional order RLC circuit integro-differential equation with nonlocal boundary conditions:

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}^{\vartheta,\beta}\mathcal{X}(\mathrm{t}) = \mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma))), \ \ \mathrm{t}\in\mathcal{J}=\lfloor 1,\mathfrak{T}\rfloor, \\ \mathcal{X}(1)=0, \ \ \mathcal{X}(\mathfrak{T}) = \sum_{\mathrm{i}=1}^{\mathfrak{l}} \eta_i \mathcal{X}(\xi_i) + \sum_{\mathrm{j}=1}^{\mathfrak{k}} \varrho_{\mathrm{j}} \mathcal{I}^{\nu_{\mathrm{j}}}\mathcal{X}(\zeta_{\mathrm{j}}), \ \ \nu_{\mathrm{j}}>0, \eta_i, \varrho_{\mathrm{j}}\in\mathbb{R}, \ \zeta_{\mathrm{j}},\xi_i\in\mathcal{J}, \end{cases} \tag{2.3}$$

where ${}^{\mathcal{HH}}\mathcal{D}^{\vartheta,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\vartheta\in(1,2)$, and type $\beta\in\lfloor 0,1\rfloor$ and $\eta_i,\varrho_j\in\mathbb{R}$ parameter $\mathcal{I}^{\nu_{\mathrm{j}}}$ is the Riemann-Liouville fractional integral of order $\nu_{\mathrm{j}}>0, \zeta_{\mathrm{j}}\in\lfloor 1,\mathfrak{T}\rfloor, \varrho_{\mathrm{j}}\in\mathbb{R}, \mathrm{j}=1,\cdots,\mathfrak{k}$.

$$\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\int_{\mathfrak{a}}^{\mathrm{t}}\mathcal{F}(\mathrm{t},\varsigma,\mathcal{X}(\varsigma)d\varsigma)) = \frac{\mathbb{E}(\mathrm{t})}{\mathbb{L}} - \frac{\mathbb{R}}{\mathbb{L}}\mathbb{I}(\mathrm{t}) - \frac{1}{\mathbb{CL}}\int_1^{\mathrm{t}}\mathbb{I}(\varsigma)d\varsigma, \tag{2.4}$$

and

$$\mathcal{H}(\mathcal{X}(\varsigma)) = \int_{\mathfrak{a}}^{\mathrm{t}}\mathcal{F}(\mathrm{t},\varsigma,\mathcal{X}(\varsigma)d\varsigma). \tag{2.5}$$

Using some fixed-point theorems, the existence and uniqueness results are established. For (2.3), we employ Banach's fixed-point and Schaefer's fixed-point theorem for uniqueness and existence results.

This section is concerned with the existence and uniqueness of solutions for the nonlinear Hilfer - Hadamard fractional derivative boundary value problem (1.3). First of all, we prove an auxiliary lemma dealing with the linear variant of the boundary value problem (1.3), which will be used to transform the problem at hand into an equivalent fixed-point problem. In the case $n=[\vartheta]+1=2$, we have $\gamma=\vartheta+(2-\vartheta)\beta$.

Lemma 2.3. *Let* $\mathfrak{h}\in C(\lfloor 1,\mathfrak{T}\rfloor,\mathbb{R})$ *and that*

$$\Pi = \Big(\log\mathfrak{T}\Big)^{\gamma-1} - \sum_{\mathrm{i}=1}^{\mathfrak{l}}\eta_i\Big(\log\xi_i\Big)^{\gamma-1} - \sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\mathcal{I}^{\nu_{\mathrm{j}}}\Big(\log\zeta_{\mathrm{j}}\Big)^{\gamma-1}\neq 0. \tag{2.6}$$

Then, $\mathcal{X}$ *is a solution of the following linear Hilfer-Hadamard fractional boundary value problem:*

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}_1^{\vartheta,\beta}\mathcal{X}(\mathrm{t}) = h(\mathrm{t}), \ \ \mathrm{t}\in\mathcal{J}=\lfloor 1,\mathfrak{T}\rfloor, \\ \mathcal{X}(1)=0, \ \ \mathcal{X}(\mathfrak{T}) = \sum_{\mathrm{i}=1}^{\mathfrak{l}} \eta_i \mathcal{X}(\xi_i) + \sum_{\mathrm{j}=1}^{\mathfrak{k}} \varrho_{\mathrm{j}} \mathcal{I}^{\nu_{\mathrm{j}}}\mathcal{X}(\zeta_{\mathrm{j}}), \ \ \nu_{\mathrm{j}}>0, \eta_i, \varrho_{\mathrm{j}}\in\mathbb{R}, \ \zeta_{\mathrm{j}},\xi_i\in\mathcal{J}, \end{cases} \tag{2.7}$$

which satisfies the following equation:

$$\begin{aligned} \mathcal{X}(\mathrm{t}) = & \frac{\Big(\log\mathrm{t}\Big)^{\gamma-1}}{\Pi}\Bigg[\sum_{\mathrm{i}=1}^{\mathfrak{l}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{h(\varsigma)}{\varsigma}d\varsigma + \sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\mathcal{I}^{\nu_{\mathrm{j}}}\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\zeta_{\mathrm{j}}}{\varsigma}\Big)^{\vartheta-1}\frac{h(\varsigma)}{\varsigma}d\varsigma \\ & - \frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}\Big(\log\frac{\mathfrak{T}}{\varsigma}\Big)^{\vartheta-1}\frac{h(\varsigma)}{\varsigma}d\varsigma\Bigg] + \frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}\Big(\log\frac{\mathrm{t}}{\varsigma}\Big)^{\vartheta-1}\frac{h(\varsigma)}{\varsigma}d\varsigma. \end{aligned} \tag{2.8}$$

Proof. Applying the Hadamard fractional integral operator of order ϑ from 1 to $\mathfrak{t}$ on both sides of Hilfer-Hadamard fractional differential equations in (2.7) and using Lemma 2.2, we find that

$$\mathcal{X}(\mathfrak{t}) - \frac{\delta({}_H\mathcal{I}_{1+}^{2-\gamma}\mathcal{X})(1)}{\Gamma(\gamma)}(\log \mathfrak{t})^{\gamma-1} - \frac{\delta({}_H\mathcal{I}_{1+}^{2-\gamma}\mathcal{X})(1)}{\Gamma(\gamma)}(\log \mathfrak{t})^{\gamma-2} = {}^{\mathcal{H}}\mathcal{I}^{\alpha}h(\mathfrak{t}),$$

and we obtain,

$$\mathcal{X}(\mathfrak{t}) = \mathfrak{c}_0\Big(\log \mathfrak{t}\Big)^{\gamma-1} + \mathfrak{c}_1\Big(\log \mathfrak{t}\Big)^{\gamma-2} + \frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{t}}\Big(\log\frac{\mathfrak{t}}{\varsigma}\Big)^{\vartheta-1}\frac{h(\varsigma)}{\varsigma}d\varsigma, \tag{2.9}$$

where $\mathfrak{c}_0$ and $\mathfrak{c}_1$ are arbitrary constants. Using the first boundary condition ($\mathcal{X}(1) = 0$) in (2.9) yields $\mathfrak{c}_1 = 0$, since $\gamma \in [\vartheta, 2]$. In consequence, (2.7) takes the following form:

$$\mathcal{X}(\mathfrak{T}) = \sum_{\mathfrak{i}=1}^{\mathfrak{l}}\eta_i\mathcal{X}(\xi_i) + \sum_{\mathfrak{j}=1}^{\mathfrak{k}}\varrho_{\mathfrak{j}}\mathcal{I}^{\nu_{\mathfrak{i}}}\mathcal{X}(\zeta_{\mathfrak{j}}), \tag{2.10}$$

and using the notation (2.10), we obtain the following :

$$\begin{aligned}\mathfrak{c}_0 =& \frac{1}{\Pi}\Bigg[\sum_{\mathfrak{i}=1}^{\mathfrak{l}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{h(\varsigma)}{\varsigma}d\varsigma - \sum_{\mathfrak{j}=1}^{\mathfrak{k}}\varrho_{\mathfrak{j}}\mathcal{I}^{\nu_{\mathfrak{i}}}\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\zeta_{\mathfrak{j}}}{\varsigma}\Big)^{\vartheta-1}\frac{h(\varsigma)}{\varsigma}d\varsigma \\ &- \frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}\Big(\log\frac{\mathfrak{T}}{\varsigma}\Big)^{\vartheta-1}\frac{h(\varsigma)}{\varsigma}d\varsigma\Bigg].\end{aligned} \tag{2.11}$$

Substituting the value of $\mathfrak{c}_0$ in (2.9) results in Eq (2.8) as desired. By direct computation, one can obtain the converse of the lemma. The proof is completed. □

3. Main results

Let $\mathcal{E} = C(\lfloor 1, \mathfrak{T}\rfloor, \mathbb{R})$ be the Banach space endowed with the norm

$$\|\mathcal{X}\| := \max_{\mathfrak{t}\in\lfloor 1,\mathfrak{T}\rfloor}|\mathcal{X}(\mathfrak{t})|.$$

Given Lemma 2.1, we introduce an operator $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ associated with the problem (2.3) as follows:

$$\begin{aligned}\mathcal{F}(\mathcal{X})(\mathfrak{t}) =& \frac{\Big(\log \mathfrak{t}\Big)^{\gamma-1}}{\Pi}\Bigg[\sum_{\mathfrak{i}=1}^{\mathfrak{l}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathfrak{t},\mathcal{X}(\mathfrak{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma \\ &+ \sum_{\mathfrak{j}=1}^{\mathfrak{k}}\varrho_{\mathfrak{j}}\mathcal{I}^{\nu_{\mathfrak{i}}}\frac{1}{\Gamma(\vartheta)}\int_1^{\zeta_i}\Big(\log\frac{\zeta_{\mathfrak{j}}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathfrak{t},\mathcal{X}(\mathfrak{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma \\ &- \frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}\Big(\log\frac{\mathfrak{T}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathfrak{t},\mathcal{X}(\mathfrak{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\Bigg] \\ &+ \frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{t}}\Big(\log\frac{\mathfrak{t}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathfrak{t},\mathcal{X}(\mathfrak{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma, \quad \mathfrak{t} \in \lfloor 1, \mathfrak{T}\rfloor.\end{aligned} \tag{3.1}$$

In the sequel, we used the following notation:

$$\Omega = \frac{\Big(\log \mathfrak{T}\Big)^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{\Big(\log \mathfrak{T}\Big)^{\gamma-1}}{|\Pi|}\Bigg[\sum_{i=1}^{\mathfrak{l}} \frac{|\eta_i|\Big(\log \xi_i\Big)^{\vartheta}}{\Gamma(\vartheta+1)} + \sum_{j=1}^{\mathfrak{k}} \frac{|\zeta_j|\Big(\log \zeta_i\Big)^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}}+1)} + \frac{\Big(\log \mathfrak{T}\Big)^{\vartheta}}{\Gamma(\vartheta+1)}\Bigg]. \tag{3.2}$$

Here, we introduce some assumptions for the following sequels.

(Q_1) The function $\mathcal{H}_1 : \mathcal{J} \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is completely continuous, and then there exists a function $\mu \in \mathcal{L}^1(\mathcal{J}, \mathbb{R})$ such that:

$$|\mathcal{H}_1(\mathrm{t}, \mathcal{X}, \mathcal{Y})| \le \mu(\mathrm{t}), \quad \mathrm{t} \in \mathcal{J},\ \mathcal{X}, \mathcal{Y} \in \mathcal{E}.$$

(Q_2) The function $\mathcal{H}_1$ is continuous, and there exist constants $\mathcal{L}_1, \mathcal{L}_2 > 0$ such that:

$$\begin{aligned}|\mathcal{H}_1(\mathrm{t}, \mathcal{X}_1, \mathcal{Y}_1) - \mathcal{H}_1(\mathrm{t}, \mathcal{X}_2, \mathcal{Y}_2)| \le& \mathcal{L}_1|\mathcal{X}_1 - \mathcal{X}_2| + \mathcal{L}_2|\mathcal{Y}_1 - \mathcal{Y}_2|,\\ &\forall\ \mathrm{t} \in \mathcal{J},\ \mathcal{X}_i, \mathcal{Y}_i \in \mathcal{E},\ i = 1, 2.\end{aligned}$$

(Q_3) The function $\mathcal{H}_1$ is continuous, and there exists a constant $\mathcal{M} > 0$ such that:

$$|\mathcal{H}_1(\mathrm{t}, \varsigma, \mathcal{X}_1) - \mathcal{H}_1(\mathrm{t}, \varsigma, \mathcal{X}_2)| \le \mathcal{M}|\mathcal{X}_1 - \mathcal{X}_2|, \ \ \forall \in \mathcal{J},\ \mathcal{X}_i \in \mathcal{E},\ i = 1, 2.$$

3.1. Existence results

In this subsection, we present different criteria for the existence of solutions for the problem (2.3). First, we prove an existence result based on Krasnoselskii's fixed-point theorem.

Theorem 3.1. *Let $\mathcal{H}_1 : \lfloor 1, \mathfrak{T} \rfloor \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (2.3). In addition, we assume that the following condition is satisfied:*

(Q_1) *There exists a continuous function $\Phi \in C(\lfloor 1, \mathfrak{T} \rfloor, \mathbb{R}^+$ such that*

$$|\mathcal{H}_1(\mathrm{t}, \mathcal{X}, \mathcal{Y})| \le \Phi(\mathrm{t}) \in \lfloor 1, \mathfrak{T} \rfloor \times \mathbb{R}.$$

Then, the nonlinear Hilfer-Hadamard fractional boundary value problem (1.3) has at least one solution on $\lfloor 1, \mathfrak{T} \rfloor$, provided that the following condition holds:

$$\left\{\frac{\Big(\log \mathfrak{T}\Big)^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{\Big(\log \mathfrak{T}\Big)^{\gamma-1}}{|\Pi|}\Bigg[\sum_{i=1}^{\mathfrak{l}} \frac{|\eta_i|\Big(\log \xi_i\Big)^{\vartheta}}{\Gamma(\vartheta+1)} + \sum_{j=1}^{\mathfrak{k}} \frac{|\zeta_j|\Big(\log \zeta_i\Big)^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}}+1)} + \frac{\Big(\log \mathfrak{T}\Big)^{\vartheta}}{\Gamma(\vartheta+1)}\Bigg]\right\}\mathcal{L} < 1. \tag{3.3}$$

Proof. By assumption (Q_1), we can fix $\rho \ \ge \ \Omega\|\Phi\|$ and consider a closed ball $\mathcal{B}_\rho = \{\mathcal{X} \in C(\lfloor 1, \mathfrak{T} \rfloor, \mathbb{R}) : \|\mathcal{X}\| \le \rho\}$, where $\|\Phi\| = \sup_{\mathrm{t} \in \lfloor 1, \mathfrak{T} \rfloor} |\Phi(\mathrm{t})|$ and Ω is given by (3.2). We verify the hypotheses of Krasnoselskii's fixed-point theorem by splitting the operator $\mathcal{F}$ defined by (3.1) on $\mathcal{B}_\rho$ to $C(\lfloor 1, \mathfrak{T} \rfloor, \mathbb{R})$ as $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined by the following:

$$(\mathcal{F}_1\mathcal{X})(\mathrm{t}) = \frac{1}{\Gamma(\vartheta)} \int_1^{\mathrm{t}} \Big(\log \frac{\mathrm{t}}{\varsigma}\Big)^{\vartheta-1} \frac{\mathcal{H}_1(\mathrm{t}, \mathcal{X}(\mathrm{t}), \mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma} d\varsigma,\ \mathrm{t} \in \lfloor 1, \mathfrak{T} \rfloor,$$

$$(\mathcal{F}_2\mathcal{X})(\mathrm{t}) = \frac{\left(\log \mathrm{t}\right)^{\gamma-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}} \eta_i \frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\left(\log\frac{\xi_i}{\varsigma}\right)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma$$
$$+\sum_{\mathrm{j}=1}^{\mathrm{f}}\varrho_{\mathrm{j}}\mathcal{I}^{\nu_{\mathrm{j}}}\frac{1}{\Gamma(\vartheta)}\int_1^{\zeta_i}\left(\log\frac{\zeta_{\mathrm{j}}}{\varsigma}\right)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma$$
$$-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varsigma}\right)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\Big],\ \mathrm{t}\in\lfloor 1,\mathfrak{T}\rfloor.$$

For any $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{B}_\rho$, we have the following:

$$|\mathcal{F}_1(\mathcal{X}_1)(\mathrm{t}) + \mathcal{F}_2(\mathcal{X}_2)(\mathrm{t})| \tag{3.4}$$
$$= \frac{\left(\log \mathrm{t}\right)^{\gamma-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}} \eta_i \frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\left(\log\frac{\xi_i}{\varsigma}\right)^{\vartheta-1}\frac{|\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)|}{\varsigma}d\varsigma$$
$$+\sum_{\mathrm{j}=1}^{\mathrm{f}}\varrho_{\mathrm{j}}\mathcal{I}^{\nu_{\mathrm{j}}}\frac{1}{\Gamma(\vartheta)}\int_1^{\zeta_i}\left(\log\frac{\zeta_{\mathrm{j}}}{\varsigma}\right)^{\vartheta-1}\frac{|\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)|}{\varsigma}d\varsigma$$
$$-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varsigma}\right)^{\vartheta-1}\frac{|\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)|}{\varsigma}d\varsigma\Big]$$
$$+\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}\left(\log\frac{\mathrm{t}}{\varsigma}\right)^{\vartheta-1}\frac{|\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)|}{\varsigma}d\varsigma,$$
$$\leq\left(\frac{\left(\log\mathfrak{T}\right)^{\vartheta}}{\Gamma(\vartheta+1)}+\frac{\left(\log\mathfrak{T}\right)^{\gamma-1}}{|\Pi|}\Big[\sum_{i=1}^{\mathrm{I}}\frac{|\eta_i|\left(\log\xi_i\right)^{\vartheta}}{\Gamma(\vartheta+1)}+\sum_{j=1}^{\mathrm{f}}\frac{|\zeta_j|\left(\log\zeta_i\right)^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}}+1)}+\frac{\left(\log\mathfrak{T}\right)^{\vartheta}}{\Gamma(\vartheta+1)}\Big]\right)\|\Phi\|,$$
$$\leq \Omega\|\Phi\| \leq \rho. \tag{3.5}$$

Hence, $\|\mathcal{F}_1(\mathcal{X}_1)(\mathrm{t}) + \mathcal{F}_2(\mathcal{X}_2)(\mathrm{t})\| \leq \rho$, which shows that $\mathcal{F}_1(\mathcal{X}_1)(\mathrm{t}) + \mathcal{F}_2(\mathcal{X}_2)(\mathrm{t}) \in \mathcal{B}_\rho$. By condition (3.3), it is easy to prove that the operator $\mathcal{F}_2$ is a contraction mapping. The operator $\mathcal{F}_1$ is continuous by the continuity of $\mathcal{H}_1$. Moreover, $\mathcal{H}_1$ is uniformly bounded on $\mathcal{B}_\rho$, since

$$\|\mathcal{F}_1\mathcal{X}\| \leq \frac{(\log\mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\|\Phi\|.$$

Finally, we prove the compactness of the operator $\mathcal{F}_1$. For $\mathrm{t}_1, \mathrm{t}_2 \in \lfloor 1, \mathfrak{T}\rfloor, \mathrm{t}_1 < \mathrm{t}_2$, we have the following case:

$$|\mathcal{F}_1\mathcal{X}(\mathrm{t}_2) - \mathcal{F}_1\mathcal{X}(\mathrm{t}_1)| \leq \frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}_1}\left[\left(\log\frac{\mathrm{t}_2}{\varsigma}\right)^{\vartheta-1}-\left(\log\frac{\mathrm{t}_1}{\varsigma}\right)^{\vartheta-1}\right]\frac{|\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)|}{\varsigma}d\varsigma$$
$$+\frac{1}{\Gamma(\vartheta)}\int_{\mathrm{t}_1}^{\mathrm{t}_2}\left(\log\frac{\mathrm{t}_2}{\varsigma}\right)^{\vartheta-1}\frac{|\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)|}{\varsigma}d\varsigma$$
$$\leq\frac{\|\Phi\|}{\Gamma(\vartheta+1)}\Big[2(\log\mathrm{t}_2-\log\mathrm{t}_1)^{\vartheta}+|(\log\mathrm{t}_2)^{\vartheta}-(\log\mathrm{t}_1)^{\vartheta}|\Big],$$

which tends to zero independently of $\mathcal{X} \in \mathcal{B}_\rho$, as $\mathrm{t}_1 \to \mathrm{t}_2$. Thus, $\mathcal{F}_1$ is equicontinuous. By the application of the Arzela–Ascoli theorem, we deduce that operator $\mathcal{F}_1$ is compact on $\mathcal{B}_\rho$. Thus, the hypotheses of Krasnoselskii's fixed-point theorem hold. In consequence, there exists at least one solution for the nonlinear Hilfer-Hadamard fractional boundary value problem (2.5) on $\lfloor 1,\mathfrak{T}\rfloor$, which completes the proof. □

Our next existence result is based on Schaefer's fixed-point theorem.

Theorem 3.2. *Assume that* $(\mathcal{Q}_1)$ *is verified. Then (2.3) admit at least one solution on* $\mathcal{J}$.

Proof. We shall use Scheafer's fixed-point theorem to prove that $\mathcal{P}$ has at least a fixed-point on $\mathcal{E}$. It is to note that $\mathcal{P}$ is continuous on $\mathcal{E}$ because of the continuity of $\mathcal{H}_1$.

Now, we shall prove that $\mathcal{P}$ maps bounded sets into bounded sets in $\mathcal{E}$. Taking $\mathrm{r}>0$, and $(\mathcal{X})\in\mathcal{B}_{\mathrm{r}}:=\{(\mathcal{X}|\mathcal{X}\in\mathcal{E}:\|\mathcal{X}\|_{\mathcal{E}}\leq\mathcal{X})\}$, then for each $\mathrm{t}\in\lfloor 1,\mathfrak{T}\rfloor$, we have

$$\begin{aligned}\mathcal{P}(\mathcal{X})(\mathrm{t})=&\frac{\left(\log\mathrm{t}\right)^{\gamma-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\\&+\sum_{\mathrm{j}=1}^{\mathrm{ŧ}}\varrho_{\mathrm{j}}\mathcal{I}^{\nu_{\mathrm{j}}}\frac{1}{\Gamma(\vartheta)}\int_1^{\zeta_i}\Big(\log\frac{\zeta_{\mathrm{j}}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\\&-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}\Big(\log\frac{\mathfrak{T}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\Big]\\&+\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}\Big(\log\frac{\mathrm{t}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma,\quad\mathrm{t}\in\lfloor 1,\mathfrak{T}\rfloor.\end{aligned}\tag{3.6}$$

Step 1. $\mathcal{P}$ is continuous.

Let $\mathcal{X}_n$ be a sequence such that $\mathcal{X}_n\to\mathcal{X}$ in $\mathcal{E}$. For each $\mathrm{t}\in\mathcal{J}$, one has

$$\begin{aligned}&|(\mathcal{P}(\mathcal{X}_n))(\mathrm{t})-(\mathcal{P}(\mathcal{X}))(\mathrm{t})|\\=&\Big|\frac{(\log\mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))\frac{d\varsigma}{\varsigma}\\&+\sum_{\mathrm{j}=1}^{\mathrm{ŧ}}\varrho_{\mathrm{j}}\frac{1}{\Gamma(\vartheta+\nu_{\mathrm{j}})}\int_1^{\zeta_{\mathrm{j}}}(\log\frac{\zeta_{\mathrm{j}}}{\varsigma})^{\vartheta+\nu_{\mathrm{j}}-1}(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))\frac{d\varsigma}{\varsigma}\\&-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}(\log\frac{\mathfrak{T}}{\varsigma})^{\vartheta-1}(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))\frac{d\varsigma}{\varsigma}\Big]\\&+\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}(\log\frac{\mathrm{t}}{\varsigma})^{\vartheta-1}(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))\frac{d\varsigma}{\varsigma}\Big|\\\leq&\frac{(\log\mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}|(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))|\frac{d\varsigma}{\varsigma}\\&+\sum_{\mathrm{j}=1}^{\mathrm{ŧ}}\varrho_{\mathrm{j}}\frac{1}{\Gamma(\vartheta+\nu_{\mathrm{j}})}\int_1^{\zeta_{\mathrm{j}}}(\log\frac{\zeta_{\mathrm{j}}}{\varsigma})^{\vartheta+\nu_{\mathrm{j}}-1}|(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))|\frac{d\varsigma}{\varsigma}\\&-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}(\log\frac{\mathfrak{T}}{\varsigma})^{\vartheta-1}|(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))|\frac{d\varsigma}{\varsigma}\Big]\end{aligned}$$

$$+\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}\Big(\log\frac{\mathrm{t}}{\varsigma}\Big)^{\vartheta-1}|(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))|\frac{d\varsigma}{\varsigma}$$

$$\leq\frac{(\log\mathrm{t})^{\vartheta-1}}{\|\Pi\|}\Big[\sum_{i=1}^{\mathrm{I}}\frac{|\eta_i|\Big(\log\xi_i\Big)^{\vartheta}}{\Gamma(\vartheta+1)}\|(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))\|_{\mathcal{E}}$$

$$+\sum_{\mathrm{j}=1}^{\mathrm{f}}\varrho_{\mathrm{j}}\frac{(\log\zeta_{\mathrm{j}})^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}})}\|(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))\|_{\mathcal{E}}$$

$$+\frac{(\log\mathcal{T})^{\vartheta}}{\Gamma(\vartheta+1)}\|(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))\|_{\mathcal{E}}\Big]$$

$$+\frac{(\log\mathcal{T})^{\vartheta}}{\Gamma(\vartheta+1)}\|(\mathcal{H}_1(\varsigma,\mathcal{X}_n(\varsigma),\mathcal{H}(\mathcal{X}_n(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma))))\|_{\mathcal{E}}.$$

Since the function $\mathcal{H}_1$ is continuous, then we obtain

$$|(\mathcal{P}(\mathcal{X}_n))(\mathrm{t})-(\mathcal{P}(\mathcal{X}))(\mathrm{t})|$$

$$\leq\frac{(\log\mathrm{t})^{\vartheta-1}}{\|\Pi\|}\Big[\sum_{i=1}^{\mathrm{I}}\frac{|\eta_i|\Big(\log\xi_i\Big)^{\vartheta}}{\Gamma(\vartheta+1)}\|(\mathcal{H}_1(\cdot,\mathcal{X}_n(\cdot),\mathcal{H}(\mathcal{X}_n(\cdot)))-\mathcal{H}_1(\cdot,\mathcal{X}(\cdot),\mathcal{H}(\mathcal{X}(\cdot))))\|_{\mathcal{E}}$$

$$+\sum_{\mathrm{j}=1}^{\mathrm{f}}\varrho_{\mathrm{j}}\frac{(\log\zeta_{\mathrm{j}})^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}})}\|(\mathcal{H}_1(\cdot,\mathcal{X}_n(\cdot),\mathcal{H}(\mathcal{X}_n(\cdot)))-\mathcal{H}_1(\cdot,\mathcal{X}(\cdot),\mathcal{H}(\mathcal{X}(\cdot))))\|_{\mathcal{E}}$$

$$+\frac{(\log\mathcal{T})^{\vartheta}}{\Gamma(\vartheta+1)}\|(\mathcal{H}_1(\cdot,\mathcal{X}_n(\cdot),\mathcal{H}(\mathcal{X}_n(\cdot)))-\mathcal{H}_1(\cdot,\mathcal{X}(\cdot),\mathcal{H}(\mathcal{X}(\cdot))))\|_{\mathcal{E}}\Big]$$

$$+\frac{(\log\mathcal{T})^{\vartheta}}{\Gamma(\vartheta+1)}\|(\mathcal{H}_1(\cdot,\mathcal{X}_n(\cdot),\mathcal{H}(\mathcal{X}_n(\cdot)))-\mathcal{H}_1(\cdot,\mathcal{X}(\cdot),\mathcal{H}(\mathcal{X}(\cdot))))\|_{\mathcal{E}}\to 0,\ \text{as}\ \ n\to\infty.$$

Therefore, the operator $\mathcal{P}$ is continuous.

Step 2. $\mathcal{P}(\mathcal{B}_{\mathrm{r}})$ is bounded.

For each $\mathrm{t}\in\mathcal{J}$ and $\mathcal{X}\in\mathcal{B}_{\mathrm{r}}$, we obtain that:

$$|(\mathcal{P}\mathcal{X})(\mathrm{t})|$$

$$=\frac{\Big(\log\mathrm{t}\Big)^{\gamma-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma$$

$$+\sum_{\mathrm{j}=1}^{\mathrm{f}}\varrho_{\mathrm{j}}\mathcal{I}^{\nu_{\mathrm{j}}}\frac{1}{\Gamma(\vartheta)}\int_1^{\zeta_i}\Big(\log\frac{\zeta_{\mathrm{j}}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma$$

$$-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\Big]$$

$$+\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}\Big(\log\frac{\mathrm{t}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma,\quad \mathrm{t}\in\lfloor 1,\mathcal{T}\rfloor.$$

$$\leq\frac{(\log\mathrm{t})^{\vartheta-1}}{|\Pi|}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{|\mu(\varsigma)|}{\varsigma}d\varsigma$$

$$
\begin{aligned}
&+\sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\frac{1}{\Gamma(\vartheta+\nu_{\mathrm{j}})}\int_1^{\zeta_{\mathrm{j}}}\Big(\log\frac{\zeta_{\mathrm{j}}}{\varsigma}\Big)^{\vartheta+\nu_{\mathrm{j}}-1}\frac{|\mu(\varsigma)|}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\vartheta-1}\frac{|\mu(\varsigma)|}{\varsigma}d\varsigma\Big]\\
&+\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}\Big(\log\frac{\mathrm{t}}{\varsigma}\Big)^{\vartheta-1}\frac{|\mu(\varsigma)|}{\varsigma}d\varsigma,\\
\leq&\frac{\|\mu(\varsigma)\|_{\mathcal{E}}(\log\mathrm{t})^{\vartheta-1}}{|\Pi|}\Big[\sum_{i=1}^{\mathfrak{l}}\frac{|\eta_i|\Big(\log\xi_i\Big)^{\vartheta}}{\Gamma(\vartheta+1)}+\sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\frac{(\log\zeta_{\mathrm{j}})^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}})}+\frac{(\log\mathcal{T})^{\vartheta}}{\Gamma(\vartheta+1)}\Big]+\frac{(\log\mathcal{T})^{\vartheta}}{\Gamma(\vartheta+1)}:=\mathfrak{L}.
\end{aligned}\tag{3.7}
$$

Thus, $\|\mathcal{P}(\mathcal{X})\|\leq\mathfrak{L}$.

Step 3. $\mathcal{P}(\mathcal{B}_{\mathrm{r}})$ is equi-continuous.

For $1\leq\mathrm{t}_1<\mathrm{t}_2\leq\mathcal{T}$, and $\mathcal{X}\in\mathcal{B}_{\mathrm{r}}$, we obtain

$$
\begin{aligned}
&|(\mathcal{P}\mathcal{X})(\mathrm{t}_2)-(\mathcal{P}\mathcal{X})(\mathrm{t}_1)|\\
\leq&\frac{(\log\mathrm{t}_1)^{\vartheta-1}-(\log\mathrm{t}_2)^{\vartheta-1}}{|\Pi|}\Big[\sum_{\mathrm{i}=1}^{\mathfrak{l}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\\
&+\sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\frac{1}{\Gamma(\vartheta+\nu_{\mathrm{j}})}\int_1^{\zeta_{\mathrm{j}}}\Big(\log\frac{\zeta_{\mathrm{j}}}{\varsigma}\Big)^{\vartheta+\nu_{\mathrm{j}}-1}\frac{|\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))|}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\vartheta-1}\frac{|\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))|}{\varsigma}d\varsigma\Big]\\
&+\frac{1}{\Gamma(\vartheta)}\left|\int_1^{\mathrm{t}_1}\left(\Big(\log\frac{\mathrm{t}_1}{\varsigma}\Big)^{\vartheta-1}-\Big(\log\frac{\mathrm{t}_2}{\varsigma}\Big)^{\vartheta-1}\right)\frac{|\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))|}{\varsigma}d\varsigma\right|\\
&-\frac{1}{\Gamma(\vartheta)}\left|\int_{\mathrm{t}_1}^{\mathrm{t}_2}\Big(\log\frac{\mathrm{t}_2}{\varsigma}\Big)^{\vartheta-1}\frac{|\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))|}{\varsigma}d\varsigma\right|,\\
\leq&\left|\frac{(\log\mathrm{t}_1)^{\vartheta-1}-(\log\mathrm{t}_2)^{\vartheta-1}}{|\Phi|}\right|\Big[\sum_{i=1}^{\mathfrak{l}}\frac{|\eta_i|\Big(\log\xi_i\Big)^{\vartheta}}{\Gamma(\vartheta+1)}+\sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\frac{(\log\zeta_{\mathrm{j}})^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}})}+\frac{(\log\mathcal{T})^{\vartheta}}{\Gamma(\vartheta+1)}\Big]\\
&+\frac{1}{\Gamma(\vartheta)}\left|\int_1^{\mathrm{t}_1}\left(\Big(\log\frac{\mathrm{t}_1}{\varsigma}\Big)^{\vartheta-1}-\Big(\log\frac{\mathrm{t}_2}{\varsigma}\Big)^{\vartheta-1}\right)\frac{|\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))|}{\varsigma}d\varsigma\right|\\
&-\frac{1}{\Gamma(\vartheta)}\left|\int_{\mathrm{t}_1}^{\mathrm{t}_2}\Big(\log\frac{\mathrm{t}_2}{\varsigma}\Big)^{\vartheta-1}\frac{|\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))|}{\varsigma}d\varsigma\right|.
\end{aligned}
$$

As $\mathrm{t}_2\to\mathrm{t}_1$, the R.H.S. of the above inequality $\to 0$. Consequently, we deduce that $\mathcal{P}$ is completely continuous.

Step 4. The priori bounds.

We need to show that the set $\Lambda=\{\mathcal{X}\in\mathcal{E}:\mathcal{X}=\Omega(\mathcal{P}(\mathcal{X}));\Omega\in(0,1)\}$ is bounded. For this, let $\mathcal{X}\in\Lambda,\mathcal{X}=\Omega(\mathcal{P}(\mathcal{X}))$ for some $\Omega\in(0,1)$. Thus, for each $\mathrm{t}\in\mathcal{J}$, one has

$$
(\mathcal{P}\mathcal{X})(\mathrm{t})=\Omega\Bigg\{\frac{\Big(\log\mathrm{t}\Big)^{\gamma-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathfrak{l}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma
$$

$$+\sum_{j=1}^{\mathfrak{k}}\varrho_j\mathcal{I}^{\nu_j}\frac{1}{\Gamma(\vartheta)}\int_1^{\zeta_i}\Big(\log\frac{\zeta_j}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(t,\mathcal{X}(t),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma$$
$$-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(t,\mathcal{X}(t),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\Big]$$
$$+\frac{1}{\Gamma(\vartheta)}\int_1^{t}\Big(\log\frac{t}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(t,\mathcal{X}(t),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\Big\},\quad t\in\lfloor 1,\mathcal{T}\rfloor. \tag{3.8}$$

This implies, by (Q_2), that:

$$\begin{aligned}
&|(\mathcal{P}\mathcal{X})(t)|\\
=&\frac{(\log t)^{\vartheta-1}}{|\Pi|}\Big[\sum_{i=1}^{\mathfrak{l}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{|\mathcal{H}_1(t,\mathcal{X}(t),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)|}{\varsigma}d\varsigma\\
&+\sum_{j=1}^{\mathfrak{k}}\varrho_j\frac{1}{\Gamma(\vartheta+\nu_j)}\int_1^{\zeta_j}\big(\log\frac{\zeta_j}{\varsigma}\big)^{\vartheta+\nu_j-1}\frac{|\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))|}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{\varsigma}\big)^{\vartheta-1}\frac{|\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))|}{\varsigma}d\varsigma\Big]\\
&+\frac{1}{\Gamma(\vartheta)}\int_1^{t}\big(\log\frac{t}{\varsigma}\big)^{\vartheta-1}\frac{|\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))|}{\varsigma}d\varsigma,\\
\leq&\frac{(\log t)^{\vartheta-1}}{|\Pi|}\Big[\sum_{i=1}^{\mathfrak{l}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{|\mu(\varsigma)|}{\varsigma}d\varsigma\\
&+\sum_{j=1}^{\mathfrak{k}}\varrho_j\frac{1}{\Gamma(\vartheta+\nu_j)}\int_1^{\zeta_j}\big(\log\frac{\zeta_j}{\varsigma}\big)^{\vartheta+\nu_j-1}\frac{|\mu(\varsigma)|}{\varsigma}d\varsigma\\
&-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{\varsigma}\big)^{\vartheta-1}\frac{|\mu(\varsigma)|}{\varsigma}d\varsigma\Big]\\
&+\frac{1}{\Gamma(\vartheta)}\int_1^{t}\big(\log\frac{t}{\varsigma}\big)^{\vartheta-1}\frac{|\mu(\varsigma)|}{\varsigma}d\varsigma,\\
\leq&\frac{(\log t)^{\vartheta-1}}{|\Pi|}\Big[\sum_{i=1}^{\mathfrak{l}}\frac{|\eta_i|\Big(\log\xi_i\Big)^{\vartheta}}{\Gamma(\vartheta+1)}\|\mu(\varsigma)\|_{\mathcal{E}}+\sum_{j=1}^{\mathfrak{k}}\varrho_j\frac{(\log\zeta_j)^{\vartheta+\nu_j}}{\Gamma(\vartheta+\nu_j)}\|\mu(\varsigma)\|_{\mathcal{E}}+\frac{(\log\mathcal{T})^{\vartheta}}{\Gamma(\vartheta+1)}\|\mu(\varsigma)\|_{\mathcal{E}}\Big]\\
&+\frac{(\log\mathcal{T})^{\vartheta}}{\Gamma(\vartheta+1)}\|\mu(\varsigma)\|_{\mathcal{E}}:=\Re.
\end{aligned}\tag{3.9}$$

Thus, $\|\mu(\varsigma)\|_{\mathcal{E}}\leq\Re$.

Therefore, the set Λ is bounded. Hence, we deduce that $\mathcal{P}$ has a fixed-point that is a solution to the presumed problem (2.3) as an outcome of Schaefer's fixed-point theorem. □

3.2. Uniqueness results

The next theorem contains the second main result in this paper, which is the uniqueness of the solution to the presumed problem (2.3).

Theorem 3.3. *Suppose that the conditions* ($\mathcal{Q}_2$) *and* ($\mathcal{Q}_3$) *are satisfied such that:*

$$(\mathcal{L}_1+\mathcal{L}_2)\mathcal{M}\Big\{\frac{\big(\log\mathfrak{T}\big)^{\vartheta}}{\Gamma(\vartheta+1)}+\frac{\big(\log\mathfrak{T}\big)^{\gamma-1}}{|\Pi|}\Big[\sum_{i=1}^{\mathrm{I}}\frac{|\eta_i|\big(\log\xi_i\big)^{\vartheta}}{\Gamma(\vartheta+1)}+\sum_{j=1}^{\mathfrak{k}}\frac{|\zeta_j|\big(\log\zeta_i\big)^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}}+1)}+\frac{\big(\log\mathfrak{T}\big)^{\vartheta}}{\Gamma(\vartheta+1)}\Big]\Big\}<1. \quad (3.10)$$

Then, the presumed problem (2.3) has a unique solution on $\mathcal{J}$.

Proof. We consider the operator $\mathcal{P}:\mathcal{E}\to\mathcal{E}$ defined as

$$\begin{aligned}\mathcal{P}(\mathcal{X})(\mathrm{t})=&\frac{\big(\log\mathrm{t}\big)^{\gamma-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\\&+\sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\mathcal{I}^{\nu_{\mathrm{i}}}\frac{1}{\Gamma(\vartheta)}\int_1^{\zeta_i}\Big(\log\frac{\zeta_{\mathrm{j}}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\\&-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}\Big(\log\frac{\mathfrak{T}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\Big]\\&+\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}\Big(\log\frac{\mathrm{t}}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma,\quad \mathrm{t}\in\lfloor 1,\mathfrak{T}\rfloor.\end{aligned} \quad (3.11)$$

We shall show that $\mathcal{P}$ is a contraction map. Let $\mathcal{X},\mathcal{Y}\in\mathcal{E}$, then one has for each $\mathrm{t}\in\mathcal{J}$

$$\begin{aligned}&|(\mathcal{P}(\mathcal{X}))(\mathrm{t})-(\mathcal{P}(\mathcal{Y}))(\mathrm{t})|\\=&|\frac{(\log\mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))\frac{d\varsigma}{\varsigma}\\&+\sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\frac{1}{\Gamma(\vartheta+\nu_{\mathrm{j}})}\int_1^{\zeta_{\mathrm{i}}}(\log\frac{\zeta_{\mathrm{j}}}{\varsigma})^{\vartheta+\nu_{\mathrm{j}}-1}(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))\frac{d\varsigma}{\varsigma}\\&-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}(\log\frac{\mathfrak{T}}{\varsigma})^{\vartheta-1}(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))\frac{d\varsigma}{\varsigma}\Big]\\&+\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}(\log\frac{\mathrm{t}}{\varsigma})^{\vartheta-1}(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))\frac{d\varsigma}{\varsigma}|\\\le&\frac{(\log\mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}(|\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma)))|)\frac{d\varsigma}{\varsigma}\\&+\sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\frac{1}{\Gamma(\vartheta+\nu_{\mathrm{j}})}\int_1^{\zeta_{\mathrm{i}}}(\log\frac{\zeta_{\mathrm{j}}}{\varsigma})^{\vartheta+\nu_{\mathrm{j}}-1}|(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))|\frac{d\varsigma}{\varsigma}\\&-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathfrak{T}}(\log\frac{\mathfrak{T}}{\varsigma})^{\vartheta-1}|(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))|\frac{d\varsigma}{\varsigma}\Big]\\&+\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}(\log\frac{\mathrm{t}}{\varsigma})^{\vartheta-1}|(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))|\frac{d\varsigma}{\varsigma}\\\le&\frac{(\log\mathrm{t})^{\vartheta-1}}{\|\Pi\|}\Big[\sum_{i=1}^{\mathrm{I}}\frac{|\eta_i|\big(\log\xi_i\big)^{\vartheta}}{\Gamma(\vartheta+1)}\|(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))\|_{\mathcal{E}}\end{aligned}$$

$$+\sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\frac{(\log\zeta_{\mathrm{j}})^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}})}\|(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))\|_{\mathcal{E}}$$
$$+\frac{(\log\mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\|(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))\|_{\mathcal{E}}\Big]$$
$$+\frac{(\log\mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\|(\mathcal{H}_1(\varsigma,\mathcal{X}(\varsigma),\mathcal{H}(\mathcal{X}(\varsigma)))-\mathcal{H}_1(\varsigma,\mathcal{Y}(\varsigma),\mathcal{H}(\mathcal{Y}(\varsigma))))\|_{\mathcal{E}},$$
$$\leq(\mathcal{L}_1+\mathcal{L}_2\mathcal{M})|\mathcal{X}(\varsigma)-\mathcal{Y}(\varsigma)|$$
$$\times\left[\frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{i=1}^{\mathfrak{l}}\frac{|\eta_i|\big(\log\xi_i\big)^{\vartheta}}{\Gamma(\vartheta+1)}+\sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\frac{(\log\zeta_{\mathrm{j}})^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}})}+\frac{(\log\mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\Big]+\frac{(\log\mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\right].$$

Therefore, we obtain

$$\|(\mathcal{P}(\mathcal{X}))(\mathrm{t})-(\mathcal{P}(\mathcal{Y}))(\mathrm{t})\|$$
$$\leq(\mathcal{L}_1+\mathcal{L}_2\mathcal{M})\left[\frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{i=1}^{\mathfrak{l}}\frac{|\eta_i|\big(\log\xi_i\big)^{\vartheta}}{\Gamma(\vartheta+1)}+\sum_{\mathrm{j}=1}^{\mathfrak{k}}\varrho_{\mathrm{j}}\frac{(\log\zeta_{\mathrm{j}})^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}})}+\frac{(\log\mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\Big]\right.$$
$$\left.+\frac{(\log\mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\right]\|\mathcal{X}(\varsigma)-\mathcal{Y}(\varsigma)\|. \tag{3.12}$$

Hence, given the condition (3.10) and the Banach contraction principle, $\mathcal{P}$ has a unique fixed-point. Thus, the existence of a unique solution to the presumed problem (2.3). □

4. Ulam stability results

In this section we will discuss Ulam-Hyers and Ulam-Hyers–Rassias stability.

Definition 4.1. *Equation (2.3) is UH stable if there exists a real number $C_{\mathrm{g}} > 0$ such that for each $\epsilon > 0$ and each $\mathfrak{z} \in C[\mathcal{J}]$ solution of the inequality:*

$$|\mathcal{D}_{0^+}^{\vartheta}\mathcal{Z}(\mathrm{t})-\mathrm{g}(\mathrm{t},\mathcal{Z}(\mathrm{t}),\mathcal{H}\mathcal{Z}(\mathrm{t}))|\leq\epsilon,\ \mathrm{t}\in\mathcal{J}, \tag{4.1}$$

there exists a solution $\mathcal{Y} \in C[\mathcal{J}]$ of Eq (2.3) such that:

$$|\mathcal{Z}(\mathrm{t})-\mathcal{Y}(\mathrm{t})|\leq C_{\mathrm{g}}\epsilon,\ \ \mathrm{t}\in\mathcal{J}.$$

Definition 4.2. *Equation (2.3) is generalized UH stable if there exists $\psi_{\mathrm{g}} \in C(\mathcal{R}_+,\mathcal{R}_+)$ with $\psi_{\mathrm{g}}(0) = 0$, such that for a solution $\mathfrak{z} \in C[\mathcal{J}]$ of the inequality:*

$$|\mathcal{D}_{0^+}^{\vartheta}\mathcal{Z}(\mathrm{t})-\mathrm{g}(\mathrm{t},\mathcal{Z}(\mathrm{t}),\mathcal{H}\mathcal{Z}(\mathrm{t}))|\leq\epsilon,\ \mathrm{t}\in\mathcal{J}, \tag{4.2}$$

there exists a solution $\mathcal{Y} \in C[\mathcal{J}]$ of Eq (2.3) such that:

$$|\mathcal{Z}(\mathrm{t})-\mathcal{Y}(\mathrm{t})|\leq\psi_{\mathrm{g}}(\epsilon),\ \ \mathrm{t}\in\mathcal{J}.$$

Definition 4.3. *Equation (2.3) is UHS stable concerning $\nu \in C(\mathcal{J}, \mathbb{R}_+)$ if there exists a real number $\mathfrak{c}_{\mathfrak{g},\nu} > 0$ such that for each $\epsilon > 0$ and for each $\epsilon > 0$ and for each $\mathfrak{z} \in C[\mathcal{J}]$ solution of the inequality:*

$$|\mathcal{D}^{\vartheta}_{0^+}\mathcal{Z}(\mathrm{t}) - \mathfrak{g}(\mathrm{t}, \mathcal{Z}(\mathrm{t}), \mathcal{H}\mathcal{Z}(\mathrm{t}))| \leq \epsilon\nu(\mathrm{t}),\ \mathrm{t} \in \mathcal{J}, \tag{4.3}$$

there exists a solution $\mathcal{Y} \in C[\mathcal{J}]$ of Eq (2.3) such that:

$$|\mathcal{Z}(\mathrm{t}) - \mathcal{Y}(\mathrm{t})| \leq \mathfrak{c}_{\mathfrak{g},\nu}\nu(\mathrm{t}),\ \ \mathrm{t} \in \mathcal{J}.$$

Definition 4.4. *Equation (2.3) is generalized UHS stable with respect to $\nu \in C(\mathcal{J}, \mathbb{R}_+)$ if there exists $\mathfrak{c}_{\mathfrak{g},\nu} > 0$ such that for each $\mathfrak{z} \in C[\mathcal{J}]$ solution of the inequality:*

$$|\mathcal{D}^{\vartheta}_{0^+}\mathcal{Z}(\mathrm{t}) - \mathfrak{g}(\mathrm{t}, \mathcal{Z}(\mathrm{t}), \mathcal{H}\mathcal{Z}(\mathrm{t}))| \leq \epsilon\nu(\mathrm{t}),\ \mathrm{t} \in \mathcal{J}, \tag{4.4}$$

there exists $\mathcal{Y} \in C[\mathcal{J}]$ solution of Eq (2.3) such that:

$$|\mathcal{Z}(\mathrm{t}) - \mathcal{Y}(\mathrm{t})| \leq \mathfrak{c}_{\mathfrak{g},\nu}\nu(\mathrm{t}),\ \ \mathrm{t} \in \mathcal{J}.$$

Remark 4.1. *A function $\mathfrak{z} \in C[\mathcal{J}]$ is a solution of the inequality:*

$$|\mathcal{D}^{\vartheta}_{0^+}\mathcal{Z}(\mathrm{t}) - \mathfrak{g}(\mathrm{t}, \mathcal{Z}(\mathrm{t}), \mathcal{H}\mathcal{Z}(\mathrm{t}))| \leq \epsilon,\ \mathrm{t} \in \mathcal{J}, \tag{4.5}$$

if there exists a function $\mathfrak{w} \in C[\mathcal{J}]$ such that:

$$\begin{aligned}&(1)|\mathfrak{w}(\mathrm{t})| \leq \epsilon,\ \mathrm{t} \in \mathcal{J},\\ &(2)\mathcal{D}^{\vartheta}_{1^+}\mathfrak{z}(\mathrm{t}) = \mathfrak{g}(\mathrm{t}, \mathcal{Z}(\mathrm{t}), \mathcal{H}\mathcal{Z}(\mathrm{t})) + \mathfrak{w}(\mathrm{t}),\ \mathrm{t} \in \mathcal{J}.\end{aligned}$$

Remark 4.2. *It is clear that:*
(1) Definition (4.1) ⇒ Definition (4.2).
(2) Definition (4.3) ⇒ Definition (4.4).

Theorem 4.1. *Assume that (Q_1) and (3.10) are satisfied, then the presumed problem (2.3) is UH stable.*

Proof. Let $\mathfrak{z} \in C[\mathcal{J}]$ be a solution of the inequality (4.1), and let $\mathcal{Y} \in C[\mathcal{J}]$ be a unique solution of the given system:

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}^{\vartheta,\beta}\mathcal{X}(\mathrm{t}) = \mathcal{H}_1(\mathrm{t}, \mathcal{X}(\mathrm{t}), \mathcal{H}(\mathcal{X}(\varsigma))),\ \ \mathrm{t} \in \mathcal{J} = \lfloor 1, \mathrm{T}\rfloor,\\ \mathcal{X}(1) = 0,\ \ \mathcal{X}(\mathrm{T}) = \displaystyle\sum_{\mathrm{i}=1}^{\mathrm{I}} \eta_i \mathcal{X}(\xi_i) + \sum_{\mathrm{j}=1}^{\mathfrak{k}} \varrho_{\mathrm{j}} \mathcal{I}^{\nu_{\mathrm{j}}}\mathcal{X}(\zeta_{\mathrm{j}}),\ \ \nu_{\mathrm{j}} > 0, \eta_i, \varrho_{\mathrm{j}} \in \mathbb{R},\ \zeta_{\mathrm{j}}, \xi_i \in \mathcal{J}. \end{cases} \tag{4.6}$$

where $1 < \vartheta < 2$,

Given Remark 4.1, we have

$$\Bigg|\mathcal{Z}(\mathrm{t}) - \frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}} \eta_i \frac{1}{\Gamma(\vartheta)} \int_1^{\xi_i} \Big(\log \frac{\xi_i}{\varsigma}\Big)^{\vartheta-1} \frac{\mathcal{H}_1(\mathrm{t}, \mathcal{Z}(\mathrm{t}), \mathcal{H}(\mathcal{Z}(\varsigma)))(\varsigma)}{\varsigma} d\varsigma$$
$$+ \sum_{\mathrm{j}=1}^{\mathfrak{k}} \varrho_{\mathrm{j}} \frac{1}{\Gamma(\vartheta + \nu_{\mathrm{j}})} \int_1^{\zeta_{\mathrm{j}}} \Big(\log \frac{\zeta_{\mathrm{j}}}{\varsigma}\Big)^{\vartheta+\nu_{\mathrm{j}}-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} d\varsigma$$

$$- \frac{1}{\Gamma(\vartheta)} \int_1^{\mathfrak{T}} \big(\log \frac{\mathfrak{T}}{\varsigma} \big)^{\vartheta-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} d\varsigma \Big] \tag{4.7}$$
$$+ \frac{1}{\Gamma(\vartheta)} \int_1^{\mathrm{t}} \big(\log \frac{\mathrm{t}}{\varsigma} \big)^{\vartheta-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} d\varsigma \Big| \leq \frac{\varepsilon \mathrm{t}^{\vartheta}}{\Gamma(\vartheta+1)}.$$

Then, for each $\mathrm{t} \in \mathcal{J}$, we obtain

$$\begin{aligned}
&|\mathcal{Z}(\mathrm{t}) - \mathcal{X}(\mathrm{t})| \\
\leq& \Big| \mathcal{Z}(\mathrm{t}) - \frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi} \Big[\sum_{i=1}^{\mathrm{I}} \eta_i \frac{1}{\Gamma(\vartheta)} \int_1^{\xi_i} \Big(\log \frac{\xi_i}{\varsigma} \Big)^{\vartheta-1} \frac{\mathcal{H}_1(\mathrm{t}, \mathcal{X}(\mathrm{t}), \mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma} d\varsigma \\
&+ \sum_{\mathrm{j}=1}^{\mathrm{f}} \varrho_{\mathrm{j}} \frac{1}{\Gamma(\vartheta+\nu_{\mathrm{j}})} \int_1^{\zeta_{\mathrm{j}}} \big(\log \frac{\zeta_{\mathrm{j}}}{\varsigma} \big)^{\vartheta+\nu_{\mathrm{j}}-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma} d\varsigma \\
&- \frac{1}{\Gamma(\vartheta)} \int_1^{\mathfrak{T}} \big(\log \frac{\mathfrak{T}}{\varsigma} \big)^{\vartheta-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma} d\varsigma \Big] \\
&+ \frac{1}{\Gamma(\vartheta)} \int_1^{\mathrm{t}} \big(\log \frac{\mathrm{t}}{\varsigma} \big)^{\vartheta-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma} d\varsigma \Big| \\
\leq& \Big| \mathcal{Z}(\mathrm{t}) - \frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi} \Big[\sum_{\mathrm{j}=1}^{\mathrm{f}} \varrho_{\mathrm{j}} \frac{1}{\Gamma(\vartheta+\nu_{\mathrm{j}})} \int_1^{\zeta_{\mathrm{j}}} \big(\log \frac{\zeta_{\mathrm{j}}}{\varsigma} \big)^{\vartheta+\nu_{\mathrm{j}}-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} d\varsigma \\
&- \frac{1}{\Gamma(\vartheta)} \int_1^{\mathfrak{T}} \big(\log \frac{\mathfrak{T}}{\varsigma} \big)^{\vartheta-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} d\varsigma \Big] \\
&+ \frac{1}{\Gamma(\vartheta)} \int_1^{\mathrm{t}} \big(\log \frac{\mathrm{t}}{\varsigma} \big)^{\vartheta-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} d\varsigma \Big| \\
&\leq \Big| \mathcal{Z}(\mathrm{t}) - \frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi} \Big[\sum_{i=1}^{\mathrm{I}} \eta_i \frac{1}{\Gamma(\vartheta)} \int_1^{\xi_i} \Big(\log \frac{\xi_i}{\varsigma} \Big)^{\vartheta-1} \\
&\times \Big(\frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma} - \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} \Big) d\varsigma \\
&+ \sum_{\mathrm{j}=1}^{\mathrm{f}} \varrho_{\mathrm{j}} \frac{1}{\Gamma(\vartheta+\nu_{\mathrm{j}})} \int_1^{\zeta_{\mathrm{j}}} \big(\log \frac{\zeta_{\mathrm{j}}}{\varsigma} \big)^{\vartheta+\nu_{\mathrm{j}}-1} \\
&\times \Big(\frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma} - \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} \Big) d\varsigma \\
&- \frac{1}{\Gamma(\vartheta)} \int_1^{\mathfrak{T}} \big(\log \frac{\mathfrak{T}}{\varsigma} \big)^{\vartheta-1} \Big(\frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma} - \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} \Big) d\varsigma \Big] \\
&+ \frac{1}{\Gamma(\vartheta)} \int_1^{\mathrm{t}} \big(\log \frac{\mathrm{t}}{\varsigma} \big)^{\vartheta-1} \Big(\frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma} - \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} \Big) d\varsigma \Big| \\
\leq& \frac{\varepsilon \mathrm{t}^{\vartheta}}{\Gamma(\vartheta+1)} + (\mathcal{L}_1 + \mathcal{L}_2 \mathcal{M})
\end{aligned}$$

$$\times\left[\frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{i=1}^{\mathrm{I}}\frac{|\eta_i|\Big(\log\xi_i\Big)^{\vartheta}}{\Gamma(\vartheta+1)}+\sum_{\mathrm{j}=1}^{\mathrm{f}}\varrho_{\mathrm{j}}\frac{(\log\zeta_{\mathrm{j}})^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}})}+\frac{(\log\mathrm{T})^{\vartheta}}{\Gamma(\vartheta+1)}\Big]+\frac{(\log\mathrm{T})^{\vartheta}}{\Gamma(\vartheta+1)}\right]|\mathcal{X}(\mathrm{t})-\mathcal{Y}(\mathrm{t})|,$$

$$\leq\frac{\varepsilon\mathrm{t}^{\vartheta}}{\Gamma(\vartheta+1)}+\Pi_1|\mathcal{X}(\mathrm{t})-\mathcal{Y}(\mathrm{t})|$$

$$\leq\frac{\varepsilon\mathrm{T}^{\vartheta}}{(1-\Pi_1)(\vartheta+1)},$$

therefore,

$$|\mathcal{Z}(\mathrm{t})-\mathcal{X}(\mathrm{t})|\leq \mathrm{c_g}\varepsilon, \tag{4.8}$$

where,

$$\Pi_1=(\mathcal{L}_1+\mathcal{L}_2\mathcal{M})\left[\frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{i=1}^{\mathrm{I}}\frac{|\eta_i|\Big(\log\xi_i\Big)^{\vartheta}}{\Gamma(\vartheta+1)}+\sum_{\mathrm{j}=1}^{\mathrm{f}}\varrho_{\mathrm{j}}\frac{(\log\zeta_{\mathrm{j}})^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}})}+\frac{(\log\mathrm{T})^{\vartheta}}{\Gamma(\vartheta+1)}\Big]+\frac{(\log\mathrm{T})^{\vartheta}}{\Gamma(\vartheta+1)}\right]. \tag{4.9}$$

This shows that (2.3) is UH stable. □

Theorem 4.2. *Assume that $\mathcal{Q}_1$–$\mathcal{Q}_3$ and (3.10) hold. Then, there exists an increasing function $\nu\in C_{1-\sigma}[\mathcal{J}]$ and a real number $\zeta_\nu>0$ such that:*

$$|\mathcal{Z}(\mathrm{t})-\mathcal{X}(\mathrm{t})|\leq\zeta_\nu\Phi(\mathrm{t}),\quad \mathrm{t}\in\mathcal{J}. \tag{4.10}$$

Then (2.3) is UHR stable.

Proof. Let $\mathcal{Z}\in C_{1-\sigma}\lfloor 1,\mathrm{T}\rfloor$ be a solution of the inequality (4.3) and let $\mathcal{X}\in C_{1-\sigma}(\mathcal{J})$ be the unique solution of the given system:

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}^{\vartheta,\beta}\mathcal{X}(\mathrm{t})=\mathcal{H}_1(\mathrm{t},\mathcal{X}(\mathrm{t}),\mathcal{H}(\mathcal{X}(\varsigma))),\quad \mathrm{t}\in\mathcal{J}=\lfloor 1,\mathrm{T}\rfloor,\\ \mathcal{X}(1)=0,\quad \mathcal{X}(\mathrm{T})=\sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\mathcal{X}(\xi_i)+\sum_{\mathrm{j}=1}^{\mathrm{f}}\varrho_{\mathrm{j}}\mathcal{I}^{\nu_{\mathrm{j}}}\mathcal{X}(\zeta_{\mathrm{j}}),\quad \nu_{\mathrm{j}}>0,\eta_i,\varrho_{\mathrm{j}}\in\mathbb{R},\ \zeta_{\mathrm{j}},\xi_i\in\mathcal{J}, \end{cases} \tag{4.11}$$

where $1<\vartheta<2$.

By Remark 4.1, we have

$$\begin{aligned}\Bigg|\mathcal{Z}(\mathrm{t})-&\frac{(\log\mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}}\eta_i\frac{1}{\Gamma(\vartheta)}\int_1^{\xi_i}\Big(\log\frac{\xi_i}{\varsigma}\Big)^{\vartheta-1}\frac{\mathcal{H}_1(\mathrm{t},\mathcal{Z}(\mathrm{t}),\mathcal{H}(\mathcal{Z}(\varsigma)))(\varsigma)}{\varsigma}d\varsigma\\ &+\sum_{\mathrm{j}=1}^{\mathrm{f}}\varrho_{\mathrm{j}}\frac{1}{\Gamma(\vartheta+\nu_{\mathrm{j}})}\int_1^{\zeta_{\mathrm{j}}}(\log\frac{\zeta_{\mathrm{j}}}{\varsigma})^{\vartheta+\nu_{\mathrm{j}}-1}\frac{\mathcal{H}_1(\varsigma,\mathcal{Z}(\varsigma),\mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma}d\varsigma\\ &-\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{T}}(\log\frac{\mathrm{T}}{\varsigma})^{\vartheta-1}\frac{\mathcal{H}_1(\varsigma,\mathcal{Z}(\varsigma),\mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma}d\varsigma\Big]\\ &+\frac{1}{\Gamma(\vartheta)}\int_1^{\mathrm{t}}(\log\frac{\mathrm{t}}{\varsigma})^{\vartheta-1}\frac{\mathcal{H}_1(\varsigma,\mathcal{Z}(\varsigma),\mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma}d\varsigma\Bigg|\leq\varepsilon\zeta_\nu\Phi(\mathrm{t}).\end{aligned} \tag{4.12}$$

Then for each $\mathrm{t} \in \mathcal{J}$, we obtain

$$
\begin{aligned}
&|\mathcal{Z}(\mathrm{t})-\mathcal{X}(\mathrm{t})| \\
\leq &\Bigg|\mathcal{Z}(\mathrm{t})-\frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}} \eta_i \frac{1}{\Gamma(\vartheta)} \int_1^{\xi_i}\Big(\log \frac{\xi_i}{\varsigma}\Big)^{\vartheta-1} \frac{\mathcal{H}_1(\mathrm{t}, \mathcal{X}(\mathrm{t}), \mathcal{H}(\mathcal{X}(\varsigma)))(\varsigma)}{\varsigma} d\varsigma \\
&+\sum_{\mathrm{j}=1}^{\mathrm{f}} \varrho_{\mathrm{j}} \frac{1}{\Gamma(\vartheta+v_{\mathrm{j}})} \int_1^{\zeta_{\mathrm{j}}}\Big(\log \frac{\zeta_{\mathrm{j}}}{\varsigma}\Big)^{\vartheta+v_{\mathrm{j}}-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma} d\varsigma \\
&-\frac{1}{\Gamma(\vartheta)} \int_1^{\mathfrak{T}}\Big(\log \frac{\mathfrak{T}}{\varsigma}\Big)^{\vartheta-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma} d\varsigma\Big] \\
&+\frac{1}{\Gamma(\vartheta)} \int_1^{\mathrm{t}}\Big(\log \frac{\mathrm{t}}{\varsigma}\Big)^{\vartheta-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma} d\varsigma\Bigg| \\
\leq &\Bigg|\mathcal{Z}(\mathrm{t})-\frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}} \eta_i \frac{1}{\Gamma(\vartheta)} \int_1^{\xi_i}\Big(\log \frac{\xi_i}{\varsigma}\Big)^{\vartheta-1} \frac{\mathcal{H}_1(\mathrm{t}, \mathcal{Z}(\mathrm{t}), \mathcal{H}(\mathcal{Z}(\varsigma)))(\varsigma)}{\varsigma} d\varsigma \\
&+\sum_{\mathrm{j}=1}^{\mathrm{f}} \varrho_{\mathrm{j}} \frac{1}{\Gamma(\vartheta+v_{\mathrm{j}})} \int_1^{\zeta_{\mathrm{j}}}\Big(\log \frac{\zeta_{\mathrm{j}}}{\varsigma}\Big)^{\vartheta+v_{\mathrm{j}}-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} d\varsigma \\
&-\frac{1}{\Gamma(\vartheta)} \int_1^{\mathfrak{T}}\Big(\log \frac{\mathfrak{T}}{\varsigma}\Big)^{\vartheta-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} d\varsigma\Big] \\
&+\frac{1}{\Gamma(\vartheta)} \int_1^{\mathrm{t}}\Big(\log \frac{\mathrm{t}}{\varsigma}\Big)^{\vartheta-1} \frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma} d\varsigma\Bigg| \\
&\leq \Bigg|\mathcal{Z}(\mathrm{t})-\frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{\mathrm{i}=1}^{\mathrm{I}} \eta_i \frac{1}{\Gamma(\vartheta)} \int_1^{\xi_i}\Big(\log \frac{\xi_i}{\varsigma}\Big)^{\vartheta-1} \\
&\times\Big(\frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma}-\frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma}\Big) d\varsigma \\
&+\sum_{\mathrm{j}=1}^{\mathrm{f}} \varrho_{\mathrm{j}} \frac{1}{\Gamma(\vartheta+v_{\mathrm{j}})} \int_1^{\zeta_{\mathrm{j}}}\Big(\log \frac{\zeta_{\mathrm{j}}}{\varsigma}\Big)^{\vartheta+v_{\mathrm{j}}-1} \\
&\times\Big(\frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma}-\frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma}\Big) d\varsigma \\
&-\frac{1}{\Gamma(\vartheta)} \int_1^{\mathfrak{T}}\Big(\log \frac{\mathfrak{T}}{\varsigma}\Big)^{\vartheta-1}\Big(\frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma}-\frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma}\Big) d\varsigma\Big] \\
&+\frac{1}{\Gamma(\vartheta)} \int_1^{\mathrm{t}}\Big(\log \frac{\mathrm{t}}{\varsigma}\Big)^{\vartheta-1}\Big(\frac{\mathcal{H}_1(\varsigma, \mathcal{X}(\varsigma), \mathcal{H}(\mathcal{X}(\varsigma)))}{\varsigma}-\frac{\mathcal{H}_1(\varsigma, \mathcal{Z}(\varsigma), \mathcal{H}(\mathcal{Z}(\varsigma)))}{\varsigma}\Big) d\varsigma\Bigg| \\
\leq &\varepsilon \zeta_v \Phi(\mathrm{t})+(\mathcal{L}_1+\mathcal{L}_2 \mathcal{M}) \\
&\times\left[\frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{i=1}^{\mathrm{I}} \frac{|\eta_i|\big(\log \xi_i\big)^{\vartheta}}{\Gamma(\vartheta+1)}+\sum_{\mathrm{j}=1}^{\mathrm{f}} \varrho_{\mathrm{j}} \frac{(\log \zeta_{\mathrm{j}})^{\vartheta+v_{\mathrm{j}}}}{\Gamma(\vartheta+v_{\mathrm{j}})}+\frac{(\log \mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\Big]+\frac{(\log \mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\right] \\
&\times|\mathcal{X}(\mathrm{t})-\mathcal{Y}(\mathrm{t})|, \\
&\leq \varepsilon \zeta_v \Phi(\mathrm{t})+\Pi_1|\mathcal{X}(\mathrm{t})-\mathcal{Y}(\mathrm{t})|,
\end{aligned}
$$

$$\leq \frac{\varepsilon \zeta_{\nu} \Phi(\mathrm{t})}{(1-\Pi_1)(\vartheta)},$$

therefore,

$$|\mathcal{Z}(\mathrm{t}) - \mathcal{X}(\mathrm{t})| \leq \mathfrak{c}_{\mathrm{g},\nu} \varepsilon \nu(\mathrm{t}). \tag{4.13}$$

Hence, (2.3) is UHR stable. □

5. Examples

Example 5.1. *Let us investigate nonlocal BVPs employing Hilfer-Hadamard fractional differential equations given by the form:*

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}^{\vartheta,\beta}\mathcal{X}(\mathrm{t}) = \dfrac{\cos^2 \mathrm{t}}{(e^{-\mathrm{t}+2})^2|\mathcal{X}(\mathrm{t})|} + \dfrac{1}{2}\displaystyle\int_1^{\mathrm{t}} e^{-1/2}\mathcal{X}(\varsigma)d\varsigma \ \ \mathrm{t} \in \chi = \lfloor 1, \mathfrak{T} \rfloor, \\ \mathcal{X}(1) = 0, \ \ \mathcal{X}(\mathfrak{T}) = \displaystyle\sum_{\mathrm{i}=1}^{\mathrm{I}} \eta_i \mathcal{X}(\xi_i) + \sum_{\mathrm{j}=1}^{\mathfrak{f}} \varrho_{\mathrm{j}} \mathcal{I}^{\nu_{\mathrm{j}}} \mathcal{X}(\zeta_{\mathrm{j}}), \ \ \nu_{\mathrm{j}} > 0, \eta_i, \varrho_{\mathrm{j}} \in \mathbb{R}, \ \zeta_{\mathrm{j}}, \xi_i \in \mathcal{J}. \end{cases} \tag{5.1}$$

$\vartheta = 6/5, \beta = 1/2, \gamma = 1.6, \mathfrak{T} = 5, \eta_1 = 1/15, \eta_2 = 1/10, \eta_3 = 2/15, \xi_1 = 5/4, \xi_2 = 3/2, \xi_3 = 7/2,$
$\varrho_1 = 6/29, \varrho_2 = 17/50, \varrho_3 = 3/25, \zeta_1 = 5/2, \zeta_2 = 5/3, \zeta_3 = 7/2, \Pi = 0.701548, \mathcal{L}_1 = \mathcal{L}_2 = 1/9,$
$\mathcal{M} = 1/8;$

Hence, the assumptions (Q_2) *and* (Q_3) *hold. We check the condition,*

$$\leq (\mathcal{L}_1 + \mathcal{L}_2)\mathcal{M}\left[\frac{(\log \mathrm{t})^{\vartheta-1}}{\Pi}\Big[\sum_{i=1}^{\mathrm{I}} \frac{|\eta_i|\big(\log \xi_i\big)^{\vartheta}}{\Gamma(\vartheta+1)} + \sum_{\mathrm{j}=1}^{\mathfrak{f}} \varrho_{\mathrm{j}} \frac{(\log \zeta_{\mathrm{j}})^{\vartheta+\nu_{\mathrm{j}}}}{\Gamma(\vartheta+\nu_{\mathrm{j}})} + \frac{(\log \mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\Big] + \frac{(\log \mathfrak{T})^{\vartheta}}{\Gamma(\vartheta+1)}\right] \approx 0.2379 < 1. \tag{5.2}$$

Hence, the problem (5.1) has a unique solution on $\lfloor 1, 2 \rfloor$.

Example 5.2. *Consider the following boundary value problem for the Hilfer-Hadamard-type fractional differential equation:*

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}^{\vartheta,\beta}\mathcal{X}(\mathrm{t}) = \dfrac{\cos^2 \mathrm{t}}{(e^{-\mathrm{t}+2})^2|\mathcal{X}(\mathrm{t})|} + \dfrac{1}{2}\displaystyle\int_1^{\mathrm{t}} e^{-1/2}\mathcal{X}(\varsigma)d\varsigma \ \ \mathrm{t} \in \chi = \lfloor 1, \mathfrak{T} \rfloor, \\ \mathcal{X}(1) = 0, \ \ \mathcal{X}(\mathfrak{T}) = \displaystyle\sum_{\mathrm{i}=1}^{\mathrm{I}} \eta_i \mathcal{X}(\xi_i) + \sum_{\mathrm{j}=1}^{\mathfrak{f}} \varrho_{\mathrm{j}} \mathcal{I}^{\nu_{\mathrm{j}}} \mathcal{X}(\zeta_{\mathrm{j}}), \ \ \nu_{\mathrm{j}} > 0, \eta_i, \varrho_{\mathrm{j}} \in \mathbb{R}, \ \zeta_{\mathrm{j}}, \xi_i \in \mathcal{J}. \end{cases} \tag{5.3}$$

$\vartheta = 6/5, \beta = 1/2, \gamma = 1.6, \mathfrak{T} = 5, \eta_1 = 1/15, \eta_2 = 1/10, \eta_3 = 2/15, \xi_1 = 5/4, \xi_2 = 3/2, \xi_3 = 7/2,$

$\varrho_1 = 6/29, \varrho_2 = 17/50, \varrho_3 = 3/25, \zeta_1 = 5/2, \zeta_2 = 5/3, \zeta_3 = 7/2, \Pi = 0.701548, \mathcal{L}_1 = \mathcal{L}_2 = 1/9;$

and

$$|\mathcal{H}_1(\mathrm{t}, \mathcal{X}, \mathcal{Y})| = \frac{1}{32}\left(\sqrt{\mathrm{t}} + \log \mathrm{t}\right)\left(\frac{|\mathcal{X}|}{2 + |\mathcal{X}|}\right) + \left(\frac{|\mathcal{Y}|}{2 + |\mathcal{Y}|}\right).$$

Clearly,

$$|\mathcal{H}_1(\mathrm{t}, \mathcal{X}, \mathcal{Y})| \leq \frac{1}{9}\left(\sqrt{\mathrm{t}} + 1\right)(|\mathcal{X}| + |\mathcal{Y}|)$$

and

$$|\mathcal{H}_1(\mathrm{t}, \mathcal{X}_1, \mathcal{Y}_1) - \mathcal{H}_1(\mathrm{t}, \mathcal{X}_2, \mathcal{Y}_2)| \leq \mathcal{L}_1|\mathcal{X}_1 - \mathcal{X}_2| + \mathcal{L}_2|\mathcal{Y}_1 - \mathcal{Y}_2|.$$

Therefore, by Theorem 3.1, the boundary value problem (1.2) has a unique solution on $(1, \mathcal{T}]$ *with* $\mathcal{L}_1$ *and* $\mathcal{L}_2 = \frac{1}{9} = 0.1111$. *We can show that* $\Omega = 1.5635$, $\mathcal{L}\Omega = 0.1735485 < 1$.

6. Conclusions

In this study, we have used the Hilfer-Hadamard derivative in conjunction with RLC circuits to investigate various aspects of fractional calculus. Using these mathematical tools, we have investigated the existence, uniqueness, and stability of solutions to fractional differential equations, especially those relevant to RLC circuits. By focusing on the Hilfer-Hadamard derivative, we have expanded our understanding of fractional calculus and its applicability in modeling complex systems such as RLC circuits.

The work described in this article is novel and considerably adds to the established literature of knowledge on the subject. When the parameters in problems (η_i, ϱ_j) were specified, our results conformed to a few special cases. Assume that we formulated the problem in Equation (2.3) by taking ϱ_j in the presented findings:

$$\begin{cases} \mathcal{X}(1) = 0, \quad \mathcal{X}(\mathrm{T}) = \sum_{\mathrm{j}=1}^{\mathrm{t}} \varrho_{\mathrm{j}} \mathcal{I}^{\nu_{\mathrm{j}}} \mathcal{X}(\zeta_{\mathrm{j}}), \quad \nu_{\mathrm{j}} > 0, \eta_i, \varrho_{\mathrm{j}} \in \mathbb{R}, \ \zeta_{\mathrm{j}}, \xi_i \in \mathcal{J}, \end{cases} \tag{6.1}$$

We can then solve the above problem (6.1) by using the methodology employed in the previous section. Future research could focus on different concepts of stability and existence concerning a neutral time-delay system/inclusion and a time-delay system/inclusion with finite delay.

Remark 6.1. *The results presented in this paper extend the theory of fractional differential equations by applying the Hilfer-Hadamard fractional derivative to RLC circuit models. This combination offers deeper insights into both the theoretical and practical aspects of such circuits, particularly through the establishment of existence, uniqueness, and stability results using advanced techniques like Schaefer's fixed-point theorem and Banach's contraction principle. Additionally, the application of Krasnoselskii's fixed-point theorem could be a valuable enhancement to further investigate the existence of solutions, particularly in the context of compact operators on Banach spaces. The inclusion of the Ulam-Hyers and Ulam-Hyers-Rassias stability criteria strengthens the relevance of these results in engineering applications. By employing both analytical techniques and numerical methods, such as the two-step Lagrange polynomial interpolation method, the study not only verifies* *theoretical findings but also demonstrates practical feasibility. This work opens new directions for the use of fractional calculus in modeling RLC circuits and provides a solid foundation for extending these results to other engineering systems and boundary value problems. Including Krasnoselskii's fixed-point theorem can deepen the mathematical rigor of your work, particularly when dealing with non-linear problems or specific functional spaces.*

Author contributions

M. Manigandan: Conceptualization; R. S. Shanthi and M. Manigandan: Methodology; R. Meganathan and M. Rhaima: Formal analysis; R. S. Shanthi and M. Rhaima: Investigation; M. Manigandan and R. S. Shanthi: Writing-original draft; M. Rhaima and R. Meganathan: Writing-review & editing; R. S. Shanthi and M. Rhaima: Supervision. All authors have read and agreed to the published version of the manuscript.

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Conflicts of interest

The authors declare no conflict of interest.

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