Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 28611–28622.
DOI: 10.3934/math.20241388
Received: 09 August 2024
Revised: 22 September 2024
Accepted: 25 September 2024
Published: 09 October 2024

Research article

Pointwise superconvergence of block finite elements for the three-dimensional variable coefficient elliptic equation

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Abstract: This study investigated the point-wise superconvergence of block finite elements for the variable coefficient elliptic equation in a regular family of rectangular partitions of the domain in three-dimensional space. Initially, the estimates for the three-dimensional discrete Greens function and discrete derivative Greens function were presented. Subsequently, employing an interpolation operator of projection type, two essential weak estimates were derived, which were crucial for superconvergence analysis. Ultimately, by combining the aforementioned estimates, we achieved superconvergence estimates for the derivatives and function values of the finite element approximation in the point-wise sense of the L^∞-norm. A numerical example illustrated the theoretical results.

Keywords: variable coefficient elliptic equation; block finite element; interpolation operator of projection type; superconvergence; weak estimate
Mathematics Subject Classification: 65N30

1. Introduction

In the realm of solving differential equations using the finite element method (FEM), it has been observed that the rate of convergence of finite element solutions at specific exceptional points within a domain surpasses the optimal global rate. This phenomenon is known as superconvergence, which has already aroused many scholars' interest. In recent decades, superconvergence has become a significant topic in the research field of the Galerkin FEM. With the advancement of research technologies, numerous superconvergence results have been obtained, and theoretical frameworks for superconvergence have also been established. Currently, several important works related to FEM superconvergence are cited in Ref. [1–5]. Depending on the partition types within a domain, the commonly used three-dimensional finite elements mainly include tetrahedral, pentahedral, and hexahedral elements. Substantial progress has been made in studying the superconvergence of these three-dimensional FEMs with numerous superconvergence results documented in various published papers or reports such as [6–11]. Recently, we also obtained some superconvergence results for the three-dimensional FEM [12–15]. This paper focuses on the superconvergence of the block finite element for variable coefficient elliptic equations, which is a topic not explored by others. The objective is to demonstrate that the error convergence rates of the finite element approximation and corresponding interpolant in the $W^{1,\infty}$-semi-norm and L^{∞}-norm are one order (or nearly one order) higher than those of the finite element approximation and the true solution (referred to as optimal global rates). It is important to note that the results presented here are generalizations of the research discussed in [14].

In the paper, the letter C is employed to represent a generic constant, which may vary in different instances. Additionally, standard notations for the Sobolev spaces and their norms are utilized.

The model problem considered is

$$\begin{cases} \mathcal{L}u \ \equiv \ -\sum_{i,j=1}^{3}\partial_j(a_{ij}\partial_i u) + \sum_{i=1}^{3} a_i\partial_i u + a_0 u = f \ \text{ in } \ \Omega, \\ u \ = \ 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathcal{R}^3$ is a rectangular block with boundary, $\partial\Omega$, consisting of faces parallel to the x-, y-, and z-axes. For simplicity, we assume a_{ij}, a_i, a_0, and f are sufficiently smooth given functions, and write $\partial_1 u = \frac{\partial u}{\partial x}$, $\partial_2 u = \frac{\partial u}{\partial y}$, and $\partial_3 u = \frac{\partial u}{\partial z}$.

Thus, the standard variational formulation of the problem (1.1) is as follows.

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u\,,\,v) = (f\,,\,v) \ \ \forall v \in H_0^1(\Omega), \end{cases} \tag{1.2}$$

where

$$a(u,v) = \int_\Omega (\sum_{i,j=1}^{3} a_{ij}\partial_i u \partial_j v + \sum_{i=1}^{3} a_i \partial_i u v + a_0 u v) dxdydz, \ (f,v) = \int_\Omega f v dxdydz.$$

We also write

$$a(u\,,\,v) = \sum_{i,j=1}^{3}(a_{ij}\partial_i u,\,\partial_j v) + \sum_{i=1}^{3}(a_i\partial_i u,\,v) + (a_0 u,\,v). \tag{1.3}$$

The existence and uniqueness of the solution to (1.2) is given by the Lax-Milgram lemma, see Ciarlet [2, Theorem 1.1.31].

In order to discretize the problem (1.2), one proceeds as follows. The domain Ω is first partitioned into a regular family of rectangular blocks $\mathcal{T}^h$ with mesh size $h \in (0,\,1)$ such that $\bar{\Omega} = \cup_{e\in\mathcal{T}^h}\bar{e}$. Then we define the finite dimensional subspace, $S_0^h(\Omega) \subset H_0^1(\Omega)$, as the standard tensor-product m-order finite element space over the partition. Thus, the discrete problem of approximating (1.2) is

$$\begin{cases} \text{Find } u_h \in S_0^h(\Omega) \text{ satisfying} \\ a(u_h\,,\,v) = (f\,,\,v) \ \ \forall v \in S_0^h(\Omega). \end{cases} \tag{1.4}$$

Obviously, from (1.2) and (1.4), the following Galerkin orthogonality relation holds.

$$a(u - u_h\,,\,v) = 0 \ \ \forall v \in S_0^h(\Omega). \tag{1.5}$$

To obtain the desired results, for every $Z \in \Omega$, and any directional unit vector $\ell \in \mathcal{R}^3$, we also need the discrete Green's function G_Z^h and discrete derivative Green's function $\partial_{Z,\ell}G_Z^h$ defined by

$$a(v, G_Z^h) = v(Z) \ \forall\, v \in S_0^h(\Omega), \tag{1.6}$$

and

$$a(v, \partial_{Z,\ell} G_Z^h) = \partial_\ell v(Z) \ \forall\, v \in S_0^h(\Omega). \tag{1.7}$$

Here, $\partial_{Z,\ell} G_Z^h$ and $\partial_\ell v(Z)$ stand for the following one-sided directional derivatives, respectively.

$$\partial_{Z,\ell} G_Z^h = \lim_{|\Delta Z|\to 0} \frac{G_{Z+\Delta Z}^h - G_Z^h}{|\Delta Z|},$$

$$\partial_\ell v(Z) = \lim_{|\Delta Z|\to 0} \frac{v(Z+\Delta Z) - v(Z)}{|\Delta Z|},\ \Delta Z = |\Delta Z|\ell.$$

As for G_Z^h and $\partial_{Z,\ell} G_Z^h$, we have [3]

$$\|\partial_{Z,\ell} G_Z^h\|_{1,1,\Omega} \le C|\ln h|^{\frac{4}{3}}, \tag{1.8}$$

$$\|\partial_{Z,\ell} G_Z^h\|_{2,1,\Omega}^h \le Ch^{-1}, \tag{1.9}$$

$$\|G_Z^h\|_{2,1,\Omega}^h \le C|\ln h|^{\frac{2}{3}}, \tag{1.10}$$

where $\|G_Z^h\|_{2,1,\Omega}^h = \sum_{e\in\mathcal{T}^h} \|G_Z^h\|_{2,1,e}$ and $\|\partial_{Z,\ell} G_Z^h\|_{2,1,\Omega}^h = \sum_{e\in\mathcal{T}^h} \|\partial_{Z,\ell} G_Z^h\|_{2,1,e}$.

The rest of the paper is arranged as follows. In Section 2, for the second-order elliptic equation with variable coefficients, we discuss two weak estimates for the finite element, which are crucial in the superconvergence analysis. Combined with (1.8)–(1.10), several superconvergence results of the finite element approximation are given in Section 3.

2. Weak estimates for the finite element

In this section, using the properties of the interpolation operator of projection type, we derive the weak estimates.

We write an element

$$e = (x_e - n_e, x_e + n_e) \times (y_e - k_e, y_e + k_e) \times (z_e - d_e, z_e + d_e) \equiv I_1 \times I_2 \times I_3. \tag{2.1}$$

Let $\{l_j(x)\}_{j=0}^\infty$, $\{\tilde{l}_j(y)\}_{j=0}^\infty$, and $\{\bar{l}_j(z)\}_{j=0}^\infty$ be the normalized orthogonal Legendre polynomial systems on $L^2(I_1)$, $L^2(I_2)$, and $L^2(I_3)$, respectively. It is easy to check that $\{l_i(x)\tilde{l}_j(y)\bar{l}_k(z)\}_{i,j,k=0}^\infty$ is the normalized orthogonal polynomial system on $L^2(e)$. Set

$$\omega_0(x) = \tilde{\omega}_0(y) = \bar{\omega}_0(z) = 1,\ \omega_{j+1}(x) = \int_{x_e-n_e}^x l_j(\xi)\,d\xi,$$

$$\tilde{\omega}_{j+1}(y) = \int_{y_e-k_e}^y \tilde{l}_j(\xi)\,d\xi,\ \bar{\omega}_{j+1}(z) = \int_{z_e-d_e}^z \bar{l}_j(\xi)\,d\xi,\ \ j \ge 0,$$

which are called Lobatto functions. Suppose $u \in H^3(e)$. Then, we have the following expansion (see [14]):

$$u(x,y,z) = \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \beta_{ijk}\omega_i(x)\tilde{\omega}_j(y)\bar{\omega}_k(z),\ (x,y,z) \in e, \tag{2.2}$$

where the coefficients β_{ijk} can be seen in [3,14] and satisfy, for $i, j, k \geq 1$,

$$\begin{aligned}\beta_{i00} = O(n_e^{i-0.5}), \beta_{0j0} = O(k_e^{j-0.5}), \beta_{00k} = O(d_e^{k-0.5}), \beta_{ij0} = O(n_e^{i-0.5}k_e^{j-0.5}),\\ \beta_{0jk} = O(k_e^{j-0.5}d_e^{k-0.5}), \beta_{i0k} = O(n_e^{i-0.5}d_e^{k-0.5}), \beta_{ijk} = O(n_e^{i-0.5}k_e^{j-0.5}d_e^{k-0.5}).\end{aligned} \tag{2.3}$$

We introduce the standard tensor-product m-order polynomial spaces denoted by T_m, namely,

$$T_m = \{q | q = \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m} b_{ijk}x^i y^j z^k\}.$$

Define the interpolation operator of projection type by Π_m^e: $H^3(e) \to T_m(e)$ such that

$$\Pi_m^e u = \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m} \beta_{ijk}\omega_i(x)\tilde{\omega}_j(y)\bar{\omega}_k(z) \equiv \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m} \lambda_{ijk}. \tag{2.4}$$

Thus, combining (2.2) and (2.4) yields

$$u - \Pi_m^e u = (\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=m+1}^{\infty} + \sum_{i=0}^{m}\sum_{j=m+1}^{\infty}\sum_{k=0}^{\infty} + \sum_{i=m+1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty})\lambda_{ijk}. \tag{2.5}$$

Further, we may define the global tensor-product m-order interpolation operator of projection type [14]

$$\Pi_m : H^3(\Omega) \cap H_0^1(\Omega) \to S_0^h(\Omega), \tag{2.6}$$

where $(\Pi_m u)|_e = \Pi_m^e u$.

Theorem 2.1. *Suppose $\{\mathcal{T}^h\}$ is a regular family of rectangular partitions of Ω, and $u \in W^{m+2,\infty}(\Omega) \cap H_0^1(\Omega)$. Then, for all $v \in S_0^h(\Omega)$, the interpolation operator Π_m, defined by (2.6), satisfies the following weak estimates:*

$$|a(u - \Pi_m u,\, v)| \leq Ch^{m+1}\|u\|_{m+2,\infty,\Omega}\, \|v\|_{1,1,\Omega},\ m \geq 1, \tag{2.7}$$

$$|a(u - \Pi_m u,\, v)| \leq Ch^{m+2}\|u\|_{m+2,\infty,\Omega}\, \|v\|_{2,1,\Omega}^h,\ m \geq 2, \tag{2.8}$$

where $\|v\|_{2,1,\Omega}^h = \sum_{e\in\mathcal{T}^h}\|v\|_{2,1,e}$.

Proof. Note that the term $u - \Pi_m^e u$ can be written as follows.

$$u - \Pi_m^e u = \lambda_{(m+1)00} + \lambda_{0(m+1)0} + \lambda_{00(m+1)} + R. \tag{2.9}$$

Now, we first bound $(a_{11}\partial_1(u-\Pi_m u),\, \partial_1 v) = \sum_e (a_{11}\partial_1(u-\Pi_m^e u),\, \partial_1 v)_e$. For $\lambda_{(m+1)00}$, by the orthogonality of Legendre functions, we have

$$(a_{11}\partial_1\lambda_{(m+1)00},\, \partial_1 v)_e = \beta_{(m+1)00}\int_e (a_{11} - a_{11}(x_e, y_e, z_e))l_m(x)\partial_1 v dxdydz. \tag{2.10}$$

Set $h_e = \text{diam}(e)$. Thus,

$$a_{11} - a_{11}(x_e, y_e, z_e) = O(h_e). \tag{2.11}$$

By (2.3), (2.10), (2.11), and the properties of Legendre functions, we get

$$|(a_{11}\partial_1\lambda_{(m+1)00},\, \partial_1 v)_e| \leq Ch_e^{m+1}\|u\|_{m+1,\infty,e}\, \|v\|_{1,1,e}. \tag{2.12}$$

It is easy to see

$$(a_{11}\partial_1\lambda_{0(m+1)0},\ \partial_1 v)_e = (a_{11}\partial_1\lambda_{00(m+1)},\ \partial_1 v)_e = 0. \tag{2.13}$$

Consider the main parts of R in (2.9), i.e.,

$$\lambda_{01(m+1)}, \lambda_{0(m+1)1}, \lambda_{1(m+1)0}, \lambda_{10(m+1)}, \lambda_{(m+1)10}, \lambda_{(m+1)01}.$$

As the symmetry, only need to discuss $\lambda_{01(m+1)}, \lambda_{1(m+1)0}, \lambda_{(m+1)10}$. Obviously,

$$(a_{11}\partial_1\lambda_{01(m+1)},\ \partial_1 v)_e = 0. \tag{2.14}$$

For $\lambda_{1(m+1)0}$,

$$(a_{11}\partial_1\lambda_{1(m+1)0},\ \partial_1 v)_e = \beta_{1(m+1)0}\int_e a_{11} l_0(x)\tilde{\omega}_{m+1}(y)\partial_1 v dxdydz. \tag{2.15}$$

By (2.3), (2.15), and the properties of Legendre and Lobatto functions, we have

$$|(a_{11}\partial_1\lambda_{1(m+1)0},\ \partial_1 v)_e| \le Ch_e^{m+1}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{1,\,1,\,e}. \tag{2.16}$$

Similarly,

$$|(a_{11}\partial_1\lambda_{(m+1)10},\ \partial_1 v)_e| \le Ch_e^{m+1}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{1,\,1,\,e}. \tag{2.17}$$

From (2.9), (2.12)–(2.14), (2.16), and (2.17),

$$|(a_{11}\partial_1(u - \Pi_m^e u),\ \partial_1 v)_e| \le Ch_e^{m+1}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{1,\,1,\,e}.$$

Summing over all elements yields

$$|(a_{11}\partial_1(u - \Pi_m u),\ \partial_1 v)| \le Ch^{m+1}\|u\|_{m+2,\,\infty,\Omega}\,\|v\|_{1,\,1,\Omega}. \tag{2.18}$$

Similarly to the arguments of the result (2.18),

$$|(a_{ii}\partial_i(u - \Pi_m u),\ \partial_i v)| \le Ch^{m+1}\|u\|_{m+2,\,\infty,\Omega}\,\|v\|_{1,\,1,\Omega},\ \ i = 2, 3. \tag{2.19}$$

Now, we bound the terms $(a_{ij}\partial_i(u-\Pi_m u),\ \partial_j v), i \ne j$. Without loss of generality, we consider $(a_{12}\partial_1(u-\Pi_m u),\ \partial_2 v) = \sum_e (a_{12}\partial_1(u - \Pi_m^e u),\ \partial_2 v)_e$. Nevertheless, from (2.9),

$$\partial_1(u - \Pi_m^e u) = \partial_1\lambda_{(m+1)00} + \partial_1 R. \tag{2.20}$$

Integration by parts results in

$$\begin{aligned} & (a_{12}\partial_1\lambda_{(m+1)00},\ \partial_2 v)_e \\ = \ & \beta_{(m+1)00}\int_e \omega_{m+1}(x)(\partial_2 a_{12}\partial_1 v - \partial_1 a_{12}\partial_2 v)dxdydz \\ & -\beta_{(m+1)00}\int_{\partial e} a_{12}\omega_{m+1}(x)\partial_1 v \cos < \vec{n}, y > dS \\ \equiv \ & A_e + B_e. \end{aligned}$$

Combined with (2.3), we get

$$|A_e| \le Ch_e^{m+1}\|u\|_{m+1,\,\infty,\,e}\,\|v\|_{1,\,1,\,e}. \tag{2.21}$$

As for B_e, we need to apply the element canceling technique. At the adjacent element $e' = (x_e - n_e, x_e + n_e) \times (y_{e'} - k_{e'}, y_{e'} + k_{e'}) \times (z_e - d_e, z_e + d_e)$ of e, where $k_{e'} - k_e = O(h_e) = O(h_{e'})$ and $y_{e'} < y_e$,

$$B_{e'} = -\beta'_{(m+1)00} \int_{\partial e'} a_{12}\omega_{m+1}(x)\partial_1 v \cos <\vec{n}, y> dS.$$

Obviously, there is the same integration factor between B_e and $B_{e'}$, the summing of which is

$$B_{e+e'} = (\beta_{(m+1)00} - \beta'_{(m+1)00}) \int_{I_1\times I_3} a_{12}(x, y_e - k_e, z)\omega_{m+1}(x)\partial_1 v(x, y_e - k_e, z)dxdz. \tag{2.22}$$

Here,

$$|\beta_{(m+1)00} - \beta'_{(m+1)00}| \le Ch_e^{m+1.5}\|u\|_{m+2,\,\infty,\,e'}. \tag{2.23}$$

Additionally,

$$\begin{aligned}
& \int_{I_1\times I_3} a_{12}(x, y_e - k_e, z)\omega_{m+1}(x)\partial_1 v(x, y_e - k_e, z)dxdz \\
= & \ \frac{1}{2k_e}\int_e a_{12}(x, y_e - k_e, z)\omega_{m+1}(x)\partial_1 v(x, y_e - k_e, z)dxdz \\
= & \ \frac{1}{2k_e}\int_e a_{12}(x, y_e - k_e, z)\omega_{m+1}(x)\partial_1 v(x, y, z)dxdydz \\
& -\frac{1}{2k_e}\int_e a_{12}(x, y_e - k_e, z)\omega_{m+1}(x)\left(\int_{y_e-k_e}^{y} \partial_2\partial_1 v(x, y, z)dy\right)dxdydz \\
\equiv & \ M + N.
\end{aligned}$$

By the inverse estimate, we have

$$|M| \le Ch_e^{-0.5}\|v\|_{1,1,e},\ \ |N| \le Ch_e^{-0.5}\|v\|_{1,1,e}.$$

Combined with (2.22) and (2.23), we obtain

$$|B_{e+e'}| \le Ch_e^{m+1}\|u\|_{m+2,\,\infty,\,e'}\,\|v\|_{1,\,1,\,e}. \tag{2.24}$$

From (2.21) and (2.24), summing over all elements yields

$$|\sum_e (a_{12}\partial_1\lambda_{(m+1)00},\ \partial_2 v)_e| \le Ch^{m+1}\|u\|_{m+2,\,\infty,\,\Omega}\,\|v\|_{1,\,1,\,\Omega}. \tag{2.25}$$

As for the main parts of $\partial_1 R$ in (2.20), it is easy to see

$$(a_{12}\partial_1\lambda_{01(m+1)},\ \partial_2 v)_e = (a_{12}\partial_1\lambda_{0(m+1)1},\ \partial_2 v)_e = 0. \tag{2.26}$$

For $\lambda_{1(m+1)0}, \lambda_{10(m+1)}, \lambda_{(m+1)10}, \lambda_{(m+1)01}$, by (2.3) and the properties of Legendre and Lobatto functions, we immediately obtain

$$(a_{12}\partial_1\lambda_{1(m+1)0},\ \partial_2 v)_e \le Ch_e^{m+1}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{1,\,1,\,e}. \tag{2.27}$$

$$(a_{12}\partial_1\lambda_{10(m+1)},\ \partial_2 v)_e \le Ch_e^{m+1}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{1,\,1,\,e}. \tag{2.28}$$

$$(a_{12}\partial_1\lambda_{(m+1)10},\ \partial_2 v)_e \le Ch_e^{m+1}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{1,\,1,\,e}. \tag{2.29}$$

$$(a_{12}\partial_1\lambda_{(m+1)01},\ \partial_2 v)_e \le Ch_e^{m+1}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{1,\,1,\,e}. \tag{2.30}$$

From (2.25)–(2.30), and summing over all elements, we obtain

$$|(a_{12}\partial_1(u-\Pi_m),\ \partial_2 v)| \leq Ch^{m+1}\|u\|_{m+2,\,\infty,\,\Omega}\,\|v\|_{1,\,1,\,\Omega}. \tag{2.31}$$

Hence,

$$|(a_{ij}\partial_i(u-\Pi_m),\ \partial_j v)| \leq Ch^{m+1}\|u\|_{m+2,\,\infty,\,\Omega}\,\|v\|_{1,\,1,\,\Omega},\ i\neq j. \tag{2.32}$$

It remains to bound the terms $(a_i\partial_i(u-\Pi_m u),\ v)$ and $(a_0(u-\Pi_m u),\ v)$. Using the integration by parts, the element canceling technique, and the interpolation error estimate, we have

$$|(a_i\partial_i(u-\Pi_m u),\ v)| = |-(u-\Pi_m u,\ \partial_i(a_i v))| \leq Ch^{m+1}\|u\|_{m+1,\,\infty,\,\Omega}\,\|v\|_{1,\,1,\,\Omega}. \tag{2.33}$$

Obviously, by the interpolation error estimate, we immediately get

$$|(a_0(u-\Pi_m u),\ v)| \leq Ch^{m+1}\|u\|_{m+1,\,\infty,\,\Omega}\,\|v\|_{1,\,1,\,\Omega}. \tag{2.34}$$

The desired result (2.7) follows from (2.18), (2.19), and (2.32)–(2.34).

Let's prove (2.8) when $m \geq 2$. From (2.20), we first bound $(a_{11}\partial_1\lambda_{(m+1)00},\ \partial_1 v)_e$. By the integration by parts and the orthogonality of Lobatto functions, we have

$$\begin{aligned} & (a_{11}\partial_1\lambda_{(m+1)00},\ \partial_1 v)_e \\ = \ & \beta_{(m+1)00}\int_e D^{-1}\omega_{m+1}(x)\partial_1(\partial_1 a_{11}\partial_1 v)dxdydz \\ & -\beta_{(m+1)00}\int_e (a_{11}-a_{11}(x_e,y_e,z_e))\omega_{m+1}(x)\partial_1^2 v dxdydz. \end{aligned}$$

Combined with (2.3), (2.11), and the properties of Lobatto functions,

$$|(a_{11}\partial_1\lambda_{(m+1)00},\ \partial_1 v)_e| \leq Ch_e^{m+2}\|u\|_{m+1,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.35}$$

Now consider the main parts of $\partial_1 R$ in (2.20). As the symmetry, we only need to discuss $\partial_1\lambda_{01(m+1)}, \partial_1\lambda_{1(m+1)0}, \partial_1\lambda_{(m+1)10}$. For $\partial_1\lambda_{1(m+1)0}$, integration by parts yields

$$\begin{aligned} & (a_{11}\partial_1\lambda_{1(m+1)0},\ \partial_1 v)_e \\ = \ & \beta_{1(m+1)0}\int_e a_{11}l_0(x)\tilde{\omega}_{m+1}(y)\partial_1 v dxdydz \\ = \ & -\beta_{1(m+1)0}\int_e l_0(x)D^{-1}\tilde{\omega}_{m+1}(y)\partial_2(a_{11}\partial_1 v)dxdydz. \end{aligned}$$

By (2.3) and the properties of Legendre and Lobatto functions, we have

$$|(a_{11}\partial_1\lambda_{1(m+1)0},\ \partial_1 v)_e| \leq Ch_e^{m+2}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.36}$$

For $\partial_1\lambda_{(m+1)10}$, integration by parts yields

$$\begin{aligned} & (a_{11}\partial_1\lambda_{(m+1)10},\ \partial_1 v)_e \\ = \ & \beta_{(m+1)10}\int_e a_{11}l_m(x)\tilde{\omega}_1(y)\partial_1 v dxdydz \\ = \ & -\beta_{(m+1)10}\int_e D^{-1}l_m(x)\tilde{\omega}_1(y)\partial_1(a_{11}\partial_1 v)dxdydz. \end{aligned}$$

By (2.3) and the properties of Legendre and Lobatto functions again, we get

$$|(a_{11}\partial_1\lambda_{(m+1)10},\ \partial_1 v)_e| \leq Ch_e^{m+2}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.37}$$

From (2.14), (2.35)–(2.37), and summing over all elements,

$$|(a_{11}\partial_1(u-\Pi_m u),\ \partial_1 v)| \leq Ch^{m+2}\|u\|_{m+2,\infty,\Omega}\,\|v\|^h_{2,1,\Omega}. \tag{2.38}$$

Similarly,

$$|(a_{ii}\partial_i(u-\Pi_m u),\ \partial_i v)| \leq Ch^{m+2}\|u\|_{m+2,\infty,\Omega}\,\|v\|^h_{2,1,\Omega},\ \ i=2,3. \tag{2.39}$$

Next, we bound the terms $(a_{ij}\partial_i(u-\Pi_m u),\ \partial_j v),\ i \neq j$. Applying integration by parts twice, we have

$$\begin{aligned}
& (a_{12}\partial_1\lambda_{(m+1)00},\ \partial_2 v)_e \\
= \ & \beta_{(m+1)00}\int_e D^{-1}\omega_{m+1}(x)(\partial_1^2 a_{12}\partial_2 v + 2\partial_1 a_{12}\partial_1\partial_2 v - \partial_2 a_{12}\partial_1^2 v)dxdydz \\
& +\beta_{(m+1)00}\int_{\partial e} a_{12}D^{-1}\omega_{m+1}(x)\partial_1^2 v\cos<\vec{n},y> dS \\
\equiv \ & C_e + D_e.
\end{aligned}$$

Combined with (2.3) and the properties of Lobatto functions,

$$|C_e| \leq Ch_e^{m+2}\|u\|_{m+1,\infty,e}\,\|v\|_{2,1,e}. \tag{2.40}$$

At the adjacent element $e' = (x_e - n_e, x_e + n_e)\times(y_{e'} - k_{e'}, y_{e'} + k_{e'})\times(z_e - d_e, z_e + d_e)$ of e, where $k_{e'} - k_e = O(h_e) = O(h_{e'})$ and $y_{e'} < y_e$,

$$D_{e'} = \beta'_{(m+1)00}\int_{\partial e'} a_{12}D^{-1}\omega_{m+1}(x)\partial_1^2 v\cos<\vec{n},y> dS.$$

Obviously, there is the same integration factor between D_e and $D_{e'}$, the summing of which is

$$D_{e+e'} = (\beta'_{(m+1)00} - \beta_{(m+1)00})\int_{I_1\times I_3} a_{12}(x,y_e-k_e,z)D^{-1}\omega_{m+1}(x)\partial_1^2 v(x,y_e-k_e,z)dxdz.$$

Similarly to the arguments of (2.24),

$$|D_{e+e'}| \leq Ch_e^{m+2}\|u\|_{m+2,\infty,e'}\,\|v\|_{2,1,e}. \tag{2.41}$$

From (2.40) and (2.41), summing over all elements yields

$$|\sum_e(a_{12}\partial_1\lambda_{(m+1)00},\ \partial_2 v)_e| \leq Ch^{m+2}\|u\|_{m+2,\infty,\Omega}\,\|v\|^h_{2,1,\Omega}. \tag{2.42}$$

Obviously,

$$(a_{12}\partial_1\lambda_{01(m+1)},\ \partial_2 v)_e = (a_{12}\partial_1\lambda_{0(m+1)1},\ \partial_2 v)_e = 0. \tag{2.43}$$

Next, we discuss $\partial_1\lambda_{1(m+1)0}, \partial_1\lambda_{10(m+1)}, \partial_1\lambda_{(m+1)10}$, and $\partial_1\lambda_{(m+1)01}$. By the integration by parts and the properties of Legendre and Lobatto functions, we immediately obtain

$$|\sum_e(a_{12}\partial_1\lambda_{1(m+1)0},\ \partial_2 v)_e| \leq Ch^{m+2}\|u\|_{m+2,\infty,\Omega}\,\|v\|^h_{2,1,\Omega}. \tag{2.44}$$

$$|\sum_e(a_{12}\partial_1\lambda_{10(m+1)},\ \partial_2 v)_e| \leq Ch^{m+2}\|u\|_{m+2,\infty,\Omega}\,\|v\|^h_{2,1,\Omega}. \tag{2.45}$$

$$|\sum_e (a_{12}\partial_1\lambda_{(m+1)10},\, \partial_2 v)_e| \le Ch^{m+2}\|u\|_{m+2,\,\infty,\,\Omega}\,\|v\|^h_{2,\,1,\,\Omega}. \tag{2.46}$$

$$|\sum_e (a_{12}\partial_1\lambda_{(m+1)01},\, \partial_2 v)_e| \le Ch^{m+2}\|u\|_{m+2,\,\infty,\,\Omega}\,\|v\|^h_{2,\,1,\,\Omega}. \tag{2.47}$$

From (2.20) and (2.42)–(2.47),

$$|(a_{12}\partial_1(u-\Pi_m u),\, \partial_2 v)| \le Ch^{m+2}\|u\|_{m+2,\,\infty,\,\Omega}\,\|v\|^h_{2,\,1,\,\Omega}.$$

Hence,

$$|(a_{ij}\partial_i(u-\Pi_m u),\, \partial_j v)| \le Ch^{m+2}\|u\|_{m+2,\,\infty,\,\Omega}\,\|v\|^h_{2,\,1,\,\Omega},\ i \ne j. \tag{2.48}$$

It remains to bound

$$\sum_{i=1}^{3}(a_i\partial_i(u-\Pi_m u),\, v) + (a_0(u-\Pi_m u),\, v).$$

As for $(a_1\partial_1(u-\Pi_m u),\, v) = \sum_e (a_1\partial_1(u-\Pi_m^e u),\, v)_e$, we first consider $(a_1\partial_1\lambda_{(m+1)00},\, v)_e$. Integration by parts twice yields

$$(a_1\partial_1\lambda_{(m+1)00},\, v)_e = \beta_{(m+1)00}\int_e D^{-1}\omega_{m+1}(x)\partial_1^2(a_1 v)dxdydz.$$

From (2.3) and the properties of Lobatto functions,

$$|(a_1\partial_1\lambda_{(m+1)00},\, v)_e| \le Ch_e^{m+2}\|u\|_{m+1,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.49}$$

In addition,

$$(a_1\partial_1\lambda_{01(m+1)},\, v)_e = (a_1\partial_1\lambda_{0(m+1)1},\, v)_e = 0. \tag{2.50}$$

Utilizing the integration by parts and the properties of Legendre and Lobatto functions, we get

$$|(a_1\partial_1\lambda_{1(m+1)0},\, v)_e| \le Ch_e^{m+2}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.51}$$

$$|(a_1\partial_1\lambda_{10(m+1)},\, v)_e| \le Ch_e^{m+2}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.52}$$

$$|(a_1\partial_1\lambda_{(m+1)10},\, v)_e| \le Ch_e^{m+2}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.53}$$

$$|(a_1\partial_1\lambda_{(m+1)01},\, v)_e| \le Ch_e^{m+2}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.54}$$

From (2.20), (2.49)–(2.54), and summing over all elements,

$$|(a_1\partial_1(u-\Pi_m u),\, v)| \le Ch^{m+2}\|u\|_{m+2,\,\infty,\,\Omega}\,\|v\|^h_{2,\,1,\,\Omega}.$$

Thus,

$$|(a_i\partial_i(u-\Pi_m u),\, v)| \le Ch^{m+2}\|u\|_{m+2,\,\infty,\,\Omega}\,\|v\|^h_{2,\,1,\,\Omega},\ i = 1,2,3. \tag{2.55}$$

For $(a_0(u-\Pi_m u),\, v)$, integration by parts results in

$$(a_0\lambda_{(m+1)00},\, v)_e = -\beta_{(m+1)00}\int_e D^{-1}\omega_{m+1}(x)\partial_1(a_0 v)dxdydz.$$

From (2.3) and the properties of Lobatto functions,

$$|(a_0\lambda_{(m+1)00},\ v)_e| \le Ch_e^{m+2}\|u\|_{m+1,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.56}$$

Similarly,

$$|(a_0\lambda_{0(m+1)0},\ v)_e| \le Ch_e^{m+2}\|u\|_{m+1,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.57}$$

$$|(a_0\lambda_{00(m+1)},\ v)_e| \le Ch_e^{m+2}\|u\|_{m+1,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.58}$$

Obviously,

$$|(a_0R,\ v)_e| \le Ch_e^{m+2}\|u\|_{m+2,\,\infty,\,e}\,\|v\|_{2,\,1,\,e}. \tag{2.59}$$

From (2.9), (2.56)–(2.59), and summing over all elements,

$$|(a_0(u-\Pi_m u),\ v)| \le Ch^{m+2}\|u\|_{m+2,\,\infty,\,\Omega}\,\|v\|^h_{2,\,1,\,\Omega}. \tag{2.60}$$

The desired result (2.8) follows from (2.38), (2.39), (2.48), (2.55), and (2.60).

3. Superconvergence of the finite element

In this section, we will use the weak estimates and the estimates for discrete Green's function and discrete derivative Green's function to obtain superconvergence estimates of the finite element.

Theorem 3.1. *Suppose $\{\mathcal{T}^h\}$ is a regular family of rectangular partitions of Ω, $u \in W^{m+2,\infty}(\Omega)\cap H_0^1(\Omega)$, and $u_h \in S_0^h(\Omega)$ and $\Pi_m u \in S_0^h(\Omega)$ are the finite element approximation and interpolant of projection type to u, respectively. Then, we have*

$$|u_h - \Pi_m u|_{1,\,\infty,\,\Omega} \le Ch^{m+1}\,|\ln h|^{\frac{4}{3}}\,\|u\|_{m+2,\,\infty,\,\Omega}\,,\ m \ge 1, \tag{3.1}$$

$$|u_h - \Pi_m u|_{1,\,\infty,\,\Omega} \le Ch^{m+1}\,\|u\|_{m+2,\,\infty,\,\Omega}\,,\ m \ge 2, \tag{3.2}$$

$$|u_h - \Pi_m u|_{0,\,\infty,\,\Omega} \le Ch^{m+2}\,|\ln h|^{\frac{2}{3}}\,\|u\|_{m+2,\,\infty,\,\Omega}\,,\ m \ge 2. \tag{3.3}$$

Proof. For every $Z \in \Omega$, applying the definitions of G_Z^h and $\partial_{Z,\ell}G_Z^h$ as well as the Galerkin orthogonality relation (1.5), we derive

$$\partial_\ell(u_h - \Pi_m u)(Z) = a(u_h - \Pi_m u,\ \partial_{Z,\ell}G_Z^h) = a(u - \Pi_m u,\ \partial_{Z,\ell}G_Z^h), \tag{3.4}$$

and

$$(u_h - \Pi_m u)(Z) = a(u_h - \Pi_m u,\ G_Z^h) = a(u - \Pi_m u,\ G_Z^h). \tag{3.5}$$

From (1.8), (2.7), and (3.4), the result (3.1) is obtained. Combining (1.9), (2.8), and (3.4) yields the result (3.2). The result (3.3) is immediately proved by using (1.10), (2,8), and (3.5).

Example 3.1. Consider the following equation:

$$\begin{cases} -\nabla\cdot(a\nabla u) = f & \text{in } \Omega = (0,1)^3, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a = e^{x+y+z}$, and the exact solution is $u = \sin \pi x \sin \pi y \sin \pi z$. Let u_h and Π_2 be the tensor-product quadratic finite element approximation and the interpolation operator of projection type, respectively. We solve Example 3.1 and obtain the following numerical results (see Table 1):

Table 1. Numerical results of superconvergence on uniform meshes.

$1/h$	$\lvert u_h - \Pi_2 u\rvert_{0,\infty,\Omega}$	reduction	$\lvert u_h - \Pi_2 u\rvert_{1,\infty,\Omega}$	reduction
2	8.696e-002	-	2.261e-001	-
4	9.838e-003	8.84	5.116e-002	4.42
8	7.861e-004	12.51	8.175e-003	6.26
16	5.322e-005	14.77	1.107e-003	7.38

The numerical results demonstrate our theoretical results.

Comments. The domain treated in the paper is a rectangular block in $\mathcal{R}^3$, which is also discussed by Goodsell [10]. Actually, as the Brandts and M. Křížek discussed [7], the results of the paper hold for a bounded polyhedral domain, which is usually presented in engineering problems.

4. Conclusions

In this paper, we generalized superconvergence results of the FEM from constant coefficient elliptic equations to variable coefficient settings in three dimensions. Applying the properties of interpolation operator of projection type, we obtained the weak estimates. Combined with the estimates for discrete Green's function and discrete derivative Green's function, superconvergence results were derived. Among the arguments, how to deal with the given variable coefficients is a challenging issue. The methods presented in the paper can also be applied to other high-dimensional second-order variable coefficient elliptic equations.

Author contributions

The first author proposed ideas of the study, and gave the theoretical analysis of Section 2 and Section 3. The second author mainly focused on the collection of literatures and the numerical example. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

This work was supported by the Special Projects in Key Fields of Guangdong Province Ordinary Universities (2022ZDZX3016), the Characteristic Innovation Project of Guangdong Province Ordinary University (2023KTSCX089), and the Projects of Talents Recruitment of Guangdong University of Petrochemical Technology (2021rc003).

Conflict of interest

The authors declare no conflicts of interest.

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