Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 28589–28610.
DOI:10.3934/math.20241387
Received: 24 August 2024
Revised: 15 September 2024
Accepted: 19 September 2024
Published: 09 October 2024

Research article

V-Moreau envelope of nonconvex functions on smooth Banach spaces

Messaoud Bounkhel*

Department of Mathematics, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: bounkhel@ksu.edu.sa; Tel: +96611467526; Fax: +96611467512.

Abstract: We continue the study of the properties of the V-Moreau envelope and generalized (f, λ)-projection that we started in [5]. In this paper, we study the differentiability and the regularity of the V-Moreau envelope and the Hölder continuity of the generalized (f, λ)-projection. Our results extend many existing results from the convex case to the nonconvex case and from Hilbert spaces to Banach spaces. Even on Hilbert spaces and for convex functions and sets, we derived new results.

Keywords: V-Moreau envelope; generalized (f, λ)-projection; p-uniformly convex Banach spaces; q-uniformly smooth Banach spaces; uniformly generalized prox-regular sets; V-prox-regular functions
Mathematics Subject Classification: 34A60, 49J53

1. Introduction and preliminaries

Let X be a Banach space with dual space X^*. The duality pairing between X and X^* will be denoted by $\langle \cdot, \cdot \rangle$. We denote by $\mathbb{B}$ and $\mathbb{B}_*$ the closed unit ball in X and X^*, respectively. The normalized duality mapping $J : X \rightrightarrows X^*$ is defined by

$$J(x) = \{j(x) \in X^* : \langle j(x), x\rangle = \|x\|^2 = \|j(x)\|^2\},$$

where $\|\cdot\|$ stands for both norms on X and X^*. Similarly, we define J^* on X^*. Many properties of J and J^* are well known and we refer the reader, for instance, to [15].

Definition 1.1. *For a fixed closed subset S of X, a fixed function $f : S \to \mathbb{R} \cup \{\infty\}$, and a fixed $\lambda > 0$, we define the following functional: $G^V_{\lambda,f} : X^* \times S \to \mathbb{R} \cup \{\infty\}$*

$$G^V_{\lambda,f}(x^*, x) = f(x) + \frac{1}{2\lambda} V(x^*, x), \quad \forall x^* \in X^*, x \in S,$$

where $V(x^*, x) = \|x^*\|^2 - 2\langle x^*, x\rangle + \|x\|^2$. Clearly, the functional V has the form $V(x^*; x) = \|x^* - x\|^2$, whenever X is a Hilbert space (i.e., $X^* = X$). This remark highlights the significance of using the functional V rather than the square of the norm, as the latter cannot generally be expressed in the form of V in Banach spaces.
Using the functional $G^V_{\lambda,f}$, we define the V-Moreau envelope of f associated with S as follows:

$$e^V_{\lambda,S} f(x^*) := \inf_{s\in S} G^V_{\lambda,f}(x^*, s)\}, \quad \text{for any } x^* \in X^*.$$

We also define the generalized (f, λ)-projection on S as follows:

$$\pi^{f,\lambda}_S(x^*) := \{x \in S : G^V_{\lambda,f}(x^*, x) = e^V_{\lambda,S} f(x^*)\}, \quad \text{for any } x^* \in X^*.$$

(•) When $f = 0$, $\lambda = \frac{1}{2}$, the generalized (f, λ)-projection $\pi^{f,\lambda}_S$ on S coincides with the generalized projection π_S over S.
(•) When $\lambda = \frac{1}{2}$, the generalized (f, λ)-projection $\pi^{f,\lambda}_S$ on S coincides with the f-generalized projection π^f_S introduced and studied in [16, 17].

We need some important results that we gather in the following proposition (see for instance [1, 5, 6]).

Proposition 1.1. *Let X be a Banach space.*

1) If X is q-uniformly convex, then for any $\alpha > 0$, there exists some constant $K_\alpha > 0$ such that

$$\langle Jx - Jy; x - y\rangle \geq K_\alpha \|x - y\|^q, \quad \forall x, y \in \alpha\mathbb{B}.$$

2) If X is p-uniformly smooth, then the dual space X^* is p'-uniformly convex with $p' = \frac{p}{p-1}$.
3) Assume that X is q-uniformly convex and let $\alpha > 0$. Then, for any $x^* \in \alpha\mathbb{B}_*$ and any $y \in \alpha\mathbb{B}$,

$$V(x^*; y) \geq \frac{2c}{4^{q-1}\alpha^{q-2}} \|J^*(x^*) - y\|^q,$$

where $c > 0$ is the constant given in the definition of q-uniform convexity of X.

We also recall many concepts and definitions as follows:

Definition 1.2.

1) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function (l.s.c. in short) and $x \in X$, where f is finite. The V-proximal subdifferential (see [8]) $\partial^\pi f$ of f at x is defined by $x^* \in \partial^\pi f(x)$ if and only if there exist $\sigma > 0, \delta > 0$ such that

$$\langle x^*, x' - x\rangle \leq f(x') - f(x) + \sigma V(J(x), x')), \forall x' \in x + \delta\mathbb{B}. \tag{1.1}$$

We notice that $\partial^\pi f(\bar{x}) \subset L\mathbb{B}_*$, whenever f is locally Lipschitz at $\bar{x}$ (see [4]).
2) The V-proximal normal cone of a nonempty closed subset S in X at $x \in S$ is defined as the V-proximal subdifferential of the indicator function of S, that is, $N^\pi(S; x) = \partial^\pi \psi_S(x)$. Note that N^π is also characterized (see [4]) via π_S as follows:

$$x^* \in N^\pi(S; \bar{x}) \Leftrightarrow \exists\alpha > 0, \text{ such that } \bar{x} \in \pi_S(J\bar{x} + \alpha x^*).$$

3) The Fréchet subdifferential and Fréchet normal cone (see for instance [3, 14]) are defined as follows: $x^* \in \partial^F f(\bar{x})$ if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\langle x^*; x - \bar{x}\rangle \leq f(x) - f(\bar{x}) + \epsilon\|x - \bar{x}\|, \quad \forall x \in \bar{x} + \delta\mathbb{B}. \tag{1.2}$$

The Fréchet normal cone $N^F(S; x)$ of a nonempty closed subset S in X at $\bar{x} \in S$ is defined as $N^F(S; \bar{x}) = \partial^F \psi_S(\bar{x})$.

4) The limiting V-proximal normal cone is defined as follows (see [7]):

$$N^{L\pi}(S; \bar{x}) = \{w - \lim_n x_n^* : x_n^* \in N^\pi(S; x_n) \text{ with } x_n \to^S \bar{x}\}.$$

Before starting our study, we state some special cases showing the importance of the study of $e^V_{\lambda,S} f$ and $\pi_S^{f,\lambda}$.

Case 1. If X is a Hilbert space and $S = X$, the functional $e^V_{\lambda,S} f$ coincides with the Moreau envelope of f with index $\lambda > 0$ and the generalized (f, λ)-projection $\pi_S^{f,\lambda}$ coincides with the proximal mapping $P_\lambda(f)$ (see for instance [13]).

Case 2. If X is a Hilbert space and $f \equiv 0$, the generalized (f, λ)-projection $\pi_S^{f,\lambda}$ coincides with the metric projection on S (see for instance [9, 10]).

Case 3. If X is a reflexive Banach space, the generalized (f, λ)-projection $\pi_S^{f,\lambda}$ coincides with the generalized projection π_S on S introduced for closed convex sets in [2,11,12] and for closed nonconvex sets in [4,6].

Motivated by the special cases presented above and their relevance (as seen in [1–4,11,12,16,17] and their references), we initially introduced and started investigating the generalized (f, λ)-projection $\pi_S^{f,\lambda}$ in [5]. There, we laid the groundwork for understanding its basic properties and potential applications. In this paper, we aim to expand upon that foundation by delving into more advanced properties of $\pi_S^{f,\lambda}$, particularly in relation to the differentiability of the functional $e^V_{\lambda,S} f$. This deeper analysis offers new perspectives on its theoretical framework and behavior. The application of these results to nonconvex variational inequalities will be addressed in a series of forthcoming papers.

2. Main results on the V-Moreau envelope $e^V_{\lambda,S} f$

In the following proposition, we prove the local Lipschitz behavior of the V-Moreau envelope $e^V_{\lambda,S} f$.

Proposition 2.1. *Let X be a reflexive Banach space. Assume that f is bounded below on S by $\beta \in \mathbb{R}$. Then, for any $x^* \in X^*$, the function $e^V_{\lambda,S} f$ is Lipschitz on every neighborhood of x^*, that is, for any $x^* \in X^*$ and for any $\delta > 0$, there exists $K_{x^*,\delta} > 0$ such that*

$$|e^V_{\lambda,S} f(y^*) - e^V_{\lambda,S} f(z^*)| \leq K_{x^*,\delta}\|y^* - z^*\|, \quad \forall y^*, z^* \in x^* + \delta\mathbb{B}_*.$$

Proof. Let $x^* \in X^*$ and fix some $\delta > 0$. Let $\epsilon \in (0, \delta)$ and $r := e^V_{\lambda,S} f(x^*) \geq \beta$. Fix now any $y^*, z^* \in x^* + \delta\mathbb{B}_*$. By definition of the infimum in the expression of $e^V_{\lambda,S} f$, there exists $s_\epsilon \in S$ such that

$$e^V_{\lambda,S} f(y^*) \leq f(s_\epsilon) + \frac{1}{2\lambda} V(y^*, s_\epsilon) < e^V_{\lambda,S} f(y^*) + \epsilon.$$

Then,

$$
\begin{aligned}
e^V_{\lambda,S} f(z^*) - e^V_{\lambda,S} f(y^*) &\leq e^V_{\lambda,S} f(z^*) - f(s_\epsilon) - \frac{1}{2\lambda} V(y^*, s_\epsilon) + \epsilon \\
&\leq f(s_\epsilon) + \frac{1}{2\lambda} V(z^*, s_\epsilon) - f(s_\epsilon) - \frac{1}{2\lambda} V(y^*, s_\epsilon) + \epsilon \\
&\leq \frac{1}{2\lambda} \left[V(z^*, s_\epsilon) - V(y^*, s_\epsilon) \right] + \epsilon \\
&\leq \frac{1}{2\lambda} \left[\|z^*\|^2 - \|y^*\|^2 - 2\langle z^* - y^*, s_\epsilon \rangle \right] + \epsilon \\
&\leq \frac{1}{2\lambda} (\|z^*\| + \|y^*\| + 2\|s_\epsilon\|) \|z^* - y^*\| + \epsilon.
\end{aligned}
$$

We need to find an upper bound of $\|s_\epsilon\|$. To do that, we use once again the definition of the infimum in $e^V_{\lambda,S} f(x^*)$ to get an element $x_\epsilon \in S$ such that

$$
f(x_\epsilon) + \frac{1}{2\lambda} V(x^*, x_\epsilon) < e^V_{\lambda,S} f(x^*) + \epsilon = r + \epsilon.
$$

From the definition of V, we have $V(y^*, s_\epsilon) = \|y^*\|^2 - 2\langle y^*, s_\epsilon \rangle + \|s_\epsilon\|^2$. This gives

$$
\begin{aligned}
V(y^*, s_\epsilon) &\geq \|y^*\|^2 - 2\|y^*\|\|s_\epsilon\| + \|s_\epsilon\|^2 \\
&\geq \left[\|s_\epsilon\| - \|y^*\| \right]^2.
\end{aligned}
$$

Hence,

$$
\|s_\epsilon\| - \|y^*\| \leq |\|s_\epsilon\| - \|y^*\|| \leq \sqrt{V(y^*, s_\epsilon)}.
$$

Then, we obtain:

$$
\begin{aligned}
\|s_\epsilon\| &\leq \sqrt{V(y^*, s_\epsilon)} + \|y^*\| \\
&\leq \sqrt{2\lambda e^V_{\lambda,S} f(y^*) - 2\lambda f(y_\epsilon) + 2\lambda\epsilon} + \|x^*\| + \delta \\
&\leq \sqrt{2\lambda \left[f(x_\epsilon) + \frac{1}{2\lambda} V(x^*, x_\epsilon) \right] - 2\lambda f(y_\epsilon) + 2\lambda\epsilon} + \|x^*\| + \delta \\
&\leq \sqrt{V(y^*, x_\epsilon) + 2\lambda f(x_\epsilon) - 2\lambda f(y_\epsilon) + 2\lambda\epsilon} + \|x^*\| + \delta \\
&\leq \sqrt{(\|y^*\| + \|x_\epsilon\|)^2 + 2\lambda f(x_\epsilon) - 2\lambda f(y_\epsilon) + 2\lambda\epsilon} + \|x^*\| + \delta.
\end{aligned}
$$

Observe that $f(x_\epsilon) < e^V_{\lambda,S} f(x^*) + \epsilon < r + \epsilon$ and $f(y_\epsilon) \geq \beta$, and so we get

$$
f(x_\epsilon) - f(y_\epsilon) < r + \epsilon - \beta.
$$

Therefore,

$$
\begin{aligned}
\|s_\epsilon\| &\le \sqrt{(\|x^*\| + \|x_\epsilon\| + \delta)^2 + 2\lambda(r + \epsilon - \beta) + 2\lambda\epsilon} + \|x^*\| + \delta \\
&\le \sqrt{\left(2\|x^*\| + \sqrt{V(x^*, x_\epsilon)} + \delta\right)^2 + 2\lambda(r + 2\epsilon - \beta)} + \|x^*\| + \delta \\
&\le \sqrt{\left(2\|x^*\| + \sqrt{2\lambda(r + \epsilon)} + \delta\right)^2 + \epsilon} + \|x^*\| + \delta \\
&\le \sqrt{\left(2\|x^*\| + \sqrt{2\lambda(r + \delta)} + \delta\right)^2 + \delta} + \|x^*\| + \delta.
\end{aligned}
$$

By taking $M_{\delta,x^*} := \sqrt{\left(2\|x^*\| + \sqrt{2\lambda(r + \delta)} + \delta\right)^2 + \delta} + \|x^*\| + \delta$, we get an upper bound of $\|s_\epsilon\|$ in terms of r, δ, and x^*. Thus, we can write

$$
\begin{aligned}
e^V_{\lambda,S} f(z^*) - e^V_{\lambda,S} f(y^*) &\le \frac{1}{2\lambda}\left(\|z^*\| + \|y^*\| + 2M_{\delta,x^*}\right)\|z^* - y^*\| + \epsilon \\
&\le \frac{1}{\lambda}\left(\|x^*\| + M_{\delta,x^*} + \delta\right)\|z^* - y^*\| + \epsilon \le K_{\delta,x^*}\|z^* - y^*\| + \epsilon,
\end{aligned}
$$

where $K_{\delta,x^*} := \frac{1}{\lambda}\left(\|x^*\| + M_{\delta,x^*} + \delta\right)$. Taking $\epsilon \to 0$ and interchanging the roles of z^* and y^*, we get

$$
|e^V_{\lambda,S} f(z^*) - e^V_{\lambda,S} f(y^*)| \le K_{\delta,x^*}\|z^* - y^*\|, \text{ for any } y^*, z^* \in x^* + \delta\mathbb{B}_*.
$$

This completes the proof. □

We recall from [5] the following result needed in the proof of the next theorem.

Proposition 2.2. *Assume that X is a reflexive Banach space with smooth dual norm, and let S be any closed nonempty set of X and $f : S \to \mathbb{R} \cup \{\infty\}$ be any l.s.c. function. Then for any $x^* \in dom\, \pi^{f,\lambda}_S$, any $\bar{x} \in \pi^{f,\lambda}_S(x^*)$, and any $t \in [0, 1)$, we have $\pi^{f,\lambda}_S(J(\bar{x}) + t(x^* - J(\bar{x}))) = \{\bar{x}\}$.*

We prove the following result ensuring the existence and uniqueness of the generalized (f, λ)-projection on closed nonempty sets under natural assumptions on the Fréchet subdifferentiability of the V-Moreau envelope.

Theorem 2.1. *Assume that X is a reflexive Banach space with smooth dual norm, and let S be any closed nonempty set of X and $f : S \to \mathbb{R} \cup \{\infty\}$ be any l.s.c. function. Then the following assertions hold.*

1) If $\partial^F e^V_{\lambda,S} f(x^*) \ne \emptyset$, then the generalized (f, λ)-projection of x^* on S exists and is unique and moreover $\partial^F e^V_{\lambda,S} f(x^*) = \{\frac{1}{\lambda}\left[J^* x^* - \pi^{f,\lambda}_S(x^*)\right]\}$;
2) If $\pi^{f,\lambda}_S(x^*) \ne \emptyset$, then $\partial^F e^V_{\lambda,S} f(x^*) \subset \{\frac{1}{\lambda}\left[J^* x^* - \pi^{f,\lambda}_S(x^*)\right]\}$;
3) $\partial^F e^V_{\lambda,S} f(x^*) \ne \emptyset$ if and only if $e^V_{\lambda,S} f$ is Fréchet differentiable at x^*.

Proof. **(1)** Assume that $\partial^F e^V_{\lambda,S} f(x^*) \neq \emptyset$ and let $y \in \partial^F e^V_{\lambda,S} f(x^*)$ and let $\epsilon > 0$. By the definition of $\partial^F e^V_{\lambda,S} f(x^*)$, there exists $\delta > 0$ such that for any $t \in (0, \delta)$ and any $v^* \in \mathbb{B}$, we have

$$\langle y; tv^* \rangle \leq e^V_{\lambda,S} f(x^* + tv^*) - e^V_{\lambda,S} f(x^*) + \epsilon t.$$

By the definition of $e^V_{\lambda,S} f(x^*)$, for any $n \geq 1$, the exists some $y_n \in S$ such that

$$e^V_{\lambda,S} f(x^*) \leq f(y_n) + \frac{1}{2\lambda} V(x^*; y_n) < e^V_{\lambda,S} f(x^*) + \frac{t}{n}. \tag{2.1}$$

Therefore,

$$\begin{aligned} \langle y; tv^* \rangle &\leq f(y_n) + \frac{1}{2\lambda} V(x^* + tv^*; y_n) - f(y_n) - \frac{1}{2\lambda} V(x^*; y_n) + \frac{t}{n} + \epsilon t \\ &\leq \frac{1}{2\lambda} \left[V(x^* + tv^*; y_n) - V(x^*; y_n) \right] + \frac{t}{n} + \epsilon t \\ &\leq \frac{1}{2\lambda} \left[\|x^* + tv^*\|^2 - \|x^*\|^2 - 2\langle tv^*; ; y_n \rangle \right] + \frac{t}{n} + \epsilon t. \end{aligned}$$

Thus,

$$\langle y + \frac{1}{\lambda} y_n; v^* \rangle \leq \frac{1}{2\lambda} \left[\frac{\|x^* + tv^*\|^2 - \|x^*\|^2}{t} \right] + \frac{1}{n} + \epsilon.$$

Since the norm of the dual space is smooth, we can take the limit $t \to 0^+$ to get

$$\langle y + \frac{1}{\lambda} y_n; v^* \rangle \leq \frac{1}{\lambda} \langle J^* x^*; v^* \rangle + \frac{1}{n} + \epsilon,$$

and hence,

$$\langle y + \frac{1}{\lambda} [y_n - J^* x^*]; v^* \rangle \leq \frac{1}{n} + \epsilon, \quad \forall v^* \in \mathbb{B}_*, \forall \epsilon > 0, \forall n \geq 1.$$

This ensures that $\lim_{n\to\infty} \|y + \frac{1}{\lambda}[y_n - J^* x^*]\| = 0$, that is, $y_n \to J^* x^* - \lambda y$ as $n \to \infty$. Set $\tilde{x} := J^* x^* - \lambda y$, and take the limit as $n \to \infty$ in the inequality (2.1), we obtain:

$$e^V_{\lambda,S} f(x^*) = f(\tilde{y}) + \frac{1}{2\lambda} V(x^*; \tilde{y}),$$

which means that $\tilde{y} \in \pi^{f,\lambda}_S(x^*)$. The uniqueness can be shown easily and so the first assertion is proved.

(2) This assertion follows directly from (1). Indeed, if $\partial^F e^V_{\lambda,S} f(x^*) = \emptyset$, then we are done. Otherwise, we assume that $\partial^F e^V_{\lambda,S} f(x^*) \neq \emptyset$, and so the assertion (1) ensures that $\partial^F e^V_{\lambda,S} f(x^*) = \{\frac{1}{\lambda}\left[J^* x^* - \pi^{f,\lambda}_S(x^*)\right]\}$. Consequently, for both cases, we have $\partial^F e^V_{\lambda,S} f(x^*) \subset \{\frac{1}{\lambda}\left[J^* x^* - \pi^{f,\lambda}_S(x^*)\right]\}$, and so the proof of (2) is complete.

(3) Obviously, the Fréchet differentiability of $e^V_{\lambda,S} f$ ensures that $\partial^F e^V_{\lambda,S} f(x^*) \neq \emptyset$. So, we have to prove the reverse implication. We assume that $\partial^F e^V_{\lambda,S} f(x^*) \neq \emptyset$, and we are going to prove that $e^V_{\lambda,S} f$ is

Fréchet differentiable at x^*. Using the assertion (1), we get a generalized (f,λ)-projection $\bar{y} \in S$, such that

$$\partial^F e^V_{\lambda,S} f(x^*) = \{\frac{1}{\lambda}\,[J^*x^* - \bar{y}]\}.$$

Thus, we have $e^V_{\lambda,S} f(x^*) = f(\bar{y}) + \frac{1}{2\lambda} V(x^*;\bar{y})$. By the definition of the Fréchet subdifferential, there exists $\delta > 0$ such that for any $t \in (0,\delta)$ and any $v^* \in \mathbb{B}_*$, we have

$$\langle \frac{1}{\lambda}\,[J^*x^* - \bar{y}]\,; tv^*\rangle \leq e^V_{\lambda,S} f(x^* + tv^*) - e^V_{\lambda,S} f(x^*) + \epsilon t.$$

Hence,

$$t^{-1}\left[e^V_{\lambda,S} f(x^* + tv^*) - e^V_{\lambda,S} f(x^*)\right] - \langle \frac{1}{\lambda}\,[J^*x^* - \bar{y}]\,; v^*\rangle \geq -\epsilon,$$

and hence, for any $v^* \in \mathbb{B}_*$ and any $\epsilon > 0$,

$$\liminf_{t\to 0^+} t^{-1}\left[e^V_{\lambda,S} f(x^* + tv^*) - e^V_{\lambda,S} f(x^*)\right] \geq \langle \frac{1}{\lambda}\,[J^*x^* - \bar{y}]\,; v^*\rangle - \epsilon. \tag{2.2}$$

On the other hand, we have, by the definition of $e^V_{\lambda,S} f$,

$$\begin{aligned} t^{-1}\left[e^V_{\lambda,S} f(x^* + tv^*) - e^V_{\lambda,S} f(x^*)\right] &\leq t^{-1}\left[\frac{1}{2\lambda} V(x^* + tv^*;\bar{y}) - \frac{1}{2\lambda} V(x^*;\bar{y})\right] \\ &\leq \frac{1}{2\lambda} t^{-1}\left[V(x^* + tv^*;\bar{y}) - V(x^*;\bar{y})\right], \end{aligned}$$

and so,

$$\limsup_{t\to 0^+} t^{-1}\left[e^V_{\lambda,S} f(x^* + tv^*) - e^V_{\lambda,S} f(x^*)\right] \leq \frac{1}{2\lambda}\left[2\langle J^*x^* - \bar{y}; v^*\rangle\right]. \tag{2.3}$$

Combining this inequality (2.3) with (2.2), and taking $\epsilon \to 0^+$, we obtain the existence of the Fréchet derivative of $e^V_{\lambda,S} f$ at x^* and $\nabla^F e^V_{\lambda,S} f(x^*) = \frac{1}{\lambda}\,[J^*x^* - \bar{y}]$. □

We prove in the next two theorems various characterizations of the continuous Fréchet differentiability of the V-Moreau envelope $e^V_{\lambda,S} f$ over open sets. We need to recall the Kadec property of the Banach space X, that is, for any sequence $(x_n)_n$ in X, we have that (x_n) is strongly convergent to some limit $\bar{x}$ if and only if (x_n) is weakly convergent to $\bar{x}$ and $\|x_n\| \to \|\bar{x}\|$.

Theorem 2.2. *Assume that X is a reflexive Banach space with Kadec property and with smooth dual norm. Let U be an open subset in X^*. Consider the following assertions:*

1) $e^V_{\lambda,S} f$ is C^1 on U;
2) $e^V_{\lambda,S} f$ is Fréchet differentiable on U;
3) $e^V_{\lambda,S} f$ is Fréchet subdifferentiable on U, that is, $\partial^F e^V_{\lambda,S} f(x^*) \neq \emptyset, \forall x^* \in U$;
4) $\pi^{f,\lambda}_S$ is single-valued and norm-to-weak continuous on U;
5) $\pi^{f,\lambda}_S$ is single-valued and norm-to-norm continuous on U.

Then, the following implications and equivalences are true.

$$\begin{array}{ccccc} (1) & \Rightarrow & (2) & \Rightarrow & (4) \\ & & \Updownarrow & & \Uparrow \\ & & (3) & & (5) \end{array}$$

Proof. The implications (1) ⇒ (2) and (5) ⇒ (4) follow directly from the definitions. The equivalence (2) ⇔ (3) follows from part (3) in Theorem 2.1. We have to prove the implication (2) ⇒ (4). We assume that $e^V_{\lambda,S} f$ is Fréchet differentiable on U, and let x^*_n be a sequence in U converging to some point $x^* \in X^*$. First, we prove that $\nabla^F e^V_{\lambda,S} f(x^*_n)$ weakly converges to $\nabla^F e^V_{\lambda,S} f(x^*)$. Observe that

$$\begin{aligned} e^V_{\lambda,S} f(x^*) &= \inf_{y\in X}\left\{-\frac{1}{\lambda}\langle x^*; y\rangle + f(y) + \frac{1}{2\lambda}\left[\|x^*\|^2 + \|y\|^2\right] + \psi_S(y)\right\} \\ &= \frac{\|x^*\|^2}{2\lambda} - h(x^*), \end{aligned}$$

with $h(x^*) := \sup_{y\in X}\left\{\frac{1}{\lambda}\langle x^*; y\rangle - f(y) - \frac{\|y\|^2}{2\lambda} - \psi_S(y)\right\}$. The function h is clearly convex Fréchet differentiable on U and so its derivative $\nabla^F h$ is norm-to-weak continuous on U, and so $\nabla^F h(x^*_n)$ weakly converges to $\nabla^F h(x^*)$. Since the norm of the dual space X^* is smooth, we have $\nabla^F \frac{\|x^*_n\|^2}{2\lambda} \to \nabla^F \frac{\|x^*\|^2}{2\lambda}$ and consequently, we get that $\nabla^F e^V_{\lambda,S} f(x^*_n) = \nabla^F \frac{\|x^*_n\|^2}{2\lambda} - \nabla^F h(x^*_n)$ weakly converges to $\nabla^F \frac{\|x^*\|^2}{2\lambda} - \nabla^F h(x^*) = \nabla^F e^V_{\lambda,S} f(x^*)$. We use Theorem 2.1 to write $\nabla^F e^V_{\lambda,S} f(x^*_n) = \frac{1}{\lambda}\left[J^* x^*_n - \pi^{f,\lambda}_S(x^*_n)\right]$ and $\nabla^F e^V_{\lambda,S} f(x^*) = \frac{1}{\lambda}\left[J^* x^* - \pi^{f,\lambda}_S(x^*)\right]$. Therefore, we obtain the weak convergence of $\pi^{f,\lambda}_S(x^*_n)$ to $\pi^{f,\lambda}_S(x^*)$, thereby satisfying the assertion (4). □

This result extends Theorem 2.10 in [6] from the case $f \equiv 0$ to $f \not\equiv 0$.

In order to get the equivalence between all the assertions in Theorem 2.2, we need an additional assumption on the function f, which is the weak lower semicontinuity of f.

Theorem 2.3. *Assume that X is a reflexive Banach space with Kadec property and with smooth dual norm. Assume further that f is weak lower semicontinuous on U. Then, all the assertions in Theorem 2.2 are equivalent, that is,*

$$\begin{array}{c} e^V_{\lambda,S} f \text{ is } C^1 \text{ on } U; \\ \Updownarrow \\ e^V_{\lambda,S} f \text{ is Fréchet differentiable on } U; \\ \Updownarrow \\ e^V_{\lambda,S} f \text{ is Fréchet subdifferentiable on } U; \\ \Updownarrow \\ e^V_{\lambda,S} f \text{ is single-valued and norm-to-norm continuous on } U; \\ \Updownarrow \\ e^V_{\lambda,S} f \text{ is single-valued and norm-to-weak continuous on } U. \end{array}$$

Proof. We have to prove the implications (2) ⇒ (1) and (4) ⇒ (5). We start with (4) ⇒ (5). Assume that $\pi^{f,\lambda}_S$ is single-valued and norm-to-weak continuous on U. Let x^*_n be a sequence in U converging to some point $x^* \in X^*$. We have to prove that $x_n := \pi_S^{f,\lambda}(x_n^*)$ converges to $\bar{x} := \pi_S^{f,\lambda}(x^*)$. By assumption (4), we have (x_n) weakly converges to $\bar{x}$. Using the local Lipschitz continuity of $e_{\lambda,S}^V f$ proved in Proposition 2.1, we can write

$$G_{\lambda,f}^V(x_n^*, x_n) = e_{\lambda,S}^V f(x_n^*) \quad \to \quad e_{\lambda,S}^V f(x^*) = G_{\lambda,f}^V(x^*, \bar{x}).$$

Observe that

$$\frac{1}{2\lambda}\|x_n\|^2 \quad = \quad G_{\lambda,f}^V(x_n^*, x_n) - f(x_n) - \frac{1}{2\lambda}\|x_n^*\|^2 + \frac{1}{2\lambda}\langle x_n^*; x_n\rangle.$$

Taking the limit superior as $n \to +\infty$ in this equality and using the weak l.s.c. of f, we get

$$\begin{aligned}\frac{1}{2\lambda}\limsup_{n\to+\infty}\|x_n\|^2 &= \limsup_{n\to+\infty}\left[G_{\lambda,f}^V(x_n^*, x_n) - f(x_n) - \frac{1}{2\lambda}\|x_n^*\|^2 + \frac{1}{2\lambda}\langle x_n^*; x_n\rangle\right] \\ &\le G_{\lambda,f}^V(x^*, \bar{x}) + \limsup_{n\to+\infty}[-f(x_n)] - \frac{1}{2\lambda}\|x^*\|^2 + \frac{1}{2\lambda}\langle x^*; \bar{x}\rangle \\ &\le G_{\lambda,f}^V(x^*, \bar{x}) - f(\bar{x}) - \frac{1}{2\lambda}\|x^*\|^2 + \frac{1}{2\lambda}\langle x^*; \bar{x}\rangle = \frac{1}{2\lambda}\|\bar{x}\|^2.\end{aligned}$$

On the other hand, we always have $\|\bar{x}\| \le \liminf_{n\to+\infty}\|x_n\|$. Thus, we obtain $\lim_{n\to+\infty}\|x_n\| = \|\bar{x}\|$. Finally, we use the fact that x_n weakly converges to $\bar{x}$ and $\|x_n\|$ converges to $\|\bar{x}\|$, and the Kadec property of the space to deduce the convergence of x_n to $\bar{x}$, and so the proof of (5) is complete. We turn to prove the implication (2) $\Rightarrow$ (1). We assume that $e_{\lambda,S}^V f$ is Fréchet differentiable on U. We have to prove that $\nabla^F e_{\lambda,S}^V f$ is continuous on U. Let x_n^* be a sequence in U converging to some point $x^* \in X^*$, and we have to prove that $\nabla^F e_{\lambda,S}^V f(x_n^*) \to \nabla^F e_{\lambda,S}^V f(x^*)$. Using Theorem 2.1, we have the existence and uniqueness of $\pi_S^{f,\lambda}(x_n^*)$ and

$$\nabla^F e_{\lambda,S}^V f(x_n^*) = \frac{1}{\lambda}\left[J^* x_n^* - \pi_S^{f,\lambda}(x_n^*)\right].$$

Similarly, we have

$$\nabla^F e_{\lambda,S}^V f(x^*) = \frac{1}{\lambda}\left[J^* x^* - \pi_S^{f,\lambda}(x^*)\right].$$

Using the implications (2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5), we get the convergence of $\pi_S^{f,\lambda}(x_n^*)$ to $\pi_S^{f,\lambda}(x^*)$. Consequently, we use the continuity of J^* to deduce the following:

$$\nabla^F e_{\lambda,S}^V f(x_n^*) = \frac{1}{\lambda}\left[J^* x_n^* - \pi_S^{f,\lambda}(x_n^*)\right] \to \frac{1}{\lambda}\left[J^* x^* - \pi_S^{f,\lambda}(x^*)\right] = \nabla^F e_{\lambda,S}^V f(x^*),$$

and so the proof of the theorem is complete.

3. Main results on the generalized (f, λ)-projection $\pi_S^{f,\lambda}$

In this section, we need more regularity assumptions on the function f and the set S to establish our main results on the generalized (f, λ)-projection. First, we start with the generalized uniform V-prox-regularity concept introduced and studied in [6].

Definition 3.1. *A nonempty closed subset S, in a reflexive smooth Banach space X, is called V-uniformly generalized prox-regular if and only if there exists $r > 0$ such that $\forall x \in S, \forall x^* \in N^{\pi}(S; x)$ (with $x^* \neq 0$), we have $x \in \pi_S \left(J(x) + r\frac{x^*}{\|x^*\|} \right)$.*

Example 3.1.

1) Any closed convex set is generalized uniformly V-prox-regular with any positive number $r > 0$.
2) We state from [6] the following nonconvex example of generalized uniformly V-prox-regular sets. Let $x_0 \in X$ with $\|x_0\| > 3$. The set $S := \mathbb{B} \cup (x_0 + \mathbb{B})$ is nonconvex but it is generalized uniformly V-prox-regular for some $r > 0$ (for its proof, we refer to Example 4.1 in [6]).

Remark 3.1.

1) It has been proved in Theorem 3.2 in [7] that for generalized uniformly V-prox-regular sets S, we have $N^{\pi}(S; x) = N^{L\pi}(S; x)$, $\forall x \in S$.
2) From Theorem 3.3 in [7], we deduce that for bounded generalized uniformly V-prox-regular sets S, there exists some $r > 0$ such that for all $x \in S$ and any $x^* \in N^{\pi}(S; x)$ with $\|x^*\| < 1$, we have

$$\langle x^*; y - x \rangle \leq \frac{1}{2r} V(Jx; y), \quad \forall y \in S. \tag{3.1}$$

Now, we state the concept of *V-prox-regular functions uniformly* over sets.

Definition 3.2. *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a l.s.c. function, and let $S \subset \mathrm{dom} f$ be a nonempty set. We say that f is V-prox-regular uniformly over S provided that there exists some $r > 0$ such that for any $x \in S$ and any $x^* \in \partial^{L\pi} f(x)$:*

$$\langle x^*; x' - x \rangle \leq f(x') - f(x) + \frac{1}{2r} V(J(x); x'), \quad \forall x' \in S. \tag{3.2}$$

Example 3.2.

1) Any l.s.c. convex function f is V-prox-regular with any positive number $r > 0$ uniformly over any closed subset $S \subset \mathrm{dom} f$.
2) The distance function d_S associated with generalized uniformly V-prox-regular set S (in the sense of Definition 3.1) is V-prox-regular uniformly over S with the same positive number $r > 0$. Indeed, for any $x \in S$ and any $x^* \in \partial^{L\pi} d_S(x)$, we have $x^* \in N^{L\pi}(S; x) = N^{\pi}(S; x)$ and $\|x^*\| \leq 1$. We set $y^* := \frac{x^*}{\|x^*\| + \epsilon}$ for $\epsilon > 0$. We have $y^* \in N^{\pi}(S; x)$ with $\|y^*\| < 1$. Then, by (3.1), we have

$$\langle \frac{x^*}{\|x^*\| + \epsilon}; y - x \rangle = \langle y^*; y - x \rangle \leq \frac{1}{2r} V(Jx; y), \quad \forall y \in S. \tag{3.3}$$

Thus,

$$\begin{aligned} \langle x^*; y - x \rangle &\leq \frac{\|x^*\| + \epsilon}{2r} V(Jx; y) \\ &\leq d_S(y) - d_S(x) + \frac{1 + \epsilon}{2r} V(Jx; y), \quad \forall y \in S. \end{aligned} \tag{3.4}$$

Taking $\epsilon \to 0^+$ gives

$$\langle x^*; y - x\rangle \quad \leq \quad d_S(y) - d_S(x) + \frac{1}{2r}V(Jx; y), \quad \forall y \in S. \tag{3.5}$$

This ensures by definition that d_S is a V-prox-regular function uniformly over S with the same constant $r > 0$.

We start by proving the following important result for this class of V-prox-regular functions uniformly over sets. It proves the r-hypomonotony of $\partial^{L\pi} f$ uniformly over sets for V-prox-regular functions f uniformly over closed sets.

Proposition 3.1. *Assume that f is V-prox-regular uniformly over $S \subset dom\, f$ with constant $r > 0$. Then for any $x_1, x_2 \in S$ and any $y_1^* \in \partial^{L\pi} f(x_1)$ and $y_2^* \in \partial^{L\pi} f(x_2)$, we have*

$$\langle y_2^* - y_1^*; x_2 - x_1\rangle \geq -\frac{1}{r}\langle J(x_2) - J(x_1); x_2 - x_1\rangle.$$

Proof. Let $x_1, x_2 \in S \subset dom\, f$ and $y_1^* \in \partial^{L\pi} f(x_1)$ and $y_2^* \in \partial^{L\pi} f(x_2)$. Then, we have, by the V-prox-regularity of f uniformly over S,

$$\langle y_1^*; x_2 - x_1\rangle \leq f(x_2) - f(x_1) + \frac{1}{2r}V(J(x_1); x_2),$$

and

$$\langle y_2^*; x_1 - x_2\rangle \leq f(x_1) - f(x_2) + \frac{1}{2r}V(J(x_2); x_1).$$

Adding these two inequalities yields

$$\langle y_1^* - y_2^*; x_2 - x_1\rangle \leq \frac{1}{2r}[V(J(x_2); x_1) + V(J(x_1); x_2)].$$

Notice that we always have

$$V(J(x_2); x_1) + V(J(x_1); x_2) = 2\langle J(x_2) - J(x_1); x_2 - x_1\rangle.$$

Thus,

$$\langle y_1^* - y_2^*; x_2 - x_1\rangle \leq \frac{1}{r}\langle J(x_2) - J(x_1); x_2 - x_1\rangle.$$

This completes the proof. □

Lemma 3.1. *Let S be any closed nonempty set in a reflexive Banach space X, and let $f : S \to \mathbb{R} \cup \{\infty\}$ be any l.s.c. function. Then for any (x^*, x) in the graph of $\pi_S^{f,\lambda}$, we have*

$$x^* \in J(x) + \lambda\partial^{L\pi} f(x) + N^{L\pi}(S; x).$$

Proof. Let $x^* \in X^*$ and $x \in \pi_S^{f,\lambda}(x^*)$. Then, by the definition of $\pi_S^{f,\lambda}$, we have

$$f(x) + \frac{1}{2\lambda}V(x^*, x) \leq f(y) + \frac{1}{2\lambda}V(x^*, y), \quad \forall y \in S.$$

Hence,

$$\frac{1}{2\lambda}\left[2\langle x^*; y-x\rangle + \|x\|^2 - \|y\|^2\right] \leq f(y) - f(x), \quad \forall y \in S. \tag{3.6}$$

Observe that

$$\begin{aligned} V(J(x), y) &= \|x\|^2 - 2\langle J(x); y\rangle + \|y\|^2 \\ &= \|y\|^2 - \|x\|^2 + 2\langle J(x); x\rangle - 2\langle J(x); y\rangle \\ &= \|y\|^2 - \|x\|^2 + 2\langle J(x); x-y\rangle. \end{aligned}$$

Hence,

$$\|x\|^2 - \|y\|^2 = -2\langle J(x); y-x\rangle - V(J(x), y),$$

and so the inequality (3.6) becomes

$$\frac{1}{2\lambda}\left[2\langle x^* - J(x); y-x\rangle - V(J(x), y)\right] \leq f(y) - f(x), \quad \forall y \in S.$$

Thus,

$$\frac{1}{\lambda}\langle x^* - J(x); y-x\rangle \leq f(y) - f(x) + \frac{1}{2\lambda}V(J(x), y), \quad \forall y \in S,$$

and so,

$$\langle \frac{1}{\lambda}[x^* - J(x)]; y-x\rangle \leq [f + \psi_S](y) - [f + \psi_S](x) + \frac{1}{2\lambda}V(J(x), y), \quad \forall y \in X.$$

This ensures, by the definition of ∂^π, that

$$\begin{aligned} \frac{1}{\lambda}[x^* - J(x)] &\in \partial^\pi[f + \psi_S](x) \subset \partial^{L\pi}[f + \psi_S](x) \\ &\subset \partial^{L\pi} f(x) + \partial^{L\pi}\psi_S(x) \subset \partial^{L\pi} f(x) + N^{L\pi}(S; x), \end{aligned}$$

and hence, $x^* \in J(x) + \lambda\partial^{L\pi} f(x) + N^{L\pi}(S; x)$. This completes the proof. □

We recall from [4] the following density theorem for the generalized (f, λ)-projection on closed nonempty sets.

Theorem 3.2. *Assume that X is a reflexive Banach space with smooth dual norm and let S be any closed nonempty set of X, and let $f : S \to \mathbb{R} \cup \{\infty\}$ be any l.s.c. function. Then, the set of points in X^* admitting unique generalized (f, λ)-projection on S is dense in X^*, that is, for any $x^* \in X^*$, there exists $x_n^* \to x^*$ with $\pi_S^{f,\lambda}(x_n^*) \neq \emptyset, \forall n$.*

Now, we are ready to prove one of the main results in this paper. We define the argmin of a function f over a given set S as the set of elements in S that achieve the global minimum of f in S, that is,

$$\arg\min_S(f) := \{x \in S : f(x) = \min_{s\in S} f(s)\}.$$

Also, we define the set:

$$U_{S,f}^{V,\lambda}(r) := \{x^* \in X^* : e_{\lambda,S}^V f(x^*) \leq r^2\}.$$

Notice that for any $x \in \arg\min_S(f)$, we have $e^V_{\lambda,S} f(J(x)) = f(x)$. Indeed, for any $x \in \arg\min_S(f)$, we have $f(x) \le f(y), \forall y \in S$, and so $\forall \lambda \ge 0$

$$f(x) = f(x) + \frac{1}{2\lambda} V(J(x); x) \le f(y) + \frac{1}{2\lambda} V(J(x); y),\ \forall y \in S.$$

This ensures that $f(x) \le e^V_{\lambda,S} f(J(x))$. Since the reverse inequality is always valid, we obtain the desired equality. We state and prove the Hölder continuity of the generalized (f, λ)-projection $\pi^{f,\lambda}_S$.

Theorem 3.3. *Let X be a q-uniformly convex and p-uniformly smooth Banach space. Assume that the following assumptions hold:*

1) S is generalized uniformly V-prox-regular with constant $r_2 > 0$;
2) f is V-prox-regular uniformly over S with constant $r_1 > 0$;
3) f is L-locally Lipschitz over S, that is, for any $\bar{x} \in S$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \le L\|x - y\|, \quad \forall x, y \in \bar{x} + \delta\mathbb{B};$$

4) f is bounded from below by some real number $\beta \in \mathbb{R}$;
5) $\arg\min_S(f) \ne \emptyset$;
6) $\lambda \in \left(0, \min\left\{\frac{r_2}{8L}, \frac{r_1}{2}\right\}\right)$.

Then, there exist $\alpha_0 \ge 0$ and $r_0 \ge 0$ such that for any $\alpha > \max\{\alpha_0, \sqrt{2\lambda(r_2^2 - \beta)}\}$, $\beta \le \frac{16c\alpha^2}{\lambda}\left(\frac{r_2}{64\alpha}\right)^{\frac{p}{p-1}}$, and any $r' \in \left(r_0, \min\left\{r_2, \sqrt{\frac{16c\alpha^2}{\lambda}\left(\frac{r_2}{64\alpha}\right)^{\frac{p}{p-1}} + \beta}\right\}\right)$, we have that the generalized (f, λ)-projection $\pi^{f,\lambda}_S$ is single-valued and Hölder continuous with coefficient $\frac{1}{q-1}$ on $U^{V,\lambda}_{S,f}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*)$, i.e., for some $\gamma > 0$, we have

$$\|\pi^{f,\lambda}_S(x^*_1) - \pi^{f,\lambda}_S(x^*_2)\| \le \gamma\|x^*_1 - x^*_2\|^{\frac{1}{q-1}}, \quad \forall x^*_1, x^*_2 \in U^{V,\lambda}_{S,f}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*). \tag{3.7}$$

Proof. First, we choose some $\alpha_0 \ge 0$ and some $r_0 \ge 0$ so that

$$U^{V,\lambda}_{S,f}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*) \ne \emptyset, \quad \forall \alpha > \alpha_0, \forall r \ge r_0.$$

Indeed, by assumption, we have $\arg\min_S(f) \ne \emptyset$, that is, there exists $z_0 \in \arg\min_S(f)$. Set $\alpha_0 := \|z_0\|$ and $r_0 := \sqrt{f(z_0)}$, if $f(z_0) > 0$, and $r_0 := 0$, if $f(z_0) \le 0$. Clearly, $J(z_0) \in U^{V,\lambda}_{S,f}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*)$, and so $U^{V,\lambda}_{S,f}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*) \ne \emptyset$, $\forall \alpha > \alpha_0, \forall r \ge r_0$.

Fix now any $\alpha > \max\{\alpha_0, \sqrt{2\lambda(r_2^2 + \beta)}\}$, $\beta \le \frac{16c\alpha^2}{\lambda}\left(\frac{r_2}{64\alpha}\right)^{\frac{p}{p-1}}$, and any $r' \in \left(r_0, \min\left\{r_2, \sqrt{\frac{16c\alpha^2}{\lambda}\left(\frac{r_2}{64\alpha}\right)^{\frac{p}{p-1}} - \beta}\right\}\right)$. Then, $U^{V,\lambda}_{S,f}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*) \ne \emptyset$. We divide our proof into two steps.

Step 1. In the first step, we prove the conclusion of the theorem for any $x^*_1, x^*_2 \in U^{V,\lambda}_{S,f}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*)$ with $\pi^{f,\lambda}_S(x^*_i) \ne \emptyset$, $i = 1, 2$, that is, $x^*_1, x^*_2 \in U^{V,\lambda}_{S,f}(r') \cap dom\,\pi^{f,\lambda}_S \cap \alpha\,\mathrm{int}(\mathbb{B}_*)$. Fix any two points $x^*_1, x^*_2 \in$

$U^{V,\lambda}_{S,f}(r') \cap dom\,\pi^{f,\lambda}_S \cap \alpha\,\text{int}(\mathbb{B}_*)$. Then, there exist $x_i \in \pi^{f,\lambda}_S(x_i^*)$, $i = 1, 2$. Without loss of generality, we assume that $x_1 \neq x_2$. We have to prove that for some $\gamma > 0$,

$$\|x_1 - x_2\| \leq \gamma \|x_1^* - x_2^*\|^{\frac{1}{q-1}}. \tag{3.8}$$

By Lemma 3.1, there exist $y_i^* \in \partial^{L\pi} f(x_i)$ $(i = 1, 2)$ such that $z_i^* := x_i^* - J(x_i) - \lambda y_i^* \in N^{L\pi}(S; x_i) = N^{\pi}(S; x_i)$, $i = 1, 2$ (by Part (1) in Remark 3.1). So, by the generalized uniform V-prox-regularity of S with ratio r_2, we have

$$\{x_i\} = \pi_S\left(Jx_i + r_2 \frac{z_i^*}{\|z_i^*\|}\right), \quad \text{for } i = 1, 2.$$

Then by the definition of π_S, we have

$$V(Jx_i + r_2 \frac{z_i^*}{\|z_i^*\|}, x_i) \leq V(Jx_i + r_2 \frac{z_i^*}{\|z_i^*\|}, z), \quad \forall z \in S,$$

and so,

$$V(w_i^*, x_i) - V(w_i^*, z) \leq 0, \quad \forall z \in S,$$

with $w_i^* := Jx_i + r_2 \frac{z_i^*}{\|z_i^*\|}$, $i = 1, 2$. Since the function $V(w_i^*; \cdot)$ is convex differentiable on X and its derivative is given by $\nabla^F V(w_i^*; \cdot)(z) = 2\left[Jz - w_i^*\right]$, then, we can write

$$2\langle Jz - w_i^*; y - z\rangle \leq V(w_i^*; y) - V(w_i^*; z), \quad \forall y, z \in X,\ i = 1, 2.$$

Taking $y = x_i$ and $z \in S$ in the previous inequality yields

$$2\langle Jz - w_i^*; x_i - z\rangle \leq V(w_i^*; x_i) - V(w_i^*; z) \leq 0.$$

Hence,

$$\langle Jz - Jx_i - r_2 \frac{z_i^*}{\|z_i^*\|}; x_i - z\rangle \leq 0, \quad \text{for } i = 1, 2, \quad \forall z \in S$$

and hence,

$$\langle Jz - Jx_i; z - x_i\rangle \geq r_2 \langle \frac{z_i^*}{\|z_i^*\|}; z - x_i\rangle \quad \text{for } i = 1, 2, \quad \forall z \in S.$$

Thus, by taking $z = x_2$ and $z = x_1$, respectively, we obtain:

$$\frac{\|z_1^*\|}{r_2} \langle Jx_2 - Jx_1; x_2 - x_1\rangle \geq \langle z_1^*; x_2 - x_1\rangle, \tag{3.9}$$

and

$$\frac{\|z_2^*\|}{r_2} \langle Jx_1 - Jx_2; x_1 - x_2\rangle \geq \langle z_2^*; x_1 - x_2\rangle. \tag{3.10}$$

Now, we turn to the bound of z_i^*, for $i = 1, 2$. First, observe that $\|x_i^*\| < \alpha$. Since f is locally Lipschitz over S with constant L, we have $\|y_i^*\| \leq L$. Also, we have for $i = 1, 2$,

$$\|x_i\| \quad \leq \quad \|x_i^*\| + \sqrt{V(x_i^*, x_i)}$$

$$\leq \alpha + \sqrt{2\lambda[e^V_{\lambda,S}f(x_i^*) - f(x_i)]}$$

$$\leq \alpha + \sqrt{2\lambda(r'^2 + \beta)}$$

$$\leq \alpha + \sqrt{2\lambda(r_2^2 + \beta)} < 2\alpha.$$

Let $M := 2\alpha$. Since X is p-uniformly smooth, the dual space X^* is p'-uniformly convex with $p' = \frac{p}{p-1}$. Thus, by Part (3) in Proposition 1.1, there exists some $c > 0$ depending on the dual space X^* such that

$$V(x^*; J^*(y^*)) \geq 8C^2 c \frac{\|x^* - y^*\|^{p'}}{(4C)^{p'}}, \quad \forall x^*, y^* \in M\mathbb{B}_*,$$

where $C := \sqrt{\frac{\|x^*\|^2+\|y^*\|^2}{2}}$. Set $\bar{c} := \frac{4^{p'-1}M^{p'-2}}{c}$. Since for $i = 1, 2$, $\|J(x_i)\| = \|x_i\| \leq M$ and $\|x_i^*\| \leq M$, we have $C = \sqrt{\frac{\|x_i^*\|^2+\|J(x_i)\|^2}{2}} \leq M$. Thus,

$$\begin{aligned}\|x_i^* - J(x_i)\|^{p'} &\leq \frac{4^{p'-1}C^{p'-2}}{2c} V(x_i^*; x_i) \leq \frac{\bar{c}}{2} V(x_i^*; x_i)\\ &\leq \bar{c}\lambda[e^V_{\lambda,S}f(x_i^*) - f(x_i)]\\ &\leq \bar{c}\lambda[r'^2 - \beta].\end{aligned}$$

Thus, for $i = 1, 2$,

$$\begin{aligned}\|z_i^*\| &= \|x_i^* - J(x_i) - \lambda y_i^*\|\\ &\leq \|x_i^* - J(x_i)\| + \lambda\|y_i^*\|\\ &\leq \left[\bar{c}\lambda(r'^2 - \beta)\right]^{\frac{1}{p'}} + \lambda L < \frac{r_2}{4},\end{aligned}$$

where the last inequality follows from our assumptions on λ and r'. Thus, the two inequalities (3.9)–(3.10) become

$$\frac{1}{4}\langle Jx_2 - Jx_1; x_2 - x_1\rangle \geq \frac{\|z_1^*\|}{r_2}\langle Jx_2 - Jx_1; x_2 - x_1\rangle \geq \langle z_1^*; x_2 - x_1\rangle;$$

$$\frac{1}{4}\langle Jx_1 - Jx_2; x_1 - x_2\rangle \geq \frac{\|z_2^*\|}{r_2}\langle Jx_1 - Jx_2; x_1 - x_2\rangle \geq \langle z_2^*; x_1 - x_2\rangle.$$

Adding these two inequalities gives

$$\frac{1}{2}\langle Jx_2 - Jx_1; x_2 - x_1\rangle \quad \geq \quad \langle z_1^* - z_2^*; x_2 - x_1\rangle$$

$$= \langle (x_1^* - J(x_1) - \lambda y_1^*) - (x_2^* - J(x_2) - \lambda y_2^*); x_2 - x_1\rangle$$

$$= \langle x_1^* - x_2^*; x_2 - x_1\rangle + \langle J(x_2) - J(x_1); x_2 - x_1\rangle$$

$$+ \lambda\langle y_2^* - y_1^*; x_2 - x_1\rangle.$$

Hence,

$$\frac{1}{2}\langle Jx_2 - Jx_1; x_2 - x_1\rangle \le \langle x_2^* - x_1^*; x_2 - x_1\rangle - \lambda\langle y_2^* - y_1^*; x_2 - x_1\rangle.$$

On the other hand, we have by the r_1-hypomonotony of f uniformly over S proved in Proposition 3.1, we have

$$-\langle y_2^* - y_1^*; x_2 - x_1\rangle \le \frac{1}{r_1}\langle J(x_2) - J(x_1); x_2 - x_1\rangle.$$

Therefore,

$$\frac{1}{2}\langle Jx_2 - Jx_1; x_2 - x_1\rangle \le \langle x_2^* - x_1^*; x_2 - x_1\rangle + \frac{\lambda}{r_1}\langle J(x_2) - J(x_1); x_2 - x_1\rangle,$$

which ensures that

$$(\frac{1}{2} - \frac{\lambda}{r_1})\langle Jx_2 - Jx_1; x_2 - x_1\rangle \le \langle x_2^* - x_1^*; x_2 - x_1\rangle \le \|x_2 - x_1\|\|x_1^* - x_2^*\|.$$

Using the assumption that X is q-uniformly convex, Part (1) in Proposition 1.1, and the fact that $\|x_i\| \le M$, we have for some positive constant $K_M > 0$

$$\langle Jx_2 - Jx_1; x_2 - x_1\rangle \ge K_M\|x_2 - x_1\|^q,$$

and so,

$$\|x_2 - x_1\|\|x_1^* - x_2^*\| \ge \frac{K_M(r_1 - 2\lambda)}{2r_1}\|x_2 - x_1\|^q.$$

Thus,

$$\|x_2 - x_1\| \le \gamma\|x_2^* - x_1^*\|^{\frac{1}{q-1}},$$

with $\gamma := \left(\frac{2r_1}{K_M(r_1-2\lambda)}\right)^{\frac{1}{q-1}} > 0$. This completes the proof of the first step.

Step 2. We are going to prove that $U_{S,f}^{V,\lambda}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*) \subset \mathrm{dom}\,\pi_S^{f,\lambda}$ with α and r' as in Step 1. Let $z^* \in U_{S,f}^{V,\lambda}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*)$, and choose $\delta > 0$ so that $z^* + \delta\mathbb{B}_* \subset U_{S,f}^{V,\lambda}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*)$. Let $\eta \in (0, \frac{\delta}{2})$ and fix any $x^* \in z^* + \eta\mathbb{B}_*$ and any $k \ge 1$. By the density theorem stated in Theorem 3.2, we can choose, for any $n \ge k$, some $x_n^* \in x^* + \frac{1}{n}\mathbb{B}_*$ and $y_n \in \pi_S^{f,\lambda}(x_n^*)$. For n sufficiently large, we have $\frac{1}{n} < \frac{\delta}{2}$, and hence, we obtain $x_n^* \in z^* + (\eta + \frac{1}{n})\mathbb{B}_* \subset z^* + \delta\mathbb{B}_* \subset U_{S,f}^{V,\lambda}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*)$. Clearly, $(x_n^*)_n$ is a sequence in $U_{S,f}^{V,\lambda}(r') \cap \alpha\,\mathrm{int}(\mathbb{B}_*) \cap dom\,\pi_S^{f,\lambda}$. Then by Step 1, we can write for any $n, m \ge k$

$$\|y_n - y_m\| \le \gamma\|x_n^* - x_m^*\|^{\frac{1}{q-1}}.$$

Since the sequence $(x_n^*)_n$ is convergent to x^*, then the sequence $(y_n)_n$ is a Cauchy sequence in X, and hence, it converges to some limit $\bar{y} \in S$. By construction, we have $y_n \in \pi_S^{f,\lambda}(x_n^*)$, that is,

$$f(y_n) + \frac{1}{2\lambda} V(x^*, y_n) \le f(s) + \frac{1}{2\lambda} V(x_n^*, s), \quad \forall s \in S.$$

Using the continuity of V and the Lipschitz continuity of f over S, and the convergence of x_n^* to x^* and y_n to $\bar{y}$, we obtain:

$$f(\bar{y}) + \frac{1}{2\lambda} V(x^*, \bar{y}) \le f(s) + \frac{1}{2\lambda} V(x^*, s), \quad \forall s \in S,$$

which means by definition that $\bar{y} \in \pi_S^{f,\lambda}(x^*)$, that is, $x^* \in dom\, \pi_S^{f,\lambda}$, and hence, $z^* + \eta \mathbb{B}_* \subset dom\, \pi_S^{f,\lambda}$. This ensures that $U_{S,f}^{V,\lambda}(r') \cap \alpha\, \mathrm{int}(\mathbb{B}_*) \subset dom \pi_S^{f,\lambda}$, that is, $U_{S,f}^{V,\lambda}(r') \cap \alpha\, \mathrm{int}(\mathbb{B}_*) \cap dom\, \pi_S^{f,\lambda} = U_{S,f}^{V,\lambda}(r') \cap \alpha\, \mathrm{int}(\mathbb{B}_*)$. This equality with Step 1 completes the proof of the Hölder continuity of $\pi_S^{f,\lambda}$ over $U_{S,f}^{V,\lambda}(r') \cap \alpha\, \mathrm{int}(\mathbb{B}_*)$.

We end the proof of the theorem by proving the single-valuedness of $\pi_S^{f,\lambda}$ on $U_{S,f}^{V,\lambda}(r') \cap \alpha\, \mathrm{int}(\mathbb{B}_*)$. Let $x^* \in U_{S,f}^{V,\lambda}(r') \cap \alpha\, \mathrm{int}(\mathbb{B}_*)$ with two generalized (f, λ)-projections $x_1, x_2 \in \pi_S^{f,\lambda}(x^*)$. Then the inequality (3.8) gives $\|x_1 - x_2\| \le \gamma \|x^* - x^*\|^{\frac{1}{q-1}} = 0$, and hence $x_1 = x_2$. This completes the proof. □

First, we derive straight-forwardly the following particular case when $f = 0$ proved in Theorem 4.4 in [6]. In this case, we have $\beta = 0$, $L = 0$, $r_1 = +\infty$, $\arg\min_S(f) = S$, and we take $\lambda = \frac{1}{2}$.

Theorem 3.4. *Let X be a q-uniformly convex and p-uniformly smooth Banach space. Assume that S is generalized uniformly V-prox-regular with constant $r_2 > 0$. Then, there exists $\alpha_0 \ge 0$ such that for any $\alpha > \max\{\alpha_0, r_2\}$ and any $r' \in \left(0, \min\left\{r_2, 4\alpha \sqrt{2c\left(\frac{r_2}{64\alpha}\right)^{\frac{p}{p-1}}}\right\}\right)$, we have that the generalized projection π_S is single-valued and Hölder continuous with coefficient $\frac{1}{q-1}$ on $U_S^V(r') \cap \alpha\, int(\mathbb{B}_*)$, i.e., for some $\gamma > 0$, we have*

$$\|\pi_S(x_1^*) - \pi_S(x_2^*)\| \le \gamma \|x_1^* - x_2^*\|^{\frac{1}{q-1}}, \quad \forall x_1^*, x_2^* \in U_S^V(r') \cap \alpha\, \mathrm{int}(\mathbb{B}_*). \tag{3.11}$$

The convex case when f is a convex function and S is a closed convex set in $\mathrm{dom}\, f$ is deduced from Theorem 3.3 as follows: First, we notice that any convex function is V-prox-regular with $r_1 = +\infty$ uniformly over any closed subset in its domain and any closed convex set is generalized uniformly V-prox-regular with $r_2 = +\infty$.

Theorem 3.5. *Let X be a q-uniformly convex and p-uniformly smooth Banach space and let $\lambda > 0$. Assume that S is a closed convex set and that f is convex L-Lipschitz over S. Assume that f is bounded from below by $\beta \in \mathbb{R}$. Then for any $\alpha \ge 0$, the generalized (f, λ)-projection $\pi_S^{f,\lambda}$ is single-valued and Hölder continuous with coefficient $\frac{1}{q-1}$ on $\alpha\, int(\mathbb{B}_*)$, i.e., for some $\gamma > 0$, we have*

$$\|\pi_S^{f,\lambda}(x_1^*) - \pi_S^{f,\lambda}(x_2^*)\| \le \gamma \|x_1^* - x_2^*\|^{\frac{1}{q-1}}, \quad \forall x_1^*, x_2^* \in \alpha\, \mathrm{int}(\mathbb{B}_*). \tag{3.12}$$

We have to mention that even in the convex case, the best result obtained on the (f, λ)-projection has been proved in Theorem 3.4 in [17] in which the authors proved the continuity (not the Hölder continuity) under the positive homogenous assumption on the function f and the compactness assumption on S. These two very strong assumptions are not needed in our proof.

Now, we are going to study the local property of the generalized (f,λ)-projection, that is, for a given $\epsilon > 0$, a closed subset S, and a given point $\bar{x} \in \arg\min_S f$, we are interested in the localization of the generalized (f,λ)-projection of the elements x^* in the ϵ-neighborhood of $J(\bar{x})$. First, we prove the following technical result.

Proposition 3.2. *Let $M > 0$ such that $\arg\min_S f \cap M\mathbb{B} \neq \emptyset$. Then for any $x^* \in M\mathbb{B}_*$, we have*

$$e^V_{\lambda,S} f(x^*) = e^V_{\lambda,S\cap 3M\mathbb{B}} f(x^*) \quad \text{and} \quad \pi_S^{f,\lambda}(x^*) = \pi^{f,\lambda}_{S\cap 3M\mathbb{B}}(x^*).$$

Proof. Let $x_0 \in \arg\min_S f \cap M\mathbb{B} \neq \emptyset$. Then, $x_0 \in S \cap M\mathbb{B}$ with

$$f(x_0) = \inf_{x\in S} f(x) \leq \inf_{x\in A} f(x), \qquad \text{for any } A \subset S.$$

Hence,

$$f(x_0) \leq \inf_{x\in S\setminus 3M\mathbb{B}} f(x). \tag{3.13}$$

Fix now any $y \in S$ with $\|y\| > 3M$. Then, we have

$$\begin{aligned} f(y) + \frac{1}{2\lambda} V(x^*; y) &\geq f(y) + \frac{1}{2\lambda} (\|y\| - \|x^*\|)^2 \\ &\geq f(y) + \frac{1}{2\lambda}(3M - M)^2 = f(y) + \frac{2M^2}{\lambda}. \end{aligned}$$

Taking the infimum over all $y \in S \setminus 3M\mathbb{B}$ and using the inequality (3.13), we obtain:

$$\begin{aligned} e^V_{\lambda,S\setminus 3M\mathbb{B}} f(x^*) &= \inf_{y\in S\setminus 3M\mathbb{B}} \left\{ f(y) + \frac{1}{2\lambda} V(x^*; y) \right\} \\ &\geq \inf_{y\in S\setminus 3M\mathbb{B}} f(y) + \frac{2M^2}{\lambda} \geq f(x_0) + \frac{2M^2}{\lambda}. \end{aligned} \tag{3.14}$$

On the other hand, we have

$$\begin{aligned} e^V_{\lambda,S\cap M\mathbb{B}} f(x^*) &= \inf_{y\in S\cap M\mathbb{B}} \left\{ f(y) + \frac{1}{2\lambda} V(x^*; y) \right\} \\ &\leq f(x_0) + \frac{1}{2\lambda} V(x^*; x_0) \\ &\leq f(x_0) + \frac{1}{2\lambda} (\|x^*\| + \|x_0\|)^2 \\ &\leq f(x_0) + \frac{1}{2\lambda}(M + M)^2 \\ &= f(x_0) + \frac{2M^2}{\lambda}. \end{aligned} \tag{3.15}$$

Combining this inequality with (3.14), we get

$$e^V_{\lambda,S\cap M\mathbb{B}}f(x^*) \leq f(x_0) + \frac{2M^2}{\lambda} \leq e^V_{\lambda,S\setminus 3M\mathbb{B}}f(x^*).$$

Therefore,

$$\begin{aligned} e^V_{\lambda,S}f(x^*) &= \inf\left\{e^V_{\lambda,S\cap 3M\mathbb{B}}f(x^*); e^V_{\lambda,S\setminus 3M\mathbb{B}}f(x^*)\right\} \\ &\geq \inf\left\{e^V_{\lambda,S\cap 3M\mathbb{B}}f(x^*); e^V_{\lambda,S\cap M\mathbb{B}}f(x^*)\right\} \\ &\geq e^V_{\lambda,S\cap 3M\mathbb{B}}f(x^*) \\ &\geq e^V_{\lambda,S}f(x^*). \end{aligned}$$

This completes the proof. □

We deduce the following proposition.

Proposition 3.3. *Assume that X is a q-uniformly convex Banach space. Let S be a closed nonempty set in X with $\bar{x} \in \arg\min_S f$ and let $\epsilon > 0$. Let $M := \|\bar{x}\| + \epsilon$, $\epsilon_1 := \frac{c\epsilon^q}{8^{q-1}M^{q-2}}$, and*

$$\mathcal{N}_{\epsilon_1,\frac{\epsilon}{2}}(J\bar{x}) := \{x^* \in X^* : V(x^*,\bar{x}) < \epsilon_1 \text{ and } \|J^*x^* - \bar{x}\| < \frac{\epsilon}{2}\}.$$

Then, for any $x^* \in \mathcal{N}_{\epsilon_1,\frac{\epsilon}{2}}(J\bar{x})$, we have

$$e^V_{\lambda,S}f(x^*) = e^V_{\lambda,S\cap(\bar{x}+\epsilon\mathbb{B})}f(x^*) \quad \text{and} \quad \pi^{f,\lambda}_S(x^*) = \pi^{f,\lambda}_{S\cap(\bar{x}+\epsilon\mathbb{B})}(x^*).$$

Proof. Fix $\epsilon > 0$ and $\bar{x} \in \arg\min_S f$ and let $M := \|\bar{x}\| + \epsilon$ and $\epsilon_1 := \frac{c\epsilon^q}{8^{q-1}M^{q-2}}$, where $c > 0$ is the constant given in the definition of the q-uniform convexity of X.
Set

$$\mathcal{N}_{\epsilon_1,\frac{\epsilon}{2}}(J\bar{x}) := \{x^* \in X^* : V(x^*,\bar{x}) < \epsilon_1 \text{ and } \|J^*x^* - \bar{x}\| < \frac{\epsilon}{2}\}.$$

Then,

$$(\bar{x} + \epsilon\mathbb{B}) \cap S \subset M\mathbb{B} \quad \text{and} \quad \mathcal{N}_{\epsilon_1,\frac{\epsilon}{2}}(J\bar{x}) \subset M\mathbb{B}_*.$$

Using Part (3) in Proposition 1.1, we have for any $x^* \in \mathcal{N}_{\epsilon_1,\frac{\epsilon}{2}}(J\bar{x})$ and any $y \in S \cap M\mathbb{B}$

$$V(x^*; y) \geq \frac{2c}{4^{q-1}M^{q-2}}\|J^*(x^*) - y\|^q. \tag{3.16}$$

Observe that $(\bar{x}+\epsilon\mathbb{B})\cap S\cap 3M\mathbb{B} = (\bar{x}+\epsilon\mathbb{B})\cap S$. Take any $x^* \in \mathcal{N}_{\epsilon_1,\frac{\epsilon}{2}}(J\bar{x})$ and any $y \in [S\cap 3M\mathbb{B}]\setminus(\bar{x}+\epsilon\mathbb{B})$. Then, we have

$$\begin{aligned} f(y) + \frac{1}{2\lambda}V(x^*; y) &\geq f(y) + \frac{1}{2\lambda}\bar{c}\|J^*(x^*) - y\|^q \\ &\geq f(y) + \frac{\bar{c}}{2\lambda}\left(\|y - \bar{x}\| - \|J^*(x^*) - \bar{x}\|\right)^q \end{aligned}$$

$$\begin{aligned} &\geq f(y) + \frac{\bar{c}}{2\lambda}\left(\epsilon - \frac{\epsilon}{2}\right)^q = f(y) + \frac{\epsilon_1}{2\lambda} \\ &\geq f(y) + \frac{1}{2\lambda}V(x^*;\bar{x}). \end{aligned} \tag{3.17}$$

Taking the infimum over all $y \in [S \cap 3M\mathbb{B}] \setminus (\bar{x} + \epsilon\mathbb{B})$, we obtain:

$$\begin{aligned} e^V_{\lambda,[S\cap 3M\mathbb{B}]\setminus(\bar{x}+\epsilon\mathbb{B})} f(x^*) &\geq \inf_{y\in[S\cap 3M\mathbb{B}]\setminus(\bar{x}+\epsilon\mathbb{B})} f(y) + \frac{1}{2\lambda}V(x^*;\bar{x}) \\ &\geq \inf_{y\in S} f(y) + \frac{1}{2\lambda}V(x^*;\bar{x}) \\ &\geq f(\bar{x}) + \frac{1}{2\lambda}V(x^*;\bar{x}). \end{aligned} \tag{3.18}$$

Hence,

$$\begin{aligned} e^V_{\lambda,S\cap 3M\mathbb{B}} f(x^*) &= \inf\left\{e^V_{\lambda,[S\cap 3M\mathbb{B}]\cap(\bar{x}+\epsilon\mathbb{B})} f(x^*); e^V_{\lambda,[S\cap 3M\mathbb{B}]\setminus(\bar{x}+\epsilon\mathbb{B})} f(x^*)\right\} \\ &\geq \inf\left\{e^V_{\lambda,S\cap(\bar{x}+\epsilon\mathbb{B})} f(x^*); f(\bar{x}) + \frac{1}{2\lambda}V(x^*;\bar{x})\right\} \\ &\geq e^V_{\lambda,S\cap(\bar{x}+\epsilon\mathbb{B})} f(x^*) \\ &\geq e^V_{\lambda,S} f(x^*). \end{aligned}$$

On the other side, since $\bar{x} \in \arg\min_S f$ and $\bar{x}\| \leq M$, we have $\arg\min_S f \cap M\mathbb{B} \neq \emptyset$. Consequently, we obtain by Proposition 3.2, $e^V_{\lambda,S\cap 3M\mathbb{B}} f(x^*) = e^V_{\lambda,S} f(x^*)$, which ensures the equality

$$e^V_{\lambda,S} f(x^*) = e^V_{\lambda,S\cap(\bar{x}+\epsilon\mathbb{B})} f(x^*), \quad x^* \in \mathcal{N}_{\epsilon_1,\frac{\epsilon}{2}}(J\bar{x}).$$

Thus, the proof is achieved. □

Observe that $\mathcal{N}_{\epsilon_1,\frac{\epsilon}{2}}(J\bar{x})$ is an open neighborhood of $J(\bar{x})$ in X^*. So, for any $\epsilon > 0$, we can find some constant $\delta > 0$ such that $J(\bar{x}) + \delta\mathbb{B} \subset \mathcal{N}_{\epsilon_1,\frac{\epsilon}{2}}(J\bar{x})$. Therefore, we can state the following localization theorem.

Theorem 3.6. *Assume that X is a q-uniformly convex Banach space. Let S be a closed nonempty set in X with $\bar{x} \in \arg\min_S f \cap M\mathbb{B}$. Then for any $\epsilon > 0$, we can find some constant $\delta > 0$ such that for any $x^* \in J(\bar{x}) + \delta\mathbb{B}$, we have*

$$e^V_{\lambda,S} f(x^*) = e^V_{\lambda,S\cap(\bar{x}+\epsilon\mathbb{B})} f(x^*) \quad \text{and} \quad \pi^{f,\lambda}_S(x^*) = \pi^{f,\lambda}_{S\cap(\bar{x}+\epsilon\mathbb{B})}(x^*).$$

4. Conclusions

In this paper, we introduced and explored an appropriate extension of the well-known Moreau envelope. Taking into account the nice and favorable properties of the functional V in uniformly smooth and uniformly convex Banach spaces, we defined the V-Moreau envelope based on V. Within the framework of reflexive Banach spaces, we established several important properties of the V-Moreau envelope. Furthermore, under the additional assumptions of uniform smoothness and convexity of the space, we demonstrated the Hölder continuity of the generalized (f, λ)-projection. Several key properties of both the V-Moreau envelope and the generalized (f, λ)-projection were also proven.

The convex case in Theorem 3.5 presents a novel result. It is noteworthy that, even in the convex case, the best result regarding the (f, λ)-projection was shown in Theorem 3.4 of [17], where the authors established continuity under two strong conditions: the positive homogeneity of the function f and the compactness of S. In contrast, our proof avoids these restrictive assumptions.

For future research, we are focusing on applying our results on the V-Moreau envelope and the generalized (f, λ)-projection to problems such as nonconvex variational inequalities and nonconvex complementarity problems in Banach spaces. Another potential research direction is extending our results to nonreflexive Banach spaces.

Acknowledgments

The author extends his appreciation to Researchers Supporting Project number (RSPD2024R1001), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The author declares that he has no conflicts of interest.

References

1. Y. Alber, I. Ryazantseva, *Nonlinear Ill-posed problems of monotone type*, Dordrecht: Springer, 2006. http://doi.org/10.1007/1-4020-4396-1
2. Y. Alber, Generalized projection operators in Banach spaces: properties and applications, *Functional Differential Equations*, **1** (1993), 1–21.
3. M. Bounkhel, *Regularity concepts in nonsmooth analysis: theory and applications*, New York: Springer, 2012. http://doi.org/10.1007/978-1-4614-1019-5
4. M. Bounkhel, Generalized projections on closed nonconvex sets in uniformly convex and uniformly smooth Banach spaces, *J. Funct. Space.*, **2015** (2015), 478437. http://doi.org/10.1155/2015/478437
5. M. Bounkhel, Generalized (f, λ)-projection operator on closed nonconvex sets and its applications in reflexive smooth Banach spaces, *AIMS Mathematics*, **8** (2023), 29555–29568. http://doi.org/10.3934/math.20231513
6. M. Bounkhel, M. Bachar, Generalized prox-regularity in reflexive Banach spaces, *J. Math. Anal. Appl.*, **475** (2019), 699–729. http://doi.org/10.1016/j.jmaa.2019.02.064
7. M. Bounkhel, M. Bachar, Primal lower nice functions in smooth Bnach spaces, *Mathematics*, **8** (2020), 2066. https://doi.org/10.3390/math8112066

8. M. Bounkhel, R. Al-Yusof, Proximal Analysis in reflexive smooth Banach spaces, *Nonlinear Anal. Theor.*, **73** (2010), 1921–1939. https://doi.org/10.1016/j.na.2010.04.077
9. M. Bounkhel, L. Tadj, A. Hamdi, Iterative schemes to solve nonconvex variational problems, *Journal of Inequalities in Pure and Applied Mathematics*, **4** (2003), 14.
10. F. H. Clarke, Y. S. Ledyaev, R. J. Stern, R. R. Wolenski, *Nonsmooth analysis and control theory*, New York: Springer, 1998. http://doi.org/10.1007/b97650
11. J. L. Li, The generalized projection operator on reflexive Banach spaces and its applications, *J. Math. Anal. Appl.*, **306** (2005), 55–71. http://doi.org/10.1016/j.jmaa.2004.11.007
12. J. L. Li, On the existence of solutions of variational inequalities in Banach spaces, *J. Math. Anal. Appl.*, **295** (2004), 115–126. http://doi.org/10.1016/j.jmaa.2004.03.010
13. R. A. Poliquin, R. T. Rockafellar, Prox-regular functions in variational analysis, *T. Am. Math. Soc.*, **348** (1996), 1805–1838.
14. R. T. Rockafellar, R. J. B. Wets, *Variational analysis*, Berlin: Springer, 1998. http://doi.org/10.1007/978-3-642-02431-3
15. W. Takahashi, *Nonlinear functional analysis*, Yokohama: Yokohama Publishers, 2000.
16. K. Q. Wu, N. J. Huang, The generalized f-projection operator with an application, *B. Aust. Math. Soc.*, **73** (2006), 307–317. http://doi.org/10.1017/S0004972700038892
17. K. Q. Wu, N. J. Huang, Properties of the generalized f-projection operator and its application in Banach spaces, *Comput. Math. Appl.*, **54** (2007), 399–406. http://doi.org/10.1016/j.camwa.2007.01.029

AIMS Press